

1. Simplify each of the following expressions.

(a)  $\sqrt{125} - 3\sqrt{80} + \sqrt{45} =$

Solution:

$$\begin{aligned}\sqrt{125} - 3\sqrt{80} + \sqrt{45} &= \sqrt{25 \cdot 5} - 3\sqrt{16 \cdot 5} + \sqrt{9 \cdot 5} \\ &= \sqrt{25}\sqrt{5} - 3\sqrt{16}\sqrt{5} + \sqrt{9}\sqrt{5} \\ &= 5\sqrt{5} - 3(4)\sqrt{5} + 3\sqrt{5} \\ &= (5 - 12 + 3)\sqrt{5} = -4\sqrt{5}\end{aligned}$$

(b)  $(\sqrt{7} - 2)^2 =$

Solution:

$$\begin{aligned}(\sqrt{7} - 2)^2 &= (\sqrt{7} - 2)(\sqrt{7} - 2) \\ &= \sqrt{7}\sqrt{7} - 2\sqrt{7} - 2\sqrt{7} + 4 \\ &= 7 - 4\sqrt{7} + 4 = 11 - 4\sqrt{7}\end{aligned}$$

(c)  $(\sqrt{3} - 1)^3 =$

Solution: We will first work out  $(\sqrt{3} - 1)^2$  and then multiply that by  $(\sqrt{3} - 1)$ .

$$\begin{aligned}(\sqrt{3} - 1)^3 &= (\sqrt{3} - 1)(\sqrt{3} - 1)(\sqrt{3} - 1) \\ &= (\sqrt{3}\sqrt{3} - 1\sqrt{3} - 1\sqrt{3} + 1)(\sqrt{3} - 1) \\ &= (3 - 2\sqrt{3} + 1)(\sqrt{3} - 1) \\ &= (4 - 2\sqrt{3})(\sqrt{3} - 1) \\ &= 4\sqrt{3} - 4 - 2\sqrt{3}\sqrt{3} + 2\sqrt{3} \\ &= 4\sqrt{3} - 4 - 2(3) + 2\sqrt{3} \\ &= 4\sqrt{3} - 4 - 6 + 2\sqrt{3} \\ &= -10 + 6\sqrt{3}\end{aligned}$$

2. Rationalize the denominator in each of the following expressions.

(a)  $\frac{3}{\sqrt{5}} =$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by  $\sqrt{5}$ .

$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

$$(b) \frac{1}{\sqrt{10}-3} =$$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is  $\sqrt{10}+3$ .

$$\frac{1}{\sqrt{10}-3} = \frac{1}{\sqrt{10}-3} \cdot \frac{\sqrt{10}+3}{\sqrt{10}+3} = \frac{\sqrt{10}+3}{1} = \sqrt{10}+3$$

The denominator is 1 since

$$\begin{aligned} (\sqrt{10}-3)(\sqrt{10}+3) &= \sqrt{10}\sqrt{10} + 3\sqrt{10} - 3\sqrt{10} - 9 \\ &= 10 - 9 = 1 \end{aligned}$$

$$(c) \frac{2}{\sqrt{7}+1} =$$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is  $\sqrt{7}-1$

$$\frac{2}{\sqrt{7}+1} = \frac{2}{\sqrt{7}+1} \cdot \frac{\sqrt{7}-1}{\sqrt{7}-1} = \frac{2(\sqrt{7}-1)}{7-1} = \frac{2(\sqrt{7}-1)}{6} = \frac{\sqrt{7}-1}{3}$$

3. Find the exact value of  $x^2 - 4x + 6$  if  $x = 2 - \sqrt{3}$ .

Solution: We work out  $x^2$  first.

$$\begin{aligned} x^2 &= (2 - \sqrt{3})^2 = (2 - \sqrt{3})(2 - \sqrt{3}) \\ &= 4 - 2\sqrt{3} - 2\sqrt{3} + \sqrt{3}\sqrt{3} \\ &= 4 - 4\sqrt{3} + 3 = 7 - 4\sqrt{3} \end{aligned}$$

Now we substitute  $x = 2 - \sqrt{3}$  into  $x^2 - 4x + 6$ .

$$\begin{aligned} x^2 - 4x + 6 &= (2 - \sqrt{3})^2 - 4(2 - \sqrt{3}) + 6 = \\ &= 7 - 4\sqrt{3} - 8 + 4\sqrt{3} + 6 \\ &= 7 - 8 + 6 = 5 \end{aligned}$$

4. Factor  $13x + 2x^2 - 24$  by completing the square.

Solution: We rearrange the terms first and then factor out the leading coefficient.

$$\begin{aligned} 13x + 2x^2 - 24 &= 2x^2 + 13x - 24 \\ &= 2\left(x^2 + \frac{13}{2}x - 12\right) \end{aligned}$$

Half of the linear coefficient is  $\frac{13}{4}$ , thus we work out  $\left(x + \frac{13}{4}\right)^2$  first to see what we need to smuggle in to complete the square.

$$\begin{aligned}\left(x + \frac{13}{4}\right)^2 &= \left(x + \frac{13}{4}\right)\left(x + \frac{13}{4}\right) = x^2 + \frac{13}{4}x + \frac{13}{4}x + \frac{169}{16} \\ &= x^2 + \frac{13}{2}x + \frac{169}{16}\end{aligned}$$

Thus we need to smuggle in  $\frac{169}{16}$

$$\begin{aligned}2x^2 + 13x - 24 &= 2\left(x^2 + \frac{13}{2}x - 12\right) \\ &= 2\left(\underbrace{x^2 + \frac{13}{2}x + \frac{169}{16}} - \frac{169}{16} - 12\right)\end{aligned}$$

We bring the last two numbers to the common denominator

$$\begin{aligned}2x^2 + 13x - 24 &= 2\left(\left(x + \frac{13}{4}\right)^2 - \frac{169}{16} - \frac{12(16)}{1(16)}\right) \\ &= 2\left(\left(x + \frac{13}{4}\right)^2 - \frac{169}{16} - \frac{192}{16}\right) \\ &= 2\left(\left(x + \frac{13}{4}\right)^2 - \frac{361}{16}\right)\end{aligned}$$

Since  $\frac{361}{16} = \left(\frac{19}{4}\right)^2$ , we factor via the difference of squares theorem.

$$\begin{aligned}2x^2 + 13x - 24 &= 2\left(\left(x + \frac{13}{4}\right)^2 - \left(\frac{19}{4}\right)^2\right) \\ &= 2\left(x + \frac{13}{4} + \frac{19}{4}\right)\left(x + \frac{13}{4} - \frac{19}{4}\right) \\ &= 2\left(x + \frac{32}{4}\right)\left(x - \frac{6}{4}\right) \\ &= 2(x + 8)\left(x - \frac{3}{2}\right)\end{aligned}$$

We may distribute 2 into the second factor. Then we get

$$\begin{aligned}2x^2 + 13x - 24 &= (x + 8)\left(2\left(x - \frac{3}{2}\right)\right) \\ &= (x + 8)(2x - 3)\end{aligned}$$

We FOIL to check:

$$\begin{aligned}(x + 8)(2x - 3) &= 2x^2 - 3x + 16x - 24 \\ &= 2x^2 + 13x - 24\end{aligned}$$

5. Solve each of the following equations. Make sure to check your solution(s).

(a)  $2x^3 = 20x^2 + 1750x$

Solution: We reduce one side to zero, then factor, and then apply the zero property.

$$\begin{aligned}2x^3 &= 20x^2 + 1750x \\ 2x^3 - 20x^2 - 1750x &= 0 && \text{factor out GCF} \\ 2x(x^2 - 10x - 875) &= 0 && \text{divide both sides by 2} \\ x(x^2 - 10x - 875) &= 0\end{aligned}$$

We will factor by completing the square. Half of the linear coefficient is  $-5$ , and thus we will work with

$$(x - 5)^2 = x^2 - 10x + 25$$

We smuggle in 25.

$$\begin{aligned}x(x^2 - 10x - 875) &= 0 \\ x(\underbrace{x^2 - 10x + 25}_{(x-5)^2} - 25 - 875) &= 0 \\ x((x - 5)^2 - 900) &= 0 \\ x((x - 5)^2 - 30^2) &= 0 \\ x(x - 5 + 30)(x - 5 - 30) &= 0 \\ x(x + 25)(x - 35) &= 0\end{aligned}$$

Applying the zero property we obtain **35, 0, and  $-25$**  as the solutions.

(b)  $\frac{3x + 17}{2} = x - 1 + \frac{x + 19}{2}$

Solution:

$$\begin{aligned}\frac{3x + 17}{2} &= x - 1 + \frac{x + 19}{2} && \text{express everything as a fraction} \\ \frac{3x + 17}{2} &= \frac{x - 1}{1} + \frac{x + 19}{2} && \text{bring everything to the common denominator} \\ \frac{3x + 17}{2} &= \frac{2(x - 1)}{2} + \frac{x + 19}{2} && \text{add fractions on right hand side} \\ \frac{3x + 17}{2} &= \frac{2(x - 1) + x + 19}{2} && \text{multiply out parentheses} \\ \frac{3x + 17}{2} &= \frac{2x - 2 + x + 19}{2} && \text{combine like terms} \\ \frac{3x + 17}{2} &= \frac{3x + 17}{2} && \text{multiply by 2} \\ 3x + 17 &= 3x + 17\end{aligned}$$

Because the left hand side is now identical to the right hand side, this equation is an identity, and all real numbers are solution.

(c)  $|3 - 2x| + 2 = 5$

Solution:

$$\begin{aligned} |3 - 2x| + 2 &= 5 && \text{subtract } 2 \\ |3 - 2x| &= 3 \\ 3 - 2x &= 3 &\text{ or }& 3 - 2x = -3 && \text{subtract } 3 \\ -2x &= 0 &\text{ or }& -2x = -6 && \text{divide by } -2 \\ x &= 0 &\text{ or }& x = 3 \end{aligned}$$

Thus the solution are 0 and 3.

(d)  $\frac{2}{3}(x - 7) = \frac{4}{5}(x + 1)$

Solution:

$$\begin{aligned} \frac{2}{3}(x - 7) &= \frac{4}{5}(x + 1) \\ \frac{2}{3} \cdot \frac{x - 7}{1} &= \frac{4}{5} \cdot \frac{x + 1}{1} \\ \frac{2(x - 7)}{3} &= \frac{4(x + 1)}{5} && \text{bring fractions to common denominator} \\ \frac{5 \cdot 2(x - 7)}{15} &= \frac{3 \cdot 4(x + 1)}{15} && \text{multiply both sides by } 15 \\ 10(x - 7) &= 12(x + 1) && \text{multiplty out parentheses} \\ 10x - 70 &= 12x + 12 && \text{subtract } 10x \\ -70 &= 2x + 12 && \text{subtract } 12 \\ -82 &= 2x && \text{divide by } 2 \\ -41 &= x \end{aligned}$$

We check:

$$\begin{aligned} \text{LHS} &= \frac{2}{3}(-41 - 7) = \frac{2}{3}(-48) = -32 \\ \text{RHS} &= \frac{4}{5}(-41 + 1) = \frac{4}{5}(-40) = -32 \end{aligned}$$

Thus our solution, -41 is correct.

(e)  $7x^2 + (x + 3)(2x - 1) = (3x + 1)^2$

Solution:

$$\begin{aligned} 7x^2 + (x + 3)(2x - 1) &= (3x + 1)^2 && \text{multiply the polynomials on both sides} \\ 7x^2 + 2x^2 + 5x - 3 &= 9x^2 + 6x + 1 && \text{combine like terms} \\ 9x^2 + 5x - 3 &= 9x^2 + 6x + 1 && \text{subtract } 9x^2 \\ 5x - 3 &= 6x + 1 && \text{subtract } 5x \\ -3 &= x + 1 && \text{subtract } 1 \\ -4 &= x \end{aligned}$$

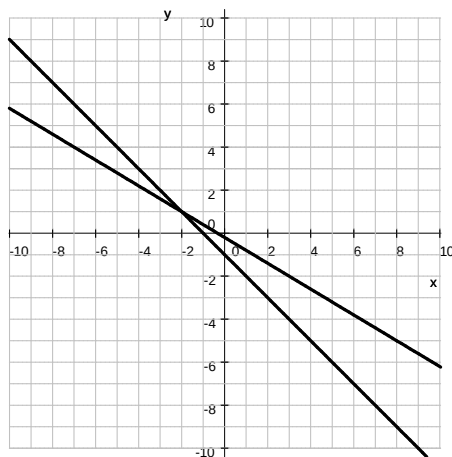
We check our result:

$$\text{LHS} = 7(-4)^2 + ((-4) + 3)(2(-4) - 1) = 7 \cdot 16 + (-1)(-9) = 112 + 9 = 121$$

$$\text{RHS} = (3(-4) + 1)^2 = (-12 + 1)^2 = (-11)^2 = 121$$

Thus the solution,  $-4$  is correct.

6. Graph the straight lines  $3x + 5y = -1$  and  $y = -x - 1$  in the same coordinate system. Use your graph to find the coordinates of the point where the lines intersect.  $(-2, 1)$



7. Graph the parabola  $y = -8x + x^2 + 15$ . Clearly label the coordinates of five points on the parabola, including vertex and intercepts.

Solution: We obtain all forms of the equation first.

$$y = x^2 - 8x + 15 \implies \text{polynomial form}$$

Half of the linear coefficient is  $-4$ , thus we will work with  $(x - 4)^2 = x^2 - 8x + 16$

$$y = x^2 - 8x + 15$$

$$y = \underbrace{x^2 - 8x + 16}_{(x-4)^2} - 16 + 15$$

$$y = (x - 4)^2 - 1 \implies \text{complete square form}$$

We factor via the difference of squares theorem

$$y = (x - 4)^2 - 1^2 \quad \text{since } 1 = 1^2$$

$$y = (x - 4 + 1)(x - 4 - 1)$$

$$y = (x - 3)(x - 5) \implies \text{factored form}$$

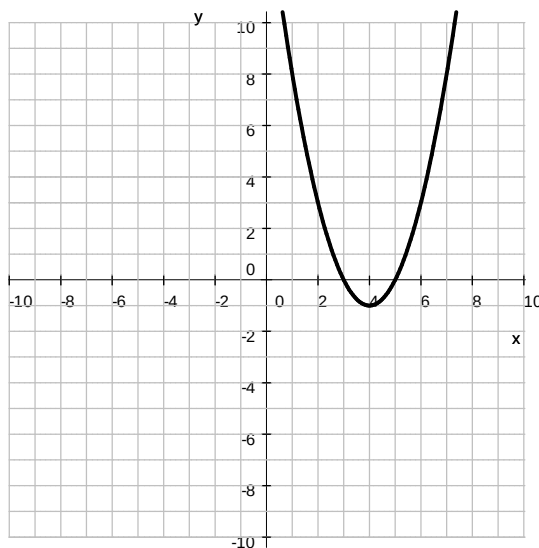
From the polynomial form we obtain the  $y$ -intercept,  $(0, 15)$ . From the complete square form, the vertex is  $(4, -1)$ . Finally, the factored form tells us that there are two  $x$ -intercepts,  $(3, 0)$  and  $(5, 0)$ . The few missing points, close to the vertex can be found by substituting values

for  $x$  into any of the three forms of the equations to find  $y$ . This time we will work with the polynomial form.

$$\text{if } x = 2, \text{ then } y = (2)^2 - 8(2) + 15 = 4 - 16 + 15 = 3$$

$$\text{if } x = 6, \text{ then } y = (6)^2 - 8(6) + 15 = 36 - 48 + 15 = 3$$

We are ready to graph:



8. There is an animal farm where chickens and cows live. All together, there are 53 heads and 174 legs. How many chickens, how many cows?

Solution: Denote the number of chickens by  $x$  and the number of cows by  $y$ . The first equation will express the number of heads, the second equation will express the number of heads.

$$\begin{aligned} x + y &= 53 \\ 2x + 4y &= 174 \end{aligned}$$

To eliminate  $x$ , we multiply the first equation by  $-1$  and divide the second equation by 2.

$$\begin{aligned} -x - y &= -53 \\ x + 2y &= 87 \end{aligned}$$

Now we add the two equations.

$$y = 34$$

We use the first equation to find  $x$ .

$$\begin{aligned} x + 34 &= 53 && \text{subtract 34} \\ x &= 19 \end{aligned}$$

Thus we have **19 chickens and 34 cows**. We check: the number of heads is  $19 + 34 = 53$ , and the number of legs is  $2(19) + 4(34) = 38 + 136 = 174$ . So our solution is correct.

9. One side of a rectangle is 4 in shorter than 3 times the other side. Find the sides of the rectangle if its perimeter is 48 in.

Solution: Let us denote the shorter side by  $x$ . Then the larger side is  $3x - 4$ . The equation expresses the perimeter.

$$\begin{aligned} 2x + 2(3x - 4) &= 48 && \text{multiply out parentheses} \\ 2x + 6x - 8 &= 48 && \text{combine like terms} \\ 8x - 8 &= 48 && \text{add 8} \\ 8x &= 56 && \text{divide by 8} \\ x &= 7 \end{aligned}$$

Thus the shorter side is 7 in, which makes the longer side  $3 \cdot 7 - 4 = 17$  in. Thus the sides are **7 in and 17 in** long.

10. One side of a rectangle is 4 in shorter than 3 times the other side. Find the sides of the rectangle if its area is 319 in<sup>2</sup>.

Solution: Let us denote the shorter side by  $x$ . Then the larger side is  $3x - 4$ . The equation expresses the area.

$$\begin{aligned} x(3x - 4) &= 319 \\ 3x^2 - 4x &= 319 \\ 3x^2 - 4x - 319 &= 0 \end{aligned}$$

We will complete the square.

$$\begin{aligned} 3x^2 - 4x - 319 &= 0 \\ 3\left(x^2 - \frac{4}{3}x - \frac{319}{3}\right) &= 0 \end{aligned}$$

Half of the linear coefficient is  $-\frac{4}{3} \div 2 = -\frac{4}{3} \cdot \frac{1}{2} = -\frac{4}{6} = -\frac{2}{3}$ , thus we work out  $\left(x - \frac{2}{3}\right)^2$ .

$$\begin{aligned} \left(x - \frac{2}{3}\right)^2 &= \left(x - \frac{2}{3}\right)\left(x - \frac{2}{3}\right) = x^2 - \frac{2}{3}x - \frac{2}{3}x + \frac{4}{9} \\ &= x^2 - \frac{4}{3}x + \frac{4}{9} \end{aligned}$$

Thus we smuggle in  $\frac{4}{9}$ .

$$\begin{aligned} 3\left(\underbrace{x^2 - \frac{4}{3}x + \frac{4}{9}}_{\left(x - \frac{2}{3}\right)^2} - \frac{4}{9} - \frac{319}{3}\right) &= 0 \\ 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{319}{3}\right) &= 0 \end{aligned}$$

We bring the last two numbers to the common denominator:

$$3 \left( \left( x - \frac{2}{3} \right)^2 - \frac{4}{9} - \frac{319(3)}{3(3)} \right) = 0$$

$$3 \left( \left( x - \frac{2}{3} \right)^2 - \frac{4}{9} - \frac{957}{9} \right) = 0$$

$$3 \left( \left( x - \frac{2}{3} \right)^2 - \frac{961}{9} \right) = 0$$

Since  $\frac{961}{9} = \left( \frac{31}{3} \right)^2$ , we factor via the difference of squares theorem.

$$3 \left( \left( x - \frac{2}{3} \right)^2 - \left( \frac{31}{3} \right)^2 \right) = 0$$

$$3 \left( x - \frac{2}{3} + \frac{31}{3} \right) \left( x - \frac{2}{3} - \frac{31}{3} \right) = 0$$

$$3 \left( x + \frac{29}{3} \right) \left( x - \frac{33}{3} \right) = 0$$

$$3 \left( x + \frac{29}{3} \right) (x - 11) = 0$$

This equation has two solutions,  $x_1 = -\frac{29}{3}$  and  $x_2 = 11$ . Since distances are positive,  $-\frac{29}{3}$  is ruled out as a solution for the shorter side. The other solution is 11 in. This makes the longer side  $3 \cdot 11 - 4 = 29$  in. We check:  $3(11) - 4 = 29$  and  $11(29) = 319$ . Thus our solution, **11 in by 29 in** is correct.

11. The population of a town has decreased from 80 000 to 68 000. What percent of a decrease does this represent?

Solution 1: We subtract 68 000 from 80 000 to determine the change.  $80\,000 - 68\,000 = 12\,000$ . Now the question is: 12 000 is what percent of 80 000? Then

$$(\text{is}) = 12\,000$$

$$F = x$$

$$(\text{of}) = 80\,000$$

We substitute the data into the formula:

$$(\text{is}) = F \cdot (\text{of})$$

$$12\,000 = x \cdot 80\,000$$

$$\frac{12\,000}{80\,000} = x$$

$$0.15 = x$$

Thus

$$x = 0.15 = \frac{0.15}{1} = \frac{0.15(100)}{1(100)} = \frac{15}{100} = 15\%$$

This is a 15% decrease.

Solution 2: The question may be re-phrased as: 68 000 is what percent of 80 000? Then

$$(\text{is}) = 68\,000$$

$$F = x$$

$$(\text{of}) = 80\,000$$

We substitute the data into the formula:

$$(\text{is}) = F \cdot (\text{of})$$

$$68\,000 = x \cdot 80\,000$$

$$\frac{68\,000}{80\,000} = x$$

$$0.85 = x$$

Thus

$$x = 0.85 = \frac{0.85}{1} = \frac{0.85(100)}{1(100)} = \frac{85}{100} = 85\%$$

Since the population has decreased to 85% of its previous count, this is a 15% decrease.

12. We have invested \$ 15 000 into two accounts. One account earns 6% interest every year, the other account earns 7% every year. How much did we invest into each account if the combined interest was \$1008 after the first year?

Solution: Let  $x$  denote the amount invested at 7%. Then we invested  $15\,000 - x$  at 6%. The equation expresses the combined interest.

$$0.07x + 0.06(15\,000 - x) = 1008 \quad \text{multiplied by 100}$$

$$7x + 6(15\,000 - x) = 100\,800$$

$$7x + 90\,000 - 6x = 100\,800$$

$$x = 10\,800$$

The other account must have  $15\,000 - 10\,800 = 4200$ . Thus we invested \$4200 at 6% and \$ 10 800 at 7%. We check:

$$10\,800 + 4200 = 15\,000 \quad \text{and}$$

$$0.07(10\,800) + 0.06(4200) = 1008$$

13. The hypotenuse of a right triangle is 68 cm. The difference between the other two sides is 28 cm. Find the sides of the triangle.

Solution: Let  $x$  denote the shorter leg. Then the other leg is  $x + 28$  cm long. We state the Pythagorean theorem for the triangle, and solve the quadratic equation for  $x$ .

$$\begin{array}{rcll}
 x^2 + (x + 28)^2 & = & 68^2 & \text{FOIL out } (x + 28)^2 \\
 x^2 + x^2 + 56x + 784 & = & 4624 & \text{combine like terms} \\
 2x^2 + 56x + 784 & = & 4624 & \text{subtract 4624} \\
 2x^2 + 56x - 3840 & = & 0 & \text{factor out 2} \\
 2(x^2 + 28x - 1920) & = & 0 & \text{divide by 2} \\
 x^2 + 28x - 1920 & = & 0 & 
 \end{array}$$

We factor by completing the square. Since half of the linear coefficient is 14, we will work with  $(x + 14)^2 = x^2 + 28x + 196$

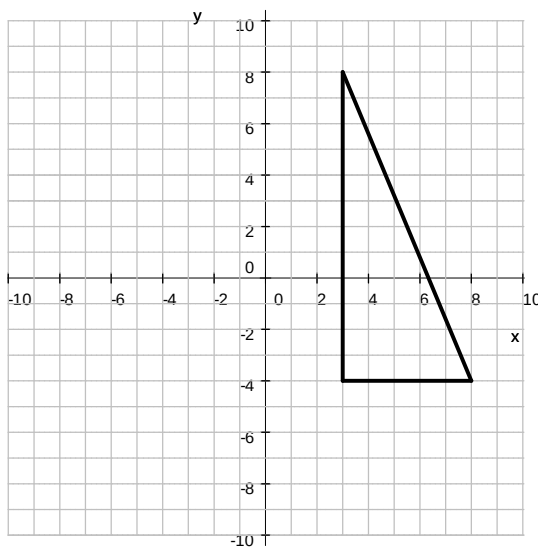
$$\begin{aligned}
 \underbrace{x^2 + 28x + 196} - 196 - 1920 &= 0 \\
 (x + 14)^2 - 2116 &= 0 \\
 (x + 14)^2 - 46^2 &= 0 \\
 (x + 14 + 46)(x + 14 - 46) &= 0 \\
 (x + 60)(x - 32) &= 0 \\
 x_1 &= -60 \quad \text{and} \quad x_2 = 32
 \end{aligned}$$

Since distances are never negative,  $-60$  is ruled out. If the shortest side is 32 cm, the other side is  $32 \text{ cm} + 28 \text{ cm} = 60 \text{ cm}$ . Thus the solution is **32 cm and 60 cm**. We check:

$$\begin{aligned}
 60 - 32 &= 28 \text{ and} \\
 60^2 + 32^2 &= 3600 + 1024 = 4624 = 68^2
 \end{aligned}$$

14. Find the distance between  $(3, 8)$  and  $(8, -4)$ .

Solution: We graph the points, they determine a right triangle as shown below. The legs are 5 and 12 units long, and we need to find the hypotenuse.



$$\begin{aligned}5^2 + 12^2 &= x^2 \\25 + 144 &= x^2 \\169 &= x^2 \\0 &= x^2 - 13^2 \\0 &= (x + 13)(x - 13) \\x_1 &= -13 \quad \text{and} \quad x_2 = 13\end{aligned}$$

Since distances are never negative, the answer is **13 units**.