1. Simplify each of the following expressions.

 (a) $\sqrt{125} - 3\sqrt{80} + \sqrt{45} =$
 Solution:

$$\begin{aligned}\sqrt{125} - 3\sqrt{80} + \sqrt{45} &= \sqrt{25\cdot 5} - 3\sqrt{16\cdot 5} + \sqrt{9\cdot 5}\\ &= \sqrt{25}\sqrt{5} - 3\sqrt{16}\sqrt{5} + \sqrt{9}\sqrt{5}\\ &= 5\sqrt{5} - 3\,(4)\,\sqrt{5} + 3\sqrt{5}\\ &= (5 - 12 + 3)\,\sqrt{5} = 4\sqrt{5}\end{aligned}$$

 (b) $\left(\sqrt{7} - 2\right)^2 =$
 Solution:

$$\begin{aligned}\left(\sqrt{7} - 2\right)^2 &= \left(\sqrt{7} - 2\right)\left(\sqrt{7} - 2\right)\\ &= \sqrt{7}\sqrt{7} - 2\sqrt{7} - 2\sqrt{7} + 4\\ &= 7 - 4\sqrt{7} + 4 = 11 - 4\sqrt{7}\end{aligned}$$

 (c) $\left(\sqrt{3} - 1\right)^3 =$
 Solution: We will first work out $\left(\sqrt{3} - 1\right)^2$ and then multiply that by $\left(\sqrt{3} - 1\right)$.

$$\begin{aligned}\left(\sqrt{3} - 1\right)^3 &= \left(\sqrt{3} - 1\right)\left(\sqrt{3} - 1\right)\left(\sqrt{3} - 1\right)\\ &= \left(\sqrt{3}\sqrt{3} - 1\sqrt{3} - 1\sqrt{3} + 1\right)\left(\sqrt{3} - 1\right)\\ &= \left(3 - 2\sqrt{3} + 1\right)\left(\sqrt{3} - 1\right)\\ &= \left(4 - 2\sqrt{3}\right)\left(\sqrt{3} - 1\right)\\ &= 4\sqrt{3} - 4 - 2\sqrt{3}\sqrt{3} + 2\sqrt{3}\\ &= 4\sqrt{3} - 4 - 2\,(3) + 2\sqrt{3}\\ &= 4\sqrt{3} - 4 - 6 + 2\sqrt{3}\\ &= -10 + 6\sqrt{3}\end{aligned}$$

2. Rationalize the denominator in each of the following expressions.

 (a) $\dfrac{3}{\sqrt{5}} =$
 Solution: To rationalize the denominator, we will multiply both the numerator and denominator by $\sqrt{5}$.

$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

(b) $\dfrac{1}{\sqrt{10}-3} =$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{10}+3$.

$$\frac{1}{\sqrt{10}-3} = \frac{1}{\sqrt{10}-3}\cdot\frac{\sqrt{10}+3}{\sqrt{10}+3} = \frac{\sqrt{10}+3}{1} = \sqrt{10}+3$$

The denominator is 1 since

$$\begin{aligned}\left(\sqrt{10}-3\right)\left(\sqrt{10}+3\right) &= \sqrt{10}\sqrt{10}+3\sqrt{10}-3\sqrt{10}-9\\ &= 10-9=1\end{aligned}$$

(c) $\dfrac{2}{\sqrt{7}+1} =$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{7}-1$

$$\begin{aligned}\frac{2}{\sqrt{7}+1} &= \frac{2}{\sqrt{7}+1}\cdot\frac{\sqrt{7}-1}{\sqrt{7}-1} = \frac{2\left(\sqrt{7}-1\right)}{7-1} = \frac{2\left(\sqrt{7}-1\right)}{6}\\ &= \frac{\sqrt{7}-1}{3}\end{aligned}$$

3. Find the exact value of x^2-4x+6 if $x=2-\sqrt{3}$.

Solution: We work out x^2 first.

$$\begin{aligned}x^2 &= \left(2-\sqrt{3}\right)^2 = \left(2-\sqrt{3}\right)\left(2-\sqrt{3}\right)\\ &= 4-2\sqrt{3}-2\sqrt{3}+\sqrt{3}\sqrt{3}\\ &= 4-4\sqrt{3}+3 = 7-4\sqrt{3}\end{aligned}$$

Now we substitute $x=2-\sqrt{3}$ into x^2-4x+6.

$$\begin{aligned}x^2-4x+6 &= \left(2-\sqrt{3}\right)^2-4\left(2-\sqrt{3}\right)+6 =\\ &= 7-4\sqrt{3}-8+4\sqrt{3}+6\\ &= 7-8+6 = 5\end{aligned}$$

4. Factor $13x+2x^2-24$ by completing the square.

Solution: We rearrange the terms first and then factor out the leading coefficient.

$$\begin{aligned}13x+2x^2-24 &= 2x^2+13x-24\\ &= 2\left(x^2+\frac{13}{2}x-12\right)\end{aligned}$$

Half of the linear coefficient is $\frac{13}{4}$, thus we work out $\left(x+\frac{13}{4}\right)^2$ first to see what we need to smuggle in to complete the square.

$$\begin{aligned}\left(x+\frac{13}{4}\right)^2 &= \left(x+\frac{13}{4}\right)\left(x+\frac{13}{4}\right) = x^2+\frac{13}{4}x+\frac{13}{4}x+\frac{169}{16}\\ &= x^2+\frac{13}{2}x+\frac{169}{16}\end{aligned}$$

Thus we need to smuggle in $\frac{169}{16}$

$$\begin{aligned}2x^2+13x-24 &= 2\left(x^2+\frac{13}{2}x-12\right)\\ &= 2\left(\underbrace{x^2+\frac{13}{2}x+\frac{169}{16}}-\frac{169}{16}-12\right)\end{aligned}$$

We bring the last two numbers to the common denominator

$$\begin{aligned}2x^2+13x-24 &= 2\left(\left(x+\frac{13}{4}\right)^2-\frac{169}{16}-\frac{12\,(16)}{1\,(16)}\right)\\ &= 2\left(\left(x+\frac{13}{4}\right)^2-\frac{169}{16}-\frac{192}{16}\right)\\ &= 2\left(\left(x+\frac{13}{4}\right)^2-\frac{361}{16}\right)\end{aligned}$$

Since $\frac{361}{16}=\left(\frac{19}{4}\right)^2$, we factor via the difference of squares theorem.

$$\begin{aligned}2x^2+13x-24 &= 2\left(\left(x+\frac{13}{4}\right)^2-\left(\frac{19}{4}\right)^2\right)\\ &= 2\left(x+\frac{13}{4}+\frac{19}{4}\right)\left(x+\frac{13}{4}-\frac{19}{4}\right)\\ &= 2\left(x+\frac{32}{4}\right)\left(x-\frac{6}{4}\right)\\ &= 2\,(x+8)\left(x-\frac{3}{2}\right)\end{aligned}$$

We may distribute 2 into the second factor. Then we get

$$\begin{aligned}2x^2+13x-24 &= (x+8)\left(2\left(x-\frac{3}{2}\right)\right)\\ &= (x+8)\,(2x-3)\end{aligned}$$

We FOIL to check:

$$\begin{aligned}(x+8)(2x-3) &= 2x^2-3x+16x-24\\ &= 2x^2+13x-24\end{aligned}$$

5. Solve each of the following equations. Make sure to check your solution(s).

(a) $2x^3 = 20x^2 + 1750x$

Solution: We reduce one side to zero, then factor, and then apply the zero property.

$$\begin{aligned}2x^3 &= 20x^2+1750x\\ 2x^3-20x^2-1750x &= 0 && \text{factor out GCF}\\ 2x\left(x^2-10x-875\right) &= 0 && \text{divide both sides by 2}\\ x\left(x^2-10x-875\right) &= 0\end{aligned}$$

We will factor by completing the square. Half of the linear coefficient is -5, and thus we will work with

$$(x-5)^2 = x^2-10x+25$$

We smuggle in 25.

$$\begin{aligned}x\left(x^2-10x-875\right) &= 0\\ x\left(\underbrace{x^2-10x+25}-25-875\right) &= 0\\ x\left((x-5)^2-900\right) &= 0\\ x\left((x-5)^2-30^2\right) &= 0\\ x(x-5+30)(x-5-30) &= 0\\ x(x+25)(x-35) &= 0\end{aligned}$$

Applying the zero property we obtain $35, 0,$ and -25 as the solutions.

(b) $\dfrac{3x+17}{2} = x-1+\dfrac{x+19}{2}$

Solution:

$$\begin{aligned}\frac{3x+17}{2} &= x-1+\frac{x+19}{2} && \text{express everything as a fraction}\\ \frac{3x+17}{2} &= \frac{x-1}{1}+\frac{x+19}{2} && \text{bring everything to the common denominator}\\ \frac{3x+17}{2} &= \frac{2(x-1)}{2}+\frac{x+19}{2} && \text{add fractions on right hand side}\\ \frac{3x+17}{2} &= \frac{2(x-1)+x+19}{2} && \text{multiply out parentheses}\\ \frac{3x+17}{2} &= \frac{2x-2+x+19}{2} && \text{combine like terms}\\ \frac{3x+17}{2} &= \frac{3x+17}{2} && \text{multiply by 2}\\ 3x+17 &= 3x+17\end{aligned}$$

Because the left hand side is now identical to the right hand side, this equation is an identity, and all real numbers are solution.

(c) $|3-2x|+2=5$
Solution:

$$\begin{aligned} |3-2x|+2 &= 5 \quad \text{subtract } 2 \\ |3-2x| &= 3 \\ 3-2x &= 3 \quad \text{or} \quad 3-2x=-3 \quad \text{subtract } 3 \\ -2x &= 0 \quad \text{or} \quad -2x=-6 \quad \text{divide by } -2 \\ x &= 0 \quad \text{or} \quad x=3 \end{aligned}$$

Thus the solution are 0 and 3.

(d) $\frac{2}{3}(x-7)=\frac{4}{5}(x+1)$
Solution:

$$\begin{aligned} \frac{2}{3}(x-7) &= \frac{4}{5}(x+1) \\ \frac{2}{3}\cdot\frac{x-7}{1} &= \frac{4}{5}\cdot\frac{x+1}{1} \\ \frac{2(x-7)}{3} &= \frac{4(x+1)}{5} \quad \text{bring fractions to common denominator} \\ \frac{5\cdot 2(x-7)}{15} &= \frac{3\cdot 4(x+1)}{15} \quad \text{multiply both sides by 15} \\ 10(x-7) &= 12(x+1) \quad \text{multiplty out parentheses} \\ 10x-70 &= 12x+12 \quad \text{subtract } 10x \\ -70 &= 2x+12 \quad \text{subtract } 12 \\ -82 &= 2x \quad \text{divide by } 2 \\ -41 &= x \end{aligned}$$

We check:

$$\begin{aligned} \text{LHS} &= \frac{2}{3}(-41-7)=\frac{2}{3}(-48)=-32 \\ \text{RHS} &= \frac{4}{5}(-41+1)=\frac{4}{5}(-40)=-32 \end{aligned}$$

Thus our solution, -41 is correct.

(e) $7x^2+(x+3)(2x-1)=(3x+1)^2$

Solution:

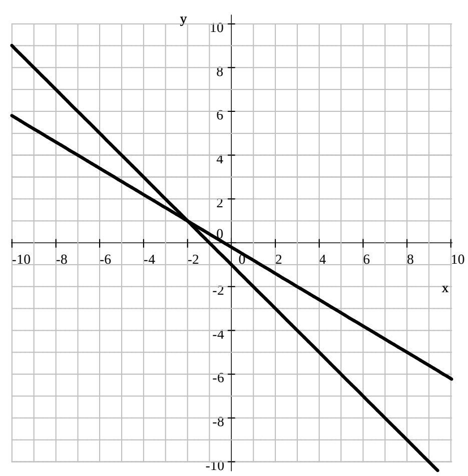
$$\begin{aligned}
7x^2 + (x+3)(2x-1) &= (3x+1)^2 && \text{multiply the polynomials on both sides} \\
7x^2 + 2x^2 + 5x - 3 &= 9x^2 + 6x + 1 && \text{combine like terms} \\
9x^2 + 5x - 3 &= 9x^2 + 6x + 1 && \text{subtract } 9x^2 \\
5x - 3 &= 6x + 1 && \text{subtract } 5x \\
-3 &= x + 1 && \text{subtract } 1 \\
-4 &= x
\end{aligned}$$

We check our result:

$$\begin{aligned}
\text{LHS} &= 7(-4)^2 + ((-4)+3)(2(-4)-1) = 7 \cdot 16 + (-1)(-9) = 112 + 9 = 121 \\
\text{RHS} &= (3(-4)+1)^2 = (-12+1)^2 = (-11)^2 = 121
\end{aligned}$$

Thus the solution, -4 is correct.

6. Graph the straight lines $3x + 5y = -1$ and $y = -x - 1$ in the same coordinate system. Use your graph to find the coordinates of the point where the lines intersect. $(-2, 1)$



7. Graph the parabola $y = -8x + x^2 + 15$. Clearly label the coordinates of five points on the parabola, including vertex and intercepts.

Solution: We obtain all forms of the equation first.

$$y = x^2 - 8x + 15 \implies \text{polynomial form}$$

Half of the linear coefficient is -4, thus we will work with $(x-4)^2 = x^2 - 8x + 16$

$$\begin{aligned}
y &= x^2 - 8x + 15 \\
y &= \underbrace{x^2 - 8x + 16}\, -16 + 15 \\
y &= (x-4)^2 - 1 \implies \text{complete square form}
\end{aligned}$$

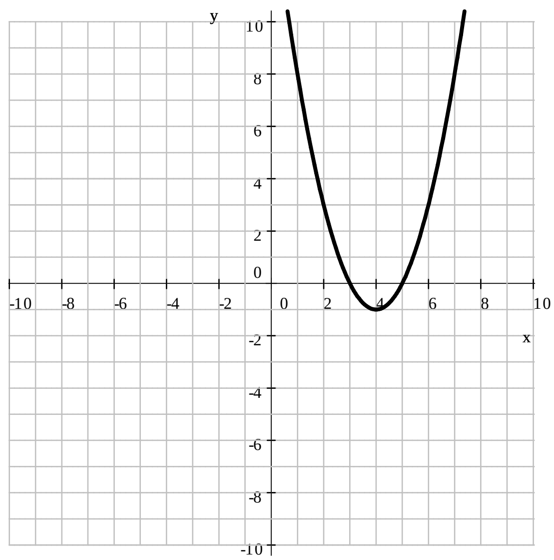
We factor via the difference of squares theorem

$$\begin{aligned} y &= (x-4)^2 - 1^2 \quad \text{since } 1 = 1^2 \\ y &= (x-4+1)(x-4-1) \\ y &= (x-3)(x-5) \implies \text{factored form} \end{aligned}$$

From the polynomial form we obtain the $y-$intecept, $(0, 15)$. From the complete square form, the vertex is $(4, -1)$. Finally, the factored form tells us that there are two $x-$intercepts, $(3, 0)$ and $(5, 0)$. The few missing points, close to the vertex can be found by substituting values for x into any of the three forms of the equations to find y. This time we will work with the polynomial form.

$$\begin{aligned} \text{if } x &= 2, \text{ then } y = (2)^2 - 8(2) + 15 = 4 - 16 + 15 = 3 \\ \text{if } x &= 6, \text{ then } y = (6)^2 - 8(6) + 15 = 36 - 48 + 15 = 3 \end{aligned}$$

We are ready to graph:



8. There is an animal farm where chickens and cows live. All together, there are 53 heads and 174 legs. How many chickens, how many cows?

   Solution: Denote the number of chickens by x and the number of cows by y. The first equation will express the number of heads, the second equation will express the number of heads.

$$\begin{aligned} x + y &= 53 \\ 2x + 4y &= 174 \end{aligned}$$

To eliminate x, we multiply the first equation by -1 and divide the second equation by 2.

$$\begin{aligned} -x - y &= -53 \\ x + 2y &= 87 \end{aligned}$$

Now we add the two equations.

$$y = 34$$

We use the first equation to find x.

$$\begin{aligned} x + 34 &= 53 \quad \text{subtract } 34 \\ x &= 19 \end{aligned}$$

Thus we have 19 chickens and 34 cows. We check: the number of heads is $19 + 34 = 53$, and the number of legs is $2(19) + 4(34) = 38 + 136 = 174$. So our solution is correct.

9. One side of a rectangle is 4 in shorter than 3 times the other side. Find the sides of the rectangle if its perimeter is 48 in.

   Solution: Let us denote the shorter side by x. Then the larger side is $3x - 4$. The equation expresses the perimeter.

$$\begin{aligned} 2x + 2(3x - 4) &= 48 \quad \text{multiply out parentheses} \\ 2x + 6x - 8 &= 48 \quad \text{combine like terms} \\ 8x - 8 &= 48 \quad \text{add } 8 \\ 8x &= 56 \quad \text{divide by } 8 \\ x &= 7 \end{aligned}$$

   Thus the shorter side is 7 in, which makes the longer side $3 \cdot 7 - 4 = 17$ in. Thus the sides are 7 in and 17 in long.

10. One side of a rectangle is 4 in shorter than 3 times the other side. Find the sides of the rectangle if its area is 319 in^2.

   Solution: Let us denote the shorter side by x. Then the larger side is $3x - 4$. The equation expresses the area.

$$\begin{aligned} x(3x - 4) &= 319 \\ 3x^2 - 4x &= 319 \\ 3x^2 - 4x - 319 &= 0 \end{aligned}$$

   We will complete the square.

$$\begin{aligned} 3x^2 - 4x - 319 &= 0 \\ 3\left(x^2 - \frac{4}{3}x - \frac{319}{3}\right) &= 0 \end{aligned}$$

   Half of the linear coefficient is $-\frac{4}{3} \div 2 = -\frac{4}{3} \cdot \frac{1}{2} = -\frac{4}{6} = -\frac{2}{3}$, thus we work out $\left(x - \frac{2}{3}\right)^2$.

$$\begin{aligned} \left(x - \frac{2}{3}\right)^2 &= \left(x - \frac{2}{3}\right)\left(x - \frac{2}{3}\right) = x^2 - \frac{2}{3}x - \frac{2}{3}x + \frac{4}{9} \\ &= x^2 - \frac{4}{3}x + \frac{4}{9} \end{aligned}$$

Thus we smuggle in $\frac{4}{9}$.

$$3\left(\underbrace{x^2-\frac{4}{3}x+\frac{4}{9}}-\frac{4}{9}-\frac{319}{3}\right) = 0$$
$$3\left(\left(x-\frac{2}{3}\right)^2-\frac{4}{9}-\frac{319}{3}\right) = 0$$

We bring the last two numbers to the common denominator:

$$3\left(\left(x-\frac{2}{3}\right)^2-\frac{4}{9}-\frac{319\,(3)}{3\,(3)}\right) = 0$$
$$3\left(\left(x-\frac{2}{3}\right)^2-\frac{4}{9}-\frac{957}{9}\right) = 0$$
$$3\left(\left(x-\frac{2}{3}\right)^2-\frac{961}{9}\right) = 0$$

Since $\frac{961}{9}=\left(\frac{31}{3}\right)^2$, we factor via the difference of squares theorem.

$$3\left(\left(x-\frac{2}{3}\right)^2-\left(\frac{31}{3}\right)^2\right) = 0$$
$$3\left(x-\frac{2}{3}+\frac{31}{3}\right)\left(x-\frac{2}{3}-\frac{31}{3}\right) = 0$$
$$3\left(x+\frac{29}{3}\right)\left(x-\frac{33}{3}\right) = 0$$
$$3\left(x+\frac{29}{3}\right)(x-11) = 0$$

This equation has two solutions, $x_1=-\frac{29}{3}$ and $x_2=11$. Since distances are positive, $-\frac{29}{3}$ is ruled out as a solution for the shorter side. The other solution is 11 in. This makes the longer side $3\cdot 11-4=29$ in. We check: $3\,(11)-4=29$ and $11\,(29)=319$. Thus our solution, 11 in by 29 in is correct.

11. The population of a town has decreased from 80 000 to 68 000. What percent of a decrease does this represent?

Solution 1: We subtract 68 000 from 80 000 to determine the change. $80\,000-68\,000=12\,000$. Now the question is: 12 000 is what percent of 80 000? Then

$$\begin{aligned}(\text{is}) &= 12\,000\\ \text{F} &= x\\ (\text{of}) &= 80\,000\end{aligned}$$

We substitute the data into the formula:

$$\begin{aligned}(\text{is}) &= \text{F}\cdot(\text{of})\\ 12\,000 &= x\cdot 80\,000\\ \frac{12\,000}{80\,000} &= x\\ 0.15 &= x\end{aligned}$$

Thus

$$x = 0.15 = \frac{0.15}{1} = \frac{0.15\,(100)}{1\,(100)} = \frac{15}{100} = 15\%$$

This is a 15% decrease.

Solution 2: The question may be re-phrased as: 68 000 is what percent of 80 000? Then

$$\begin{aligned}(\text{is}) &= 68\,000\\ \text{F} &= x\\ (\text{of}) &= 80\,000\end{aligned}$$

We substitute the data into the formula:

$$\begin{aligned}(\text{is}) &= \text{F}\cdot(\text{of})\\ 68\,000 &= x\cdot 80\,000\\ \frac{68\,000}{80\,000} &= x\\ 0.85 &= x\end{aligned}$$

Thus

$$x = 0.85 = \frac{0.85}{1} = \frac{0.85\,(100)}{1\,(100)} = \frac{85}{100} = 85\%$$

Since the population has decreased to 85% of its previous count, this is a 15% decrease.

12. We have invested \$ 15 000 into two accounts. One account earns 6% interest every year, the other account earns 7% every year. How much did we invest into each account if the combined interest was \$1008 after the first year?

Solution: Let x denote the amount invested at 7%. Then we invested $15\,000 - x$ at 6%. The equation expresses the combined interest.

$$\begin{aligned}0.07x + 0.06\,(15\,000 - x) &= 1008 \qquad \text{multipliy by 100}\\ 7x + 6\,(15\,000 - x) &= 100\,800\\ 7x + 90\,000 - 6x &= 100\,800\\ x &= 10\,800\end{aligned}$$

The other account must have $15\,000 - 10\,800 = 4200$. Thus we invested \$4200 at 6% and \$ 10 800 at 7%. We check:

$$\begin{aligned}10\,800 + 4200 &= 15\,000 \quad \text{and}\\ 0.07\,(10\,800) + 0.06\,(4200) &= 1008\end{aligned}$$

13. The hypotenuse of a right triangle is 68 cm. The difference between the other two sides is 28. cm. Find the sides of the triangle.

Solution: Let x denote the shorter leg. Then the other leg is $x+28$ cm long. We state the Pythagorean theorem for the triangle, and solve the quadratic equation for x.

$$\begin{aligned} x^2+(x+28)^2 &= 68^2 && \text{FOIL out } (x+28)^2 \\ x^2+x^2+56x+784 &= 4624 && \text{combine like terms} \\ 2x^2+56x+784 &= 4624 && \text{subtract 4624} \\ 2x^2+56x-3840 &= 0 && \text{factor out 2} \\ 2\left(x^2+28x-1920\right) &= 0 && \text{divide by 2} \\ x^2+28x-1920 &= 0 \end{aligned}$$

We factor by completing the square. Since half of the linear coefficient is 14, we will work with $(x+14)^2 = x^2+28x+196$

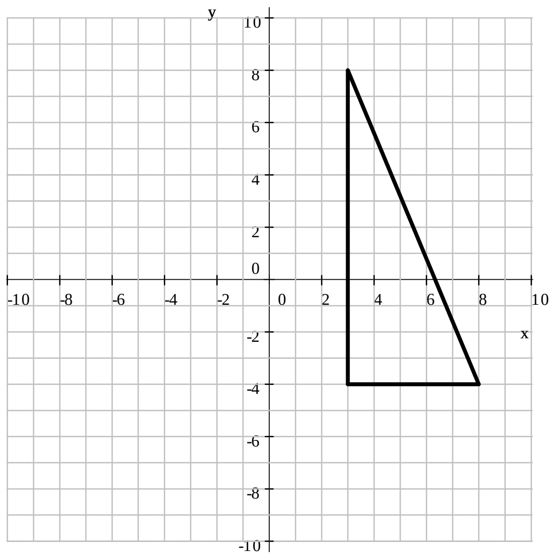
$$\begin{aligned} \underbrace{x^2+28x+196}-196-1920 &= 0 \\ (x+14)^2-2116 &= 0 \\ (x+14)^2-46^2 &= 0 \\ (x+14+46)(x+14-46) &= 0 \\ (x+60)(x-32) &= 0 \\ x_1 &= -60 \quad \text{and} \quad x_2 = 32 \end{aligned}$$

Since distances are never negative, -60 is ruled out. If the shortest side is 32 cm, the other side is $32\,\text{cm}+28\,\text{cm} = 60\,\text{cm}$. Thus the solution is 32 cm and 60 cm. We check:

$$\begin{aligned} 60-32 &= 28 \text{ and} \\ 60^2+32^2 &= 3600+1024 = 4624 = 68^2 \end{aligned}$$

14. Find the distance between $(3,8)$ and $(8,-4)$.

Solution: We graph the points, they determine a right triangle as shown below. The legs are 5 and 12 units long, and we need to find the hypotenuse.



$$
\begin{aligned}
5^2 + 12^2 &= x^2 \\
25 + 144 &= x^2 \\
169 &= x^2 \\
0 &= x^2 - 13^2 \\
0 &= (x+13)(x-13) \\
x_1 &= -13 \quad \text{and} \quad x_2 = 13
\end{aligned}
$$

Since distances are never negative, the answer is 13 units.