1. Simplify each of the following.

   a) $\frac{1}{2}\log_3 45x^2 + \log_3 \sqrt{20} - \log_3 30x + \log_3 6 - \log_3 2$

   b) $\ln(\tan 45°) \cdot \ln(\cos 75°) \cdot \ln(\sin 60°)$

2. Solve each of the following equations.

   a) $\log_2(x+3) + \log_2(x-3) = \log_2(x-9)$

   b) $(2x-5)^3(x+5)^8 - (2x-5)^4(x+5)^7 = 0$

   c) $\sqrt{1-x^2} - \dfrac{x^2}{\sqrt{1-x^2}} = 0$

   d) $\frac{1}{2}x^{-1/2}(5-x)^{2/3} - \frac{2}{3}x^{1/2}(5-x)^{-1/3} = 0$

   e) $4^{2\log x} \cdot 5^{\log x} = 6400$

3. Sketch the graph of the function $f(x) = -2(x+3)(x+2)^2 x^3 (x-3)^4 (x-4)$

4. State and prove the product rule for derivatives.

5. Differentiate each of the following functions.

   a) $f(x) = x^3 \cos x$

   b) $f(x) = \sin x \cos x$

   c) $f(x) = \sqrt{x}\left(x^2 - \sin x\right)$

   d) $f(x) = \sin^2 x + \cos^2 x$

6. Compute each of the following indefinite integrals.

   a) $\displaystyle\int (\cos x - 6x + 1)\, dx$ b) $\displaystyle\int \left(\sqrt{x} + 6x^3 - 2x\right) dx$ c) $\displaystyle\int \left(\frac{1}{x^3} + x^3 + \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right) dx$

7. We know the following. $f''(x) = 12x - 8$ and $f'(3) = 20$ and $f(2) = -9$. Find the equation for f.

8. Find a third degree polynomial with the following properties: $f(0) = 1$, $f'(0) = -2$, $f''(0) = 10$ and $f'''(0) = -6$.

9. Find all relative maximums and minimums for each of the following functions. Use the second derivative test to determine whether f has a maximum or a minimum at the points you found.

   a) $f(x) = \dfrac{x^2 - 2x - 9}{x^4}$ b) $f(x) = \dfrac{3x^2 - 9x + 5}{x^3}$

10. Find two non-negative numbers a and b whose sum is 6 and the product a^2b^5 is the greatest possible.

11. We would like to make an open rectangular box with square base. We also want the box's volume to be $60\,\text{ft}^3$. The material for the bottom of the box costs 12 cents per ft^2 and the material for the sides costs cents per $10\,\text{ft}^2$.

    a) What dimensions will result in the box with the lowest possible cost?

    b) Use the second derivative test to verify that you have indeed found a minimum and not a maximum.

12. We are designing a book. Each page should contain $27\,\text{cm}^2$ text. The left margin needs to be 2 cm wide, and the other margins (right, top, and bottom) need to be 1 cm wide each.

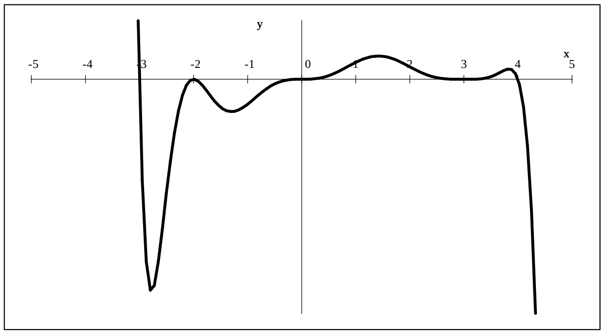
    a) What dimensions (length and width) would guarantee the smallest area for each page?

    b) Prove that your result is indeed a minimum and not a maximum.

Answers

1.) a) 1 b) 0 2.) a) no solution b) $\frac{5}{2}, -5, 10$ c) $\pm\frac{\sqrt{2}}{2}$ d) $\frac{15}{7}$ e) 100

3.) $f(x) = -2(x+3)(x+2)^2 x^3 (x-3)^4 (x-4)$ 4.) see handout



5.) a) $f'(x) = 3x^2\cos x - x^3 \sin x$ b) $f'(x) = \cos^2 x - \sin^2 x = \cos 2x$

c) $f'(x) = \sqrt{x}(2x - \cos x) + \frac{1}{2\sqrt{x}}\left(x^2 - \sin x\right)$ d) $f'(x) = 0$

6.) a) $\sin x - 3x^2 + x + C$ b) $\frac{3}{2}x^4 - x^2 + \frac{2}{3}x^{3/2} + C$ c) $-\frac{1}{2x^2} + \frac{1}{4}x^4 + \frac{3}{4}\sqrt[3]{x^4} + \frac{3}{2}\sqrt[3]{x^2} + C$

7.) $f(x) = 2x^3 - 4x^2 - 10x + 11$ 8.) $f(x) = -x^3 + 5x^2 - 2x + 1$

9.) a) relative maximums at $x = -3$ and $x = 6$ b) relative minimum at $x = 1$ and a maximum at $x = 5$

10.) $b = \frac{30}{7} = 4\frac{2}{7}$ and $a = 1\frac{5}{7}$

11.) a) base is $\sqrt[3]{100}$ by $\sqrt[3]{100}$ and the height is $\frac{3}{5}\sqrt[3]{100}$ b) see solutions

12.) a) Optimal dimensions are $3 + \frac{9\sqrt{2}}{2} \approx 9.36396$ and $3\sqrt{2} + 2 \approx 6.242641$ centimeters b) see solutions

Solutions

2. b) $(2x-5)^3(x+5)^8 - (2x-5)^4(x+5)^7 = 0$ $\frac{5}{2}, -5, 10$

$$\begin{aligned}
(2x-5)^3(x+5)^8 - (2x-5)^4(x+5)^7 &= 0 \qquad \text{factor out } (2x-5)^3(x+5)^7 \\
(2x-5)^3(x+5)^7\left[x+5-(2x-5)\right] &= 0 \\
(2x-5)^3(x+5)^7\left[x+5-2x+5\right] &= 0 \\
(2x-5)^3(x+5)^7(-x+10) &= 0
\end{aligned}$$

$$x_1 = \frac{5}{2} \qquad x_2 = -5 \quad \text{and} \quad x_3 = 10$$

c) $\sqrt{1-x^2}-\dfrac{x^2}{\sqrt{1-x^2}}=0$ $\qquad \pm\dfrac{\sqrt{2}}{2}$

$$\begin{aligned}
\sqrt{1-x^2}-\frac{x^2}{\sqrt{1-x^2}} &= 0\\
\sqrt{1-x^2} &= \frac{x^2}{\sqrt{1-x^2}} \qquad \text{multiply by } \sqrt{1-x^2}\\
1-x^2 &= x^2\\
1 &= 2x^2\\
\frac{1}{2} &= x^2\\
\pm\frac{1}{\sqrt{2}} &= x
\end{aligned}$$

d) $\dfrac{1}{2}x^{-1/2}(5-x)^{2/3}-\dfrac{2}{3}x^{1/2}(5-x)^{-1/3}=0$ $\qquad \dfrac{15}{7}$

$$\begin{aligned}
\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\left(\sqrt[3]{5-x}\right)^2-\frac{2}{3}\sqrt{x}\frac{1}{\sqrt[3]{5-x}} &= 0\\
\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\left(\sqrt[3]{5-x}\right)^2 &= \frac{2}{3}\sqrt{x}\frac{1}{\sqrt[3]{5-x}} \qquad \text{mutliply by } \sqrt{x}\\
\frac{1}{2}\left(\sqrt[3]{5-x}\right)^2 &= \frac{2}{3}\left(\sqrt{x}\right)^2\frac{1}{\sqrt[3]{5-x}} \qquad \text{multiply by } \sqrt[3]{5-x}\\
\frac{1}{2}\left(\sqrt[3]{5-x}\right)^3 &= \frac{2}{3}x\\
\frac{1}{2}(5-x) &= \frac{2}{3}x \qquad \text{multiply by } 6\\
3(5-x) &= 4x\\
15-3x &= 4x\\
15 &= 7x\\
\frac{15}{7} &= x
\end{aligned}$$

e) $4^{2\log x}\cdot 5^{\log x}=6400$ $\qquad 100$

$$\begin{aligned}
4^{2\log x}\cdot 5^{\log x} &= 6400\\
\left(4^2\right)^{\log x}\cdot 5^{\log x} &= 6400\\
(16)^{\log x}\cdot 5^{\log x} &= 6400\\
(16\cdot 5)^{\log x} &= 6400\\
80^{\log x} &= 80^2\\
\log x &= 2\\
x &= 100
\end{aligned}$$

9\. a) $f(x)=\dfrac{x^2-2x-9}{x^4}=\dfrac{1}{x^2}-\dfrac{2}{x^3}-\dfrac{9}{x^4}$

$$f'(x)=\frac{d}{dx}\left(\frac{1}{x^2}-\frac{2}{x^3}-\frac{9}{x^4}\right)=-\frac{2}{x^3}+\frac{6}{x^4}+\frac{36}{x^5}=\frac{-2x^2+6+36}{x^5}=\frac{-2(x+3)(x-6)}{x^5}$$

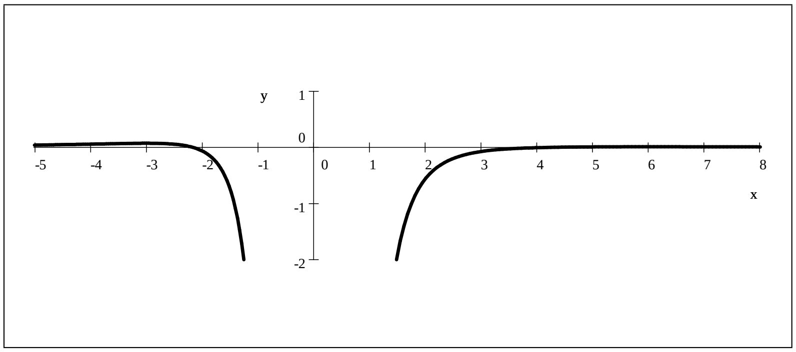
zero is a critical number but f has a vertical asymptote at $x=0$ so it is not a maximum or a minimum.

relative max/min possibly at $x=-3,6$

$$f''(x) = \frac{d}{dx}\left(-\frac{2}{x^3} + \frac{6}{x^4} + \frac{36}{x^5}\right) = \frac{6}{x^4} - \frac{24}{x^5} - \frac{180}{x^6}$$

$$f''(x) = \frac{6x^2 - 24x - 180}{x^6} = \frac{6\left(x^2 - 4x - 30\right)}{x^6}$$

$f''(-3) = -\dfrac{2}{27}$ indicating a maximum for f at $x = -3$ $\quad f''(6) = -\dfrac{1}{432}$ also indicating a maximum for f at $x = 6$

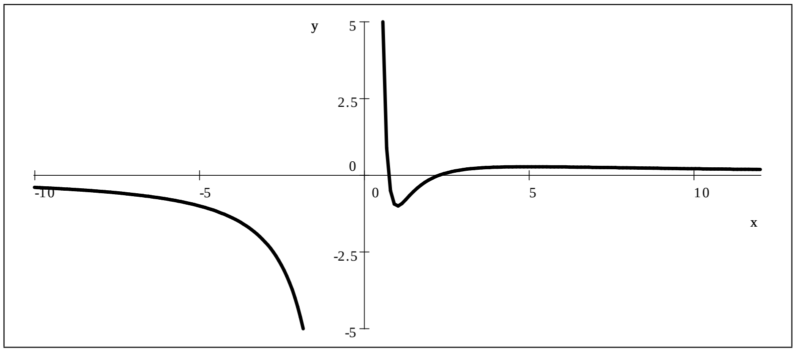


b) $f(x) = \dfrac{3x^2 - 9x + 5}{x^3}$ $\quad f'(x) = \dfrac{-3x^2 + 18x - 15}{x^4} = \dfrac{-3(x-1)(x-5)}{x^4}$

zero is a critical number but f has a vertical asymptote at $x = 0$ so it is not a maximum or a minimum.

relative max/min possibly at $x = 1, 5$ $\quad f''(x) = \dfrac{6x^2 - 54x + 60}{x^5} = \dfrac{6}{x^3} - \dfrac{54}{x^4} + \dfrac{60}{x^5}$

$f''(1) = 12$ indicating a minimum for f at $x = 1$ $\quad f''(5) = -\dfrac{12}{625}$ indicating a maximum for f at $x = 5$



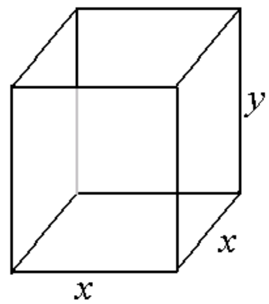
10. $a = 6 - b$ $\quad f(b) = (6-b)^2 b^5 = b^7 - 12b^6 + 36b^5$ $\quad f'(b) = 7b^6 - 72b^5 + 180b^4 = b^4(7b - 30)(b - 6)$

$b = 0$ not a relative maximum or a minimum but an absolute minimum on the domain $[0, 6]$

$b = 6$ $\quad a = 0$ a relative minimum (also an absolute minimum on the domain $[0, 6]$

$b = \dfrac{30}{7} = 4\dfrac{2}{7}$ and $a = 1\dfrac{5}{7}$ a maximum

11. a) Let us denote the sides as shown on the picture.



$60 = x^2 y \implies y = \dfrac{60}{x^2}$

$C = 0.12x^2 + 0.1\,(4xy)$

$C\,(x) = 0.12x^2 + 0.4x\left(\dfrac{60}{x^2}\right) = 0.12x^2 + \dfrac{24}{x}$

$C'\,(x) = \dfrac{d}{dx}\left(0.12x^2 + \dfrac{24}{x}\right) = 0.24x - \dfrac{24}{x^2}$

Solve for x with $C'\,(x) = 0$

$$\begin{aligned} 0.24x - \frac{24}{x^2} &= 0 \\ 0.24x^3 &= 24 \\ x^3 &= 100 \\ x &= \sqrt[3]{100} \approx 4.641\,588\,833\,612\,78 \end{aligned}$$

$y = \dfrac{60}{x^2} = \dfrac{60}{\left(\sqrt[3]{100}\right)^2} = \dfrac{60\sqrt[3]{100}}{100} = \dfrac{3}{5}\sqrt[3]{100}$

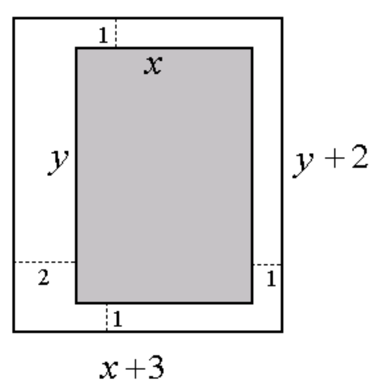
base is $\sqrt[3]{100}$ by $\sqrt[3]{100}$ and the height is $\dfrac{3}{5}\sqrt[3]{100}$

b) $C''\,(x) = \dfrac{48}{x^3} + 0.24$ We evaluate this at $x = \sqrt[3]{100}$

$C''\left(\sqrt[3]{100}\right) = \dfrac{48}{\left(\sqrt[3]{100}\right)^3} + 0.24 = 0.72$

Since C'' is positive here, the extremum is indeed a minimum

12. a) Let us denote the dimensions of the printed part by x and y. The page is then $x+3$ by $y+2$. We know that $xy = 27 \implies y = \dfrac{27}{x}$



Need to find: minimum of $(x+3)\,(y+2)$

$$\begin{aligned} A &= (x+3)\,(y+2) = xy + 2x + 3y + 6 \\ &= 27 + 2x + 3y + 6 \\ &= 2x + 3y + 33 \\ A\,(x) &= 2x + 3\left(\frac{27}{x}\right) + 33 = 2x + \frac{81}{x} + 33 \end{aligned}$$

$A'\,(x) = 2 - \dfrac{81}{x^2}$

We solve for the zero of the derivative:

$$\begin{aligned} 2 - \frac{81}{x^2} &= 0 \\ 2 &= \frac{81}{x^2} \\ x^2 &= \frac{81}{2} \\ x &= \pm \frac{9}{\sqrt{2}} = \pm \frac{9}{2}\sqrt{2} \end{aligned}$$

We discard the negative solution and so $x = \frac{9}{2}\sqrt{2}$. Then $y = \frac{27}{x} = \frac{27}{\frac{9}{\sqrt{2}}} = 3\sqrt{2}$. This means that the optimal page is $x + 3$ by $y + 2$ i.e. $\frac{9}{2}\sqrt{2} + 3$ by $3\sqrt{2} + 2$

So the optimal dimensions are $3 + \frac{9\sqrt{2}}{2} \approx 9.363\,96$ and $3\sqrt{2} + 2 \approx 6.242\,641$ centimeters

b) We will ise the second derivative test.

$A''(x) = \frac{d}{dx}\left(2 - \frac{81}{x^2}\right) = \frac{162}{x^3}$

$A''\left(\frac{9}{\sqrt{2}}\right) = \frac{162}{\left(\frac{9}{\sqrt{2}}\right)^3}$ is positive indicating a relative minimum for A at $x = \frac{9}{2}\sqrt{2}$.