1. Simplify each of the following expressions.

(a) $(5a-1)^2 = 25a^2 - 10a + 1$

Solution: to square something means to write it down twice and multiply. Then it is a FOIL problem. F stand for first with first, O for outer terms, I for inner terms, and L for last terms.

$$\begin{aligned}(5a-1)^2 &= (5a-1)(5a-1)\\ &= \underbrace{25a^2}_{\text{F}}\ \underbrace{-5a}_{\text{O}}\ \underbrace{-5a}_{\text{I}}\ \underbrace{+1}_{\text{L}} \quad \text{combine like terms}\\ &= 25a^2 - 10a + 1\end{aligned}$$

(b) $(3x^5+4y)(3x^5-4y) = 9x^{10} - 16y^2$

$$\begin{aligned}\left(3x^5+4y\right)\left(3x^5-4y\right) &= \underbrace{9x^{10}}_{\text{F}}\ \underbrace{-12x^5}_{\text{O}}\ \underbrace{+12x^5}_{\text{I}}\ \underbrace{-16y^2}_{\text{L}} \quad \text{combine like terms}\\ &= 9x^{10} - 16y^2\end{aligned}$$

The expressions $3x^5+4y$ and $3x^5-4y$ are called conjugates. Becuase of the same terms and alternating signs, O and I cacel out when "FOIL"-ing, giving us the difference of two squares.

(c) $\dfrac{3a-8}{8-3a} = -1$

Solution: We need to notice that the numerator and denominator are opposites of each other. Indeed,

$$-1(8-3a) = -8+3a = 3a-8$$

Thus

$$\frac{3a-8}{8-3a} = \frac{-1(8-3a)}{8-3a} = -1$$

(d) $\dfrac{2x+1}{4x^2-1} = \dfrac{1}{2x-1}$

Solution: We factor the denominator via the difference of squares theorem, and then cancel.

$$\frac{2x+1}{4x^2-1} = \frac{2x+1}{(2x+1)(2x-1)} = \frac{1}{2x-1}$$

(e) $(x-y)(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5) = x^6 - y^6$

Solution: We apply the law of distributivity

$$\begin{aligned}&(x-y)\left(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5\right) =\\ &= x\cdot x^5 + x\cdot x^4y + x\cdot x^3y^2 + x\cdot x^2y^3 + x\cdot xy^4 + x\cdot y^5\\ &\qquad - y\cdot x^5 - y\cdot x^4y - y\cdot x^3y^2 - y\cdot x^2y^3 - y\cdot xy^4 - y\cdot y^5\\ &= x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5\\ &\qquad - x^5y - x^4y^2 - x^3y^3 - x^2y^4 - xy^5 - y^6\\ &= x^6 - y^6\end{aligned}$$

(f) $\dfrac{ab-a-b+1}{b^2-1} = \dfrac{a-1}{b+1}$
Solution: We will facotr both numerator and denominator and then cancel. The numerator can be factored by grouping

$$\begin{aligned} \underbrace{ab-a}\ \underbrace{-b+1} &= a(b-1)-1(b-1) \\ &= (a-1)(b-1) \end{aligned}$$

The denominaotr factors by the difference of squares theorem.

$$b^2-1=(b+1)(b-1)$$

Thus the fraction can be simplified as

$$\frac{ab-a-b+1}{b^2-1} = \frac{(a-1)(b-1)}{(b+1)(b-1)} = \frac{a-1}{b+1}$$

(g) $\dfrac{5x-30}{x^2-36} \cdot \dfrac{3x+18}{5} = 3$
Solution: we will factor whatever we can and then cancel.

$$\frac{5x-30}{x^2-36} \cdot \frac{3x+18}{5} = \frac{5(x-6)}{(x+6)(x-6)} \cdot \frac{3(x+6)}{5} = 3$$

(h) $\dfrac{3x}{x-2} - \dfrac{x+4}{x-2} = 2$
Solution: This is a subtraction of fractions. The denominators are the same, the only difficulty is that we are subtracting expressions instead of numbers. The second pair of parentheses is essential.

$$\frac{3x}{x-2} - \frac{x+4}{x-2} = \frac{(3x)-(x+4)}{x-2} = \frac{3x-x-4}{x-2} = \frac{2x-4}{x-2} = \frac{2(x-2)}{x-2} = 2$$

(i) $\sqrt{125} - 3\sqrt{80} + \sqrt{45} = -4\sqrt{5}$
Solution:

$$\begin{aligned} \sqrt{125} - 3\sqrt{80} + \sqrt{45} &= \sqrt{25 \cdot 5} - 3\sqrt{16 \cdot 5} + \sqrt{9 \cdot 5} \\ &= \sqrt{25}\sqrt{5} - 3\sqrt{16}\sqrt{5} + \sqrt{9}\sqrt{5} \\ &= 5\sqrt{5} - 3(4)\sqrt{5} + 3\sqrt{5} \\ &= (5-12+3)\sqrt{5} = -4\sqrt{5} \end{aligned}$$

(j) $\left(\sqrt{7}-2\right)^2 = \ 11 - 4\sqrt{7}$
Solution:

$$\begin{aligned} \left(\sqrt{7}-2\right)^2 &= \left(\sqrt{7}-2\right)\left(\sqrt{7}-2\right) \\ &= \sqrt{7}\sqrt{7} - 2\sqrt{7} - 2\sqrt{7} + 4 \\ &= 7 - 4\sqrt{7} + 4 = 11 - 4\sqrt{7} \end{aligned}$$

(k) $\left(\sqrt{3}-1\right)^3 = -10+6\sqrt{3}$

Solution: We will first work out $\left(\sqrt{3}-1\right)^2$ and then multiply that by $\left(\sqrt{3}-1\right)$.

$$\begin{aligned}
\left(\sqrt{3}-1\right)^3 &= \left(\sqrt{3}-1\right)\left(\sqrt{3}-1\right)\left(\sqrt{3}-1\right)\\
&= \left(\sqrt{3}\sqrt{3}-1\sqrt{3}-1\sqrt{3}+1\right)\left(\sqrt{3}-1\right)\\
&= \left(3-2\sqrt{3}+1\right)\left(\sqrt{3}-1\right)\\
&= \left(4-2\sqrt{3}\right)\left(\sqrt{3}-1\right)\\
&= 4\sqrt{3}-4-2\sqrt{3}\sqrt{3}+2\sqrt{3}\\
&= 4\sqrt{3}-4-2(3)+2\sqrt{3}\\
&= 4\sqrt{3}-4-6+2\sqrt{3}\\
&= -10+6\sqrt{3}
\end{aligned}$$

2. Rationalize the denominator in each of the following expressions.

(a) $\dfrac{3}{\sqrt{5}} = \dfrac{3\sqrt{5}}{5}$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by $\sqrt{5}$.

$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

(b) $\dfrac{1}{\sqrt{10}-3} = \sqrt{10}+3$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{10}+3$.

$$\frac{1}{\sqrt{10}-3} = \frac{1}{\sqrt{10}-3}\cdot\frac{\sqrt{10}+3}{\sqrt{10}+3} = \frac{\sqrt{10}+3}{1} = \sqrt{10}+3$$

The denominator is 1 since

$$\begin{aligned}
\left(\sqrt{10}-3\right)\left(\sqrt{10}+3\right) &= \sqrt{10}\sqrt{10}+3\sqrt{10}-3\sqrt{10}-9\\
&= 10-9=1
\end{aligned}$$

(c) $\dfrac{2}{\sqrt{7}+1} = \dfrac{\sqrt{7}-1}{3}$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{7}-1$

$$\frac{2}{\sqrt{7}+1} = \frac{2}{\sqrt{7}+1}\cdot\frac{\sqrt{7}-1}{\sqrt{7}-1} = \frac{2\left(\sqrt{7}-1\right)}{7-1} = \frac{2\left(\sqrt{7}-1\right)}{6} = \frac{\sqrt{7}-1}{3}$$

3. Find the exact value of $x^2 - 4x + 6$ if $x = 2 - \sqrt{3}$. 5

Solution: We work out x^2 first.

$$\begin{aligned} x^2 &= \left(2-\sqrt{3}\right)^2 = \left(2-\sqrt{3}\right)\left(2-\sqrt{3}\right) \\ &= 4 - 2\sqrt{3} - 2\sqrt{3} + \sqrt{3}\sqrt{3} \\ &= 4 - 4\sqrt{3} + 3 = 7 - 4\sqrt{3} \end{aligned}$$

Now we substitute $x = 2 - \sqrt{3}$ into $x^2 - 4x + 6$.

$$\begin{aligned} x^2 - 4x + 6 &= \left(2-\sqrt{3}\right)^2 - 4\left(2-\sqrt{3}\right) + 6 = \\ &= 7 - 4\sqrt{3} - 8 + 4\sqrt{3} + 6 \\ &= 7 - 8 + 6 = 5 \end{aligned}$$

4. Factor $13x + 2x^2 - 24$ by completing the square. $2(x+8)\left(x - \frac{3}{2}\right) = (x+8)(2x-3)$

Solution: We rearrange the terms first and then factor out the leading coefficient.

$$\begin{aligned} 13x + 2x^2 - 24 &= 2x^2 + 13x - 24 \\ &= 2\left(x^2 + \frac{13}{2}x - 12\right) \end{aligned}$$

Half of the linear coefficient is $\frac{13}{4}$, thus we work out $\left(x + \frac{13}{4}\right)^2$ first to see what we need to smuggle in to complete the square.

$$\begin{aligned} \left(x + \frac{13}{4}\right)^2 &= \left(x + \frac{13}{4}\right)\left(x + \frac{13}{4}\right) = x^2 + \frac{13}{4}x + \frac{13}{4}x + \frac{169}{16} \\ &= x^2 + \frac{13}{2}x + \frac{169}{16} \end{aligned}$$

Thus we need to smuggle in $\frac{169}{16}$

$$\begin{aligned} 2x^2 + 13x - 24 &= 2\left(x^2 + \frac{13}{2}x - 12\right) \\ &= 2\left(\underbrace{x^2 + \frac{13}{2}x + \frac{169}{16}} - \frac{169}{16} - 12\right) \end{aligned}$$

We bring the last two numbers to the common denominator

$$\begin{aligned} 2x^2 + 13x - 24 &= 2\left(\left(x + \frac{13}{4}\right)^2 - \frac{169}{16} - \frac{12(16)}{1(16)}\right) \\ &= 2\left(\left(x + \frac{13}{4}\right)^2 - \frac{169}{16} - \frac{192}{16}\right) \\ &= 2\left(\left(x + \frac{13}{4}\right)^2 - \frac{361}{16}\right) \end{aligned}$$

Since $\dfrac{361}{16} = \left(\dfrac{19}{4}\right)^2$, we factor via the difference of squares theorem.

$$\begin{aligned}
2x^2 + 13x - 24 &= 2\left(\left(x + \frac{13}{4}\right)^2 - \left(\frac{19}{4}\right)^2\right) \\
&= 2\left(x + \frac{13}{4} + \frac{19}{4}\right)\left(x + \frac{13}{4} - \frac{19}{4}\right) \\
&= 2\left(x + \frac{32}{4}\right)\left(x - \frac{6}{4}\right) \\
&= 2(x+8)\left(x - \frac{3}{2}\right) = (x+8)(2x-3)
\end{aligned}$$

We may distribute 2 into the second factor. Then we get

$$\begin{aligned}
2x^2 + 13x - 24 &= (x+8)\left(2\left(x - \frac{3}{2}\right)\right) \\
&= (x+8)(2x-3)
\end{aligned}$$

We FOIL to check:

$$\begin{aligned}
(x+8)(2x-3) &= 2x^2 - 3x + 16x - 24 \\
&= 2x^2 + 13x - 24
\end{aligned}$$

5. Factor completely each of the following:

(a) $4a^2mn - 15abm^2 - 6abmn + 10a^2m^2 = am(2n+5m)(2a-3b)$
Solution:

$$\begin{aligned}
4a^2mn - 15abm^2 - 6abmn + 10a^2m^2 &= \quad \text{the GCF is } am \\
am(4an - 15bm - 6bn + 10am) &= \quad \text{rearrange} \\
am\left(\underbrace{4an - 6bn}\ \underbrace{+10am - 15bm}\right) &= \\
am(2n(2a-3b) + 5m(2a-3b)) &= am(2n+5m)(2a-3b)
\end{aligned}$$

(b) $a^2x^3 - b^2x - a^2x + b^2x^3 = x(a^2+b^2)(x+1)(x-1)$
Solution:

$$\begin{aligned}
a^2x^3 - b^2x - a^2x + b^2x^3 &= \quad \text{the GCF is } x \\
x\left(a^2x^2 - b^2 - a^2 + b^2x^2\right) &= \quad \text{rearrange} \\
x\left(\underbrace{a^2x^2 - a^2}\ \underbrace{+b^2x^2 - b^2}\right) &= \\
x\left(a^2\left(x^2-1\right) + b^2\left(x^2-1\right)\right) &= x\left(a^2+b^2\right)\left(x^2-1\right)
\end{aligned}$$

We are not done yet since $\left(x^2-1\right) = \left(x^2-1^2\right)$ further factors via the difference of squares theorem. Thus the answer is

$$\begin{aligned}
x\left(a^2+b^2\right)\left(x^2-1\right) &= x\left(a^2+b^2\right)\left(x^2-1^2\right) \\
&= x\left(a^2+b^2\right)(x+1)(x-1)
\end{aligned}$$

(c) $162a+162b-2ax^4-2bx^4 = 2(9+x^2)(3+x)(3-x)(a+b)$
Solution:

$$\begin{aligned} 162a+162b-2ax^4-2bx^4 &= \qquad \text{the GCF is } 2 \\ 2\left(\underbrace{81a+81b}\ \underbrace{-ax^4-bx^4}\right) &= \\ 2\left(81(a+b)-x^4(a+b)\right) &= 2\left(81-x^4\right)(a+b) \end{aligned}$$

We are not done yet, since $81-x^4 = 9^2-\left(x^2\right)^2$ further factors via the difference of squares theorem.

$$\begin{aligned} 2\left(81-x^4\right)(a+b) &= 2\left(9^2-\left(x^2\right)^2\right)(a+b) \\ &= 2\left(9+x^2\right)\left(9-x^2\right)(a+b) \end{aligned}$$

One factor still further factors: $9-x^2 = 3^2-x^2 = (3+x)(3-x)$. Thus the final answer is

$$\begin{aligned} &= 2\left(9+x^2\right)\left(9-x^2\right)(a+b) \\ &= 2\left(9+x^2\right)\left(3^2-x^2\right)(a+b) \\ &= 2\left(9+x^2\right)(3+x)(3-x)(a+b) \end{aligned}$$

(d) $x^2-6x+8 = (x-2)(x-4)$
Solution: we will factor by grouping. First we conduct the "$pq-$game".

$$\begin{aligned} pq &= 8 && \text{1st coefficient times 3rd coefficient} \\ p+q &= -6 && \text{2nd coefficient} \end{aligned}$$

We start by expressing 8 as a product of two numbers. there are only two pairs, 1 with 8 and 2 with 4. Since the product pq is positive, p and q have to have the same sign. Since the sum $p+q$ is negative, they both have to be negative. We only need to consider -1 with -8 and -2 with -4. Clearly -2 with -4 work as p and q. We use these these numbers to express the second term as the sum of two terms, and then factor by grouping.

$$\begin{aligned} x^2-6x+8 &= \underbrace{x^2-2x}\ \underbrace{-4x+8} \\ &= x(x-2)-4(x-2) = (x-2)(x-4) \end{aligned}$$

We check by multiplication:

$$(x-2)(x-4) = x^2-4x-2x+8 = x^2-6x+8$$

Thus our result is correct.

(e) $3a^2-5a-2 = (a-2)(3a+1)$
Solution: we will factor by grouping. First we conduct the "$pq-$game".

$$\begin{aligned} pq &= -6 && \text{1st coefficient times 3rd coefficient} \\ p+q &= -5 && \text{2nd coefficient} \end{aligned}$$

We start by expressing 6 as a product of two numbers. there are only two pairs, 1 with 6 and 2 with 3. Since the product pq is negative, one number must be positive, the other one must be positive.. Since the sum $p+q$ is negative, the negative sign has to be in front of the larger number. We only need to consider 1 with -6 and 2 with -3. Clearly 1 with -6 work as p and q. We use these these numbers to express the second term as the sum of two terms, and then factor by grouping.

$$\begin{aligned} 3a^2-5a-2 &= \underbrace{3a^2+a}\ \underbrace{-6a-2} \\ &= a(3a+1)-2(3a+1)=(a-2)(3a+1) \end{aligned}$$

We check by multiplication:

$$(a-2)(3a+1)=3a^2+a-6a-2=3a^2-5a-2$$

Thus our result is correct.

(f) $4b^2-b-5=$ $(4b-5)(b+1)$

Solution: we will factor by grouping. First we conduct the "$pq-$game".

$$\begin{aligned} pq &= -20 && \text{1st coefficient times 3rd coefficient} \\ p+q &= -1 && \text{2nd coefficient} \end{aligned}$$

We start by expressing 20 as a product of two numbers. the possible pairs are, 1 with 20, 2 with 10, and 4 with 5. Since the product pq is negative, one number must be positive, the other one must be positive.. Because the sum $p+q$ is negative, the negative sign has to be in front of the larger number. We only need to consider 1 with -20, 2 with -10, and 4 with -5. Clearly 4 with -5 work as p and q. We use these these numbers to express the second term as the sum of two terms, and then factor by grouping.

$$\begin{aligned} 4b^2-b-5 &= \underbrace{4b^2+4b}\ \underbrace{-5b-5} \\ &= 4b(b+1)-5(b+1)=(4b-5)(b+1) \end{aligned}$$

We check by multiplication:

$$(4b-5)(b+1)=4b^2+4b-5b-5=4b^2-b-5$$

Thus our result is correct.

6. Solve each of the following equations. Make sure to check your solution(s).

(a) $2x^3=20x^2+1750x$ $35, 0,$ and -25

Solution: We reduce one side to zero, then factor, and then apply the zero property.

$$\begin{aligned} 2x^3 &= 20x^2+1750x \\ 2x^3-20x^2-1750x &= 0 && \text{factor out GCF} \\ 2x\left(x^2-10x-875\right) &= 0 && \text{divide both sides by 2} \\ x\left(x^2-10x-875\right) &= 0 \end{aligned}$$

We will factor by completing the square. Half of the linear coefficient is -5, and thus we will work with

$$(x-5)^2 = x^2 - 10x + 25$$

We smuggle in 25.

$$\begin{aligned}
x\left(x^2 - 10x - 875\right) &= 0 \\
x\left(\underbrace{x^2 - 10x + 25} - 25 - 875\right) &= 0 \\
x\left((x-5)^2 - 900\right) &= 0 \\
x\left((x-5)^2 - 30^2\right) &= 0 \\
x(x-5+30)(x-5-30) &= 0 \\
x(x+25)(x-35) &= 0
\end{aligned}$$

Applying the zero property we obtain $35, 0,$ and -25 as the solutions.

(b) $\dfrac{3x+17}{2} = x - 1 + \dfrac{x+19}{2}$ identity, all real numbers are solution.

Solution:

$$\begin{aligned}
\frac{3x+17}{2} &= x - 1 + \frac{x+19}{2} && \text{express everything as a fraction} \\
\frac{3x+17}{2} &= \frac{x-1}{1} + \frac{x+19}{2} && \text{bring everything to the common denominator} \\
\frac{3x+17}{2} &= \frac{2(x-1)}{2} + \frac{x+19}{2} && \text{add fractions on right hand side} \\
\frac{3x+17}{2} &= \frac{2(x-1)+x+19}{2} && \text{multiply out parentheses} \\
\frac{3x+17}{2} &= \frac{2x-2+x+19}{2} && \text{combine like terms} \\
\frac{3x+17}{2} &= \frac{3x+17}{2} && \text{multiply by 2} \\
3x+17 &= 3x+17
\end{aligned}$$

Because the left hand side is now identical to the right hand side, this equation is an identity, and all real numbers are solution.

(c) $|3-2x| + 2 = 5$

Solution:

$$\begin{aligned}
|3-2x| + 2 &= 5 \quad \text{subtract } 2 \\
|3-2x| &= 3 \\
3 - 2x &= 3 \quad \text{or} \quad 3 - 2x = -3 \quad \text{subtract } 3 \\
-2x &= 0 \quad \text{or} \quad -2x = -6 \quad \text{divide by } -2 \\
x &= 0 \quad \text{or} \quad x = 3
\end{aligned}$$

Thus the solution are 0 and 3.

(d) $\frac{2}{3}(x-7)=\frac{4}{5}(x+1)$ -41
Solution:

$$\begin{aligned}
\frac{2}{3}(x-7) &= \frac{4}{5}(x+1) \\
\frac{2}{3}\cdot\frac{x-7}{1} &= \frac{4}{5}\cdot\frac{x+1}{1} \\
\frac{2(x-7)}{3} &= \frac{4(x+1)}{5} && \text{bring fractions to common denominator} \\
\frac{5\cdot 2(x-7)}{15} &= \frac{3\cdot 4(x+1)}{15} && \text{multiply both sides by 15} \\
10(x-7) &= 12(x+1) && \text{multiplty out parentheses} \\
10x-70 &= 12x+12 && \text{subtract } 10x \\
-70 &= 2x+12 && \text{subtract 12} \\
-82 &= 2x && \text{divide by 2} \\
-41 &= x
\end{aligned}$$

We check:

$$\begin{aligned}
\text{LHS} &= \frac{2}{3}(-41-7)=\frac{2}{3}(-48)=-32 \\
\text{RHS} &= \frac{4}{5}(-41+1)=\frac{4}{5}(-40)=-32
\end{aligned}$$

Thus our solution, -41 is correct.

(e) $7x^2+(x+3)(2x-1)=(3x+1)^2$ -4
Solution:

$$\begin{aligned}
7x^2+(x+3)(2x-1) &= (3x+1)^2 && \text{multiply the polynomials on both sides} \\
7x^2+2x^2+5x-3 &= 9x^2+6x+1 && \text{combine like terms} \\
9x^2+5x-3 &= 9x^2+6x+1 && \text{subtract } 9x^2 \\
5x-3 &= 6x+1 && \text{subtract } 5x \\
-3 &= x+1 && \text{subtract 1} \\
-4 &= x
\end{aligned}$$

We check our result:

$$\begin{aligned}
\text{LHS} &= 7(-4)^2+((-4)+3)(2(-4)-1)=7\cdot 16+(-1)(-9)=112+9=121 \\
\text{RHS} &= (3(-4)+1)^2=(-12+1)^2=(-11)^2=121
\end{aligned}$$

Thus the solution, -4 is correct.

(f) $8a+2a^2=42$ $-7,3$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the special zero property.

$$\begin{aligned} 8a+2a^2 &= 42 && \text{subtract 42, rearrange} \\ 2a^2+8a-42 &= 0 && \text{the GCF is } 2 \\ 2\left(a^2+4a-21\right) &= 0 \end{aligned}$$

We will factor $a^2+4a-21$ by grouping. First we conduct the "$pq-$game".

$$\begin{aligned} pq &= -21 && \text{1st coefficient times 3rd coefficient} \\ p+q &= 4 && \text{2nd coefficient} \end{aligned}$$

We start by expressing 21 as a product of two numbers. the only possible pairs are, 1 with 21 and 3 with 7. Since the product pq is negative, one number must be positive, the other one must be positive.. Because the sum $p+q$ is positive, the negative sign has to be in front of the smaller number. We only need to consider -1 with 20, and -3 with 7. Clearly -3 with 7 work as p and q. We use these these numbers to express the second term as the sum of two terms, and then factor by grouping.

$$\begin{aligned} 2\left(a^2+4a-21\right) &= 0 \\ 2\left(\underbrace{a^2+7a}\quad\underbrace{-3a-21}\right) &= 0 \\ 2\left(a\left(a+7\right)-3\left(a+7\right)\right) &= 0 \\ 2\left(a-3\right)\left(a+7\right) &= 0 \end{aligned}$$

Thus our equation is

$$2\left(a-3\right)\left(a+7\right)=0$$

We now apply the special zero property. If this product is zero, then either $2=0$ or $a-3=0$ or $a+7=0$. We solve these equations for a.

$$\begin{array}{ccccc} a-3=0 & \text{or} & a+7=0 & \text{or} & 2=0 \\ a=3 & \text{or} & a=-7 & \text{or} & \text{no solution here} \end{array}$$

We check both solutions. If $a=3$, then

$$\text{LHS}=8\left(3\right)+2\left(3\right)^2=8\cdot 3+2\cdot 9=24+18=42=\text{RHS} \quad \checkmark$$

If $a=-7$, then

$$\text{LHS}=8\left(-7\right)+2\left(-7\right)^2=8\cdot\left(-7\right)+2\cdot 49=-56+98=42=\text{RHS} \quad \checkmark$$

Thus both solutions, -7 and 3 are correct.

(g) $8x^3 = 50x^2$ $\frac{25}{4}, 0$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the special zero property.

$$\begin{aligned} 8x^3 &= 50x^2 && \text{subtract } 50x^2 \\ 8x^3 - 50x^2 &= 0 && \text{the GCF is } 2x^2 \\ 2x^2(4x-25) &= 0 \end{aligned}$$

We now apply the special zero property. If this product is zero, then either $2x^2 = 0$ or $4x - 25 = 0$. We solve these equations for x.

$$\begin{aligned} 2x^2 &= 0 && \text{or} && 4x - 25 = 0 \\ 2 \cdot x \cdot x &= 0 && \text{or} && 4x = 25 \\ x &= 0 && \text{or} && x = \frac{25}{4} \end{aligned}$$

We check both solutions. If $x = 0$, then

$$\begin{aligned} \text{LHS} &= 8 \cdot 0^3 = 8 \cdot 0 = 0 \\ \text{RHS} &= 50 \cdot 0^2 = 50 \cdot 0 = 0 \quad \checkmark \end{aligned}$$

If $x = \frac{25}{4}$, then

$$\begin{aligned} \text{LHS} &= 8\left(\frac{25}{4}\right)^3 = \frac{8}{1} \cdot \frac{15\,625}{64} = \frac{15\,625}{8} \\ \text{RHS} &= 50\left(\frac{25}{4}\right)^2 = \frac{50}{1} \cdot \frac{625}{16} = \frac{15\,625}{8} \quad \checkmark \end{aligned}$$

Thus both solutions, 0 and $\frac{25}{4}$ are correct.

(h) $8p^3 = 50p$ $-\frac{5}{2}$, 0, $\frac{5}{2}$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the special zero property.

$$\begin{aligned} 8p^3 &= 50p && \text{subtract } 50p \\ 8p^3 - 50p &= 0 && \text{the GCF is } 2p \\ 2p\left(4p^2 - 25\right) &= 0 \\ 2p\left((2p)^2 - 5^2\right) &= 0 && \text{factor via difference of squares theorem} \\ 2p(2p+5)(2p-5) &= 0 \end{aligned}$$

We now apply the special zero property. If this product is zero, then either $2p = 0$ or $2p + 5 = 0$ or $2p - 5 = 0$. We solve these equations for p.

$$\begin{aligned} 2p + 5 &= 0 && \text{or} && 2p - 5 = 0 && \text{or} && 2p = 0 \\ 2p &= -5 && \text{or} && 2p = 5 && \text{or} && p = 0 \\ p &= -\frac{5}{2} && \text{or} && p = \frac{5}{2} \end{aligned}$$

We check all three solutions. If $p = -\frac{5}{2}$, then

$$\begin{aligned} \text{LHS} &= 8\left(-\frac{5}{2}\right)^3 = \frac{8}{1} \cdot \frac{-125}{8} = -125 \\ \text{RHS} &= 50\left(-\frac{5}{2}\right) = \frac{50}{1} \cdot \frac{-5}{2} = \frac{-250}{2} = -125 \quad \checkmark \end{aligned}$$

If $p = \frac{5}{2}$, then

$$\begin{aligned} \text{LHS} &= 8\left(\frac{5}{2}\right)^3 = \frac{8}{1} \cdot \frac{125}{8} = 125 \\ \text{RHS} &= 50\left(\frac{5}{2}\right) = \frac{50}{1} \cdot \frac{5}{2} = \frac{250}{2} = 125 \quad \checkmark \end{aligned}$$

and if $p = 0$, then

$$\begin{aligned} \text{LHS} &= 8 \cdot 0^3 = 8 \cdot 0 = 0 \\ \text{RHS} &= 50 \cdot 0 = 0 \quad \checkmark \end{aligned}$$

Thus all three solutions, $-\frac{5}{2}$, 0, and $\frac{5}{2}$ are correct.

(i) $2 - (3 - x)(2x + 5) = (x - 1)(2x - 1)$ 7

Solution: We have to use the FOIL method on both sides to perform the multiplications. It is very important, however, to keep the expressions in a parentheses since we are dealing with subtraction between *algebraic expressions*.

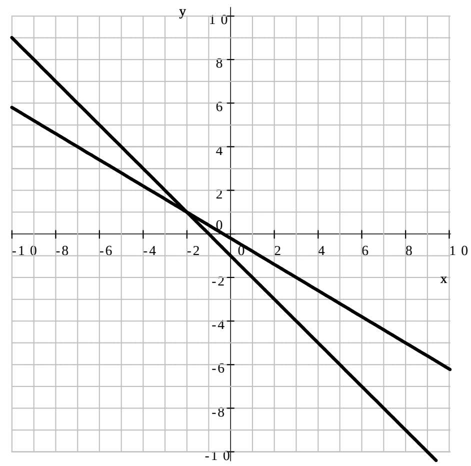
$$\begin{aligned} 2 - (3 - x)(2x + 5) &= (x - 1)(2x - 1) && \text{FOIL} \\ 2 - \left(6x + 15 - 2x^2 - 5x\right) &= 2x^2 - x - 2x + 1 && \text{combine like terms} \\ 2 - \left(-2x^2 + x + 15\right) &= 2x^2 - 3x + 1 && \text{perform subtraction} \\ 2 + 2x^2 - x - 15 &= 2x^2 - 3x + 1 && \text{combine like terms} \\ 2x^2 - x - 13 &= 2x^2 - 3x + 1 && \text{subtract } 2x^2 \quad \text{(the equation is linear!)} \\ -x - 13 &= -3x + 1 && \text{add } 3x \\ 2x - 13 &= 1 && \text{add } 13 \\ 2x &= 14 && \text{divide by } 2 \\ x &= 7 \end{aligned}$$

We check: if $x = 7$, then

$$\begin{aligned} \text{LHS} &= 2 - (3 - 7)(2(7) + 5) = 2 - (-4)(14 + 5) = 2 - (-4)19 = 2 - (-76) = 78 \\ \text{RHS} &= (7 - 1)(2(7) - 1) = 6(14 - 1) = 6 \cdot 13 = 78 \quad \checkmark \end{aligned}$$

Thus our solution, 7 is correct.

7. Graph the straight lines $3x+5y=-1$ and $y=-x-1$ in the same coordinate system. Use your graph to find the coordinates of the point where the lines intersect. $(-2,1)$



8. Graph the parabola $y=-8x+x^2+15$. Clearly label the coordinates of five points on the parabola, including vertex and intercepts.

Solution: We obtain all forms of the equation first.

$$y=x^2-8x+15 \implies \text{polynomial form}$$

Half of the linear coefficient is -4, thus we will work with $(x-4)^2=x^2-8x+16$

$$\begin{aligned} y &= x^2-8x+15 \\ y &= \underbrace{x^2-8x+16}-16+15 \\ y &= (x-4)^2-1 \implies \text{complete square form} \end{aligned}$$

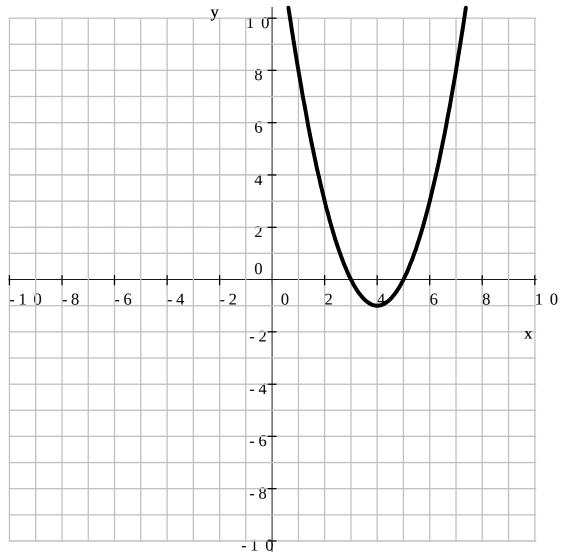
We factor via the difference of squares theorem

$$\begin{aligned} y &= (x-4)^2-1^2 \quad \text{since } 1=1^2 \\ y &= (x-4+1)(x-4-1) \\ y &= (x-3)(x-5) \implies \text{factored form} \end{aligned}$$

From the polynomial form we obtain the $y-$intecept, $(0,15)$. From the complete square form, the vertex is $(4,-1)$. Finally, the factored form tells us that there are two $x-$intercepts, $(3,0)$ and $(5,0)$. The few missing points, close to the vertex can be found by substituting values for x into any of the three forms of the equations to find y. This time we will work with the polynomial form.

$$\begin{aligned} \text{if } x &= 2, \text{ then } y=(2)^2-8(2)+15=4-16+15=3 \\ \text{if } x &= 6, \text{ then } y=(6)^2-8(6)+15=36-48+15=3 \end{aligned}$$

We are ready to graph:



9. One side of a rectangle is 4 ft shorter than three times the other side. Find the sides if the perimeter is 64 ft. 9 ft and 23 ft

   Solution: Let us denote the shorter side by x. Then the other side is $3x-4$. The equation expresses the perimeter of the rectangle.

$$\begin{aligned} 2(x)+2(3x-4) &= 64 && \text{multiply out parentheses} \\ 2x+6x-8 &= 64 && \text{combine like terms} \\ 8x-8 &= 64 && \text{add} \\ 8x &= 72 && \text{divide by 8} \\ x &= 9 \end{aligned}$$

   If the shorter side was denoted by x, we now know it is 9 ft. The longer side was denoted by $3x-4$, so it must be $3(9\text{ ft})-4\text{ ft}=23\text{ ft}$. Thus the sides of the rectangle are 9 ft and 23 ft. We check: $P=2(9\text{ ft})+2(23\text{ ft})=64\text{ ft}$ and $23\text{ ft}=3(9\text{ ft})-4\text{ ft}$. Thus our solution is correct.

10. One side of a rectangle is 4 ft shorter than three times the other side. Find the sides if the area is 84 ft^2. 6 ft and 14 ft

    Solution: Let us denote the shorter side by x. Then the other side is $3x-4$. The equation expresses the area of the rectangle.

$$\begin{aligned} x(3x-4) &= 84 && \text{multiply out parentheses} \\ 3x^2-4x &= 84 && \text{subtract 84} \\ 3x^2-4x-84 &= 0 \end{aligned}$$

    Because the equation is quadratic, we need to factor the left-hand side and then apply the zero property. We will factor by grouping. First we conduct the "$pq-$game". The sum of

p and q has to be the linear coefficient (the number in front of x, with its sign), so it is -4. The product of p and q has to be the product of the other coefficients, $3(-84) = -252$.

$$\begin{aligned} pq &= -252 \\ p+q &= -4 \end{aligned}$$

Now we need to find p and q. Because the product is negative, we re looking for a positive and a negative number. Becuse the sum is negtive, the larger number must carry the negative sign. We enter $\sqrt{252} = 15.874\,507\,87$ into the calculator and get a decimal:

$$\sqrt{252} = 15.874....$$

So we start looking for factors of 252, starting at 15, and moving down. We soon find 14 and -18. These are our values for p and q. We use these numbers to express the linear term:

$$-4x = 14x - 18x$$

and factor by grouping.

$$\begin{aligned} 3x^2 - 4x - 84 &= 0 \\ \underbrace{3x^2 + 14x}\underbrace{-18x - 84} &= 0 \\ x(3x+14) - 6(3x+14) &= 0 \\ (x-6)(3x+14) &= 0 \end{aligned}$$

We now apply the zero property. Either $x - 11 = 0$ or $3x + 14 = 0$. We solve both these equations for x.

$$\begin{aligned} x - 6 &= 0 \\ x &= 6 \end{aligned}$$

and

$$\begin{aligned} 3x + 14 &= 0 \\ 3x &= -14 \\ x &= -\frac{14}{3} \end{aligned}$$

Since distances can not be negative, the second solution for x, $-\frac{14}{3}$ is ruled out. Thus $x = 6$. Then the longer side is $3(6 \text{ ft}) - 4 \text{ ft} = 14 \text{ ft}$, and so the rectangle's sides are 6 ft and 14 ft long. We check: $6 \text{ ft}(14 \text{ ft}) = 84 \text{ ft}^2$ and $14 \text{ ft} = 3(6 \text{ ft}) - 4 \text{ ft}$. Thus our solution is correct.

11. One side of a rectangle is 4 in shorter than 3 times the other side. Find the sides of the rectangle if its area is 319 in^2. 11 in by 29 in

Solution: Let us denote the shorter side by x. Then the larger side is $3x - 4$. The equation expresses the area.

$$\begin{aligned} x(3x-4) &= 319 \\ 3x^2 - 4x &= 319 \\ 3x^2 - 4x - 319 &= 0 \end{aligned}$$

We will complete the square. (It can be done by the pq-game as well)

$$\begin{aligned} 3x^2 - 4x - 319 &= 0 \\ 3\left(x^2 - \frac{4}{3}x - \frac{319}{3}\right) &= 0 \end{aligned}$$

Half of the linear coefficient is $-\frac{4}{3} \div 2 = -\frac{4}{3} \cdot \frac{1}{2} = -\frac{4}{6} = -\frac{2}{3}$, thus we work out $\left(x - \frac{2}{3}\right)^2$.

$$\begin{aligned} \left(x - \frac{2}{3}\right)^2 &= \left(x - \frac{2}{3}\right)\left(x - \frac{2}{3}\right) = x^2 - \frac{2}{3}x - \frac{2}{3}x + \frac{4}{9} \\ &= x^2 - \frac{4}{3}x + \frac{4}{9} \end{aligned}$$

Thus we smuggle in $\frac{4}{9}$.

$$\begin{aligned} 3\left(\underbrace{x^2 - \frac{4}{3}x + \frac{4}{9}} - \frac{4}{9} - \frac{319}{3}\right) &= 0 \\ 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{319}{3}\right) &= 0 \end{aligned}$$

We bring the last two numbers to the common denominator:

$$\begin{aligned} 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{319\,(3)}{3\,(3)}\right) &= 0 \\ 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{957}{9}\right) &= 0 \\ 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{961}{9}\right) &= 0 \end{aligned}$$

Since $\frac{961}{9} = \left(\frac{31}{3}\right)^2$, we factor via the difference of squares theorem.

$$\begin{aligned} 3\left(\left(x - \frac{2}{3}\right)^2 - \left(\frac{31}{3}\right)^2\right) &= 0 \\ 3\left(x - \frac{2}{3} + \frac{31}{3}\right)\left(x - \frac{2}{3} - \frac{31}{3}\right) &= 0 \\ 3\left(x + \frac{29}{3}\right)\left(x - \frac{33}{3}\right) &= 0 \\ 3\left(x + \frac{29}{3}\right)(x - 11) &= 0 \end{aligned}$$

This equation has two solutions, $x_1 = -\frac{29}{3}$ and $x_2 = 11$. Since distances are positive, $-\frac{29}{3}$ is ruled out as a solution for the shorter side. The other solution is 11 in. This makes the longer side $3 \cdot 11 - 4 = 29$ in. We check: $3(11) - 4 = 29$ and $11(29) = 319$. Thus our solution, 11 in by 29 in is correct.

12. The population of a town has decreased from 80 000 to 68 000. What percent of a decrease does this represent? 15% decrease

Solution 1: We subtract 68 000 from 80 000 to determine the change. $80\,000 - 68\,000 = 12\,000$. Now the question is: 12 000 is what percent of 80 000? Then

$$\begin{aligned} (\text{is}) &= 12\,000 \\ \text{F} &= x \\ (\text{of}) &= 80\,000 \end{aligned}$$

We substitute the data into the formula:

$$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 12\,000 &= x \cdot 80\,000 \\ \frac{12\,000}{80\,000} &= x \\ 0.15 &= x \end{aligned}$$

Thus

$$x = 0.15 = \frac{0.15}{1} = \frac{0.15\,(100)}{1\,(100)} = \frac{15}{100} = 15\%$$

This is a 15% decrease.

Solution 2: The question may be re-phrased as: 68 000 is what percent of 80 000? Then

$$\begin{aligned} (\text{is}) &= 68\,000 \\ \text{F} &= x \\ (\text{of}) &= 80\,000 \end{aligned}$$

We substitute the data into the formula:

$$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 68\,000 &= x \cdot 80\,000 \\ \frac{68\,000}{80\,000} &= x \\ 0.85 &= x \end{aligned}$$

Thus

$$x = 0.85 = \frac{0.85}{1} = \frac{0.85\,(100)}{1\,(100)} = \frac{85}{100} = 85\%$$

Since the population has decreased to 85% of its previous count, this is a 15% decrease.

13. The hypotenuse of a right triangle is 68 cm. The difference between the other two sides is 28. cm. Find the sides of the triangle. 32 cm and 60 cm

Solution: Let x denote the shorter leg. Then the other leg is $x+28$ cm long. We state the Pythagorean theorem for the triangle, and solve the quadratic equation for x.

$$\begin{aligned} x^2+(x+28)^2 &= 68^2 && \text{FOIL out } (x+28)^2 \\ x^2+x^2+56x+784 &= 4624 && \text{combine like terms} \\ 2x^2+56x+784 &= 4624 && \text{subtract } 4624 \\ 2x^2+56x-3840 &= 0 && \text{factor out } 2 \\ 2\left(x^2+28x-1920\right) &= 0 && \text{divide by } 2 \\ x^2+28x-1920 &= 0 \end{aligned}$$

We factor by completing the square. Since half of the linear coefficient is 14, we will work with $(x+14)^2 = x^2+28x+196$

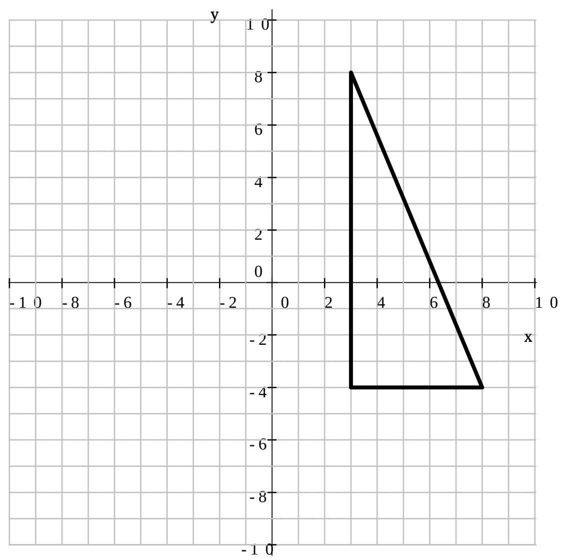
$$\begin{aligned} \underbrace{x^2+28x+196}-196-1920 &= 0 \\ (x+14)^2-2116 &= 0 \\ (x+14)^2-46^2 &= 0 \\ (x+14+46)(x+14-46) &= 0 \\ (x+60)(x-32) &= 0 \\ x_1 &= -60 \quad \text{and} \quad x_2 = 32 \end{aligned}$$

Since distances are never negative, -60 is ruled out. If the shortest side is 32 cm, the other side is $32\,\text{cm} + 28\,\text{cm} = 60\,\text{cm}$. Thus the solution is 32 cm and 60 cm. We check:

$$\begin{aligned} 60-32 &= 28 \text{ and} \\ 60^2+32^2 &= 3600+1024 = 4624 = 68^2 \end{aligned}$$

14. Find the distance between $(3,8)$ and $(8,-4)$. 13 units

Solution: We graph the points, they determine a right triangle as shown below. The legs are 5 and 12 units long, and we need to find the hypotenuse.



$$\begin{aligned}
5^2+12^2 &= x^2\\
25+144 &= x^2\\
169 &= x^2\\
0 &= x^2-13^2\\
0 &= (x+13)(x-13)\\
x_1 &= -13 \quad \text{and} \quad x_2 = 13
\end{aligned}$$

Since distances are never negative, the answer is 13 units.