1. Differentiate each of the following by evaluating the limit of the difference quotient.

   a) $f(x) = \dfrac{1}{3x-1}$ b) $f(x) = \sqrt{1-x^2}$

2. Prove that if a function f is differentiable at a number a, then it is also continuous there.

3. Prove that in case of the unit circle, the tangent line and the radius drawn to the point of tangency are perpendicular.

4. a) Prove that if $f(x) = mx + b$, then $f'(x) = m$.

   b) Prove that if $f(x)$ is a linear function with slope m, then the inverse function, $f^{-1}(x)$ is also linear with slope $\dfrac{1}{m}$.

5. Differentiate each of the following.

   a) $f(x) = 3x^2 - \dfrac{5}{x}$ b) $g(x) = -\sqrt{x} + \dfrac{1}{\sqrt{x}} - 5e^3$ c) $h(x) = \sin x + \cos x$

6. Match each of the following graphs to the descriptions. Assume that the graphs you see are the location functions of an obejct moving along a vertical line.

   (a) The object is moving upward and its speed is increasing.

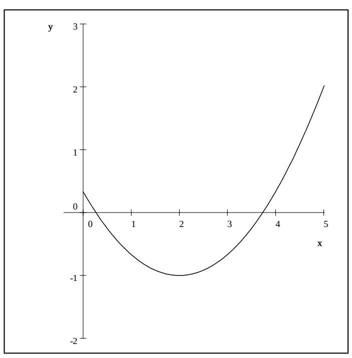
   (b) The object is moving upward and its speed is decreasing.

   (c) The object is moving downward and its speed is increasing.

   (d) The object is moving downward and its speed is decreasing.

   (e) The obejct has been moving upward but later is moving downward.

   (f) The obejct has been moving downward but later is moving upward.



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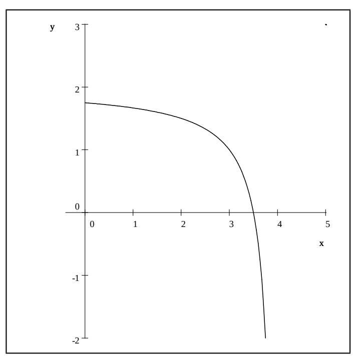

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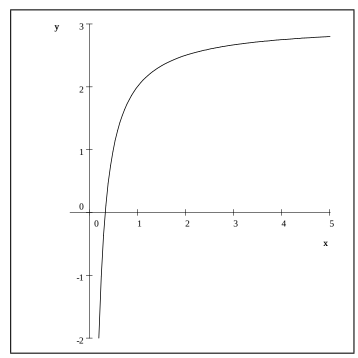

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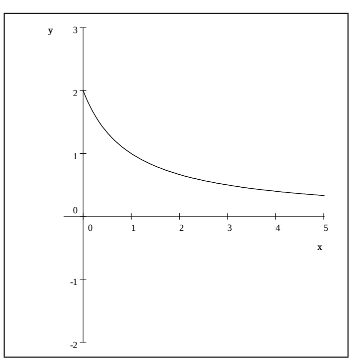

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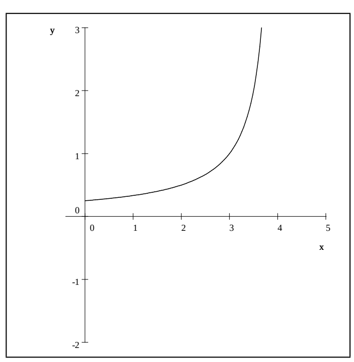

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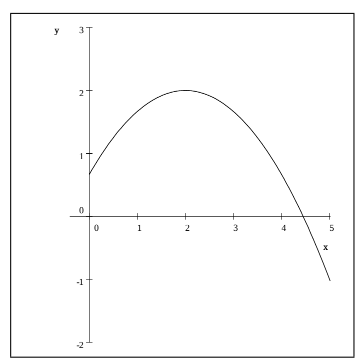


6.

7. Match each of the following graphs to the descriptions. Assume that the graphs you see are the velocity functions of an obejct moving along a vertical line.

(a) The object is moving upward and its speed is increasing.

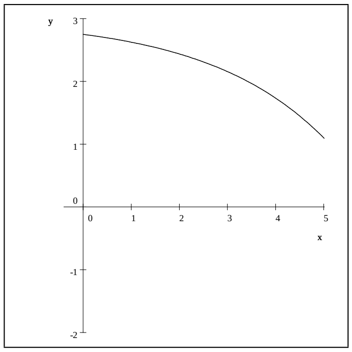
(b) The object is moving upward and its speed is decreasing.

(c) The object is moving downward and its speed is increasing.

(d) The object is moving downward and its speed is decreasing.

(e) The obejct has been moving upward but later is moving downward.

(f) The obejct has been moving downward but later is moving upward.



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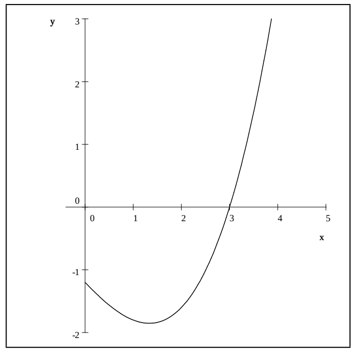

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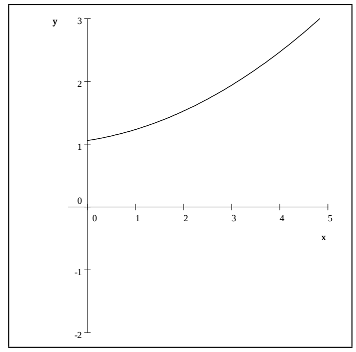

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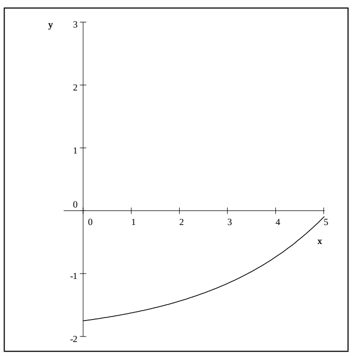

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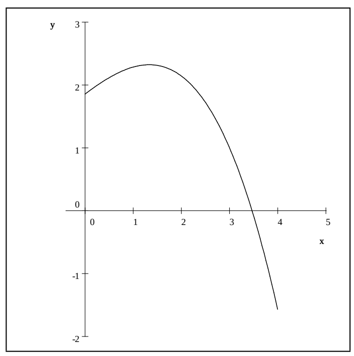

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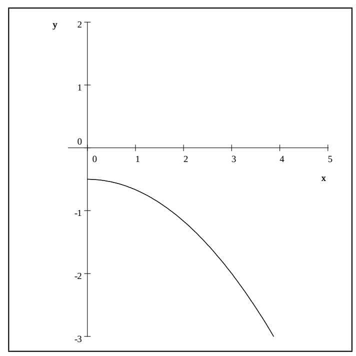


6.

8. Find each of the following limits.

a) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}}$

b) $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}}$

c) $\displaystyle\lim_{x\to\infty}\frac{\sin x}{x}$

d) $\displaystyle\lim_{x\to 6^-}\frac{\frac{1}{x}-\frac{1}{6}}{x-6}$

e) $\displaystyle\lim_{x\to 6^+}\log_6(x-6)$

f) $\displaystyle\lim_{x\to 0}\frac{\left(\sqrt{x+25}-5\right)^2}{x^2}$

g) $\displaystyle\lim_{x\to -3^-}\frac{x^2-9}{x^2-2x-15}$

h) $\displaystyle\lim_{x\to 5^-}\frac{x^2-9}{x^2-2x-15}$

i) $\displaystyle\lim_{x\to\infty}\frac{3^{2x-1}}{2^{3x+1}}$

j) $\displaystyle\lim_{x\to\infty}\frac{3^{2x+1}}{2^{4x-1}}$

k) $\displaystyle\lim_{x\to\infty}\frac{2^{x-1}-2^{-x}}{2^{x+1}+2^{-x}}$

l) $\displaystyle\lim_{x\to\infty} x\left(\frac{1}{2}-\frac{1}{2+\frac{1}{x}}\right)$

m) $\displaystyle\lim_{x\to\infty}\left(5^{x+1}-5^x\right)$

n) $\displaystyle\lim_{x\to\infty}\frac{\cos^2 x}{x}$

o) $\displaystyle\lim_{x\to\infty}\left(\sqrt{2x}-\sqrt{x}\right)$

p) $\displaystyle\lim_{x\to\infty}\frac{3\cdot 2^{2x}-2^x+1}{-4^x+2^x-1}$

q) $\displaystyle\lim_{x\to\pi}\frac{\sin x}{x-\pi}$

9. Find an equation of the tangent line drawn to the graph of the functions given at the point given.

a) $f(x) = 2x^3 - 7x + 2$ at the point $(-1, 7)$

b) $g(x) = \sin x$ at the point $\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}}{2}\right)$

c) $h(x) = -\dfrac{1}{x^3}$ at the point $\left(2, -\dfrac{1}{8}\right)$

10. Find all relative maximums and minimums for each of the following functions.

a) $f(x) = 3x^5 - 20x^3 - 6$

b) $f(x) = 2x^6 - 3x^4 + 12$

c) $f(x) = -3x^4 - 8x^3 + 48x^2 + 60$

d) $f(x) = 2x^5 - 5x^4 - 10x^3 - 20$

e) $f(x) = x^3\left(6x^2 - 1\right)$

11. Consider the function $f(x) = (x + 10)\left(x^2 - 19x + 70\right)$.

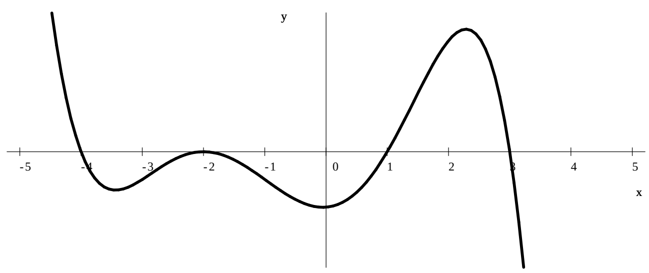
a) Find all $x-$intercepts of f.

b) Find all relative maxima and minima of f.

c) Find all absolute maxima and minima of f.

d) Sketch the graph of f.

12. The graph below shows f', the derivative of a function f.

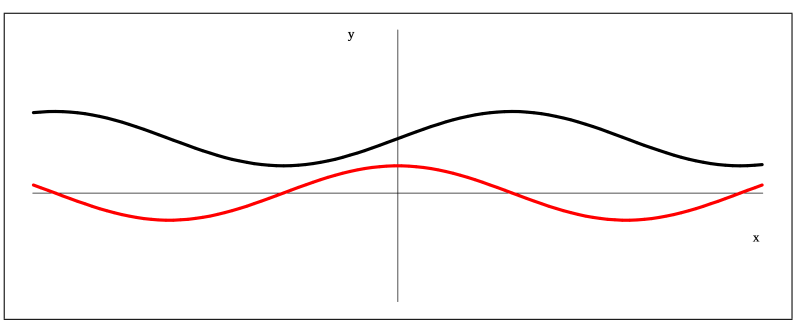


a) Find all values of x for which f has a relative maximum at x.

b) Find all values of x for which f has a relative minimum at x.

13. Sketch the graph and give a complete analysis for $f(x) = \sqrt{x+1} - 2$.

14. The picture below shows a function f and its derivative, f'. Which is which?



15. a) Suppose that $f(x) = x^3 - 6x^2 + 3x - 10$. Find all values of x where the tangent line drawn to the graph of f is perpendicular to the line $x - 6y = 7$.

b) Suppose that an object's location function is given by $L(t) = t^3 - 6t^2 + 3t - 10$. Find the moment when the object is moving downward with the greatest speed. What is that greatest speed?

16. Suppose that P is a polynomial with degree 3. Then we can write the polynomial as $P(x) = ax^3 + bx^2 + cx + d$ where a, b, c, and d are real numbers. Find the values of a, b, c, and d if we kow that $P(0) = -2$, $P'(0) = 2$, $P''(0) = 10$ and $P'''(0) = -24$.

17. Find the values of a and b so that $f(x) = 2x^3 - ax^2 + bx + 4$ has a relative maximum at $x = -2$ and a relative minimum at $x = 5$.

18. Suppose that f is a function with derivative f' defined as

$$f'(x) = -3(x+2)^3(x+1)^4 x(x-5)^3(x-8)^6$$

Classify the zeroes of f' as places where f has a relative maximum, a relative minimum, or neither.

19. Find the value of a if we know that the line $y = x$ is tangent to the parabola $y = ax^2 + 6$.

20. a) Prove the sum rule for derivatives. b) Prove that $\dfrac{d}{dx}(\sin x) = \cos x$

Answers

1.) a) $f(x) = \dfrac{1}{3x-1}$

$$\begin{aligned} f'(x) &= \lim_{x \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to 0} \frac{\dfrac{1}{3(x+h)-1} - \dfrac{1}{3x-1}}{h} = \lim_{x \to 0} \frac{1}{h} \cdot \frac{(3x-1) - (3(x+h)-1)}{[3(x+h)-1][3x-1]} \\ &= \lim_{x \to 0} \frac{1}{h} \cdot \frac{3x - 1 - 3x - 3h + 1}{[3(x+h)-1][3x-1]} = \lim_{x \to 0} \frac{1}{h} \cdot \frac{-3h}{[3(x+h)-1][3x-1]} = \lim_{x \to 0} \frac{-3}{[3(x+h)-1][3x-1]} \\ &= \frac{-3}{(3x-1)(3x-1)} = \frac{-3}{(3x-1)^2} \end{aligned}$$

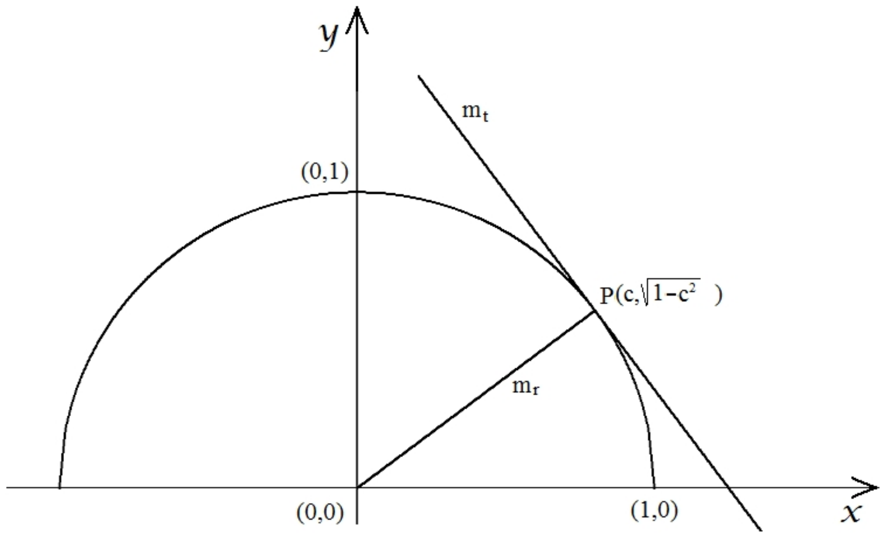
b) see handout Differentiating by Finding the Limits

2.) Suppose that f is differentiable at a number a. Then $f'(a)$ exists which means that $f(a)$ exists and the limit $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ also exists and is finite. Let us start with the true statement that $0 = 0 \cdot f'(a)$.

$$\begin{aligned} 0 &= 0 \cdot f'(a) \\ 0 &= \lim_{h \to 0} h \cdot \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} && \text{by the product rule of limits} \\ 0 &= \lim_{h \to 0} \left(h \cdot \frac{f(a+h) - f(a)}{h} \right) && \text{cancel out } h \\ 0 &= \lim_{h \to 0} (f(a+h) - f(a)) && \text{by the difference rule of limits} \\ 0 &= \lim_{h \to 0} f(a+h) - \lim_{h \to 0} f(a) \\ \lim_{h \to 0} f(a) &= \lim_{h \to 0} f(a+h) && \text{by the constant rule of limits} \\ f(a) &= \lim_{h \to 0} f(a+h) \end{aligned}$$

and $f(a) = \displaystyle\lim_{h \to 0} f(a+h)$ means that f is continuous at a.

3.) Let $f(x) = \sqrt{1-x^2}$ - the upper half of the unit circle. Let c be a number between 0 and 1. We will look at the tangent line drawn at the point $\left(c, \sqrt{1-c^2}\right)$.



First, let us figure out the slope of the radius. That can be easily done via the slope formula between $\left(c, \sqrt{1-c^2}\right)$ and $(0,0)$.

$$m_r = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sqrt{1-c^2} - 0}{c - 0} = \frac{\sqrt{1-c^2}}{c}$$

Recall that the derivative measures the slope of the tangent line. The derivative of $f(x) = \sqrt{1-x^2}$ is $-\dfrac{x}{\sqrt{1-x^2}}$. (We just proved it in 1b.) So, the tangent line drawn at $x = c$ will have slope $-\dfrac{c}{\sqrt{1-c^2}}$ and so

$$m_t = -\frac{c}{\sqrt{1-c^2}}$$

Now the product of the two slopes is

$$m_r \cdot m_t = \frac{\sqrt{1-c^2}}{c} \cdot \left(-\frac{c}{\sqrt{1-c^2}}\right) = -1$$

and so the two lines are perpendicular.

4.) a)

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to 0} \frac{m(x+h) + b - (mx+b)}{h} = \lim_{x \to 0} \frac{mx + mh + b - mx - b}{h} = \lim_{x \to 0} \frac{mh}{h} = \lim_{x \to 0} m = m$$

b) see Handout Inverse Functions

5.) a) $f'(x) = 6x + \dfrac{5}{x^2}$ b) $g'(x) = -\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x^{3/2}}$ c) $h'(x) = \cos x - \sin x$

6.) 1-(f) 2-(d) 3-(c) 4-(a) 5-(b) 6-(e) 7.) 1-(b) 2-(d) 3-(f) 4-(e) 5-(a) 6-(c)

8.) a) $\dfrac{1}{1-\sqrt{2}} = -\sqrt{2} - 1$ b) undefined c) 0 d) $-\dfrac{1}{36}$ e) $-\infty$ f) $\dfrac{1}{100}$ g) $\dfrac{3}{4}$ h) $-\infty$

i) ∞ j) 0 k) $\dfrac{1}{4}$ l) $\dfrac{1}{4}$ m) ∞ n) 0 o) ∞ p) -3 q) -1

Solution for q) : Let $y = x - \pi$. Then $y \to 0$ as $x \to \pi$ and also $x = y + \pi$

$$\lim_{x\to\pi} \frac{\sin x}{x-\pi} = \lim_{y\to 0} \frac{\sin(y+\pi)}{y} = \lim_{y\to 0} \frac{\sin y \cos \pi + \cos y \sin \pi}{y} = \lim_{y\to 0} \frac{\sin y\,(-1) + \cos y \cdot 0}{y} = \lim_{y\to 0} \frac{-\sin y}{y} = -\lim_{y\to 0} \frac{\sin y}{y} = -1$$

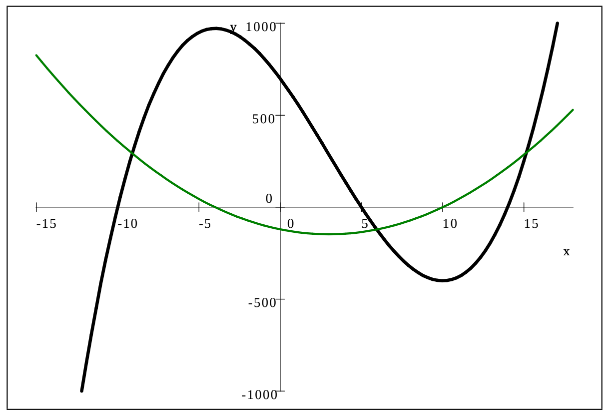
9.) a) $y = -x + 6$ b) is: $y = \frac{1}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$ c) $y = \frac{3}{16}x - \frac{1}{2}$

10.) a) rel. max at $x = -2$ and rel. min at $x = 2$ b) rel. max at $x = 0$ and rel. min at $x = -1$ and $x = 1$

c) rel max at $x = -4, 2$ and rel. min at $x = 0$ d) rel max at $x = -1$ and rel. min at $x = 3$

e) rel max at $\frac{1}{\sqrt{10}}$ and rel. min at $x = -\frac{1}{\sqrt{10}}$

11.) a) $(-10, 0), (5, 0), (14, 0)$
b) $(-4, 972)$ is a relative maximum, $(10, -400)$ is a relative minimum.
c) there is none



d)

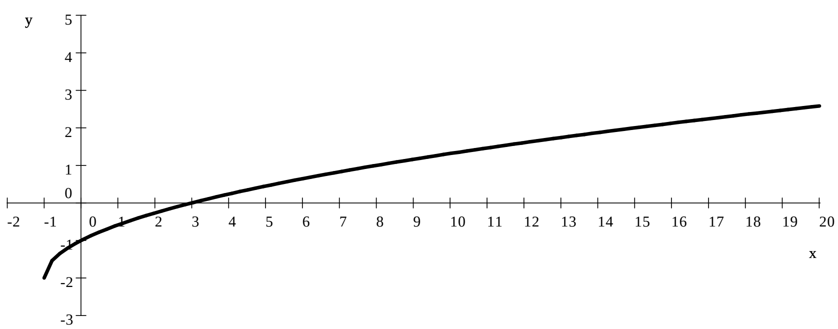
12.) a) $-4, 3$ b) 1

13.) $f(x) = \sqrt{x+1} - 2$

domain: $[-1, \infty)$
range: $[-2, \infty)$
no horizontal asymptote
no vertical asymptote
$y-$intercept: $(0, -1)$

$x-$intercept: $(3, 0)$
one-to-one
strictly increasing on on $[-1, \infty)$
no relative maximum
no absolute maximum

no relative minimum
absolute minimum: $(-1, -2)$
end behavior:
$\lim_{x\to-\infty} f(x) =$ undefined and $\lim_{x\to\infty} f(x) = \infty$



14.) The red graph is f'. 15.) a) 1, 3 b) $f'(2) = -9$ 16.) $a = -4$, $b = 5$, $c = 2$, $d = -2$

17.) $a = 9, b = -60$ 18.) rel. max at $x = -2$ and 5; rel. min. at $x = 0$; neither at $x = -1$ and 8 19.) $\frac{1}{24}$

20.) a) see handout Differentiation 1 (Proofs) b) see handout Differentiating sinx and cosx

