1. Consider the function $f(x) = (x+1)^7 (x-3)^8$
   a) Sketch the graph of f.
   b) Compute the derivative of f and factor it as much as possible.
   c) Sketch the graph of the derivative.
   d) Find the $x-$coordinate of all relative extrema for $f(x)$.

2. Find each of the following limits.

   a) $\lim\limits_{x\to 1+} \dfrac{x^2-1}{x^2-3x+2}$

   b) $\lim\limits_{x\to 2+} \dfrac{x^2-1}{x^2-3x+2}$

   c) $\lim\limits_{x\to \infty} \dfrac{2x^2+3x+1}{3x^2-5x+2}$

   d) $\lim\limits_{x\to \infty} \dfrac{-x^3+2x+1}{x-3}$

   e) $\lim\limits_{x\to 3} \dfrac{x^2-3x}{x^2-9}$

   f) $\lim\limits_{x\to 3+} \dfrac{x^2+3x}{x^2-9}$

   g) $\lim\limits_{x\to 0} \dfrac{\sqrt{16+x}-4}{x}$

   h) $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$

   i) $\lim\limits_{x\to 7} \dfrac{\sqrt{x+2}-3}{x-7}$

   j) $\lim\limits_{x\to \infty} \left(1+\dfrac{2}{x}\right)^x$

   k) $\lim\limits_{x\to \infty} \dfrac{3+2\ln(x^3)}{2-\ln 7x}$

   l) $\lim\limits_{x\to \infty} \dfrac{3\cdot 2^{2x-1}}{5\cdot 3^{x+2}}$

   m) $\lim\limits_{x\to \infty} \tan^{-1} x$

   n) $\lim\limits_{x\to 0} \dfrac{x^2}{1-\cos 3x}$

   o) $\lim\limits_{x\to \infty} \left(1-\dfrac{2}{x}\right)^x$

   p) $\lim\limits_{x\to \infty} \dfrac{x}{\sqrt{x^2-1}}$

   q) $\lim\limits_{x\to 0} \dfrac{\tan 3x}{\sin 2x}$

3. Find the inverse for each of the following functions.
   a) $f(x) = \sqrt[7]{3x-5}$ b) $f(x) = 2^{3x-1}+1$ c) $f(x) = 2\ln(x+8)$

4. Differentiate each of the following.

   a) $f(x) = 7^x - x^7 + \sqrt{7x} - e^7$

   b) $f(x) = \dfrac{e^x - e^{-x}}{2}$

   c) $f(x) = \dfrac{1}{4}xe^{4x} - \dfrac{1}{16}e^{4x}$

   d) $f(x) = \dfrac{8x^2-20x+1}{4x-10}$

   e) $f(x) = \dfrac{e^x}{e^{2x}+1}$

   f) $f(x) = \ln\left(\ln\left(x^5\right)\right)$

   g) $f(x) = \log_5\left(\sqrt[3]{5x^3-7x^2+8}\right)$

   h) $f(x) = \tan(3x)$

   i) $f(x) = \dfrac{1}{1+x^6}$

   j) $f(x) = \dfrac{5^{2x-1}}{5^{x-4}}$

   k) $f(x) = \tan\left(2\pi x - \dfrac{\pi}{4}\right)$

   l) $f(x) = \sin^2 x + \cos^2 x$

   m) $g(x) = -\dfrac{1}{4x^2} - \dfrac{\ln x}{2x^2}$

   n) $f(\theta) = \dfrac{x^2}{x^2+1}$

5. Find the second derivative for the function $f(x) = \dfrac{5x-1}{5x+1}$.

6. Find all values of x for which the tangent line drawn to the graph of $f(x) = \dfrac{1}{2}x^2 - \ln x + 8$ is perpendicular to $x + 2y = -7$.

7. For each of the following functions, answer the following questions.
   i) For what interval(s) is f increasing and decreasing?
   ii) Find both coordinates of all relative extrema.
   iii) Find both coordinates of all absolute extrema.

   a) $f(x) = x^3 - 3x^2 - 9x + 5$ on $[-4, 5]$
   b) $f(x) = x(x-4)^4$ on $[-1, 5]$
   c) $f(x) = x^3 e^{2x}$ on $[-2, 1]$

8. Compute each of the following indefinite integrals.

a) $\int \sqrt{x}\,dx$

b) $\int x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2}\,dx$

c) $\int x^5 - 2ax - a^2\,dx$

d) $\int x^5 - 2ax - a^2\,da$

e) $\int (\sin x - \cos x)\,dx$

f) $\int (x-2)(x+4)\,dx$

g) $\int 5\,dx$

h) $\int 5\,dt$

i) $\int \frac{3}{\sqrt[7]{x^4}}\,dx$

9. Find the equation of $F(x)$ if $F'(x) = 10x^4 - 6x^2 + 4x - 5$ and $F(-1) = 8$.

10. An object is moving along a vertical line. Its acceleration, as a function of time, is $a(t) = 24t - 10$. After 5 seconds, the velocity of the object is $v(5) = -20$ and its height is $h(5) = 25$.
a) Find the velocity function $v(t)$. b) Find the location function $h(t)$.

11. A box manufacturer plans to create an open box with a surface area of $30\,\mathrm{ft}^2$. What is the maximum size volume that can be formed by bending this material into a box? The box is to have a square base and rectangular sides.

12. A box manufacturer plans to create an open box with a volume of $2000\,\mathrm{cm}^3$. What are the dimensions that allow us to use the least material for box? The box is to have a square base and rectangular sides.

13. We want to build a flowerbed in shape of a circular sector. If the area is to be $800\,\mathrm{ft}^2$, what radius and central angle would guarantee the maximal area for this flowerbed?

14. Find all values of c that satisfy the statement of the Main Value Theorem for the given function.

a) $f(x) = x^3 - 2x + 7$ on domain $[-3, -1]$ b) $f(x) = x + \frac{3}{x}$ on $[1, 5]$ c) $f(x) = \sin x$ on $\left[0, \frac{\pi}{4}\right]$

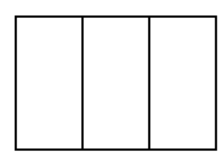
15. Let $P(x, y)$ be a point on the graph of $y = -x^2 + 1$ with $0 \le x \le 1$. Let $PQRS$ be a rectangle with one side on the $x-$axis and two vertices on the graph, as shown on the picture below. Find the exact value of the greatest possible area of such a rectangle.

16. Find two non-negative numbers x and y for which $2x + y = 30$, such that x^5y^3 is maximized.

17. Find an equation of the tangent line drawn to the graph of $f(x) = \frac{1}{(3x-1)^2}$ at $x = 1$.

18. Which point on $y = 1 - x^2$ is closest to the origin?

19. We have P meters of fencing and want to create three adjacent rectangular enclosings as shown on the figure. What is the maximal area we can enclose this way?



20. Suppose that g is a function with $g(5) = 2$ and $g'(5) = -3$. Compute $f'(5)$ if $f(x)$ is defined as

a) $f(x) = [g(x)]^4$ b) $f(x) = \ln(g(x))$ c) $f(x) = \frac{1}{g(x)}$ d) $f(x) = \sqrt{g(x)}$

21. Find the exact values of a and b so that the curve $y = x^2 + ax + b$ will be tangent to the line $y = 2x + 1$ at the point $(1, 3)$.

22. Find a polynomial function $P(x)$ such that P is of degree four (or less) and $f(0) = P(0)$, $f'(0) = P'(0)$, $f''(0) = P''(0)$ $f'''(0) = P'''(0)$, and $f^{(4)}(0) = P^{(4)}(0)$ if $f(x)$ is defined as
a) $f(x) = \sin x$ b) $f(x) = e^{-x/2}$

23. Suppose that f is a function with derivative $f'(x) = (9 - x^2)(x+2)^2(x-3)(1-x)x$

a) Graph f' b) Find all x for which f has a relative maximum. c) Find all x for which f has a relative minimum.

24. Find all values of a and b for which the given function is differentiable on $\mathbb{R}$.

$$f(x)=\begin{cases} -3x+b & \text{if} \quad x<2 \\ x^2+ax+5 & \text{if} \quad x\geq 2 \end{cases}$$

25. Let $f(x)=\sin x$. Compute the exact value of $f^{(2017)}(0)$. (That is, evaluate the 2017th derivative of f at $x=0$).

26. The vertical position of an object is given as $s(t)=t^3-12t^2+18t$ on the interval $[0,6]$. (Distance is measured in meters, time in second)

a) When does the object have the greatest velocity?
b) When does the object have the greatest speed?

27. Give a complete analysis of the function $f(x)=x\sqrt{1-x^2}$

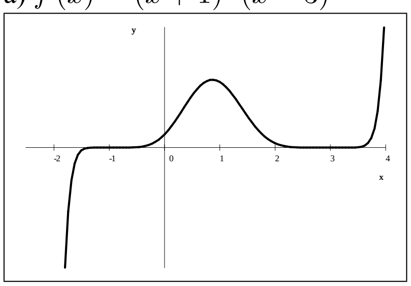
28. An underground telephone cable is to be laid between two boat docks on opposite banks of a straight river. One boathouse is 300 meters downstream from the other. The river is 200 meters wide. If the cost of laying the cable is \$30 per meter under water and \$20 per meter on land, how should the cable to be laid to minimize cost?

29. State the Least Upper Bound Axiom (aka the Axiom of Completeness)

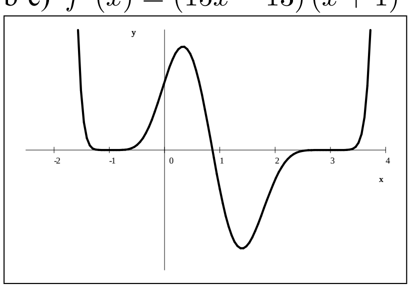
30. Prove the Mean Value Theorem

Answers

1. a) $f(x)=(x+1)^7(x-3)^8$



b-c) $f'(x)=(15x-13)(x+1)^6(x-3)^7$



d) relative min at $x=3$ and relative max at $x=\dfrac{13}{15}$

2. a) -2 b) ∞ c) $\dfrac{2}{3}$ d) $-\infty$ e) $\dfrac{1}{2}$ f) ∞

g) $\dfrac{1}{8}$ h) 1 i) $\dfrac{1}{6}$ j) e^2 h) -6 l) ∞

m) $\dfrac{\pi}{2}$ n) $\dfrac{2}{9}$ o) $\dfrac{1}{e^2}$ p) 1 q) $\dfrac{3}{2}$

3. a) $f^{-1}(x)=\dfrac{x^7+5}{3}$

b) $f^{-1}(x)=\dfrac{1}{3}(\log_2(x-1)+1)$

c) $f^{-1}(x)=e^{x/2}-8$

4. a) $7^x\ln 7-7x^6+\dfrac{7}{2\sqrt{7x}}$ b) $\dfrac{e^x+e^{-x}}{2}$

c) xe^{4x} d) $\dfrac{8x^2-40x+49}{4x^2-20x+25}=2-\dfrac{1}{(2x-5)^2}$

e) $\dfrac{e^x-e^{3x}}{(e^{2x}+1)^2}$ f) $\dfrac{1}{x\ln x}$ g) $\dfrac{15x^2-14x}{3\ln 5(5x^3-7x^2+8)}$

h) $3\tan^2 3x+3=\dfrac{3}{\cos^2(3x)}$ i) $\dfrac{-6x^5}{(x^6+1)^2}$

j) $\ln 5\cdot 5^{x+3}$ k) $2\pi\sec^2\left(2\pi x-\dfrac{\pi}{4}\right)$ l) 0

m) $\dfrac{\ln x}{x^3}$ n) $\dfrac{2x}{(x^2+1)^2}$

5. $-\dfrac{100}{(5x+1)^3}$

6. $1+\sqrt{2}$

7. a) $f(x)=x^3-3x^2-9x+5$ on the interval $[-4,5]$

i) f increases on $(-4,-1)$, decreases on $(-1,3)$, increases on $(3,5)$
ii) rel. min: $(3,-22)$, rel. max: $(-1,10)$
iii) absolute minimum: $(-4,-71)$
absolute maximum: $(-1,10)$ and $(5,10)$

b) $f(x) = x(x-4)^4$ on the interval $[-1, 5]$

i) increases on $\left(-1, \frac{4}{5}\right)$, decreases on $\left(\frac{4}{5}, 4\right)$, and increases on $(4, 5)$

ii) rel. min: $(4, 0)$, rel. max: $\left(\frac{4}{5}, 83.886\right)$

iii) absolute minimum: $(-1, -625)$, absolue maximum: $\left(\frac{4}{5}, 83.886\right)$

c) $f(x) = x^3 e^{2x}$ on $[-2, 1]$

i) f decreases on $\left(-2, -\frac{3}{2}\right)$ and increases on $\left(-\frac{3}{2}, 1\right)$

ii) rel. min: $\left(-\frac{3}{2}, -\frac{27}{8}e^{-3}\right)$ there is no rel. max.

iii) absolute minimum: $\left(-\frac{3}{2}, -\frac{27}{8}e^{-3}\right)$ abs maximum: $(1, e^2)$

8. a) $\frac{2}{3}x\sqrt{x} + C$ b) $\frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x| - \frac{1}{x} + C$
c) $\frac{x^6}{6} - ax^2 - a^2x + C$ d) $x^5a - xa^2 - \frac{a^3}{3} + C$
e) $-\cos x - \sin x + C$ f) $\frac{1}{3}x^3 + x^2 - 8x + C$
g) $5x + C$ h) $5t + C$ i) $7x^{3/7} + C$

9. $F(x) = 2x^2 - 5x - 2x^3 + 2x^5 + 1$

10. a) $v(t) = 12t^2 - 10t - 270$
b) $h(t) = 4t^3 - 5t^2 - 270t + 1000$

11. $5\sqrt{10}\,\text{ft}^3 \approx 15.81139\,\text{ft}^3$

12. base: $10\sqrt[3]{4}\,\text{cm}$ by $10\sqrt[3]{4}\,\text{cm}$, height: $5\sqrt[3]{4}\,\text{cm}$

13. $r = 20\sqrt{2}$ and $\alpha = 2\text{rad}$

14. a) $-\frac{\sqrt{39}}{3}$ b) $\sqrt{5}$ c) $\cos^{-1}\left(\frac{2\sqrt{2}}{\pi}\right) \approx 0.45030$

15. $\frac{4}{9}\sqrt{3}$ 16. $x = \frac{75}{8}$ $y = \frac{45}{4}$ 17. $y = -\frac{3}{4}x + 1$

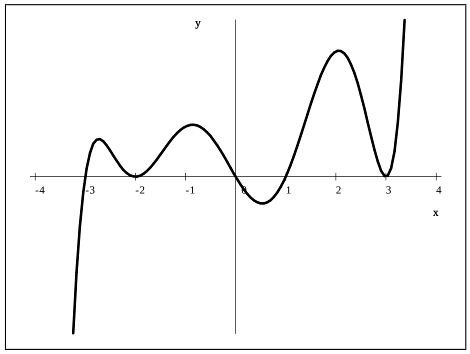
18. $\left(-\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ 19. $\frac{P^2}{32}$

20. a) -96 b) $-\frac{3}{2}$ c) $\frac{3}{4}$ d) $-\frac{3}{4}\sqrt{2}$

21. $a = 0, b = 2$

22. a) $x - \frac{1}{6}x^3$ b) $1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4$

23. a) see below b) 0 c) $-3, 1$



24. $a = -7, b = 1$ 25. 1

26. a) at $t = 0$, $v(0) = 18$ b) at $t = 4$, $v(4) = -30$

27. $f(x) = x\sqrt{1-x^2}$
$f'(x) = \frac{-2x^2+1}{\sqrt{1-x^2}}$ and $f''(x) = \frac{2x^3 - 3x}{(1-x^2)^{3/2}}$

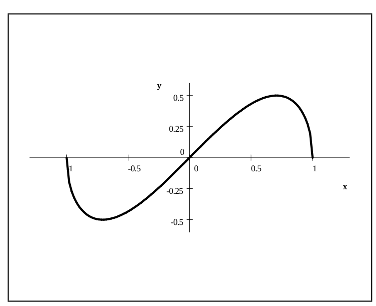
domain: $[-1, 1]$ range: $\left[-\frac{1}{2}, \frac{1}{2}\right]$;
continuous on $[-1, 1]$;
$y-$intercept: $(0, 0)$;
x-intercepts: $(-1, 0)$, $(0, 0)$, and $(1, 0)$

decreasing on $\left(-1, -\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, 1\right)$,

increasing on $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

relative and absolute minimum: $\left(-\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$,

relative and absolute maximum: $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$



28. $\left(300 - 80\sqrt{5}\right)$ m $\approx 121.11456\,\text{m}$ on dry land, along the river, and then straight line

29-30. see handout

