1. Simplify (or rationalize) each of the following.

   a) $\dfrac{x^3-9x}{x^2-7x+12}$ b) $\dfrac{x^2-2x-8}{x^3+x^2-2x}$ c) $\dfrac{\frac{1}{x}-\frac{1}{5}}{\frac{1}{x^2}-\frac{1}{25}}$ d) $\dfrac{9-x^{-2}}{3+x^{-1}}$ e) $\dfrac{5+\sqrt{5}}{5-\sqrt{5}}$

   f) $\log_{10}\left(120a^3\right)-\left(\log_{10}3a+2\log_{10}2a\right)$

2. Assume that all variables represent positive numbers. Write each of the following expressions in the form $c\,a^p b^q$ where c, p, q are numbers:

   a) $\dfrac{\left(2a^2\right)^3}{b}$ b) $\sqrt{9ab^3}$ c) $\dfrac{a\left(\frac{2}{b}\right)}{\frac{3}{a}}$ d) $\dfrac{ab-a}{b^2-b}$ e) $\dfrac{a^{-1}}{\left(b^{-1}\right)\sqrt{a}}$ f) $\left(\dfrac{a^{2/3}}{b^{1/2}}\right)^2\left(\dfrac{b^{3/2}}{a^{1/2}}\right)$

3. Solve the quadratic equation $3x^2+4x-1=0$, **by completing the square**. Check your solutions using exact values.

4. Solve each of the following inequalities.

   a) $x^2+36\le 12x$ b) $x^2-4x-5x<0$ c) $\dfrac{2x-3}{x+5}\ge -11$

5. Perform each of the following divisions.

   a) $2x^5+x^4-10x^3-2x^2+14x-7$ by x^2+x-2 b) x^5-1 by $x+3$

6. Suppose that $x=\log_2 5$ and $y=\log_{10}9$. Express $\log_2 3$ in terms of x and y.

7. Suppose that a and b are numbers such that $a+b=20$. Find the

   a) smallest value of $3a^2+2b^2$ b) greatest value of a^2-3b^2 c) greatest value of $a-b^2$.

8. If we set the price of a ticket to \$20, we can sell 600 tickets. If we raise the price by x dollar, $4x$ less people will buy the ticket. What is the highest possible revenue that we can obtain?

9. Let $f(x)=x^3-x$.

   a) Graph $f(x)$ b) Graph $f(x+1)$ c) Graph $\dfrac{1}{f(x)}$ d) Graph $\dfrac{f(x)+|f(x)|}{2}$

10. Graph each of the following.

    a) $f(x)=(3x+24)(x+5)(x+8)(x+1)(5-x)^2(7-x)$

    b) $10x+x^2+y^2=6(y-5)$

    c) $f(x)=\dfrac{49-x^2}{2x+x^2-35}$

    d) $f(x)=\dfrac{3(x+1)^2(x-5)}{(x-1)(x+1)^8}$

    e) $f(x)=\dfrac{-2(x+2)x(x-2)^3(x-3)^2}{(x+1)^2x^2(x-2)^2(x-3)^2}$

    f) $f(x)=x^5-5x^3$

11. Simplify each of the following expressions.

    a) $\log_9\left(\dfrac{1}{27}\right)$ b) $\log_{16}4$ c) $\log_3\left(3^{21}\right)$ d) $\log_3\left(9^k\right)$ e) $\log_{64}\left(\dfrac{1}{16}\right)$ f) $1+2\log_2 3-\log_2 36$ g) $25^{\log_5 7}$ h) $\log_{\sqrt{27}}\left(\dfrac{1}{9}\right)$ i) $e^{2\ln 5}$ j) $e^{-2\ln 7}$ k) $3^{-2\log_3 2}$ l) $\log_2 5-\log_2 40$

12. Which of the following is NOT equivalent to $\log_8\left(\frac{50}{3}\right)$?

A) $\dfrac{\ln\left(\frac{50}{3}\right)}{\ln 8}$ B) $\dfrac{\ln 50 - \ln 3}{\ln 8}$ C) $\dfrac{\ln 50 - \ln 3}{3\ln 2}$ D) $\dfrac{2\ln 5 + \ln 2 - \ln 3}{3\ln 2}$ E) $\dfrac{2\ln 5 - \ln 3}{3}$

13. Perform the indicated operations on the complex numbers.

a) $|6-8i|$ b) $(3+2i)-(7-i)$ c) $(3+2i)(7-i)$ d) $\dfrac{6-8i}{7-i}$ e) $(1-i)^6$

14. Find all values of z for which $z^2 = 8-6i$

15. Find all complex solutions of the equation $x^2+13=6x$

16. Suppose that $f(x)=3-x^2$ and $g(x)=2x-1$. Compute each of the following.

a) $f(4)+g(4)$ b) $\dfrac{g(2)}{f(2)}$ c) $f(g(-1))$ d) $g(f(-1))$ e) $f(g(x))$ f) $g(f(x))$

17. Find the domain for each of the following functions.

a) $f(x)=2^{x-1}$ b) $f(x)=\sqrt{10-x^2}$ c) $f(x)=\ln\left(10-x^2\right)$ d) $f(x)=\dfrac{1}{\ln(10-x^2)}$

18. Find an equation for the inverse of each of the following functions.

a) $f(x)=3^{5x-1}$ b) $f(x)=\dfrac{x+4}{3x-5}$ c) $f(x)=\ln(2x-1)$

19. Solve each of the following equations over the real numbers. Use exact values, and show all steps. Make sure to check your solution(s).

a) $x^2+59=16x$
b) $125x+5x^3=40x^2$
c) $\log_2(x-3)-\log_2(x+1)=1$
d) $\sqrt{2x+10}+\sqrt{x+7}=4$
e) $5^{x+2}=2^{2x-3}$
f) $4^{x+1}-9\cdot 2^{x+1}=-8$
g) $3\cdot 2^{2x-1}=5^{2-x}$
h) $-\cos 2x=\sin x$
i) $\dfrac{2}{5}\ln(3x-1)=-2$
j) $\sin 3x\cos 3x=\dfrac{\sqrt{3}}{4}$

20. Consider the functions $f(x)=\log_3 x$ and $g(x)=\log_{1/3} x$.

a) Graph these functions in the same coordinate system. b) What kind of a symmetry do you notice?
c) What is the connection between these two functions? Justify your answer using algebra.

21. Redo problem 20. with the functions $f(x)=2^x$ and $g(x)=\log_2 x$.

22. Redo problem 20. with the functions $f(x)=2^x$ and $g(x)=\left(\dfrac{1}{2}\right)^x$.

23. Find the equation of the straight line passing through the intersection of the circles $(x+2)^2+(y+2)^2=50$ and $(x-2)^2+(y-1)^2=25$.

24. Find an equation of the tangent line drawn to the graph of $6y+x^2+y^2+33=14x$ at the point $(10,-7)$.

25. Let C_1 and C_2 be circles defined by $x^2+y^2=64$ and $(x-10)^2+y^2=9$, respectively. Let t_1 and t_2 be the common tangent lines drawn to the circles.

a) Find the coordinates of the point where t_1 and t_2 intersect each other.
b) Find an approximation for the acute angle formed by t_1 and t_2.
c) Compute the exact value of the length of the line segment $\overline{PQ}$ where P and Q are the points of tangency on t_1.

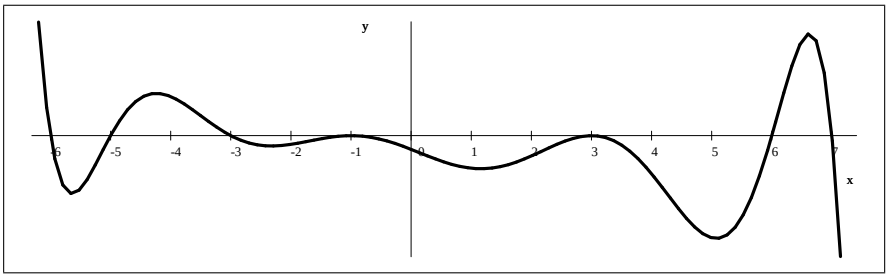
26. Let A_1 and A_2 denote the area of two circles, C_1 and C_2, respectively. Find the ratio $\dfrac{A_1}{A_2}$ if we know that an arc subtended by a central angle of 45° in C_1 is as long as an arc subtended by a central angle of 30° in C_2.

27. Find an equation for all tangent lines drawn to $y = -\dfrac{1}{2}x^2 + x - 1$ from $P(-1, 2)$.

28. Suppose that at time t, (where t is measured in hours) a sample contains $Q(t) = 4.5\left(0.95^{3t}\right)$ grams of a certain substance. How long does it take for this substance to decrease to half of its original quantity?

29. Find the exact value of $\sin\alpha$ where α is the angle formed by the common tangent lines drawn to the graphs of $(x-4)^2 + y^2 = 16$ and $x^2 + y^2 = 25$.

30. The picture below shows the graph of a polynomial function, $f(x)$.



a) What can be the degree of f?
b) Is the leading coefficient positive or negative?
c) Write a possible equation for f.

31. Consider the function $f(x) = \dfrac{x-6}{2x+5}$.

a) Find all horizontal asymptotes of the graph of f.
b) Find all vertical asymptotes of the graph of f.
c) Compute the intercepts of f.
d) Graph $f(x)$.
e) Find the inverse of f.
f) Find x for which $f(x) = -\dfrac{4}{5}$.
g) Solve: $\dfrac{x-6}{2x+5} \le 1$

32. Graph each of the following functions.

a) $f(x) = -\dfrac{1}{2}\sin(2x - \pi) + 1$ on $[-2\pi, 2\pi]$
b) $f(x) = -3\cos\left(\dfrac{\pi x}{3}\right) - 2$ on $[-9, 9]$
c) $f(x) = \tan^{-1} x$
d) $f(x) = \sec x$

33. Find the exact value for each of the following expressions.

a) $\cos 22.5^\circ$ b) $\cos 15^\circ \cos 75^\circ$ c) $\dfrac{\tan 65^\circ - \tan 5^\circ}{1 + (\tan 65^\circ)\tan 5^\circ}$ d) $\sin^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$ d) $\cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$

34. Prove each of the following identities.

a) $1 - \left(\cos\dfrac{x}{2} - \sin\dfrac{x}{2}\right)^2 = \sin x$ b) $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$ c) $\sin 2x = \dfrac{1 - \tan^2\left(\frac{\pi}{4} - x\right)}{1 + \tan^2\left(\frac{\pi}{4} - x\right)}$

35. Find the exact value of all solutions for each of the following equations. Present your answer in radians.

a) $\sin x = \sin 2x$ b) $7\sin x + 1 = 6\cos^2 x$ c) $\sin x + 1 = 2\cos^2 x$

36. Suppose that $\sin\alpha = -\frac{5}{13}$ and α is not in the fourth quadrant; $\cos\beta = \frac{7}{25}$ and β is not in the first quadrant. Find the exact value for each of the following.

    a) $\tan(\alpha - \beta)$ b) $\cos(\alpha + \beta)$ c) $\cos 2\alpha$ d) $\tan\frac{\alpha}{2}$

37. Let x and y be angles such that $\sin x = -\frac{3}{5}$, $\cos y = -\frac{20}{29}$. In addition, we know that $180° \le x \le 270°$ and $90° \le y \le 180°$. Find the exact value of each of the following.

    a) $\cos(x + y)$ b) $\sin(3x)$ c) $\tan(x - y)$

38. Express each of the following as a sum or difference.

    a) $\sin 35° \cos 25°$ b) $\cos 25° \cos 75°$ c) $\cos 4x \cos 2x$

39. Express each of the following as a product.

    a) $\sin 50° + \sin 20°$ b) $\sin 75° - \sin 35°$ c) $\cos 7x + \cos 3x$

40. Suppose that $\tan 2x = \frac{3}{4}$. Compute the exact value of a) $\cos 2x$ b) $\sin x$

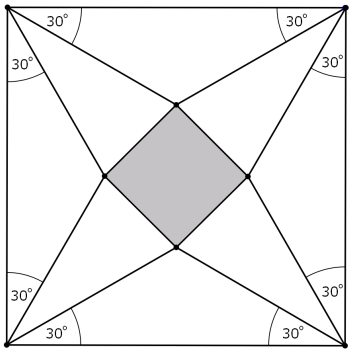
41. Find $\tan\beta$ if β is the acute angle formed by $y = \frac{2}{3}x - 5$ and $y = -x + 1$.

42. Solve each of the following triangles.

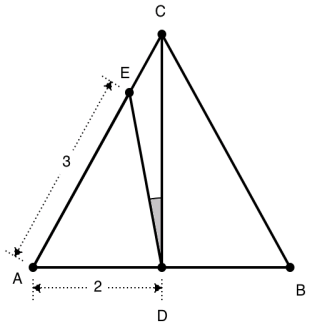
    a) $b = 248.6$, $c = 186.2$, and $\gamma = 43.1°$ b) $\gamma = 42°$, $a = 122\,\text{m}$, and $c = 70\,\text{m}$ c) $a = 5$, $b = 12$, and $c = 8$

43. Triangle ABC has sides of length $6, 7$, and 8. Find the exact value of $\cos\alpha + \cos\beta + \cos\gamma$.

44. Consider the square with sides 1 meter shown on the picture. Find the exact value of the area of the shaded region.



45. Consider the regular triangle with sides 4 meter shown on the picture.



    a) Find the exact value of the length of line segment CD.
    b) Find the exact value of the length of line segment ED.
    c) Find the exact value of $\cos\delta$ if δ is the shaded angle $\angle EDC$.

46. Re-write the decimal $0.\overline{25}$ as a quotient of two integers.

47. Prove that $\sin 70° - \sin 50° = \sin 10°$

Answers

1.) a) $\dfrac{x(x+3)}{x-4}$ b) $\dfrac{x-4}{x(x-1)}$ c) $\dfrac{5x}{x+5}$ d) $\dfrac{3x-1}{x}$ e) $\dfrac{3+\sqrt{5}}{2}$ f) 1

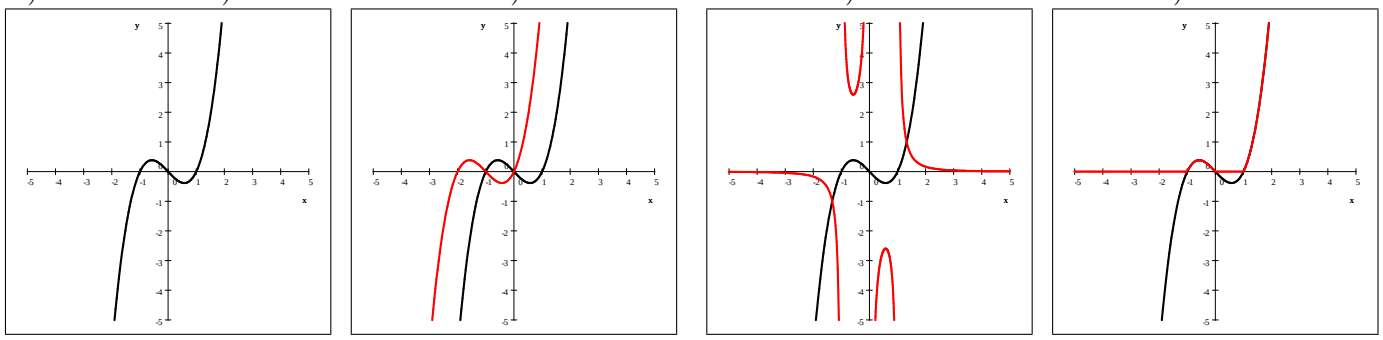
2.) a) $8a^6b^{-1}$ b) $3a^{\frac{1}{2}}b^{\frac{3}{2}}$ c) $\dfrac{2}{3}a^2b^{-1}$ d) ab^{-1} e) $a^{-\frac{3}{2}}b$ f) $a^{\frac{5}{6}}b^{\frac{1}{2}}$

3.) $\dfrac{-2\pm\sqrt{7}}{3}$ 4.) a) $x=6$ b) $0<x<9$ c) $x<-5$ or $x\geq -4$

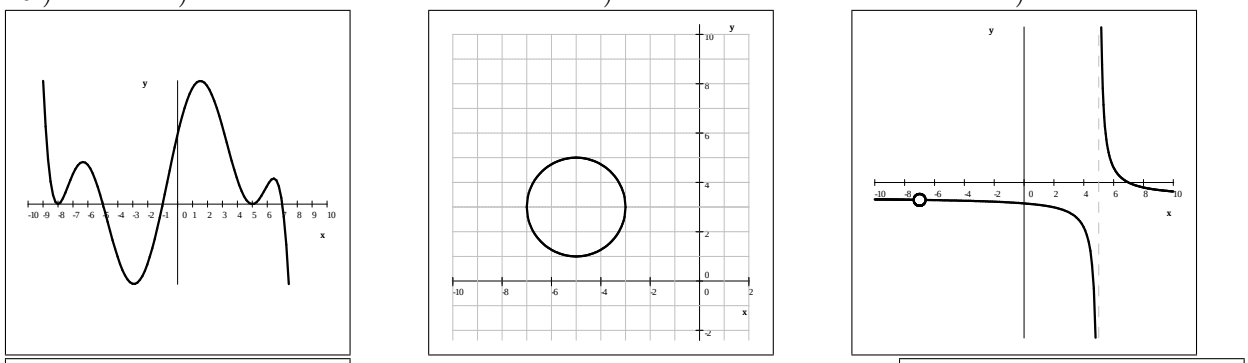
5.) a) $2x^3-x^2-5x+1$ R $3x-5$ b) $x^4-3x^3+9x^2-27x+81$ R -244

6.) $\dfrac{1}{2}y(x+1)$ 7.) a) 480 b) 600 c) $\dfrac{81}{4}$ 8.) \$28 900 when the price is \$85

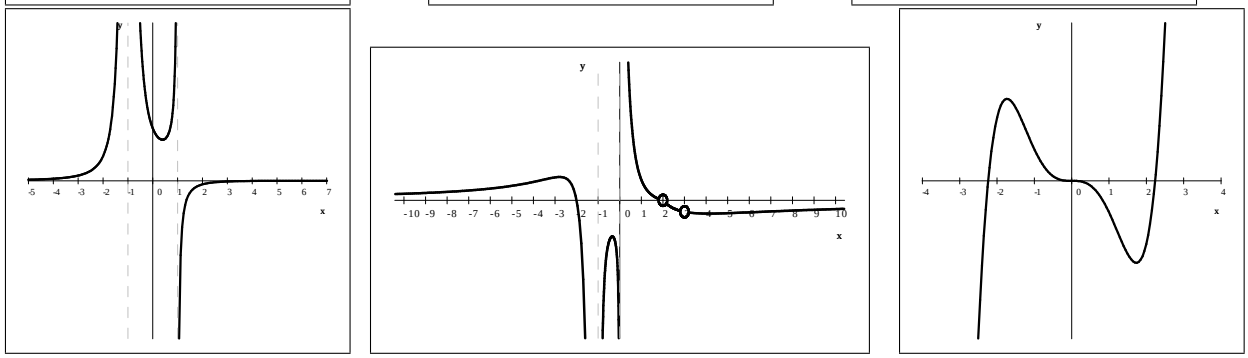
9.) a) b) c) d)



10.) a) b) c)



d) e) f)



11.) a) $-\dfrac{3}{2}$ b) $\dfrac{1}{2}$ c) 21 d) $2k$ e) $-\dfrac{2}{3}$ f) -1 g) 49 h) $-\dfrac{4}{3}$ i) 25 j) $\dfrac{1}{49}$ k) $\dfrac{1}{4}$ l) -3

12.) E 13.) a) 10 b) $-4+3i$ c) $23+11i$ d) $1-i$ e) $8i$ 14.) $3-i$ and $-3+i$

15.) $3-2i$ and $3+2i$ 16.) a) -6 b) -3 c) -6 d) 3 e) $-4x^2+4x+2$ f) $-2x^2+5$

17.) a) $\mathbb{R}$ b) $[-\sqrt{10},\sqrt{10}]$ c) $f(x)=(-\sqrt{10},\sqrt{10})$ d) $f(x)=(-\sqrt{10},\sqrt{10})\setminus\{3,-3\}$

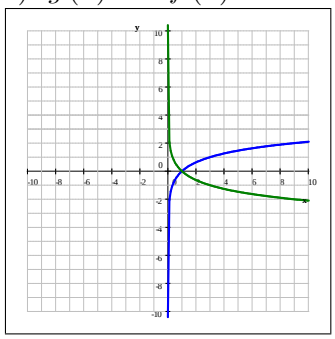
18.) a) $f^{-1}(x) = \frac{1}{5}(\log_3 x + 1)$ b) $f^{-1}(x) = \frac{5x+4}{3x-1}$ c) $f^{-1}(x) = \frac{1}{2}(e^x + 1)$

19.) a) $8 \pm \sqrt{5}$ b) 0 c) no solution d) -3 e) $\log_{4/5} 200 = \frac{\ln 200}{\ln 4 - \ln 5}$ f) $2, -1$

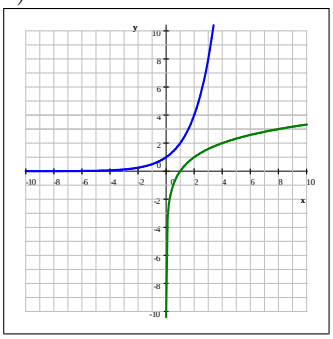
g) $\log_{20}\left(\frac{50}{3}\right) = \frac{\ln 50 - \ln 3}{\ln 20}$ h) $\frac{\pi}{2} + 2k\pi,\ -\frac{\pi}{6} + 2k\pi,\ -\frac{5\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$ i) $\frac{1}{3} + \frac{1}{3e^5}$

j) $\frac{\pi}{9} + \frac{k\pi}{3}$ or $\frac{\pi}{18} + \frac{k\pi}{3}$ where $k \in \mathbb{Z}$

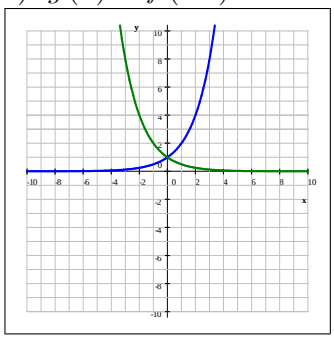
20.) a) f is the blue graph, g is the green graph
b) symmetry through the x axis
c) $g(x) = -f(x)$



21.) a) f is the blue graph, g is the green graph
b) symmetry through the line $y = x$
c) inverse functions



22.) a) f is the blue graph, g is the green graph
b) symmetry through the y axis
c) $g(x) = f(-x)$

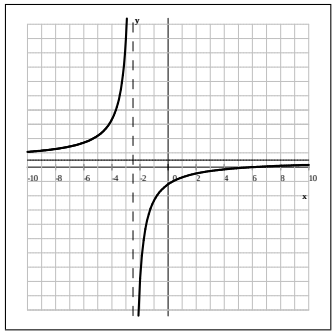


23.) $4x + 3y = 11$ or $y = -\frac{4}{3}x + \frac{11}{3}$ 24.) $\frac{3}{4}(x - 10) = y + 7$ 25.) a) $(16, 0)$ b) $60°$ c) $5\sqrt{3}$

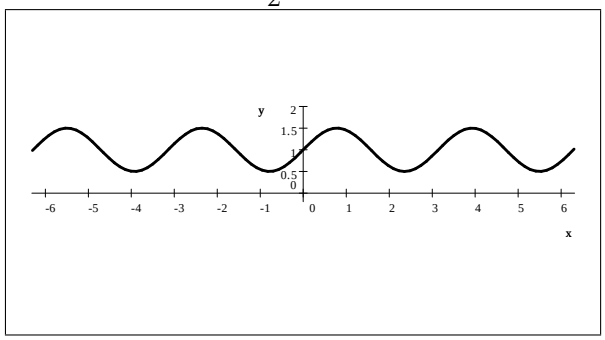
26.) $\frac{4}{9}$ 27.) $y = -x + 1$ and $y = 5x + 7$ 28.) $t = \frac{-\ln 2}{3 \ln 0.95}$ hours $\approx 4.504\,47$ hours 29.) $\frac{\sqrt{15}}{8}$

30.) a) $9, 11, 13...$ b) negative c) $f(x) = -(x+6)(x+5)(x+3)(x+1)^2(x-3)^2(x-6)(x-7)$

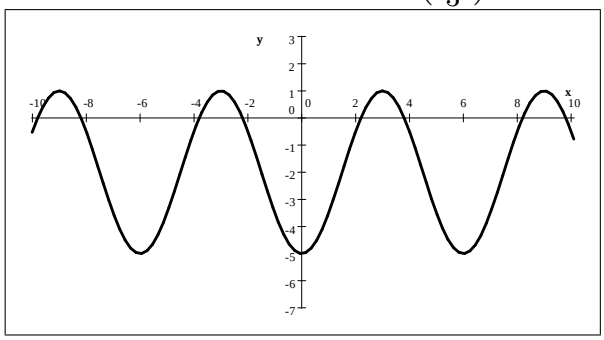
31.) a) $y = \frac{1}{2}$ b) $x = -\frac{5}{2}$ c) $(6, 0)$ and $(0, -2.5)$ d) e) $f^{-1}(x) = \frac{5x+6}{-2x+1}$ f) $\frac{10}{13}$ g) $x \le -11$ or $x > -\frac{5}{2}$



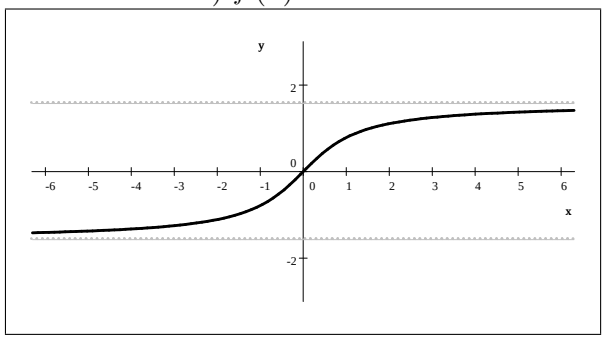
32.) a) $f(x) = -\frac{1}{2}\sin(2x-\pi)+1$ on $[-2\pi, 2\pi]$



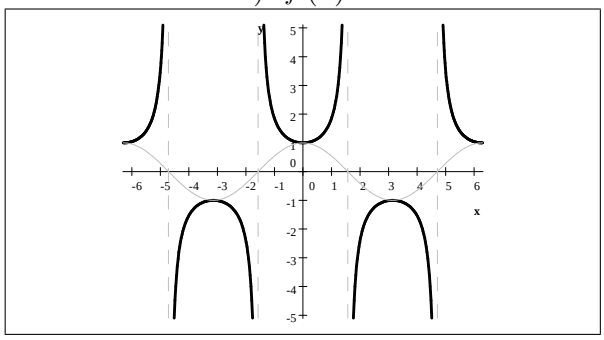
b) $f(x) = -3\cos\left(\frac{\pi x}{3}\right) - 2$ on $[-9, 9]$



c) $f(x) = \tan^{-1} x$



d) $f(x) = \sec x$



33.) a) $\frac{1}{2}\sqrt{\sqrt{2}+2}$ b) $\frac{1}{4}$ c) $\sqrt{3}$ d) $-\frac{\pi}{4}$ e) $\frac{3\pi}{4}$

34.) a) $1 - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2 = \sin x$

$$\text{LHS} = 1 - \left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right) = 1 - (1 - \sin x) = \text{RHS}$$

b) $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

$$\begin{aligned}\cos 4x &= \cos(2\cdot 2x) = 2\cos^2(2x) - 1 = 2\left(2\cos^2 x - 1\right)^2 - 1 = 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1 \\ &= 8\cos^4 x - 8\cos^2 x + 1\end{aligned}$$

c) $\sin 2x = \dfrac{1 - \tan^2\left(\frac{\pi}{4} - x\right)}{1 + \tan^2\left(\frac{\pi}{4} - x\right)}$

$$\text{RHS} = \frac{1 - \tan^2\left(\frac{\pi}{4} - x\right)}{1 + \tan^2\left(\frac{\pi}{4} - x\right)} = \frac{1 - \left(\dfrac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right)^2}{1 + \left(\dfrac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right)^2} = \frac{1 - \left(\dfrac{1 - \tan x}{1 + \tan x}\right)^2}{1 + \left(\dfrac{1 - \tan x}{1 + \tan x}\right)^2} = \frac{1 - \dfrac{(1-\tan x)^2}{(1+\tan x)^2}}{1 + \dfrac{(1-\tan x)^2}{(1+\tan x)^2}}$$

To clear the denominators, we multiply both numerator and denominator by $(1+\tan x)^2$

$$\begin{aligned}
\text{RHS} &= \frac{1-\dfrac{(1-\tan x)^2}{(1+\tan x)^2}}{1+\dfrac{(1-\tan x)^2}{(1+\tan x)^2}} = \frac{1-\dfrac{(1-\tan x)^2}{(1+\tan x)^2}}{1+\dfrac{(1-\tan x)^2}{(1+\tan x)^2}} \cdot \frac{(1+\tan x)^2}{(1+\tan x)^2} = \frac{(1+\tan x)^2-(1-\tan x)^2}{(1+\tan x)^2+(1-\tan x)^2} \\
&= \frac{1+\tan^2 x+2\tan x-\left(1+\tan^2 x-2\tan x\right)}{1+\tan^2 x+2\tan x+\left(1+\tan^2 x-2\tan x\right)} = \frac{1+\tan^2 x+2\tan x-1-\tan^2 x+2\tan x}{1+\tan^2 x+2\tan x+1+\tan^2 x-2\tan x} \\
&= \frac{4\tan x}{2+2\tan^2 x} = \frac{2\tan x}{1+\dfrac{\sin^2 x}{\cos^2 x}} = \frac{2\tan x}{\dfrac{\cos^2 x}{\cos^2 x}+\dfrac{\sin^2 x}{\cos^2 x}} = \frac{2\tan x}{\dfrac{\sin^2 x+\cos^2 x}{\cos^2 x}} = \frac{2\tan x}{\dfrac{1}{\cos^2 x}} = 2\tan x\cos^2 x \\
&= 2\frac{\sin x}{\cos x}\cos^2 x = 2\sin x\cos x = \sin 2x = \text{LHS}
\end{aligned}$$

35.) a) $k\pi, \quad \pm\frac{1}{3}\pi+2k\pi \quad$ where $\quad k\in\mathbb{Z}$ b) $\frac{\pi}{6}+2k\pi, \quad \frac{5\pi}{6}+2k\pi \quad$ where $\quad k\in\mathbb{Z}$

c) $-\frac{\pi}{2}+2k\pi, \quad \frac{\pi}{6}+2k\pi, \quad \frac{5\pi}{6}+2k\pi \quad k\in\mathbb{Z}$ 36.) a) $-\frac{323}{36}$ b) $-\frac{204}{325}$ c) $\frac{119}{169}$ d) -5

37.) a) $\frac{143}{145}$ b) $-\frac{117}{125}$ c) $\frac{144}{17}$

38.) a) $\frac{1}{2}(\sin 60^\circ+\sin 10^\circ)$ b) $\frac{1}{2}(\cos 50^\circ+\cos 100^\circ)$ c) $\frac{1}{2}(\cos 6x+\cos 2x)$

39.) a) $2\sin 35^\circ\cos 15^\circ$ b) $2\cos 55^\circ\sin 20^\circ$ c) $2\cos 5x\cos 2x$ 40.) a) $\pm\frac{4}{5}$ b) $\pm\frac{\sqrt{10}}{10}, \pm\frac{3\sqrt{10}}{10}$ 41.) 5

42.) a) $\beta_1=65.819^\circ, \quad \alpha_1=71.081^\circ, \quad a_1=257.790$ and $\beta_2=114.181^\circ, \quad \alpha_2=22.719^\circ \quad a_2=105.247$

b) no solution c) $\alpha=17.612^\circ \quad \beta=133.433^\circ \quad \gamma=28.955^\circ$ 43.) $\frac{47}{32}$ 44.) $\frac{2-\sqrt{3}}{3}$

45.) a) $\sqrt{12}$ b) $\sqrt{7}$ c) $\frac{3\sqrt{21}}{14}$ 46.) $\frac{25}{99}$

47) by the product-sum identities, $\sin 70^\circ-\sin 50^\circ=2\cos 60^\circ\sin 10^\circ=\sin 10^\circ$

