1. Compute each of the following limits.

a) $\lim\limits_{n\to\infty}\left(1-\dfrac{2}{n}\right)^{n-2}$

b) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{5n}$

c) $\lim\limits_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)$

d) $\lim\limits_{x\to\infty}\left(\dfrac{\sqrt{x^2-1}}{x}\right)$

e) $\lim\limits_{x\to-\infty}\dfrac{\sin x}{x}$

f) $\lim\limits_{x\to-\infty}\dfrac{12-x^2}{3x^2-7x+2}$

g) $\lim\limits_{x\to\infty}\log_2\left(\dfrac{40x^2-x+1}{5x^2+20x-3}\right)$

h) $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+1}\right)^{x}$

i) $\lim\limits_{x\to\infty}\dfrac{-x^3+2x+1}{5x-3x^3}$

j) $\lim\limits_{x\to\infty}\log_{0.2}x$

k) $\lim\limits_{x\to\infty}\left(\log_2\left(-x^3+5x^2\right)\right)$

l) $\lim\limits_{x\to\infty}\dfrac{2-\sqrt{x}}{3+7\sqrt{x-1}}$

m) $\lim\limits_{x\to\infty}\dfrac{5\cdot 2^{3x-1}}{3^{2x-1}}$

n) $\lim\limits_{x\to\infty}\dfrac{\dfrac{1}{4-x}-\dfrac{1}{4+x}}{\dfrac{1}{x}}$

o) $\lim\limits_{x\to\infty}\dfrac{3-\sqrt{9+\dfrac{1}{x}}}{\dfrac{1}{x}}$

p) $\lim\limits_{x\to\infty}\left(\log_2\left(\dfrac{1}{x^2+1}\right)\right)$

q) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}}$

r) $\lim\limits_{x\to\infty}\dfrac{2^x+2^{-x}}{2^x-2^{-x}}$

s) $\lim\limits_{x\to-\infty}\dfrac{2^x+2^{-x}}{2^x-2^{-x}}$

t) $\lim\limits_{x\to\infty}\left(\ln\left(2x^2+3x+1\right)-\ln\left(3x^2-5x+2\right)\right)$

2. Compute each of the following limits.

a) $\lim\limits_{x\to0}\dfrac{\tan x}{3x}$

b) $\lim\limits_{x\to4^-}\dfrac{|2x-8|}{x-4}$

c) $\lim\limits_{x\to4^+}\dfrac{|2x-8|}{x-4}$

d) $\lim\limits_{x\to0^-}\dfrac{\cos x-1}{x^2}$

e) $\lim\limits_{x\to(\pi/2)^+}\tan x$

f) $\lim\limits_{x\to0}\dfrac{\dfrac{1}{2-x}-\dfrac{1}{2+x}}{x}$

g) $\lim\limits_{x\to4}\dfrac{3-\sqrt{5+x}}{x-4}$

h) $\lim\limits_{x\to4^+}\dfrac{\sqrt{x-4}}{x-4}$

i) $\lim\limits_{x\to4^-}\dfrac{\sqrt{x-4}}{x-4}$

j) $\lim\limits_{x\to0}\dfrac{\sin 2x}{\sin 3x}$

k) $\lim\limits_{x\to1/\sqrt{3}}\tan^{-1}x$

l) $\lim\limits_{a\to0}\dfrac{\sqrt{1+5a}-1}{a}$

m) $\lim\limits_{y\to1^-}\dfrac{1-\dfrac{1}{y^2}}{\left|1-\dfrac{1}{y}\right|}$

n) $\lim\limits_{x\to0^-}\dfrac{\sin x}{\sqrt{1-\cos x}}$

o) $\lim\limits_{x\to5}\dfrac{x^2-4x-5}{x^2-25}$

p) $\lim\limits_{x\to10}\dfrac{\sqrt{x^2-100}}{x-10}$

q) $\lim\limits_{x\to0}\dfrac{(2-x)^3-8}{x}$

r) $\lim\limits_{x\to0^-}\dfrac{\sin 3x}{x}$

s) $\lim\limits_{x\to6}\dfrac{x^2-6x}{x^2-36}$

t) $\lim\limits_{x\to5}\dfrac{x^2-25}{2-\sqrt{x-1}}$

u) $\lim\limits_{x\to\pi/4}\dfrac{\tan x-1}{x-\dfrac{\pi}{4}}$

v) $\lim\limits_{x\to0}\dfrac{1-\cos^2(3x)}{x^2}$

3. a) Prove that $\lim\limits_{x\to0^+}\dfrac{\sin x}{x}=1$. b) Prove that if $f(x)=\cos x$, then $f'(x)=-\sin x$.

4. Differentiate each of the following, using the definition of the derivative.

a) $f(x)=\sqrt{2x-1}$ b) $f(x)=\dfrac{1}{x^2-1}$ c) $f(x)=\sqrt{1-x^2}$

5. Differentiate each of the following functions.

a) $f(\theta)=\sin\theta\cos\theta$

b) $f(x)=\dfrac{\sin x}{\sqrt{x}}$

c) $f(x)=\cos^2 x+\sin^2 x$

d) $f(x)=\dfrac{x^4-3x^2+x-2}{x^3}$

e) $f(y)=\dfrac{1}{y^2}+\dfrac{1}{y}+\dfrac{1}{\sqrt{y}}+\sqrt{y}$

f) $f(x)=3x^5-2\sqrt[7]{x^3}+8\sqrt[3]{x^7}-\pi^5$

For Problems 6.-7. Suppose that a small object is moving along a vertical line. Distances are measured in meters, time in seconds. The motion of the object can be described one of the following.

A) The object is moving upward and its speed is increasing.

B) The object is moving upward and its speed is decreasing.

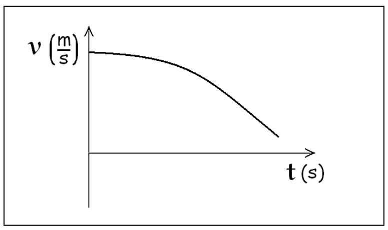
C) The object is moving downward and its speed is increasing.

D) The object is moving downward and its speed is decreasing.

E) At first the obejct has been moving upward but later it is moving downward.

F) At first the obejct has been moving downward but later it is moving upward.

6. Match each of the following graphs to one of the descriptions given above, assuming that the graphs are depicting the velocity function, $v(t)$ of the object as a function of time.



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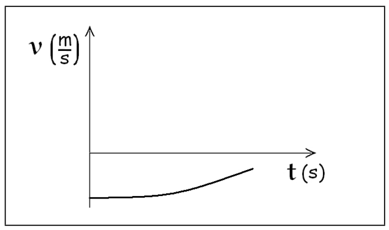

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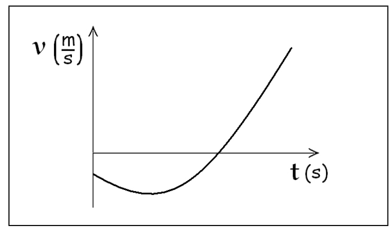

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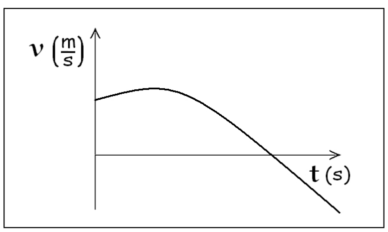

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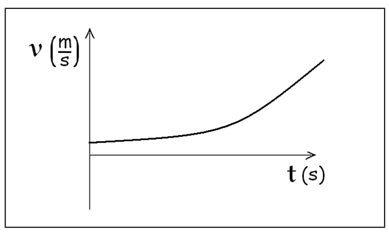

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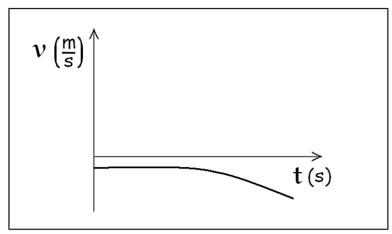

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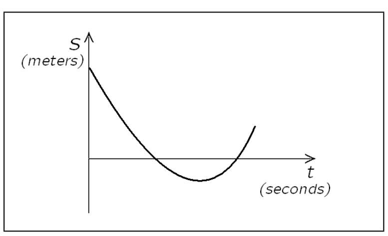
7. Match each of the following graphs to one of the descriptions given above, assuming that the graphs are depicting the position function, $s(t)$ of the object as a function of time.



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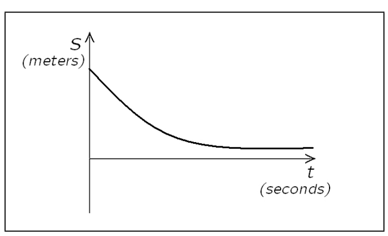

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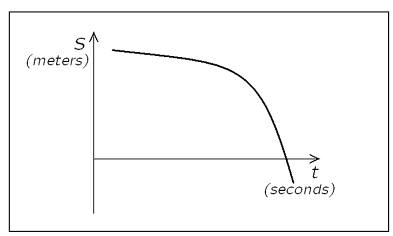

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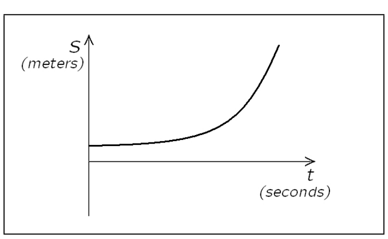

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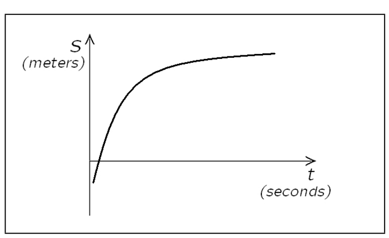

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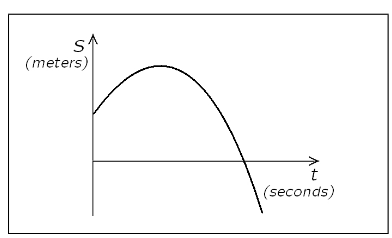

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8. Find an equation for the tangent line drawn to the graph of $f(x) = x^2 - \dfrac{2}{x}$ at $x = -2$.

9. We know the following things about a function f. $f'(x) = 20x^3 - 3$ and $f(-1) = 16$. Find a possible equation for f.

10. Find the value of c for which the following is true: the tangent line drawn to the graph of $y = \dfrac{1}{x}$ has $x-$intercept $(6, 0)$.

11. Suppose that $f(x) = 12x^3 + bx^2 + cx + 7$. Find the exact values of b and c if we know that f has a relative maximum at $x = -2$ and a relative minimum at $x = 3$.

12. Prove that the function given is one-to-one: $f(x) = 3x^5 - 50x^3 + 390x - 1200$.

13. Find the value of a so that the line $y = 2x$ is a tangent line to the parabola $y = ax^2 + 5$.

14. Find a third degree polynomial $P(x)$ such that $P(0) = -5$, $P'(0) = 3$, $P''(0) = -6$ and $P'''(0) = 60$.

15. Suppose that a small object is moving along a vertical line. (Distances are measured in meters, time in seconds.) The position function of the object is given by $s(t) = -t^3 + 18t^2 - 60t$.

    a) Compute the velocity function, $v(t)$ and sketch its graph.

    b) Describe the object's motion at $t = 0$.

    c) At what time is the object moving upward?

    d) At what time is the object moving fastest upward?

16. Find all values of p for which the given function is continuous on $\mathbb{R}$.

$$f(x) = \begin{cases} 2x - 1 & \text{if} \quad x < -3 \\ 2x^2 + px + 5 & \text{if} \quad x \geq 3 \end{cases}$$

17. Find all values of p and q for which the given function is differentiable on $\mathbb{R}$.

$$f(x) = \begin{cases} -3x + 13 & \text{if} \quad x < 2 \\ x^2 + px + q & \text{if} \quad x \geq 2 \end{cases}$$

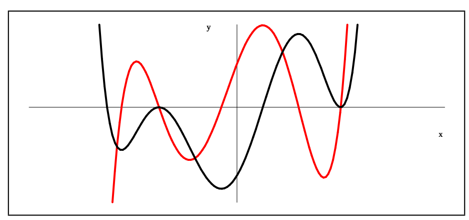
18. Find the $x-$coordinate of all relative maximums and minimums of each of the functions given below.

    a) $f(x) = x^3(5x - 2)$ b) $f(x) = 2x + \dfrac{18}{x}$ c) $f(x) = x^3 + 3x^2 - 24x + 24$

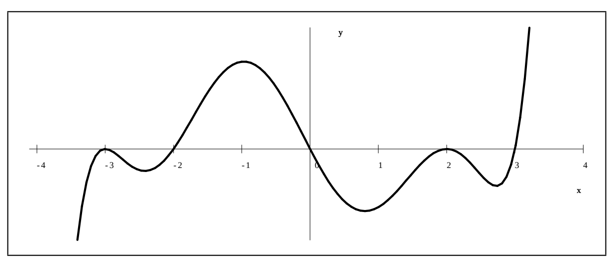
19. Find all relative and absolute maximums and minimums for the $f(x) = 6x^5 - 15x^4 + 5x^6 + 60$ on $[-3, 3]$. Sketch the graph of both f and f' on this domain.

20. Prove that the function $f(x) = \sin x - x$ does not have any relative minimums or maximums.

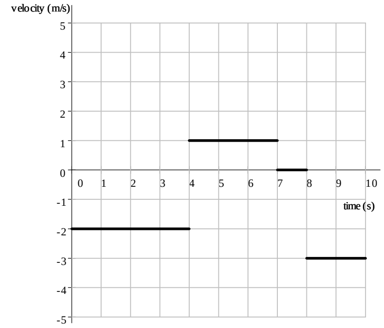
21. The graph shows a function f and its first derivative, f'. Which is which?



22. The graph shows f', the first derivative of a function f.
    a) Find all values of x for which the function f has a local maximum at x.
    b) Find all values of x for which the function f has a local minimum at x.



23. The picture shows the velocity function, $v(t)$ of an object. (Time is measured in seconds, distance in meters, velocity in $\frac{\mathrm{m}}{\mathrm{s}}$. Positive direction is upward.).
a) Suppose that the object starts at a height of $5\,\mathrm{m}$. Where is the object at $t = 10$?
b) Suppose that the object starts at a height of $9\,\mathrm{m}$. Where is the object at $t = 10$?



24. We shoot a small object upward, from the top of a tower. The acceleration function of the object is $a(t) = -10$. (Location is measured in meters, velocity in $\frac{\mathrm{m}}{\mathrm{s}}$, acceleration in $\frac{\mathrm{m}}{\mathrm{s}^2}$.)
a) Given that $v(0) = 160$, find $v(t)$, the velocity function of the object.
b) Given that $h(0) = 525$, find $h(t)$, the location function of the object.
c) Find the maximum height that the object reaches.

25. State both versions of the Intermediate Value Theorem.

26. Use the intermediate value theorem to prove that the function $f(x) = x^4 - 9x^3 - 9x^2 + 81x + 10$ has a zero in the interval $[-1, 1]$.

27. Assume the product rule of derivatives, i.e $(fg)' = f'g + fg'$ and derive from it the product rule of derivatives for three functions.

Answers

1. a) $\frac{1}{e^2}$ b) e^5 c) 0 d) 1 e) 0 f) $-\frac{1}{3}$ g) 3
   h) $\frac{1}{e}$ i) $\frac{1}{3}$ j) $-\infty$ k) undefined l) $-\frac{1}{7}$ m) 0
   n) -2 o) $-\frac{1}{6}$ p) $-\infty$ q) $\frac{1}{1-\sqrt{2}}$ r) 1
   s) -1 t) $\ln\left(\frac{2}{3}\right)$

2. a) $\frac{1}{3}$ b) -2 c) 2 d) $-\frac{1}{2}$ e) $-\infty$ f) $\frac{1}{2}$ g) $-\frac{1}{6}$
   h) ∞ i) undefined j) $\frac{2}{3}$ k) $\frac{\pi}{6}$ l) $\frac{5}{2}$ m) -2
   n) $-\sqrt{2}$ o) $\frac{3}{5}$ p) undefined q) -12 r) 3
   s) $\frac{1}{2}$ t) -40 u) 2 v) 9 3., 4. see handout

5. a) $\cos^2\theta - \sin^2\theta = \cos 2\theta$ b) $-\frac{\sqrt{x}\cos x}{x} - \frac{\sqrt{x}\sin x}{2x^2}$
   c) 0 d) $1+\frac{3}{x^2}-\frac{2}{x^3}+\frac{6}{x^4}$ e) $-\frac{2}{y^3}-\frac{1}{y^2}-\frac{1}{2y\sqrt{y}}+\frac{1}{2\sqrt{y}}$
   f) $15x^4 - \frac{6}{7x}\sqrt[7]{x^3} + \frac{56}{3x}\sqrt[3]{x^7}$

6. 1B 2D 3F 4E 5A 6C 7. 1F 2D 3C 4A 5B 6E

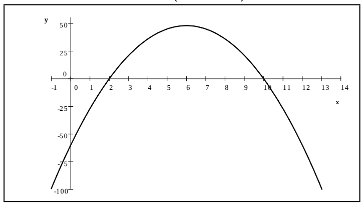
8. $y = -\frac{7}{2}x - 2$ 9. $f(x) = 5x^4 - 3x + 8$

10. 3 11. $b = -18, c = -216$

12. $f'(x) = 15\left(\left(x^2-5\right)^2 + 1\right)$
    f' is always positive, hence f is always strictly increasing. Thus one-to-one.

13. $\frac{1}{5}$ 14. $P(x) = 10x^3 - 3x^2 + 3x - 5$

15. a) $v(t) = -3t^2 + 36t - 60 = -3(t-2)(t-10)^3$
    $= -3(t-6)^2 + 48$

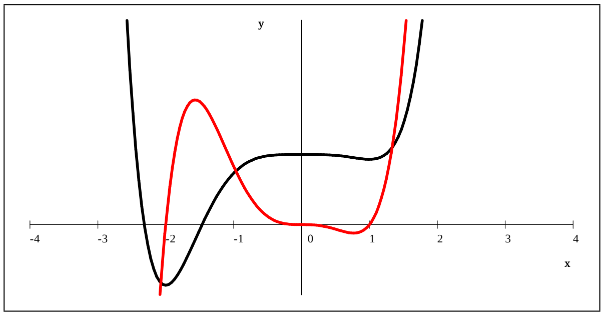


    b) moving downward with a speed of $60\frac{\text{m}}{\text{s}}$
    c) between $t = 2\,\text{s}$ and $t = 10\,\text{s}$
    d) at $t = 6\,\text{s}$, the greatest upward speed is $48\frac{\text{m}}{\text{s}}$

16. $p = 10$ 17. $p = -\frac{7}{2}, q = 10$

18. a) no relative max, relative min at $x = \frac{3}{10}$
    b) relative max at $x = -3$ relative min at $x = 3$
    c) relative max at $x = -4$, relative min at $x = 2$

19. $f'(x) = 30x^5 + 30x^4 - 60x^3$
    $= 30(x+2)x^3(x-1)$
    rel min: $(-2,-52)$ and $(1,56)$
    abs min: $(-2,-52)$
    rel max: $(0,60)$
    abs max: $(-3,332)$



20. $f'(x) = \cos x - 1$ is always negative or zero.
    Thus f' never changes sign, so f has no relative max or min.

21. the black graph is f, the red graph is f'

22. a) $x = 0$ b) $x = -2$ and $x = 3$

23. a) -6 b) -2

24. a) $v(t) = -10t + 160$
    b) $h(t) = -5t^2 + 160t + 525$
    c) $h_{\text{max}} = 1805\,\text{m}$

25. see handout

26. f is a polynomial, thus continuous everywhere, including the closed interval $[-1,1]$.
    Since $f(-1) = -70$ is negative and $f(1) = 74$ is positive, by the intermediate value theorem, there exists c in $(-1,1)$ with $f(c) = 0$.

27. $(fgh)' = f'(gh) + f(gh)' = f'(gh) + f(g'h + gh')$
    $= f'gh + fg'h + fgh'$