Part I

1. Find the equation of the straight line passing through the points $(2,-5)$ and $(5,4)$.

   (a) $y=-3x+1$
   (b) $y=3x-11$
   (c) $y=\frac{1}{3}x-\frac{17}{3}$
   (d) $y=-\frac{1}{3}x-\frac{13}{3}$

   Solution: We apply the slope formula to find the slope determined by the points. If the first point is $(2,-5)$ and the second one is $(5,4)$, then the slope formula gives us

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{4-(-5)}{5-2}=\frac{9}{3}=3$$

   We know that the slope is $m=3$.

$$\begin{aligned} y &= mx+b && \text{slope is 3} \\ y &= 3x+b && \text{line passes through } (5,4) \\ 4 &= 3(5)+b && \text{solve for } b \\ 4 &= 15+b && \text{subtract 15} \\ -11 &= b \end{aligned}$$

   Thus the solution is $y=3x-11$, which is b)

2. Perform the operation and simplify. $\frac{x^2-5x+78}{18x+x^2-208}-\frac{x}{x+26}$

   (a) $\frac{8}{x+26}$
   (b) $\frac{-2x}{x-26}$
   (c) $\frac{x}{x-8}$
   (d) $\frac{3}{x-8}$

   Solution:

$$\begin{aligned} \frac{x^2-5x+78}{18x+x^2-208}-\frac{x}{x+26} &= \frac{x^2-5x+78}{(x+26)(x-8)}-\frac{x}{x+26} \\ &= \frac{x^2-5x+78}{(x+26)(x-8)}-\frac{x\cdot(x-8)}{(x+26)(x-8)} \\ &= \frac{\left(x^2-5x+78\right)-x(x-8)}{(x+26)(x-8)}=\frac{x^2-5x+78-x^2+8x}{(x+26)(x-8)} \\ &= \frac{x^2-5x+78-x^2+8x}{(x+26)(x-8)}=\frac{3x+78}{(x+26)(x-8)} \\ &= \frac{3(x+26)}{(x+26)(x-8)}=\frac{3}{(x-8)} \qquad \text{which is d).} \end{aligned}$$

3. Solve the system of linear equations shown below.

$$\begin{aligned} \frac{1}{2}x - \frac{2}{3}y &= 25 \\ \frac{1}{3}x + \frac{1}{6}y &= 2 \end{aligned}$$

(a) $(18, -24)$
(b) $(-18, -51)$
(c) $(24, -36)$
(d) $(-6, -42)$

Solution: We will multiply the equations by 6 to clear the denominators. The first equation:

$$\begin{aligned} 6\left(\frac{1}{2}x - \frac{2}{3}y\right) &= 6 \cdot 25 \\ 6 \cdot \frac{1}{2}x - 6 \cdot \frac{2}{3}y &= 150 \\ 3x - 4y &= 150 \end{aligned}$$

The second equation:

$$\begin{aligned} 6\left(\frac{1}{3}x + \frac{1}{6}y\right) &= 6 \cdot 2 \\ 6 \cdot \frac{1}{3}x + 6 \cdot \frac{1}{6}y &= 12 \\ 2x + y &= 12 \end{aligned}$$

Thus we will need to solve

$$\begin{aligned} 3x - 4y &= 150 \\ 2x + y &= 12 \end{aligned}$$

We will use elimination. We will multiply the second equation by 4 to eliminate y.

$$\begin{aligned} 3x - 4y &= 150 \\ 8x + 4y &= 48 \end{aligned}$$

We add the two equations now:

$$\begin{aligned} 11x &= 198 \quad \text{divide by 11} \\ x &= 18 \end{aligned}$$

We substitute $x = 18$ into the second equation to obtain the value of y.

$$\begin{aligned} 2(18) + y &= 12 \quad \text{Solve for } y. \\ 36 + y &= 12 \quad \text{subtract 36} \\ y &= -24 \end{aligned}$$

Thus the solution is $(18, -24)$, which is choice a).

4. Simplify $\dfrac{2ax+6ay-bx-3by}{6ax-2ay-3bx+by}$

(a) $\dfrac{x+3y}{3x-y}$
(b) $\dfrac{x-3y}{3x+y}$
(c) $\dfrac{x+y}{x-y}$
(d) $\dfrac{x-y}{x+y}$

Solution: We will factor the numerator and denominator by grouping. The numerator:

$$\begin{aligned} \underbrace{2ax+6ay}\underbrace{-bx-3by} &= \\ 2a(x+3y)-b(x+3y) &= (2a-b)(x+3y) \end{aligned}$$

Now the denominator:

$$\begin{aligned} \underbrace{6ax-2ay}\underbrace{-3bx+by} &= \\ 2a(3x-y)-b(3x-y) &= (2a-b)(3x-y) \end{aligned}$$

Now we are ready to simplify the fraction

$$\frac{2ax+6ay-bx-3by}{6ax-2ay-3bx+by}=\frac{(2a-b)(x+3y)}{(2a-b)(3x-y)}=\frac{x+3y}{3x-y}$$

which is choice a).

5. Find the equation of the straight line that passes through the point $(3,3)$ and is perpendicular to the line $2x+y=7$.

(a) $y=-2x+9$
(b) $y=\dfrac{1}{2}x+\dfrac{13}{2}$
(c) $y=-2x-\dfrac{7}{2}$
(d) $y=\dfrac{1}{2}x+\dfrac{3}{2}$

Solution: We will first obtain the slope of the line given. We can do so by solving for y.

$$\begin{aligned} 2x+y &= 7 \quad \text{subtract } 2x \\ y &= -2x+7 \end{aligned}$$

Now we can see that the slope is -2. Since our slope is perpendicular to this one, its slope has to be the negative reciprocal of -2, which is $\dfrac{1}{2}$. Thus we are looking for the equation of the line whose slope is $\frac{1}{2}$ and passes through $(3,3)$. We are looking for the slope-intercept form of the line.

$$\begin{aligned} y &= mx+b & m=\frac{1}{2} \\ y &= \frac{1}{2}x+b & \text{passes through } (3,3) \\ 3 &= \frac{1}{2}(3)+b & \text{solve for } b \\ 3 &= \frac{3}{2}+b & \text{subtract } \frac{3}{2} \qquad 3-\frac{3}{2}=\frac{6}{2}-\frac{3}{2}=\frac{3}{2} \\ \frac{3}{2} &= b \end{aligned}$$

Thus the solution is $y=\frac{1}{2}x+\frac{3}{2}$, which is choice d).

6. Simplify $1-\dfrac{1}{1-\dfrac{1}{x-3}}$.

(a) $\dfrac{1}{x-2}$

(b) $\dfrac{2x-7}{x-4}$

(c) $-\dfrac{1}{x-4}$

(d) $-\dfrac{1}{x+2}$

Solution:

$$\begin{aligned} 1-\frac{1}{1-\frac{1}{x-3}} &= 1-\frac{1}{\frac{1}{1}-\frac{1}{x-3}}=1-\frac{1}{\frac{x-3}{x-3}-\frac{1}{x-3}} \\ &= 1-\frac{1}{\frac{x-3-1}{x-3}}=1-\frac{1}{\frac{x-4}{x-3}}=1-\frac{x-3}{x-4} \\ &= \frac{x-4}{x-4}-\frac{x-3}{x-4}=\frac{(x-4)-(x-3)}{x-4} \\ &= \frac{x-4-x+3}{x-4}=\frac{-1}{x-4}, \text{ which is choice c).} \end{aligned}$$

7. The solution set of the equation $x^3=24x^2+217x$

(a) $\{-7,0,7\}$

(b) $\{-31,31\}$

(c) $\{-7,0,31\}$

(d) there is no solution, the solution set is $\varnothing$.

Solution: Since the equation is of degree 3, our only option is to reduce one side to zero, factor the other side, and apply the zero porperty.

$$\begin{aligned} x^3 &= 24x^2+217x && \text{subtract } 24x^2+217x \\ x^3-24x^2-217x &= 0 && \text{Factor out the GCF} \\ x\left(x^2-24x-217\right) &= 0 && \text{Factor } x^2-24x-217 \end{aligned}$$

We need to find two numbers p and q such that

$$\begin{aligned} pq &= -217 \\ p+q &= -24 \end{aligned}$$

217 can be expressed as a product as $1 \cdot 217$ and $7 \cdot 31$. Since pq is negative, one of the numbers, p or q has to be positive and the other has to be negative. The sum $p+q$ is negative, that tells us that the larger nunber has to carry the negative sign. Thus we only need to consider 1 with -217 and 7 with -31. Clearly 7 and -31 will work. We proceed to factor by grouping.

$$\begin{aligned} x^2-24x-217 &= && \text{take apart the middle term, using } p \text{ and } q \\ \underbrace{x^2+7x}\underbrace{-31x-217} &= \\ x(x+7)-31(x+7) &= (x-31)(x+7) \end{aligned}$$

Thus our equation is

$$x(x-31)(x+7)=0$$

By the special zero property, either $x=0$, or $x-31=0$, or $x+7=0$. Thus there are three solutions, $x=0$, $x=-7$, and $x=31$, which is choice c).

8. Perform the operation and simplify. $\dfrac{(2p)^3-27}{x^2-49} \div \dfrac{6p+4p^2+9}{14p-3x+2px-21}$

(a) $\dfrac{1}{x+7}$

(b) $\dfrac{(2p-3)^2}{x-7}$

(c) $2p-3$

(d) $x-7$

Solution: We proceed to factor everything we can. The first numerator factors via the difference of cubes theorem.

$$\begin{aligned} (2p)^3-27 &= (2p)^3-3^3 \\ &= ((2p)-3)\left((2p)^2+(2p)(3)+(3)^2\right) \\ &= (2p-3)\left(4p^2+6p+9\right) \end{aligned}$$

The first denominator factors via the difference of squares theorem.

$$x^2-49=(x+7)(x-7)$$

The second numerator does not factor but we can already see that it will be cancelled out. The second numerator can be factored by grouping.

$$\begin{aligned} 14p - 3x + 2px - 21 &= \\ \underbrace{14p + 2px}\ \underbrace{-3x - 21} &= \\ 2p(7+x) - 3(x+7) &= \quad \text{addition is commutative} \\ 2p(x+7) - 3(x+7) &= (2p-3)(x+7) \end{aligned}$$

We are now ready to perform the division and simplify. To divide is to multiply by the reciprocal.

$$\begin{aligned} \frac{(2p)^3 - 27}{x^2 - 49} \div \frac{6p + 4p^2 + 9}{14p - 3x + 2px - 21} &= \frac{(2p)^3 - 27}{x^2 - 49} \cdot \frac{14p - 3x + 2px - 21}{4p^2 + 6p + 9} \\ &= \frac{(2p-3)\left(4p^2 + 6p + 9\right)}{(x+7)(x-7)} \cdot \frac{(2p-3)(x+7)}{4p^2 + 6p + 9} \\ &= \frac{(2p-3)\left(4p^2 + 6p + 9\right)(2p-3)(x+7)}{(x+7)(x-7)\left(4p^2 + 6p + 9\right)} \\ &= \frac{(2p-3)^2}{x-7}, \text{ which is choice b)} \end{aligned}$$

Part II

1. Simplify each of the following expressions.

(a) $\dfrac{5 - \dfrac{1}{a}}{\dfrac{1}{a^2} - 25} =$

Solution: Start with the subtractions. We need to work with common denominators. The numerator:

$$5 - \frac{1}{a} = \frac{5}{1} - \frac{1}{a} = \frac{5a}{a} - \frac{1}{a} = \frac{5a - 1}{a}$$

The denominator:

$$\frac{1}{a^2} - 25 = \frac{1}{a^2} - \frac{25}{1} = \frac{1}{a^2} - \frac{25a^2}{a^2} = \frac{1 - 25a^2}{a^2}$$

Notice that the numerator here factors via the difference of squares theorem.

$$\frac{1 - 25a^2}{a^2} = \frac{1^2 - (5a)^2}{a^2} = \frac{(1-5a)(1+5a)}{a^2}$$

Now we are ready to perform the division. To divide is to multiply by the reciprocal.

$$\frac{5-\dfrac{1}{a}}{\dfrac{1}{a^2}-25} = \frac{\dfrac{5a-1}{a}}{\dfrac{(1-5a)(1+5a)}{a^2}} = \frac{5a-1}{a}\cdot\frac{a^2}{(1-5a)(1+5a)} \quad \text{cancel out } a = \frac{a(5a-1)}{(1-5a)(1+5a)}$$

There is one more cancellation: $1-5a$ and $5a-1$ are opposites. We re-write $5a-1$ as $-(1-5a)$. Then

$$\frac{a(5a-1)}{(1-5a)(1+5a)} = \frac{-(1-5a)(a)}{(1-5a)(1+5a)} = \frac{-a}{1+5a} \quad \text{or} \quad -\frac{a}{5a+1}$$

(b) $\dfrac{30ax^2-2ax-4a}{4b-22bx+30bx^2}=$

Solution: We will factor the numeratior and the denominator and then simplify. The factor out the GCF from the numerator first

$$30ax^2-2ax-4a = 2a\left(15x^2-x-2\right)$$

Now we play the 'pq'-game. We are looking for two numbers, p and q such that

$$\begin{aligned} pq &= -30 \\ p+q &= -1 \end{aligned}$$

The different ways to write 30 as a product is $1\cdot 30$, $2\cdot 15$, and $5\cdot 6$. Since the product has to be negative, one of the numbers has to be negative. Since the sum $p+q$ has to be negative, we know that the smaller number in the pair has to carry the negative sign. Thus we need to consider -1 with 30, -2 with 15, and -5 with 6. Clearly, only -5 with 6 works. Now we re-write the linear term and factor by grouping.

$$\begin{aligned} 15x^2-x-2 &= \\ \underbrace{15x^2-6x}\ \underbrace{+5x-2} &= \\ 3x(5x-2)+(5x-2) &= (3x+1)(5x-2) \end{aligned}$$

Thus the numerator is $2a(3x+1)(5x-2)$. Now for the denominator:

$$\begin{aligned} 4b-22bx+30bx^2 &= \\ 2b\left(2-11x+15x^2\right) &= \quad \text{rearrange terms} \\ &= 2b\left(15x^2-11x+2\right) \end{aligned}$$

Now for the pq-game: we need two numbers, p and q such that

$$\begin{aligned} pq &= 30 \\ p+q &= -11 \end{aligned}$$

The different ways to write 30 as a product is $1 \cdot 30$, $2 \cdot 15$, and $5 \cdot 6$. Since the product has to be positive, either both of the numbers have to be positive, or both have to be negative. Since the sum $p+q$ has to be negative, we know that both numbers are negative. Thus we need to consider -1 with -30, -2 with -15, and -5 with -6. Clearly, only -5 with -6 works. Now we re-write the linear term and factor by grouping.

$$\begin{aligned} 15x^2 - 11x + 2 &= \\ \underbrace{15x^2 - 6x}\underbrace{-5x + 2} &= \\ 3x(5x-2) - (5x-2) &= (3x-1)(5x-2) \end{aligned}$$

Thus the denominator is $2b(3x-1)(5x-2)$. We now are ready to simplify

$$\frac{30ax^2 - 2ax - 4a}{4b - 22bx + 30bx^2} = \frac{2a(3x+1)(5x-2)}{2b(3x-1)(5x-2)} = \frac{a(3x+1)}{b(3x-1)}$$

(c) $\dfrac{a^3x - a^3y + 125b^3x - 125b^3y}{a^2 - 5ab + 25b^2} \cdot \dfrac{3x+3}{a + 5b + ax + 5bx} =$

Solution: We will factor everything we can and then cancel. We factor the first numerator by grouping.

$$\begin{aligned} \underbrace{a^3x - a^3y}\underbrace{+125b^3x - 125b^3y} &= \\ a^3(x-y) + 125b^3(x-y) &= \qquad \text{Now } x-y \text{ is the GCF} \\ \left(a^3 + 125b^3\right)(x-y) &= \end{aligned}$$

$a^3 + 125b^3$ factors via the sum of cubes theorem

$$\begin{aligned} \left(a^3 + 125b^3\right)(x-y) &= \\ \left(a^3 + (5b)^3\right)(x-y) &= \\ (a + (5b))\left(a^2 - a(5b) + (5b)^2\right)(x-y) &= \\ &= (a+5b)\left(a^2 - 5ab + 25b^2\right)(x-y) \end{aligned}$$

The first denominator does not factor, but it will cancel out. The second numerator is clearly $3(x+1)$. We factor the second denominator by grouping.

$$\begin{aligned} a + 5b + ax + 5bx &= \\ \underbrace{a + ax}\underbrace{+5bx + 5b} &= \\ a(1+x) + 5b(x+1) &= (5b+a)(x+1) \end{aligned}$$

Now we are ready to perform the operations indicated

$$\begin{aligned} \frac{a^3x - a^3y + 125b^3x - 125b^3y}{a^2 - 5ab + 25b^2} \cdot \frac{3x+3}{a + 5b + ax + 5bx} &= \\ \frac{(a+5b)\left(a^2 - 5ab + 25b^2\right)(x-y)}{a^2 - 5ab + 25b^2} \cdot \frac{3(x+1)}{(5b+a)(x+1)} &= \qquad \text{cancel out } a^2 - 5ab + 25b^2 \\ \frac{3(5a+b)(x-y)(x+1)}{(5b+a)(x+1)} &= \frac{3(x-y)}{1} = 3(x-y) \end{aligned}$$

2. Completely factor each of the following.

(a) $ax^4 - 9ay^2 + 18by^2 - 2bx^4 =$
Solution: Since there is no GCF, we start by grouping.

$$\begin{aligned} ax^4 - 9ay^2 + 18by^2 - 2bx^4 &= \\ \underbrace{ax^4 - 9ay^2}\underbrace{-2bx^4 + 18by^2} &= \\ a\left(x^4 - 9y^2\right) - 2b\left(x^4 - 9y^2\right) &= (a-2b)\left(x^4 - 9y^2\right) \end{aligned}$$

We are not done yet since $x^4 - 9y^2$ factors via the difference of squares theorem.

$$\begin{aligned} (a-2b)\left(x^4 - 9y^2\right) &= (a-2b)\left(\left(x^2\right)^2 - (3y)^2\right) \\ &= (a-2b)\left(x^2 + 3y\right)\left(x^2 - 3y\right) \end{aligned}$$

(b) $5p^4t^2 - 5q^4t^2 =$
Solution: We start with the GCF.

$$5p^4t^2 - 5q^4t^2 = 5t^2\left(p^4 - q^4\right)$$

The second factor factors via the difference of squares theorem.

$$\begin{aligned} 5t^2\left(p^4 - q^4\right) &= 5t^2\left(\left(p^2\right)^2 - \left(q^2\right)^2\right) \\ &= 5t^2\left(p^2 + q^2\right)\left(p^2 - q^2\right) \end{aligned}$$

The last factor further factors via the difference of squares theorem. Thus the answer is $5t^2\left(p^2 + q^2\right)(p+q)(p-q)$

(c) $3a^3m - a^3n + 3b^3m - b^3n =$
Solution: since there is no GCF, we start by factoring by grouping.

$$\begin{aligned} \underbrace{3a^3m - a^3n}\underbrace{+3b^3m - b^3n} &= \\ a^3(3m-n) + b^3(3m-n) &= \left(a^3 + b^3\right)(3m-n) \end{aligned}$$

The first factor factors via the sum of cubes theorem. Thus the answer is $(a+b)\left(a^2 - ab + b^2\right)(3m-n)$

(d) $15a^2cd - 33abcd + 6b^2cd =$
Solution: We start with the GCF.

$$15a^2cd - 33abcd + 6b^2cd = 3cd\left(5a^2 - 11ab + 2b^2\right)$$

Now we need the pq game. We are looking for two numbers, p and q, such that

$$\begin{aligned} pq &= 10 \\ p+q &= -11 \end{aligned}$$

There is only two ways of writing 10 as a product of two numbers, $1 \cdot 10$ and $2 \cdot 5$. Since the product is positive, the numbers p and q have to be either both positive, or both negative. The sum $p+q$ tells us that they both have to be negative. Thus we only need to consider -1 with -10 and -2 with -5. Clearly -10 with -1 works. Now we proceed to factor by grouping.

$$\begin{aligned} 3cd\left(5a^2-11ab+2b^2\right) &= \\ 3cd\left(\underbrace{5a^2-10ab}\underbrace{-ab+2b^2}\right) &= \\ 3cd\left(5a\left(a-2b\right)-b\left(a-2b\right)\right) &= 3cd\left(a-2b\right)\left(5a-b\right) \end{aligned}$$

3. Solve each of the following equations. Make sure to check your solution.

(a) $35x^3-65x^2=10x$

Solution: We reduce one side to zero and factor the other side.

$$\begin{aligned} 35x^3-65x^2 &= 10x && \text{subtract } 10x \\ 35x^3-65x^2-10x &= 0 && \text{factor out GCF} \\ 5x\left(7x^2-13x-2\right) &= 0 \end{aligned}$$

To further factor, we need the pq-game:

$$\begin{aligned} pq &= -14 \\ p+q &= -13 \end{aligned}$$

Clearly -14 and 1 will work only. We proceed by grouping.

$$\begin{aligned} 7x^2-13x-2 &= \underbrace{7x^2+x}\underbrace{-14x-2} \\ &= x\left(7x+1\right)-2\left(7x+1\right) \\ &= \left(x-2\right)\left(7x+1\right) \end{aligned}$$

Thus our equation is now

$$5x\left(x-2\right)\left(7x+1\right)=0$$

We now apply the special zero property.

$$\begin{aligned} 5x &= 0 \quad \text{or} \quad x-2=0 \quad \text{or} \quad 7x+1=0 \\ x &= 0 \quad \text{or} \quad x=2 \quad \text{or} \quad 7x=-1 \\ x &= 0 \quad \text{or} \quad x=2 \quad \text{or} \quad x=-\frac{1}{7} \end{aligned}$$

We check: If $x=0$, then

$$\begin{aligned} \text{LHS} &= 35\left(0\right)^3-65\left(0\right)^2=0 \\ \text{RHS} &= 10\left(0\right)=0 \end{aligned}$$

If $x=2$, then

$$\begin{aligned} \text{LHS} &= 35\left(2\right)^3-65\left(2\right)^2=35\left(8\right)-65\left(4\right)=280-260=20 \\ \text{RHS} &= 10\left(2\right)=20 \end{aligned}$$

If $x=-\frac{1}{7}$, then

$$\begin{aligned}\text{LHS} &= 35\left(-\frac{1}{7}\right)^3-65\left(-\frac{1}{7}\right)^2=35\left(-\frac{1}{343}\right)-65\left(\frac{1}{49}\right)=-\frac{5}{49}-\frac{65}{49}=\frac{-70}{49}=-\frac{10}{7}\\ \text{RHS} &= 10\left(-\frac{1}{7}\right)=-\frac{10}{7}\end{aligned}$$

Thus all three solutions, $2,0,-\frac{1}{7}$ are correct.

(b) $\frac{3-x}{4}-\frac{10-3x}{5}=x+2$

Solution:

$$\begin{aligned}\frac{3-x}{4}-\frac{10-3x}{5} &= x+2 && \text{make everything a fraction}\\ \frac{3-x}{4}-\frac{10-3x}{5} &= \frac{x+2}{1} && \text{common denominator}\\ \frac{5(3-x)}{20}-\frac{4(10-3x)}{20} &= \frac{20(x+2)}{20} && \text{multiply by 20}\\ 5(3-x)-4(10-3x) &= 20(x+2) && \text{distribute}\\ 15-5x-40+12x &= 20x+40 && \text{combine like terms}\\ 7x-25 &= 20x+40 && \text{subtract } 7x\\ -25 &= 13x+40 && \text{subtract } 40\\ -65 &= 13x && \text{divide by 13}\\ -5 &= x\end{aligned}$$

We check:

$$\begin{aligned}\text{LHS} &= \frac{3-(-5)}{4}-\frac{10-3(-5)}{5}=\frac{8}{4}-\frac{25}{5}=2-5=-3\\ \text{RHS} &= -5+2=-3\end{aligned}$$

Thus our solution, -5 is correct.

(c) $|3x+1|-7=1$

Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned}|3x+1|-7 &= 1 && \text{add } 7\\ |3x+1| &= 8\\ 3x+1 &= 8 \quad \text{or} \quad 3x+1=-8 && \text{solve for } x\\ 3x &= 7 \quad \text{or} \quad 3x=-9\\ x &= \frac{7}{3} \quad \text{or} \quad x=-3\end{aligned}$$

We check. If $x=\frac{7}{3}$, then

$$\begin{aligned}\text{LHS} &= \left|3\left(\frac{7}{3}\right)+1\right|-7=|7+1|-7=|8|-7=8-7=1\\ \text{RHS} &= 1\end{aligned}$$

We check. If $x=-3$, then

$$\begin{aligned} \text{LHS} &= |3(-3)+1|-7=|-9+1|-7=|-8|-7=8-7=1 \\ \text{RHS} &= 1 \end{aligned}$$

Thus both solutions, -3 and $\frac{7}{3}$ are correct.

(d) $|3x+1|-1=-11$
Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} |3x+1|-1 &= -11 \quad \text{add } 1 \\ |3x+1| &= -10 \end{aligned}$$

Since absolute values are never negative, there is no solution.

(e) $(x+4)(1-2x)=3x-2(x-3)^2$
Solution: We re-rwite $(x-3)^2$ as $(x-3)(x-3)$

$$\begin{aligned} (x+4)(1-2x) &= 3x-2(x-3)(x-3) && \text{multiply out parentheses} \\ x-2x^2+4-8x &= 3x-2\left(x^2-3x-3x+9\right) && \text{combine like terms} \\ -2x^2-7x+4 &= 3x-2\left(x^2-6x+9\right) && \text{distribute } -2 \\ -2x^2-7x+4 &= 3x-2x^2+12x-18 && \text{combine like terms} \\ -2x^2-7x+4 &= -2x^2+15x-18 && \text{add } 2x^2 \\ -7x+4 &= 15x-18 && \text{add } 7x \\ 4 &= 22x-18 && \text{add } 18 \\ 22 &= 22x && \text{divide by } 22 \\ 1 &= x \end{aligned}$$

We check:

$$\begin{aligned} \text{LHS} &= (1+4)(1-2(1))=5(1-2)=5(-1)=-5 \\ \text{RHS} &= 3(1)-2(1-3)^2=3(1)-2(-2)^2=3(1)-2(4) \\ &= 3-8=-5 \end{aligned}$$

Thus our solution, 1 is correct.

(f) $3(x-5)-5(x-1)=-2x+1$
Solution:

$$\begin{aligned} 3(x-5)-5(x-1) &= -2x+1 && \text{multiply out parentheses} \\ 3x-15-5x+45 &= -2x+1 && \text{combine like terms} \\ -2x+30 &= -2x+1 && \text{add } 2x \\ 30 &= 1 \end{aligned}$$

Since x disappeared from the equation and we are left with an unconditionally false statement, there is no solution for this equation. This type of an equation is called a contradiction.

(g) $\left|\frac{1}{2}x-3\right|-2=-23$
Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} \left|\frac{1}{2}x-3\right|-2 &= -23 \qquad \text{add } 2 \\ \left|\frac{1}{2}x-3\right| &= -21 \end{aligned}$$

Since absolute values are always non-negative (i.e. positive or zero), this condition will not hold for any x. Thus there is no solution for this equation.

(h) $\left|\frac{1}{2}x-3\right|-2=23$
Solution: We first isolate the expression within the absolute value sign.

$$\begin{aligned} \left|\frac{1}{2}x-3\right|-2 &= 23 \qquad \text{add } 2 \\ \left|\frac{1}{2}x-3\right| &= 25 \end{aligned}$$

We translate the one equation involving absolute values into two linear equations and solve them separately.

$$\begin{aligned} \frac{1}{2}x-3 &= -25 \qquad \text{or} \qquad \frac{1}{2}x-3=25 \qquad \text{add } 3 \\ \frac{1}{2}x &= -22 \qquad \text{or} \qquad \frac{1}{2}x=28 \qquad \text{multiply by } 2 \\ x &= -44 \qquad \text{or} \qquad x=56 \end{aligned}$$

The solutions are -44 and 56.
We check: If $x=-44$, then

$$\text{LHS}=\left|\frac{1}{2}(-44)-3\right|-2=|-22-3|-2=|-25|-2=25-2=23=\text{RHS}$$

and if $x=56$, then

$$\text{LHS}=\left|\frac{1}{2}(56)-3\right|-2=|28-3|-2=|25|-2=25-2=23=\text{RHS}$$

The our solutions, -44 and 56 are correct.

4. Solve each of the following inequalities.

(a) $-3<-\frac{1}{2}x+7\leq 5$

Solution: Notice first that $-\frac{1}{2}x$ is the same as $\frac{x}{-2}$. (To divide is to multiply by the reciprocal...)
A computation to show this:

$$-\frac{1}{2}x=\frac{-1}{2}\cdot\frac{x}{1}=\frac{-x}{2}=\frac{-x(-1)}{2(-1)}=\frac{x}{-2}$$

When multiplying by a negative number, we must reverse the inequality sign.

$$\begin{array}{rcll} -3 & < & -\frac{1}{2}x+7 \le 5 & \text{subtract } 7 \\ -10 & < & \frac{x}{-2} \le -2 & \text{multiply by } -2 \\ 20 & > & x \ge 4 & \end{array}$$

The statement $20 > x \ge 4$ is the same as $4 \le x < 20$. Thus the solution, in interval notation, is $[4, 20)$.

(b) $-7 > -2x - 11 \ge -31$
Solution:

$$\begin{array}{rcll} -7 & > & -2x-11 \ge -31 & \text{add } 11 \\ 4 & > & -2x \ge -20 & \text{divide by } -2 \\ -2 & < & x \le 10 & \end{array}$$

The solution, in interval notation is $(-2, 10]$.

5. Graph the straight lines $2x - 3y = -18$ and $2x + y = -2$ in the same coordinate system.

(a) Use your graph to find the coordinates of the point where the lines intersect. $(-3, 4)$
Solution: Let us start with $2x - 3y = -18$. We will first transform this equation to slope-intercemt form. This takes a little work, but will make graphing very easy.

$$\begin{array}{rcll} 2x-3y & = & -18 & \text{add } 3y \\ 2x & = & 3y-18 & \text{add } 18 \\ 2x+18 & = & 3y & \text{divide by } 3 \\ \frac{2x+18}{3} & = & y & \\ y & = & \frac{2x+18}{3} = \frac{2}{3}x + \frac{18}{3} & \\ y & = & \frac{2}{3}x+6 & \end{array}$$

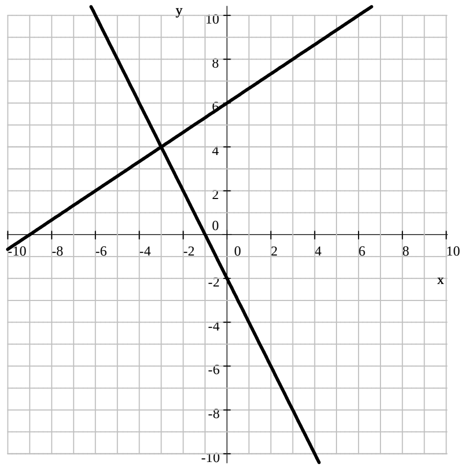
Why is the step $y = \frac{2x+18}{3} = \frac{2}{3}x + \frac{18}{3}$ correct? This is because we distribute in division as well. To divide is to multiply by the reciprocal, and so

$$y = \frac{2x+18}{3} = \frac{1}{3}(2x+18) = \frac{1}{3}(2x) + \frac{1}{3}(18) = \frac{2}{3}x + 6$$

Now grahping the line $y = \frac{2}{3}x + 6$ is easy. The $y-$intercept is $(0, 6)$. The slope is $\frac{2}{3}$, and so from the point $(0, 6)$, we step 3 to the right, 2 up. Then we get to $(3, 8)$. And so on, we can get many points.
Now for the line $2x + y = -2$. We solve for y to obtain the slope-intercept form.

$$\begin{array}{rcll} 2x+y & = & -2 & \text{subtract } 2x \\ y & = & -2x-2 & \end{array}$$

Now it is easy to graph the line. We start with the $y-$intercept, which is $(0, -2)$. From there, we will step 1 to the right, 2 down, since the slope is $-2 = \frac{-2}{1}$.



(b) Use algebraic methods to check your solution.

$2(-3) - 3(4) = -6 - 12 = -18 \implies$ the point $(-3, 4)$ is on the first line

$2(-3) + (4) = -6 + 4 = -2 \implies$ the point $(-3, 4)$ is on the second line

6. Solve the system of linear equations. Make sure to check your solution.

$$\begin{aligned} 3x - 5y &= -31 \\ 2x + 3y &= 11 \end{aligned}$$

Solution: We will eliminate y by multiplying the first equation by 3 and the second by 5.

$$\begin{aligned} 9x - 15y &= -93 \\ 10x + 15y &= 55 \end{aligned}$$

We add the two equations:

$$\begin{aligned} 19x &= -38 \quad \text{divide by 19} \\ x &= -2 \end{aligned}$$

We substitute $x = -2$ into the second equation to obtain the value of y.

$$\begin{aligned} 2(-2) + 3y &= 11 \\ -4 + 3y &= 11 \quad \text{add 4} \\ 3y &= 15 \quad \text{divide by 3} \\ y &= 5 \end{aligned}$$

We check: $x = -2$ and $y = 5$. The first equation:

$$\begin{aligned} \text{LHS} &= 3(-2) - 5(5) = -6 - 25 = -31 \\ \text{RHS} &= -31 \end{aligned}$$

The second equation:

$$\begin{aligned} \text{LHS} &= 2(-2) + 3(5) = -4 + 15 = 11 \\ \text{RHS} &= 11 \end{aligned}$$

Thus the solution is indeed. $(-2, 5)$.