Exam 3 will cover the following topics. All topics covered by Quizzes 1-8 and Exams 1,2. These topics include quadratic inequalities, functions and their graphs, inverse functions, exponents and logarithms, limits, differentiation by using the definition, the sum rule, the constant multiplier rule, the generalized power rule, the product rule, chain rule and quotient rule, derivative of trigonometric, exponential and logarithmic functions , and applications of the derivative: increasing and decreasing functions, relative and absolute extrema, concavity and points of inflection, optimization, tangent lines, related rates, and antiderivatives.

The statement and proof of theorems will also be covered on the exam.

Students must be able to correctly state the following theorems:
Intermediate Value Theorem, Rolle's Theorem, Mean Value Theorem, Second Derivative Test.

Students must be able to prove the following theorems: differentiating functions using the definition (limit of the differential quotient). These include $\sin x$, $\cos x$, $\log_a x$, a^x. If a function is differentiable at a number x, then it is continuous there. The product rule for derivatives.

Review Problems

1. a) Consider $f(x) = \dfrac{x+4}{3x-5}$. Find the equation for the inverse of f, $f^{-1}(x)$.

   b) Graph $y = f(x)$. c) Find the range of f.

2. Find each of the following limits.

   a) $\displaystyle\lim_{x\to 1+} \frac{x^2-1}{x^2-3x+2}$ b) $\displaystyle\lim_{x\to 2+} \frac{x^2-1}{x^2-3x+2}$ c) $\displaystyle\lim_{x\to\infty} \frac{2x^2+3x+1}{3x^2-5x+2}$

   d) $\displaystyle\lim_{x\to\infty} \frac{-x^3+2x+1}{x-3}$ e) $\displaystyle\lim_{x\to 3} \frac{x^2-3x}{x^2-9}$ f) $\displaystyle\lim_{x\to 3+} \frac{x^2+3x}{x^2-9}$

   g) $\displaystyle\lim_{x\to 0} \frac{\sqrt{16+x}-4}{x}$ h) $\displaystyle\lim_{x\to 0} \frac{\tan x}{x}$ i) $\displaystyle\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$

   j) $\displaystyle\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x$ k) $\displaystyle\lim_{x\to\infty} \frac{3+2\ln\left(x^3\right)}{2-\ln 7x}$ l) $\displaystyle\lim_{x\to\infty} \frac{3\cdot 2^{2x-1}}{5\cdot 3^{x+2}}$

3. Find the inverse for each of the following functions.

   a) $f(x) = \sqrt[7]{3x-5}$ b) $f(x) = 2^{3x-1}+1$ c) $f(x) = 2\ln(x+8)$

4. Differentiate each of the following.

   a) $f(x) = 7^x - x^7 + \sqrt{7x} - e^7$

   b) $f(x) = \dfrac{e^x - e^{-x}}{2}$

   c) $f(x) = \dfrac{1}{4}xe^{4x} - \dfrac{1}{16}e^{4x}$

   d) $f(x) = \dfrac{8x^2-20x+1}{4x-10}$

   e) $f(x) = \log_x\left(x^2+1\right)$

   f) $f(x) = \ln\left(\ln\left(x^5\right)\right)$

   g) $f(x) = \log_5\left(\sqrt[3]{5x^3-7x^2+8}\right)$

   h) $f(x) = \tan(3x)$

   i) $f(x) = \dfrac{1}{1+x^6}$

   j) $f(x) = \dfrac{5^{2x-1}}{5^{x-4}}$

5. Find the second derivative for the function $f(x) = \dfrac{5x-1}{5x+1}$.

6. Find all values of x for which the tangent line drawn to the graph of $f(x) = \dfrac{1}{2}x^2 - \ln x + 8$ is perpendicular to $x + 2y = -7$.

7. For each of the following functions, answer the following questions.
   i) For what interval(s) is f increasing and decreasing?
   ii) Find both coordinates of all relative extrema.
   iii) Find both coordinates of all absolute extrema.
   iv) For what interval(s) is f concave up and concave down?
   v) Find both coordinates of all points of inflection.
   vi) Sketch the graph of the function.

   a) $f(x) = x^3 - 3x^2 - 9x + 5$ on $[-4, 5]$
   b) $f(x) = x(x-4)^4$ on $[-1, 5]$
   c) $f(x) = x^3 e^{2x}$ on $[-2, 1]$

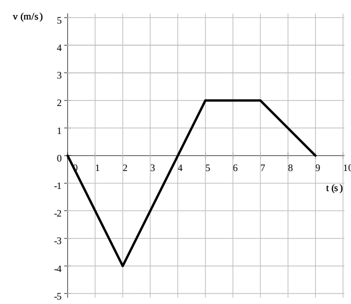
8. Compute each of the following indefinite integrals.

   a) $\displaystyle\int \frac{x-3}{x+2}\,dx$
   b) $\displaystyle\int x^2+x+1+\frac{1}{x}+\frac{1}{x^2}\,dx$
   c) $\displaystyle\int x^5 - 2ax - a^2\,dx$
   d) $\displaystyle\int x^5 - 2ax - a^2\,da$
   e) $\displaystyle\int e^{2x}dx$
   f) $\displaystyle\int 5^x - x^5\,dx$
   g) $\displaystyle\int \sin 5x\,dx$
   h) $\displaystyle\int \frac{2x-1}{x+3}\,dx$
   i) $\displaystyle\int 5^{2x-3}dx$
   j) $\displaystyle\int (3x-5)^{10}\,dx$
   k) $\displaystyle\int \sqrt{7x+1}\,dx$
   l) $\displaystyle\int \frac{3}{\sqrt[7]{x^4}}\,dx$

9. Find the equation of $F(x)$ if $F'(x) = 10x^4 - 6x^2 + 4x - 5$ and $F(-1) = 8$.

10. An object is moving along a vertical line. Its acceleration, as a function of time, is $a(t) = 24t - 10$. After 5 seconds, the velocity of the object is $v(5) = -20$ and its height is $h(5) = 25$.
    a) Find the velocity function $v(t)$. b) Find the location function $h(t)$.

11. Graph the location function $h(t)$ of an object if $h(0) = 0$ and the velocity function, $v(t)$ is given on the graph given. Compute $h(2)$, $h(4)$, $h(5)$, $h(7)$, and $h(9)$.



12. We want to build a flowerbed in shape of a circular sector. If the perimeter is to be 20 feet, what radius and central angle would guarantee the maximal area for this flowerbed?

13. A box manufacturer desires to create an open box with a surface area of $30\,\mathrm{ft}^2$. What is the maximum size volume that can be formed by bending this material into a box? The box is to have a square base and rectangular sides.

14. Let $ABCD$ be a unit square. Find the coordinates of point P on line segment CD so that the perimeter of triangle ABP is a) minimal b) maximal.

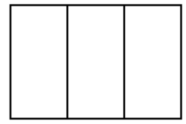
15. Let $P(x, y)$ be a point on the graph of $y = -x^2 + 1$ with $0 \le x \le 1$. Let $PQRS$ be a rectangle with one side on the $x-$axis and two vertices on the graph, as shown on the picture below. Find the exact value of the greatest possible area of such a rectangle.

16. Find two non-negative numbers x and y for which $2x + y = 30$, such that x^5y^3 is maximized.

17. Find an equation of the tangent line drawn to the graph of $f(x) = \dfrac{1}{(3x-1)^2}$ at $x = 1$.

18. Which point on $y = 1 - x^2$ is closest to the origin?

19. We have P meters of fencing and want to create three adjacent rectangular enclosings as shown on the figure. What is the maximal area we can enclose this way?



20. Suppose that g is a function with $g(5) = 2$ and $g'(5) = -3$. Compute $f'(5)$ if $f(x)$ is defined as
a) $f(x) = [g(x)]^4$ b) $f(x) = \ln(g(x))$ c) $f(x) = \dfrac{1}{g(x)}$ d) $f(x) = \sqrt{g(x)}$

21. Find the exact values of a and b so that the curve $y = x^2 + ax + b$ will be tangent to the line $y = 2x + 1$ at the point $(1, 3)$.

22. Find a polynomial function $P(x)$ such that P is of degree four (or less) and $f(0) = P(0)$, $f'(0) = P'(0)$, $f''(0) = P''(0)$ $f'''(0) = P'''(0)$, and $f^{(4)}(0) = P^{(4)}(0)$ if $f(x)$ is defined as
a) $f(x) = \sin x$ b) $f(x) = e^{-x/2}$

23. Suppose that f is a function with derivative $f'(x) = (9 - x^2)(x + 2)^2(x - 3)(1 - x)x$
a) Graph f'
b) Find all x for which f has a relative maximum.
c) Find all x for which f has a relative minimum.
d) How many points of inflection does f have?
e) Plot the graph of f in the same coordinate system with f'.
f) Is it possible that f does not have any x−intercept?
g) Graph f'' in the same coordinate system with f'.

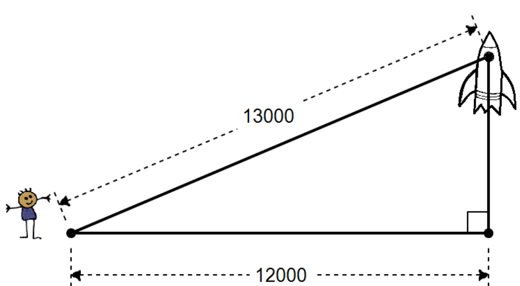
24. Consider the function $f(x) = (x - 1)^{10}(4 - x)^5$. Find all values of x for which f has a
a) a relative maximum b) a relative minimum c) a point of inflection.

25. Car A is traveling west at $40 \dfrac{\text{mi}}{\text{h}}$ and car B is traveling north at $45 \dfrac{\text{mi}}{\text{h}}$. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

26. A company finds that if x is the amount of money spent on research and y is the amount of money spent on advertising, then they can make a profit of $\sqrt[4]{xy^3}$. The company has \$50 000 to spend on these two activities. How much money should the company spend on research and on advertising to maximize its profit?

27. A company finds that if they produce q units of a product, then they can sell them all at a price of $p(q) = \sqrt{6300 - 0.02q}$. What production level would guarantee the maximum revenue?

28. At a distance of 12 000 meters from the launch site, a spectator is observing a rocket being launched vertically. What is the speed of the rocket at the instant when the distance of the rocket from the spectator is 13 000 meters and is increasing at the rate of 480 meters per second?



29. Give a complete analysis of the function $f(x) = x\sqrt{1 - x^2}$

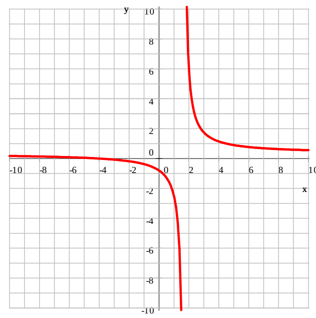
30. Each side of a square is increasing at a rate of $6 \dfrac{\text{cm}}{\text{s}}$. At what rate is the area of the square increasing when the area of the square is 25 cm?

Answers

1.) a) $f^{-1}(x) = \dfrac{5x+4}{3x-1}$

b) see next

c) $y \neq \dfrac{1}{3}$



2.) a) -2 b) ∞ c) $\dfrac{2}{3}$ d) $-\infty$ e) $\dfrac{1}{2}$ f) ∞ g) $\dfrac{1}{8}$ h) 1 j) e^2 h) -6 l) ∞

3.) a) $f^{-1}(x) = \dfrac{x^7+5}{3}$ b) $f^{-1}(x) = \dfrac{1}{3}\left(\log_2(x-1)+1\right)$ c) $f^{-1}(x) = e^{x/2} - 8$

4.) a) $7^x \ln 7 - 7x^6 + \dfrac{7}{2\sqrt{7x}}$ b) $\dfrac{e^x + e^{-x}}{2}$ c) xe^{4x} d) $\dfrac{8x^2-40x+49}{4x^2-20x+25} = 2 - \dfrac{1}{(2x-5)^2}$

e) $\dfrac{\dfrac{2x\ln x}{x^2+1} - \ln\left(x^2+1\right)\left(\dfrac{1}{x}\right)}{(\ln x)^2} = \dfrac{2x}{\ln x\,(x^2+1)} - \dfrac{\ln\left(x^2+1\right)}{x(\ln x)^2}$ f) $\dfrac{1}{x\ln x}$ $\dfrac{15x^2-14x}{3\ln 5\,(5x^3-7x^2+8)}$

h) $3\tan^2 3x + 3 = \dfrac{3}{\cos^2(3x)}$ i) $\dfrac{-6x^5}{(x^6+1)^2}$ j) $\ln 5 \cdot 5^{x+3}$ 5.) $-\dfrac{100}{(5x+1)^3}$ 6.) $1+\sqrt{2}$

7.) a) $f(x) = x^3 - 3x^2 - 9x + 5$ on the interval $[-4, 5]$

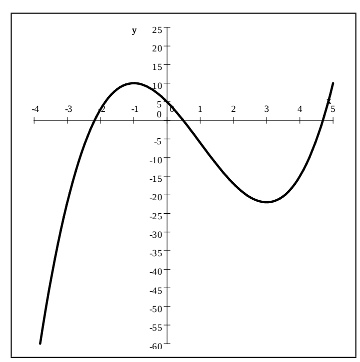
i) f increases on $(-4,-1)$, decreases on $(-1,3)$, and increases on $(3,5)$

ii) rel. min: $(3,-22)$, rel. max: $(-1,10)$

iii) abs. min: $(-4,-71)$ abs max: $(-1,10)$ and $(5,10)$

iv) concave down on $(-4,1)$ and concave up on $(1,5)$

v) point of inflection: $(1,-6)$



b) $f(x) = x(x-4)^4$ on the interval $[-1, 5]$

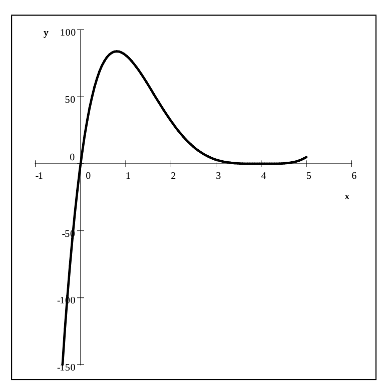
i) increases on $\left(-1, \dfrac{4}{5}\right)$, decreases on $\left(\dfrac{4}{5}, 4\right)$, and increases on $(4,5)$

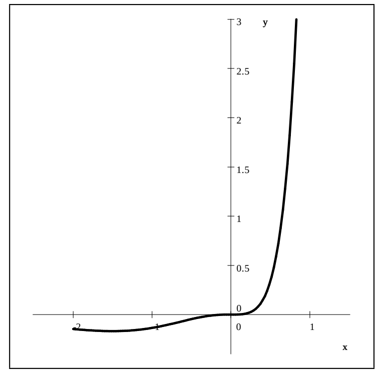
ii) rel. min: $(4,0)$, rel. max: $\left(\dfrac{4}{5}, 83.886\right)$

iii) abs. min: $(-1,-625)$, abs. max: $\left(\dfrac{4}{5}, 83.886\right)$

iv) concave down on $\left(-1, \dfrac{8}{5}\right)$ and concave up on $\left(\dfrac{8}{5}, 5\right)$

v) point of inflection: $\left(\dfrac{8}{5}, 53.084\right)$





c) $f(x) = x^3 e^{2x}$ on $[-2, 1]$

i) f decreases on $\left(-2, -\frac{3}{2}\right)$ and increases on $\left(-\frac{3}{2}, 1\right)$

ii) rel. min: $\left(-\frac{3}{2}, -\frac{27}{8}e^{-3}\right)$ there is no rel. max.

iii) abs. min: $\left(-\frac{3}{2}, -\frac{27}{8}e^{-3}\right)$ abs max: $(1, e^2)$

iv) concave up on $\left(-2, \frac{-3+\sqrt{3}}{2}\right)$ concave down on $\left(\frac{-3+\sqrt{3}}{2}, 0\right)$, and concave up on $(0, 1)$

v) pts of inflection: $\left(\frac{-3+\sqrt{3}}{2}, -0.071705\right)$, and $(0, 0)$

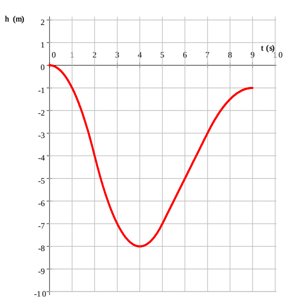
8.) a) $x - 5\ln|x+2| + C$ b) $\frac{x^3}{3} + \frac{x^2}{2} + x + \ln x - \frac{1}{x} + C$ c) $\frac{x^6}{6} - ax^2 - a^2x + C$

d) $x^5a - xa^2 - \frac{a^3}{3} + C$ e) $\frac{1}{2}e^{2x} + C$ f) $\frac{5^x}{\ln 5} - \frac{x^6}{6} + C$ g) $-\frac{1}{5}\cos 5x + C$

h) $2x - 7\ln|x+3| + C$ i) $\frac{5^{2x-3}}{2\ln 5} + C$ j) $\frac{(3x-5)^{11}}{33} + C$ k) $\frac{2}{21}(7x+1)^{3/2} + C$ l) $7x^{3/7} + C$

9.) $F(x) = 2x^2 - 5x - 2x^3 + 2x^5 + 1$ 10.) a) $v(t) = 12t^2 - 10t - 270$ b) $h(t) = 4t^3 - 5t^2 - 270t + 1000$

11.) $h(2) = -4$, $h(4) = -8$, $h(5) = -7$, $h(7) = -3$, $h(9) = -1$ 12.) $\frac{P}{4} = r$ and $\theta = 2$ rad 13.) $5\sqrt{10}$

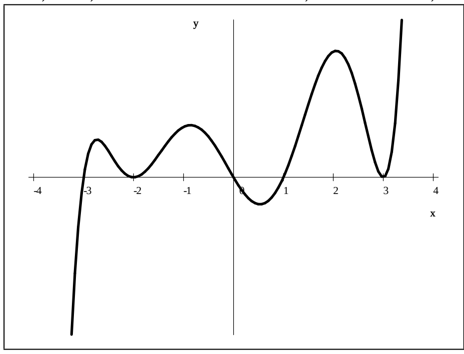


14.) a) $\left(\frac{1}{2}, 1\right)$ b) $(1, 0)$ and $(1, 1)$ 15.) $\frac{4}{9}\sqrt{3}$ 16.) $x = \frac{75}{8}$ $y = \frac{45}{4}$ 17.) $y = -\frac{3}{4}x + 1$

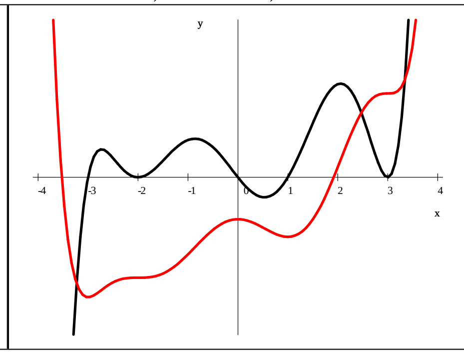
18.) $\left(-\frac{\sqrt{6}}{2}, -\frac{1}{2}\right)$ and $\left(\frac{\sqrt{6}}{2}, -\frac{1}{2}\right)$ 19.) $\frac{P^2}{32}$ 20.) a) -96 b) $-\frac{3}{2}$ c) $\frac{3}{4}$ d) $-\frac{3\sqrt{5}}{10}$

21.) $a = 0, b = 2$ 22.) a) $x - \frac{1}{6}x^3$ b) $1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4$

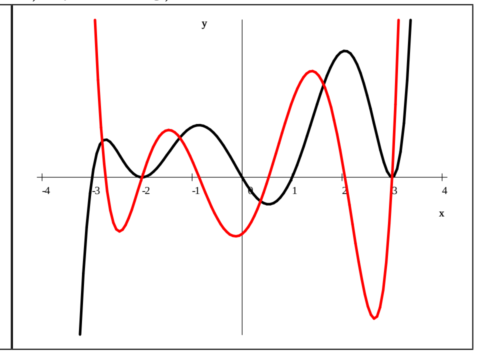
23.) a) see below b) 0 c) $-3, 1$ d) 6 e) see below f) yes g) see below



a)



e)



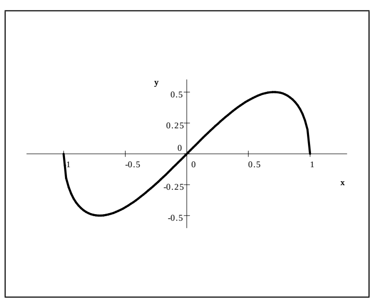
g)

24.) a) 3 b) 1 c) $3-\dfrac{1}{\sqrt{7}}, 3+\dfrac{1}{\sqrt{7}}, 4$ 25.) $60\dfrac{\text{mi}}{\text{h}}$

26.) \$12 500 on research and \$37 500 on advertising. 27.) 210 000 units 28.) 1248 meters per second

29.) $f(x) = x\sqrt{1-x^2}$ $f'(x) = \dfrac{-2x^2+1}{\sqrt{1-x^2}}$ $f''(x) = \dfrac{2x^3-3x}{(1-x^2)^{3/2}}$

domain: $[-1, 1]$ range: $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$; continuous on $[-1, 1]$; bounded; no asymptotes; $y-$intercept: $(0, 0)$;
x-intercepts: $(-1, 0)$, $(0, 0)$, and $(1, 0)$
decreasing on $\left(-1, -\dfrac{\sqrt{2}}{2}\right)$ and $\left(\dfrac{\sqrt{2}}{2}, 1\right)$, increasing on $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$
relative and absolute minimum: $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{1}{2}\right)$, relative and absolute maximum: $\left(\dfrac{\sqrt{2}}{2}, \dfrac{1}{2}\right)$
concave up on $(-1, 0)$, concave down on $(0, 1)$, point of inflection: $(0, 0)$
odd; $\displaystyle\lim_{x\to-\infty} f(x)$ and $\displaystyle\lim_{x\to\infty} f(x)$ are both undefined



30.) $60\dfrac{\text{cm}^2}{\text{s}}$