1. Consider the function $f(x) = (x+1)^7 (x-3)^8$
   a) Sketch the graph of f.
   b) Compute the derivative of f and factor it as much as possible.
   c) Sketch the graph of the derivative.
   d) Find the $x-$coordinate of all relative extrema for $f(x)$.

2. Find each of the following limits.

   a) $\displaystyle\lim_{x\to 1+} \frac{x^2-1}{x^2-3x+2}$

   b) $\displaystyle\lim_{x\to 2+} \frac{x^2-1}{x^2-3x+2}$

   c) $\displaystyle\lim_{x\to\infty} \frac{2x^2+3x+1}{3x^2-5x+2}$

   d) $\displaystyle\lim_{x\to\infty} \frac{-x^3+2x+1}{x-3}$

   e) $\displaystyle\lim_{x\to 3} \frac{x^2-3x}{x^2-9}$

   f) $\displaystyle\lim_{x\to 3+} \frac{x^2+3x}{x^2-9}$

   g) $\displaystyle\lim_{x\to 0} \frac{\sqrt{16+x}-4}{x}$

   h) $\displaystyle\lim_{x\to 0} \frac{\tan x}{x}$

   i) $\displaystyle\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$

   j) $\displaystyle\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x$

   k) $\displaystyle\lim_{x\to\infty} \frac{3+2\ln\left(x^3\right)}{2-\ln 7x}$

   l) $\displaystyle\lim_{x\to\infty} \frac{3\cdot 2^{2x-1}}{5\cdot 3^{x+2}}$

   m) $\displaystyle\lim_{x\to\infty} \tan^{-1} x$

   n) $\displaystyle\lim_{x\to 0} \frac{x^2}{1-\cos 3x}$

   o) $\displaystyle\lim_{x\to\infty} \left(1-\frac{2}{x}\right)^x$

   p) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}}$

   q) $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{\sin 2x}$

3. Find the inverse for each of the following functions.
   a) $f(x) = \sqrt[7]{3x-5}$ b) $f(x) = 2^{3x-1}+1$ c) $f(x) = 2\ln(x+8)$

4. Differentiate each of the following.

   a) $f(x) = 7^x - x^7 + \sqrt{7x} - e^7$

   b) $f(x) = \dfrac{e^x - e^{-x}}{2}$

   c) $f(x) = \dfrac{1}{4}xe^{4x} - \dfrac{1}{16}e^{4x}$

   d) $f(x) = \dfrac{8x^2-20x+1}{4x-10}$

   e) $f(x) = \dfrac{e^x}{e^{2x}+1}$

   f) $f(x) = \ln\left(\ln\left(x^5\right)\right)$

   g) $f(x) = \log_5\left(\sqrt[3]{5x^3-7x^2+8}\right)$

   h) $f(x) = \tan(3x)$

   i) $f(x) = \dfrac{1}{1+x^6}$

   j) $f(x) = \dfrac{5^{2x-1}}{5^{x-4}}$

   k) $f(x) = \tan\left(2\pi x - \dfrac{\pi}{4}\right)$

   l) $f(x) = \sin^2 x + \cos^2 x$

   m) $g(x) = -\dfrac{1}{4x^2} - \dfrac{\ln x}{2x^2}$

   n) $f(\theta) = \dfrac{x^2}{x^2+1}$

5. Find the second derivative for the function $f(x) = \dfrac{5x-1}{5x+1}$.

6. Find all values of x for which the tangent line drawn to the graph of $f(x) = \dfrac{1}{2}x^2 - \ln x + 8$ is perpendicular to $x+2y=-7$.

7. For each of the following functions, answer the following questions.
   i) For what interval(s) is f increasing and decreasing?
   ii) Find both coordinates of all relative extrema.
   iii) Find both coordinates of all absolute extrema.

   a) $f(x) = x^3 - 3x^2 - 9x + 5$ on $[-4, 5]$
   b) $f(x) = x(x-4)^4$ on $[-1, 5]$
   c) $f(x) = x^3 e^{2x}$ on $[-2, 1]$

8. Compute each of the following indefinite integrals.

a) $\int \sqrt{x}\ dx$ d) $\int x^5 - 2ax - a^2\ da$ g) $\int 5\ dx$

b) $\int x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2}\ dx$ e) $\int (\sin x - \cos x)\, dx$ h) $\int 5\ dt$

c) $\int x^5 - 2ax - a^2\ dx$ f) $\int (x-2)(x+4)\ dx$ i) $\int \frac{3}{\sqrt[7]{x^4}}\ dx$

9. Find the equation of $F(x)$ if $F'(x) = 10x^4 - 6x^2 + 4x - 5$ and $F(-1) = 8$.

10. An object is moving along a vertical line. Its acceleration, as a function of time, is $a(t) = 24t - 10$. After 5 seconds, the velocity of the object is $v(5) = -20$ and its height is $h(5) = 25$.
a) Find the velocity function $v(t)$. b) Find the location function $h(t)$.

11. A box manufacturer plans to create an open box with a surface area of $30\,\mathrm{ft}^2$. What is the maximum size volume that can be formed by bending this material into a box? The box is to have a square base and rectangular sides.

12. A box manufacturer plans to create an open box with a volume of $2000\,\mathrm{cm}^3$. What are the dimensions that allow us to use the least material for box? The box is to have a square base and rectangular sides.

13. We want to build a flowerbed in shape of a circular sector. If the perimeter is to be 20 feet, what radius and central angle would guarantee the maximal area for this flowerbed?

14. Find all values of c that satisfy the statement of the Main Value Theorem for the given function.

a) $f(x) = x^3 - 2x + 7$ on domain $[-3, -1]$ b) $f(x) = x + \frac{3}{x}$ on $[1, 5]$ c) $f(x) = \sin x$ on $\left[0, \frac{\pi}{4}\right]$

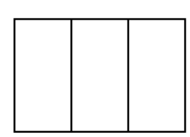
15. Let $P(x, y)$ be a point on the graph of $y = -x^2 + 1$ with $0 \leq x \leq 1$. Let $PQRS$ be a rectangle with one side on the $x-$axis and two vertices on the graph, as shown on the picture below. Find the exact value of the greatest possible area of such a rectangle.

16. Find two non-negative numbers x and y for which $2x + y = 30$, such that x^5y^3 is maximized.

17. Find an equation of the tangent line drawn to the graph of $f(x) = \frac{1}{(3x-1)^2}$ at $x = 1$.

18. Which point on $y = 1 - x^2$ is closest to the origin?

19. We have P meters of fencing and want to create three adjacent rectangular enclosings as shown on the figure. What is the maximal area we can enclose this way?



20. Suppose that g is a function with $g(5) = 2$ and $g'(5) = -3$. Compute $f'(5)$ if $f(x)$ is defined as

a) $f(x) = [g(x)]^4$ b) $f(x) = \ln(g(x))$ c) $f(x) = \frac{1}{g(x)}$ d) $f(x) = \sqrt{g(x)}$

21. Find the exact values of a and b so that the curve $y = x^2 + ax + b$ will be tangent to the line $y = 2x + 1$ at the point $(1, 3)$.

22. Find a polynomial function $P(x)$ such that P is of degree four (or less) and $f(0) = P(0)$, $f'(0) = P'(0)$, $f''(0) = P''(0)$ $f'''(0) = P'''(0)$, and $f^{(4)}(0) = P^{(4)}(0)$ if $f(x)$ is defined as
a) $f(x) = \sin x$ b) $f(x) = e^{-x/2}$

23. Suppose that f is a function with derivative $f'(x) = (9 - x^2)(x+2)^2(x-3)(1-x)x$

a) Graph f' b) Find all x for which f has a relative maximum. c) Find all x for which f has a relative minimum.

24. Find all values of a and b for which the given function is differentiable on $\mathbb{R}$.

$$f(x) = \begin{cases} -3x + b & \text{if } x < 2 \\ x^2 + ax + 5 & \text{if } x \geq 2 \end{cases}$$

25. Let $f(x) = \sin x$. Compute the exact value of $f^{(2017)}(0)$. (That is, evaluate the 2017th derivative of f at $x = 0$).

26. The vertical position of an object is given as $s(t) = t^3 - 12t^2 + 18t$ on the interval $[0, 6]$. (Distance is measured in meters, time in second)

a) When does the object have the greatest velocity?
b) When does the object have the greatest speed?

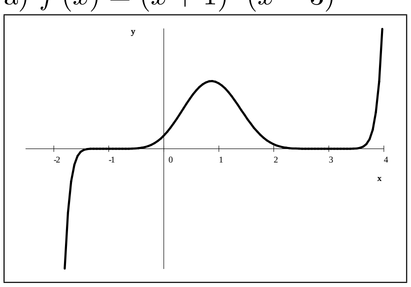
27. Give a complete analysis of the function $f(x) = x\sqrt{1 - x^2}$

28. State the Least Upper Bound Axiom (aka the Axiom of Completeness)

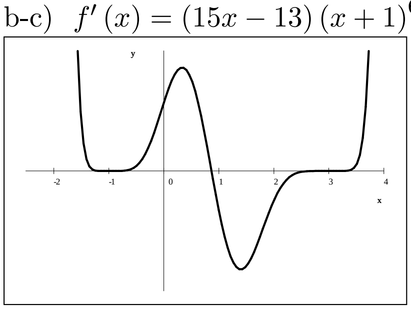
29. Prove the Mean Value Theorem.

Answers

1. a) $f(x) = (x+1)^7(x-3)^8$



b-c) $f'(x) = (15x - 13)(x+1)^6(x-3)^7$



d) relative min at $x = 3$ and relative max at $x = \dfrac{13}{15}$

2. a) -2 b) ∞ c) $\dfrac{2}{3}$ d) $-\infty$ e) $\dfrac{1}{2}$ f) ∞

g) $\dfrac{1}{8}$ h) 1 i) $\dfrac{1}{6}$ j) e^2 h) -6 l) ∞

m) $\dfrac{\pi}{2}$ n) $\dfrac{2}{9}$ o) $\dfrac{1}{e^2}$ p) 1 q) $\dfrac{3}{2}$

3. a) $f^{-1}(x) = \dfrac{x^7 + 5}{3}$

b) $f^{-1}(x) = \dfrac{1}{3}(\log_2(x-1) + 1)$

c) $f^{-1}(x) = e^{x/2} - 8$

4. a) $7^x \ln 7 - 7x^6 + \dfrac{7}{2\sqrt{7x}}$ b) $\dfrac{e^x + e^{-x}}{2}$

c) xe^{4x} d) $\dfrac{8x^2 - 40x + 49}{4x^2 - 20x + 25} = 2 - \dfrac{1}{(2x-5)^2}$

e) $\dfrac{e^x - e^{3x}}{(e^{2x}+1)^2}$ f) $\dfrac{1}{x \ln x}$ g) $\dfrac{15x^2 - 14x}{3 \ln 5 (5x^3 - 7x^2 + 8)}$

h) $3\tan^2 3x + 3 = \dfrac{3}{\cos^2(3x)}$ i) $\dfrac{-6x^5}{(x^6+1)^2}$

j) $\ln 5 \cdot 5^{x+3}$ k) $2\pi \sec^2\left(2\pi x - \dfrac{\pi}{4}\right)$ l) 0

m) $\dfrac{\ln x}{x^3}$ n) $\dfrac{2x}{(x^2+1)^2}$

5. $-\dfrac{100}{(5x+1)^3}$ 6. $1 + \sqrt{2}$

7. a) $f(x) = x^3 - 3x^2 - 9x + 5$ on the interval $[-4, 5]$

i) f increases on $(-4, -1)$, decreases on $(-1, 3)$, increases on $(3, 5)$

ii) rel. min: $(3,-22)$, rel. max: $(-1,10)$

iii) absolute minimum: $(-4,-71)$
absolute maximum: $(-1,10)$ and $(5,10)$

b) $f(x)=x(x-4)^4$ on the interval $[-1,5]$

i) increases on $\left(-1,\frac{4}{5}\right)$, decreases on $\left(\frac{4}{5},4\right)$, and increases on $(4,5)$

ii) rel. min: $(4,0)$, rel. max: $\left(\frac{4}{5},83.886\right)$

iii) absolute minimum: $(-1,-625)$,
absolue maximum: $\left(\frac{4}{5},83.886\right)$

c) $f(x)=x^3e^{2x}$ on $[-2,1]$

i) f decreases on $\left(-2,-\frac{3}{2}\right)$ and
increases on $\left(-\frac{3}{2},1\right)$

ii) rel. min: $\left(-\frac{3}{2},-\frac{27}{8}e^{-3}\right)$
there is no rel. max.

iii) absolute minimum: $\left(-\frac{3}{2},-\frac{27}{8}e^{-3}\right)$
abs maximum: $(1,e^2)$

8. a) $\frac{2}{3}x\sqrt{x}+C$ b) $\frac{x^3}{3}+\frac{x^2}{2}+x+\ln|x|-\frac{1}{x}+C$
c) $\frac{x^6}{6}-ax^2-a^2x+C$ d) $x^5a-xa^2-\frac{a^3}{3}+C$
e) $-\cos x-\sin x+C$ f) $\frac{1}{3}x^3+x^2-8x+C$
g) $5x+C$ h) $5t+C$ i) $7x^{3/7}+C$

9. $F(x)=2x^2-5x-2x^3+2x^5+1$

10. a) $v(t)=12t^2-10t-270$
b) $h(t)=4t^3-5t^2-270t+1000$

11. $5\sqrt{10}\,\text{ft}^3\approx 15.81139\,\text{ft}^3$

12. base: $10\sqrt[3]{4}\,\text{cm}$ by $10\sqrt[3]{4}\,\text{cm}$, height: $5\sqrt[3]{4}\,\text{cm}$

13. $r=5$ and $\alpha=2\text{rad}$

14. a) $-\frac{\sqrt{39}}{3}$ b) $\sqrt{5}$ c) $\cos^{-1}\left(\frac{2\sqrt{2}}{\pi}\right)\approx 0.45030$

15. $\frac{4}{9}\sqrt{3}$ 16. $x=\frac{75}{8}$ $y=\frac{45}{4}$ 17. $y=-\frac{3}{4}x+1$

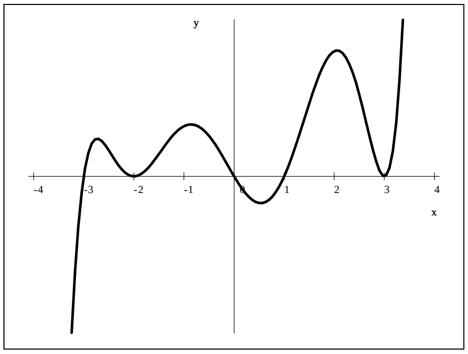
18. $\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)$ and $\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)$ 19. $\frac{P^2}{32}$

20. a) -96 b) $-\frac{3}{2}$ c) $\frac{3}{4}$ d) $-\frac{3}{4}\sqrt{2}$

21. $a=0,b=2$

22. a) $x-\frac{1}{6}x^3$ b) $1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{48}x^3+\frac{1}{384}x^4$

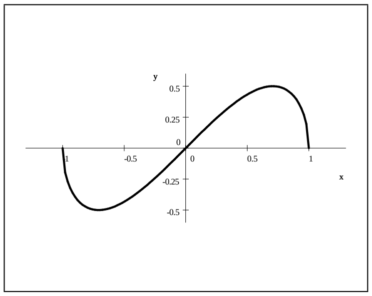
23. a) see below b) 0 c) $-3,1$



24. $a=-7,b=1$ 25. 1

26. a) at $t=0$, $v(0)=18$ b) at $t=4$, $v(4)=-30$

27. $f(x)=x\sqrt{1-x^2}$
$f'(x)=\frac{-2x^2+1}{\sqrt{1-x^2}}$ and $f''(x)=\frac{2x^3-3x}{(1-x^2)^{3/2}}$
domain: $[-1,1]$ range: $\left[-\frac{1}{2},\frac{1}{2}\right]$;
continuous on $[-1,1]$;
$y-$intercept: $(0,0)$;
x-intercepts: $(-1,0)$, $(0,0)$, and $(1,0)$
decreasing on $\left(-1,-\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2},1\right)$,
increasing on $\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
relative and absolute minimum: $\left(-\frac{\sqrt{2}}{2},-\frac{1}{2}\right)$,
relative and absolute maximum: $\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)$



28-29. see handout

