Beginning Algebra with Geometry – Weeks 1-4

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September, 2018

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Contents

Chapter 1

Class 1

1.1 The Language of Mathematics – Optional

Part 1- What is Mathematics?

As a graduate student, I had the annoying habit of asking my teachers and peer students what they think mathematics is. To my surprise, I received many different answers, and, to this day, I agree with many of them. In my eyes, mathematics is many things. In mathematics, we will be talking a lot about things being true or being not true. Although this probably happens in every course in every discipline, mathematical truth can be objectively established and agreed upon. To achieve such an objective approach, we have to develop a language that is objectively understood. In this sense, mathematics is also a language.

Definition: **Mathematics** is a collection of *true statements* that are developed, expressed, and interpreted using an objective *language* and rules of *logic*.

To understand mathematics, we need to first agree on an objective language. Reading and writing mathematical notation correctly will be important.

'Then you should say what you mean', the March Hare went on. 'I do,' Alice hastily replied; 'at least - at least I mean what I say - that's the same thing, you know.' 'Not the same thing a bit!' said the Hatter. Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see!'

Lewis Carroll

Alice's Adventures in Wonderland

Part 2 - The Language of Mathematics

Mathematical statements are much like English sentences. As English sentences are built from different kinds of words, mathematical statements usually contain three types of components: **objects, operations, and relations**.

> **Definition:** The concept of an **object** (very much like nouns in English sentences) is usually clearly understood and needs no explanation.

Examples of objects from algebra include the number 2, the number 3, and numbers in general. Objects from geometry include lines, points, triangles, circles, line segments, etc.

> **Definition:** An **operation** is an action (very much like verbs in English sentences) that can be applied to objects and usually result in new objects.

Examples of operations from algebra include addition, subtraction, multiplication, and division. Operations from geometry include reflection to a line, rotation, or translation that can be performed on points, triangles, circles, line segments, etc.

> **Definition:** A **relation** is something we use to compare two objects.

Unlike operations, relations do not produce new things; we use relations to compare already existing objects. For example, the operation addition produces 7 if applied to 2 and 5. Relations in algebra are equal, or less, or greater. Relations from geometry are how geometric objects can be compared to each other: similar, congruent, parallel, perpendicular.

Many statements use at least one of each of these three components. In the statement $2+5=7$ the numbers 2, 5, and 7 are the objects, addition (denoted by $+$) is the operation, and being equal (denoted by $=$) is the relation.

It is a common misconception to think of mathematics as the study of *only* numbers. Numbers are only certain types of objects. As we progress in the study of mathematics, we will find that there are many other types of interesting objects. For example, sets are objects we will soon study. Furthermore, the study of operations and relations is also interesting and fruitful.

So, what kind of true statements can be established in mathematics? There are three types of true statements in mathematics: **definitions**, **axioms**, and **theorems**.

> **Definition:** A **definition** is a labeling statement in which we agree to use an expression to refer to an object, operation, or relation in mathematics.

Definitions are all true statements, because they simply reflect an agreement in the terms of the language. To be precise, these decisions were made without consultating any of us, often decades (if not centuries) before any of us were ever born. An example of a definition would be if we pointed to a clear sky and said: "From now on, let’s call this color blue".

> **Definition:** A **theorem** is a statement that we insist on proving before believing that it is true. To **prove** a theorem means to derive it from previously established true statements, using logically correct steps.

If you think about that last definition a little, you will see that no theorem can exist, unless we agree on accepting a few statements to be true, without proving them. These are our "starting true statements".

> **Definition:** An **axiom** is a statement we agree to accept to be true without proof.

Axioms are usually simple, basic statements that are in agreement with our intuition. For example, the statement "*It is possible to draw a straight line from any point to any other point.*" is an axiom. It has been a constant effort to keep the number of axioms to a minimum. We prove a theorem by deriving its statement from statements already established to be true.

To be precise, when we prove our first theorem, we derive its statement from the axioms. When we prove our second theorem, we derive its statement from the axioms and the first theorem. When proving the third theorem, we can use all the axioms, and the first and second theorems. And so on. For our tenth theorem, we have all the axioms and the first nine theorems at our disposal. At this point, we are building a logically sound theory, a unified discipline within mathematics. It is one thing to suspect, to feel, or to have a hunch that something is true. It is entirely different from proving it, with unescapable force of logic.

The ancient Greek mathematician Euclid discussed mathematics in this manner, i.e stating axioms and building a theory by deriving a sequence of theorems from the axioms. (He called axioms postulates.) Mathematicians immediately accepted and embraced this logical approach to the study of mathematics - and it is how it is done still today. Although Euclid has contributed to several parts of mathematics (including geometry and number theory), he completely axiomatized of what we now call classical geometry or Euclidean geometry. He stated five postulates, accepted them to be true and derived most basic theorems of classical geometry.

Enrichment

1. Look up Euclid's book *Elements* on the internet. (Start at Wikipedia). List Euclid's five postulates. Explain the significance of these five statements.
2. What is the parallel postulate? Look up the history of Euclid's parallel postulate on the internet. (Start with Wikipedia.) How would we go about proving that an axiom is really a theorem? List statements from geometry that are logically equivalent to the parallel postulate. What exactly could it mean for two axioms to be equivalent to each other?

Euclid (~400 BC - ~300 BC)

1.2 The words *And* and *Or*

The words *and* and *or* are used differently in mathematics from every day use. Their meaning in mathematics is more restricted than in the English language. First of all, statements or questions in mathematics that use these words are all yes or no questions.

Imagine a cold Monday morning when Sylvia arrives late to her first class, a mathematics class. To make things worse, it is exam day. The teacher stops her at the door and asks: ”Did you bring a pen or a pencil?” What does an answer of yes mean? What does an answer of no mean? An answer of yes means that she either has a pen only, or a pencil only, or both. An answer of no means that she has neither.

Her next class is a drawing class where both pen and pencils are needed. There the teacher might ask her: ”Did you bring a pen and pencil?” What does an answer of yes mean? What does an answer of no mean? An answer of yes means that she has both a pen and a pencil. An answer of no means that she has only pen, or only pencil, or neither. This is the only allowed use of the words *and* and *or* in mathematics.

Definition: Suppose that A and B are statements. The statement **A or B** is true when A is true, or when B is true, or when both A and B are true. A or B is false if both A and B are false.

The statement **A and B** is true when both A and B is true. A and B is false when either A is false, or B is false, or both A and B are false. We can express this using truth tables.

Truth table for A or B

A	B	A or B
true	true	true
true	false	true
false	true	true
false	false	false

Truth table for A and B

A	B	A and B
true	true	true
true	false	false
false	true	false
false	false	false

When a single statement is formed by connetcting two or more statements with *and* or *or*, we call such a statement a **compound statement**.

Example 1. Determine whether the given statements are true or false.

a) *The sky is blue or the Earth is flat.* b) *The sky is blue and the Earth is flat.*

Solution: These compound statements are made by connecting two statements. These statements are:

The sky is blue. - this is true. and *The Earth is flat.* - this is false.

a) *The sky is blue* **or** *the Earth is flat.*
This is true because one true statement is enough for an ’or’ statement to be true.

b) *The sky is blue* **and** *the Earth is flat..*
This is false because for an ’and’ statement to be true, both statements must be true.

Example 2. Determine whether the given statements are true or false.

a) *The number* 8 *is greater than* 8 *and is equal to* 8. b) *The number* 8 *is greater than* 8 *or is equal to* 8.

Solution: *The number* 8 *is greater than* 8 - this is false. *The number* 8 *is equal to* 8 - this is true.

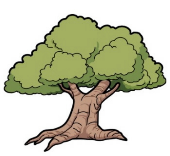
a) When we connect a true and a false statement with ’and’, the compound statement is false.
So, *The number* 8 *is greater than* 8 *and is equal to* $8. -$ is false.

b) When we connect a true and a false statement with ’or’, the compound statement is true.
The number 8 *is greater than* 8 *or is equal to* $8. -$is true, and we write it as $8 \geq 8$.

Practice Problems

Label each of the given statements as true or false.

1. A week consists of seven days.
2. Every month consist of 31 days.
3. Water is a liquid at room temperature.
4. Water is frozen solid at a temperature of 5F°.
5. $2+2=5$
6. 5 is an odd number.
7. 3 is an even number.
8. 2 is less than 10.
9. 2 is greater than 10.
10. 2 is equal to 10.
11. A week consists of seven days, or every month consist of 31 days.
12. A week consists of seven days, and every month consist of 31 days.
13. $2+2=5$, or water is a liquid at room temperature.
14. $2+2=5$, and water is a liquid at room temperature.
15. 2 is less than 10, or 2 is greater than 10.
16. 3 is an even number or $2+2=5$.
17. 2 is equal to 10 or 5 is an odd number.
18. 2 is less than 10 and water is frozen solid at a temperature of 5F°.
19. 7 is less than 7 or 7 is equal to 7. (We write this as $7 \leq 7$)
20. 7 is less than 7 and 7 is equal to 7.



Enrichment

1. Interpret A or B or C as (A or B) or C and create a truth table for this compound statement. How many different cases are there?
2. Interpret A and B and C as (A and B) and C and create a truth table for this compound statement. How many different cases are there?
3. Create a truth table for the compound statement (A and B) or C. How many different cases are there?
4. Create a truth table for the compound statement A and (B or C). How many different cases are there?

Chapter 2

Class 2

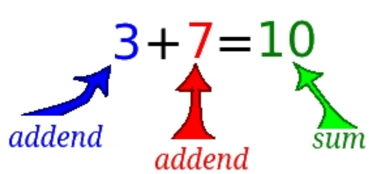
2.1 The Set of All Natural Numbers and Basic Operations

The **set of all natural numbers**, (sometimes also called the set of all counting numbers), denoted by $\mathbb{N}$, is the set

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

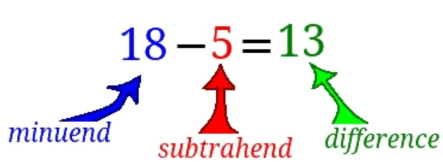
It is easy to imagine that the set of all natural numbers was the first set of numbers at which human beings looked. The four basic operations can be defined and performed on natural numbers as follows.

Addition, denoted by $+$, is defined as we usually think of addition: the addition of two natural numbers is obtaining the total amount of those quantities combined. In the statement $3+7=10$, we say that 3 and 7 are **addend**s and 10 is called the **sum** of 3 and 7.



If we add two natural numbers, the sum is always a natural number. This is called **closure**. The set of all natural numbers is closed under addition: If n and m are natural numbers, then $n+m$ is also a natural number.

Subtraction, denoted by $-$, is a mathematical operation that represents the operation of removing objects from a collection. In the statement $18-5=13$, we say that 18 is the **minuend** and 5 is the **subtrahend**, and 13 is called the **difference** of 18 and 5.

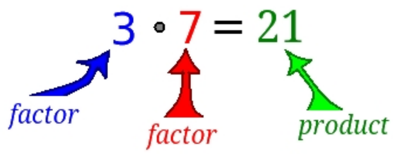


If we subtract a natural number from another natural number, the difference may or may not exits within $\mathbb{N}$. For example, the subtraction $8-5$ results in a natural number but the subtraction $3-11$ does not. In other words, the set of all natural numbers is NOT closed under subtraction.

Multiplication, denoted by $\cdot$, or by $\times$, or by nothing at all between two objects, is defined as we usually think of multiplication

$$3\cdot 7=21 \quad \text{or} \quad 3\times 7=21 \quad \text{or} \quad (3)\,7=21 \quad \text{or} \quad 3\,(7)=21$$

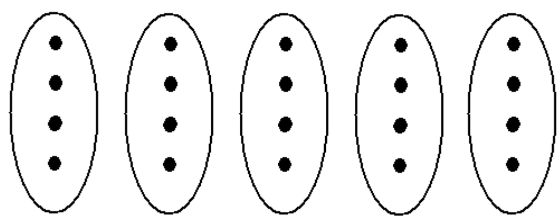
Please note that in the last two equations, the **parentheses do not indicate multiplication**. The parentheses helps us interpret 3 and 7 as two separate numbers and not the number 37. Once we see two numbers with no operation sign between them, that NOTHING indicates multiplication. In the statement $3\cdot 7=21$, we say that 21 is the **product** of 3 and 7. We also say that 3 and 7 are **divisors** or factors of 21.



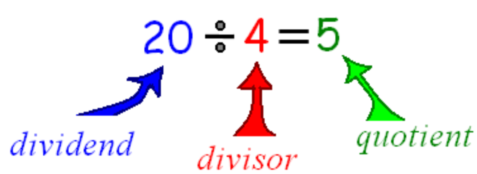
If we multiply two natural numbers, the product is always a natural number. In other words, the set of all natural numbers is closed under multiplication: if n and m are natural numbers, then nm is also a natural number.

Division, denoted by $\div$ or by $/$ is defined as we usually think of division. One example is: if we have 20 dots and we circle together every four dots, how many packages of four do we obtain? The answer is clearly five, because five packages of four will account for 20 dots.

$$20\div 4=5 \quad \text{or} \quad \frac{20}{4}=5$$



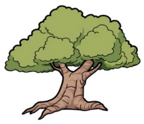
In the statement $20\div 4=5$, we say that 20 is the **dividend**, 4 is the **divisor**, and 5 is the **quotient** of 20 and 4.



If we divide a natural number by another natural number, the quotient may or may not exist within $\mathbb{N}$. For example, the division $12\div 3$ results in a natural number, but the division $20\div 7$ does not. In other words, the set of all natural numbers is NOT closed under division.

Discussion:

1. Consider the multiplication $4\cdot 9=36$ and the division $30\div 6=5$. While in the division all of 30, 6, and 5 have different names, both 4 and 9 are simply called factors in the multiplication. Can you explain why this will not cause any problems?
2. Under which of the four basic operations (addition, subtraction, multiplication, division) is $\mathbb{N}$ closed?



Enrichment

If you haven't already done so, read the very first section, because you need to understand the concepts of axioms and theorems for this exercise.

1. The set of all natural numbers, $\mathbb{N}$ was not defined rigorously. Instead, we gave an intuitive description of $\mathbb{N}$. However, mathematicians insisted on axiomatizing $\mathbb{N}$. This means that they established a set of axioms from which many theorems can be derived. The resulting collection of true statements is the same as what we get starting with the intuitive definition.

   Research the Peano axioms. Feel free to start at Wikipedia. What are the Peano axioms?

   Note: when you look at the Peano axioms, you may notice that according to Wikipedia, 0 is a natural number. In our class, 0 was not defined as a natural number. Do not let yourself be annoyed or confused by the difference. Both conventions are very common.

2.2 The Order of Operations Agreement

The order of operations rule is an agreement among mathematicians, it simplifies notation. P stands for parentheses, E for exponents, M and D for multiplication and division, A and S for addition and subtraction. Notice that M and D are written next to each other. This is to suggest that multiplication and division are equally strong. Similarly, A and S are positioned to suggest that addition and subtraction are equally strong.

P
E
M D
A S

This is the hierarchy, and there are two basic rules.

1. **Between two operations that are on different levels of the hierarchy, we start with the operation that is higher.** For example, between a division and a subtraction, we start with the division since it is higher in the hierarchy than subtraction.

2. **Between two operations that are on the same level of the hierarchy, we start with the operation that comes first from reading left to right.**

Example 1. Perform the indicated operations. $20-3\cdot 4$

Solution: We observe two operations, a subtraction and a multiplication. Multiplication is higher in the hierarchy than subtraction, so we start there.

$$\begin{aligned} 20-3\cdot 4 &= && \text{multiplication} \\ 20-12 &= && \text{subtraction} \\ &= \boxed{8} \end{aligned}$$

Example 2. Perform the indicated operations. $36 \div 3 \cdot 2$

Solution: It is a common mistake to start with the multiplication. The letters M and D are in the same line because they are equally strong; among them we proceed left to right. From left to right, the division comes first

$$\begin{aligned} 36 \div 3 \cdot 2 &= && \text{division} \\ 12 \cdot 2 &= && \text{multiplication} \\ &= \boxed{24} \end{aligned}$$

Example 3. Perform the indicated operations. $36 \div 2 \div 2$

Solution: It is essential to perform these two divisions left to right. If we proceeded differently, we would get a different result.

$$\begin{aligned} 36 \div 2 \div 2 &= && \text{first division from left} \\ 18 \div 2 &= && \text{division} \\ &= \boxed{9} \end{aligned}$$

Example 4. Perform the indicated operations. $(8-5)^2 - 5 + 2$

Solution: We start with the subtraction within the parentheses. Then we can drop the parentheses.

$$\begin{aligned} (8-5)^2 - 5 + 2 &= 3^2 - 5 + 2 && \text{exponentiation} \\ &= 9 - 5 + 2 && \text{subtraction} \\ &= 4 + 2 && \text{addition} \\ &= \boxed{6} \end{aligned}$$

Notations for Multiplication

There are different ways we denote multiplication. Suppose we wanted to express the multiplication 2 times 3, our options are $\cdot$ (dot), or $\times$ (cross), or *nothing*.

$2 \cdot 3$	or	2×3	or	$(2)(3)$ or $2(3)$ or $(2)3$
This is the preferred notation		We will almost never use this. It's a dinosaur, it is going away....		We will almost never use this with positive numbers

It is a common misconception to believe that parentheses mean multiplication. This is not true. **Parentheses never denote multiplication.** In the expression $2(3)$, the parentheses tell us that we are not looking at the two-digit number 23, but rather at the separate numbers 2 and 3, with *nothing* between them. In written notation, multiplication is the default operation. In other words, if we see two numbers with nothing between them, the *nothing* indicates multiplication. This becomes more clear if we consider expressions such as $2x$ or ab. There is no operation sign (or parentheses) in these expressions and yet we know that the operation is mutiplication. It is the *nothing* between 2 and x and between a and b means multiplication.

Understanding Parentheses

Parentheses have no meaning on their own. They merely help us to understand the context in the written language, like punctuation does in written English. Some parentheses are **grouping symbols** overwriting order of operations. Others clarify boundaries of numbers. For example, it is possible to write $2(3)$ instead of $2 \cdot 3$. The parentheses in $2(3)$ is not a grouping symbol. Rather, it helps us understand that we are not looking at the two-digit number 23. We will refer to this as **clarifying parentheses**. One easy way to tell the difference between the two types of parentheses is that if there is no operation inside a pair of parentheses, then it cannot be a grouping symbol.

We will often see expressions with several pairs of parentheses. Luckily, there are only two possibilities.

$4(3-1)-(7\cdot 2-9)$	$20-\left(12-2\left(5\cdot 8-6^2\right)+1\right)$
one pair of parentheses follows the other	one pair of parentheses is nested inside the other

When faced with an order of operations problem, *we must pair the parentheses before starting the computation*. Students are encouraged to use colored pencils to make the pairing more visual such as in $20-\left(12-2\left(5\cdot 8-6^2\right)+1\right)$. Sometimes authors aim to make notation easier to read by using pairs of parentheses with different shapes, such as $[\ ]$ or $\{\ \}$. These different looking parentheses serve as regular grouping symbols, they only made to look differently. For example,
$20-\left(12-2\left[5\cdot 8-6^2\right]+1\right)$ might be easier to read than $20-\left(12-2\left(5\cdot 8-6^2\right)+1\right)$.

When we have several pairs of parentheses following each other, we perform them left to right. When we are dealing with parentheses nested inside others, we start with the innermost one.

Sometimes we have a case of an **invisible parentheses**. When an operation sign encloses entire expressions, it serves as a grouping symbol. For example, there is no parentheses in sight in the expression $\dfrac{32-4}{9-2}$, and division comes before substraction. However, the division bar stretching over the entire expression indicates that we must perform the subtraction on the top and on the bottom, and only then divide. This becomes obvious if we switch the notation from the division bar to the symbol $\div$.

$$\frac{32-4}{9-2}=(32-4)\div(9-2)$$

In this exmple, the division bar is easier to read, so it will be the notation of choice. Note however, that students often make a mistake when entering things like this in the calculator. The correct answer is 7 but the calculator will give us the wrong answer if we enter it incorrectly as $32-4\div 9-2$. In these cases, the parentheses must become visible.

Example 5. Consider the expression $4(3-1)-(7\cdot 2-9)$

a) How many operations are there in the expression?

b) Simplify the expression by applying the order of operations agreement. For each step, write a separate line.

Solution: a) We "scan" the expression left to right, and count the operations.

1. multiplication between 4 and $(3-1)$
2. subtraction $3-1$
3. subtraction between the two expressions within the parentheses
4. multiplication between 7 and 2
5. subtracting 9

So there are five operations.

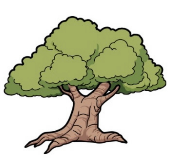
b) We have two pairs of parentheses. We will work them out, left to right. We start with the subtraction $3-1$. Once this subtraction is performed, we no longer need the parentheses. Instead, we will denote the multiplication using the dot notation.

$$\begin{aligned}
4(3-1)-(7\cdot 2-9) &= 4\cdot 2-(7\cdot 2-9) && \text{Now we work on the other parentheses.}\\
& && \text{Multiplication before subtraction}\\
&= 4\cdot 2-(14-9) && \text{Subtraction within parentheses}\\
& && \text{Drop parentheses}\\
&= 4\cdot 2-5 && \text{Multiplication}\\
&= 8-5 && \text{Subtraction}\\
&= \boxed{3}
\end{aligned}$$

Example 6. Evaluate the expression $\left(7-\left(5-\left(11-(7-4)^2\right)^2\right)^2\right)^2$

Solution: This problem is a good example for why it is a good thing to make only one step in each line. Instead of sorting out all the things to do, we just need to find the one operation to do. Once we write down our next line, we have the same goal, but this time we are looking at a simpler problem. The first operation to perform is the subtraction $7-4$ in the innermost parentheses. In this example, we will immediately drop parentheses that are no longer necessary. This is the recommended practice.

$$\begin{aligned}
&\left(7-\left(5-\left(11-(7-4)^2\right)^2\right)^2\right)^2 =\\
&= \left(7-\left(5-\left(11-3^2\right)^2\right)^2\right)^2 && \text{exponentiation in the innermost parentheses}\\
&= \left(7-\left(5-(11-9)^2\right)^2\right)^2 && \text{subtraction in the innermost parentheses}\\
&= \left(7-\left(5-2^2\right)^2\right)^2 && \text{exponentiation in the innermost parentheses}\\
&= \left(7-(5-4)^2\right)^2 && \text{subtraction in the innermost parentheses}\\
&= \left(7-1^2\right)^2 && \text{exponentiation in parentheses}\\
&= (7-1)^2 && \text{subtraction in parentheses}\\
&= 6^2 && \text{exponentiation}\\
&= \boxed{36}
\end{aligned}$$



Enrichment

Place one or more pairs of parentheses into the expression on the left-hand side to make the equation true.

$$12-2\cdot 3-1^2+2-3+4=20$$

Sample Problems

Simplify each of the following expressions by applying the order of operations agreement.

1. $2 \cdot 3^2 - \left(6^2 - 2 \cdot 5\right) \div 2$
2. $18 - 7 - 3$
3. $5^2 - 2\left(10 - 2^2\right)$
4. $8^2 - 3^2$
5. $(8-3)^2$
6. $\left(3^3 - 4 \cdot 5 + 2\right)^2$
7. $\dfrac{3 + 2\left(20 - 3^2 - 5\right)}{3^2 - 2^2} + 4^1$

Practice Problems

Simplify each of the following expressions by applying the order of operations agreement.

1. $2 \cdot 5^2 - \left(6 \cdot 5 - 3^2\right) \div 3$
2. $10^2 - 7^2$
3. $(10-7)^2$
4. $20 - 7 - 1$
5. $2^3 - \left(11 - 3^2\right)^2$
6. $\dfrac{5^2 - 3^2}{2^2}$
7. $\dfrac{(5-3)^2}{2^2}$
8. $120 \div 6 \cdot 2$
9. $\left((7-4)^2 - 5\right)^2 - 1$
10. $\dfrac{22 - 3^2 + 2\left(20 - 3^2 - 5\right)}{3^2 - 2^2}$
11. $\dfrac{5 + \left(5^2 - 3^2\right)}{3^2 - 2 \cdot 1^8}$
12. $30 - \left(2\left(15 - 2^3\right) - 2^2\right)$
13. $4\left(3\left(2(2^2 - 1) + 1\right) - 1\right) + 5$
14. $\dfrac{2\left(3^3 - 4 \cdot 5\right) - 2^2}{4^2 - \left(3^2 + 2\right)}$
15. $\left(\left(\left(19 - \left(8 - 2^2\right)^2\right)^2 - 7\right)^2 - 3\right)^2$

2.3 Problem Set 1

1. Label each of the following statements as true or false.

   a) 3 is odd and the set of all natural numbers is closed under addition.

   b) 12 is even and -7 is a natural number.

   c) 5 is greater than 7 or 5 is equal to 7. (We write this as $5 \geq 7$ read: 5 is greater than or equal to 7)

   d) 8 is less than 8 or 8 is equal to 8. (We write this as $8 \leq 8$ read: 8 is less than or equal to 8)

2. List all natural numbers x with the given property.

   a) $x < 6$ b) $x < 7$ and $x > 3$ c) $x < 7$ or $x > 3$ d) $x \leq 10$ and x is even

3. Simplify each of the following expressions by applying the order of operations agreement. **Show all steps. For each step, write a separate line!**

   a) $8-2+3$

   b) $\dfrac{15-2^3+3}{3^2-2^3}$

   c) $120 \div \left(4+3\left(5\cdot 2^2-2\left(5+2^2\right)\right)\right)$

   d) $\dfrac{3^2-2^2}{(3-2)^2}$

   e) $\left(3-\left(10-3^2\right)^2\right)^2$

   f) $\dfrac{100\div 5\cdot 2}{25-10+5}$

   g) $32-3\left(28-2^2(20-5\cdot 3)\right)$

   h) $\left(\left((10-8)^2-1\right)^2-2\right)^2$

   i) $1^2+1^3-1^4+1^5$

   j) 2^2+2^3

   k) $2^5-3\left(12-3^2\right)^2$

   l) $5\cdot 2^3-\left(10-(7-2\cdot 3+1)\div 2+2^2\right)$

   m) $\left(2-\left(2-\left(10-3^2\right)^2\right)^2\right)^2$

4. Insert parentheses in the expression on the left-hand side to make the equation true.

$$36-2\cdot 5-2^2+4=10$$

Chapter 3

Class 3

3.1 Introduction to Set Theory – Definitions and Set Relations

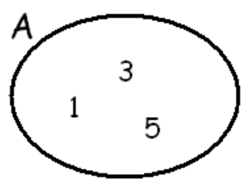
> **Definition:** A **set** is a collection of objects. Two sets are equal if they contain the same objects.

Sets are usually denoted by uppercase letters. There are several ways a set could be given. We can describe a set using English language. In case of small sets, we can also simply list its elements separated by commas and enclosed in braces $\{\ \ \}$.

Example 1. Let M be the set of all one-digit natural numbers. Re-write this set by listing its elements.

We use the braces and list all one-digit natural numbers. $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

We can also describe a set using a diagram.
We depicted $A = \{1, 3, 5\}$.



Some famous sets have their own set theory label. For example, we already know the infinite set $\{1, 2, 3, 4, ...\}$. This is the set of all natural numbers or counting numbers, and it is denoted by $\mathbb{N}$.

Sometimes we need to be more descriptive when specifying sets.

Example 2. $S = \{x^2 : x \text{ is a natural number and } x \leq 5\}$.

We read this as S is a set containing x^2, where x is a natural number and x is less than or equal to 5. Of course, such a small set can be expressed much simpler, by listing its elements, as $S = \{1, 4, 9, 16, 25\}$. This notation is often very useful when describing infinite sets.

> **Definition:** A set is a collection of objects. The objects that make up the set are called the **elements** or **members** of the set, or that it **belongs** to the set.
>
> Notation: If x is an element of a set S, we write by $x \in S$. If y is not an element of S, we write $y \notin S$.

Suppose that $A = \{1, 3, 5\}$. The following statements are all true.

$3 \in A$	read as: 3 belongs to A	$4 \notin A$	read as: 4 does not belong to A
$5 \in A$	read as: 5 is an element of A	$-6 \notin A$	read as: -6 is not an element of A

Example 3. Suppose that $A = \{1, 3, 5\}$ and that $\mathbb{N}$ is the set of all natural numbers, in short, $\mathbb{N} = \{1, 2, 3, 4, ...\}$. Determine whether the given statements are true or false.

a) $1 \in A$ b) $2 \in A$ c) $-1 \in \mathbb{N}$ d) $5 \notin \mathbb{N}$ e) $4 \notin A$

Solution: a) The statement $1 \in A$ reads: 1 *belongs to set* A. This is true as 1 is an element of set A.

b) The statement $2 \in A$ reads: 2 *belongs to set* A. This is not true as 2 is not an element of set A.

c) The statement $-1 \in \mathbb{N}$ reads: -1 *belongs to the set of all natural numbers*. (In short, -1 is a natural number). This statement is false.

d) The statement $5 \notin \mathbb{N}$ reads: 5 *does not belong to the set of all natural numbers*. (In short, 5 is not a natural number). This statement is false.

e) The statement $4 \notin A$ reads: 4 *does not belong to set* A. This statement is true.

Definition: Two sets are **equal** if they contain the same objects. We use the symbol $=$ to denote equal sets.

When writing a set, the order of listing and repetition of its elements does not change a set.

Example 4. Let A be the set of odd natural numbers between 0 and 6. Suppose further that $B = \{5, 1, 3\}$, and that $C = \{1, 5, 1, 5, 1, 1, 1, 3, 3\}$. Then all three sets are equal to each other, i.e $A = B = C$.

Since we are free to list the elements of a set any way we want to, it is often startegic to keep things organized by listing elements in increasing order. In this case, $A = B = C = \{1, 3, 5\}$. Sometimes we have reasons to part from this convention.

Definition: The **set of all integers**, denoted by $\mathbb{Z}$, is the set containing all natural numbers, their opposites, and zero.

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots\}$$

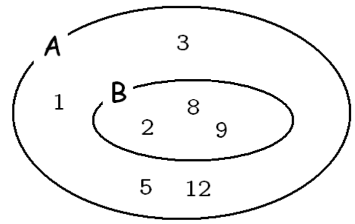
Some people prefer to present the set of all integers sort of organized, as $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The disadvantage here is that both the beginning and the end of this infinite list goes on forever. More precisely, there is no beginning and no end. Both presentations are commonly used.

Definition: There is a unique set that contains no elements. It is called the **empty set** and is denoted by $\varnothing$ or by $\{\quad\}$.

Definition: Set B is a **subset** of set A if all elements of B also belong to A. Notation: $B \subseteq A$

There is another way to express this relationship. B is a subset of A if for all things x in the world, if x is an element of B, then x is also an element of A. This approach might be helpful later.

Example 5. Suppose that $A = \{1, 2, 3, 5, 8, 9, 12\}$ and
$B = \{2, 8, 9\}$. Then B is a subset of A.



Example 6. Suppose that $X = \{a, b, d, f\}$ and $Y = \{a, b, c, d, e, f, g\}$. Then $X \subseteq Y$.

Example 7. Suppose that $S = \{1, 4, 9, 16\}$. Then $S \subseteq S$.

While this might look strange at first, the definition of subset applies. Every element of S is an element of S. Perhaps this statement is similar to $5 \leq 5$. For every number x, the statement $x \leq x$ is true.

Even more interestingly, the empty set is also a subset of every set. This is because the definition applies, even if strangely so. For every object x in the world, if x is in the empty set (it's not), then it is also in set A. We say that this statement is **vacuously true**.

Theorem: For all sets S, the following are both true: $\varnothing \subseteq S$ and $S \subseteq S$.

Example 8. If $\mathbb{N}$ is the set of all natural numbers and $\mathbb{Z}$ is the set of all integers, then $\mathbb{N} \subseteq \mathbb{Z}$.

Example 9. Suppose that E is the set of all even natural numbers, $E = \{2, 4, 6, 8, 10, \ldots\}$, and recall that $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. Then $E \subseteq \mathbb{N}$.

Example 10. Suppose that L is the set of all letters in the English alphabet, and V is the set of vowels in the English alphabet. Then $V \subseteq L$.

Example 11. Let M be the set of all mammals and D the set of all dogs. Then D is a subset of M, or, in short, $D \subseteq M$.

Example 12. List all subsets of the given set if

a) $A = \{1\}$ b) $B = \{1, 2\}$ c) $C = \{1, 2, 3\}$

Solution: a) The set $\{1\}$ has two subsets: $\varnothing$ and $\{1\}$.

b) The set $\{1, 2\}$ has four subsets: $\varnothing$, $\{1\}$, $\{2\}$, and $\{1, 2\}$.

c) As the given sets become larger, it becomes important to be systematic when finding the subsets. Let us list the subsets of C by organizing by the number of its elements.

0-element subsets:	$\varnothing$	
1-element subsets:	$\{1\}, \{2\}, \{3\}$	
2-element subsets:	$\{1, 2\}, \{1, 3\}, \{2, 3\}$	
3-element subsets:	$\{1, 2, 3\}$	So there are 8 subsets.

Practice Problems

1. Suppose that S is a set defined as $S = \{-2, 4, 5, 16\}$ and recall that $\mathbb{N} = \{1, 2, 3, 4, ...\}$. Determine whether the given statements are true or false.

   a) $-2 \in S$ b) $-2 \in \mathbb{N}$ c) $-3 \notin \mathbb{N}$ d) $5 \notin S$ e) $1 \in \mathbb{N}$

2. Let $A = \{1, 2, 5, 8, 9\}$ and $B = \{2, 4, 6, 8\}$. Draw a Venn diagram depicting these sets.

3. Suppose that $P = \{1, 7, 8\}$ and $Q = \{1, 2, 5, 7, 8\}$. Label each of the following statements as true or false.

   a) $P \subseteq \mathbb{Z}$ b) $Q \subseteq P$ c) $P \subseteq Q$ d) $\mathbb{N} \subseteq Q$ e) $\varnothing \subseteq P$

4. Find each of the following sets and if possible, present them by listing their elements.

   a) $S = \{x : x$ is a natural number such that $x < 4$ and $x < 7\}$

   b) $A = \{a : a$ is a natural number such that $a < 4$ or $a < 7\}$

   c) $P = \{y : y$ is an integer such that $y > 3$ and $y < 8\}$

   d) $M = \{y : y$ is an integer such that $y > 3$ or $y < 8\}$

5. Suppose that $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 3, 4\}$. Explain why $Y \in X$ is a false statement.

6. Find two sets A and B so that both $A \subseteq B$ and $B \subseteq A$ are true.

7. List all subsets of $A = \{1, 2, 3, 4\}$.

Enrichment

1. Suppose that A and B are sets such that $A \cap B = \{1, 2, 5\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$. How many different sets are possible for A?

2. Our junior class had 60 students. If 42 students took history, 35 students took French, and 19 took both history and French, how many students in the junior class took neither French nor history? (Hint: this is the kind of problem in which a Venn diagram can be very helpful.)

3.2 Introduction To Number Theory

> **Definition:** Suppose that N and m are any two integers. If there exists an integer k such that $N = mk$, then we say that m is a **factor** or **divisor** of N. We also say that N is a **multiple** of m or that N is **divisible** by m.
>
> Notation: $m|N$

For example, 3 is a factor of 15 because there exists another integer (namely 5) so that $3 \cdot 5 = 15$. Notation: $3|15$.

Example 1. Label each of the following statements as true or false.

a) 2 is a factor of 10 c) 14 is a factor of 7 e) every integer is divisible by 1
b) 3 is divisible by 3 d) 0 is a multiple of 5 f) every integer n is divisible by n

Solution: a) $10 = 2 \cdot 5$ and so 2 is a factor of 10. This statement is true.

b) $3 = 3 \cdot 1$ and so 3 is divisible by 3. This statement is true.

c) $14 = 7 \cdot 2$ and so 14 is a multiple of 7, not a factor. Can we find an integer k so that $7 = 14 \cdot k$? This is not possible. $k = \frac{1}{2}$ would work, but $\frac{1}{2}$ is not an integer. This statement is false.

d) Since $0 = 5 \cdot 0$, it is indeed true that 0 is a multiple of 5. This statement is true.

e) For any integer n, $n = n \cdot 1$ and so every integer n is divisible by 1. This statement is true.

f) For any integer n, $n = 1 \cdot n$ and so every integer n is divisible by n. This statement is true.

Example 2. List all positive factors of the number 28.

Solution: We start counting, starting at 1.
Is 1 a divisor of 28?
Yes, because $28 = 1 \cdot 28$.
We note both factors we found.

28	
1	28

We continue counting. Is 2 a divisor of 28?
Yes, because $28 = 2 \cdot 14$.
We note both factors we found.

28	
1	28
2	14

We continue counting. Is 3 a divisor of 28? No. We can divide 28 by 3 and the answer is not an integer.

We continue counting. Is 4 a divisor of 28? Yes, because $28 = 4 \cdot 7$. We note both factors we found.

28	
1	28
2	14
4	7

We continue counting. Is 5 a divisor of 28? No. (We can check with the calculator.) Is 6 a divisor of 28? No. Now we arrive to 7, a number that is already listed as a factor. That's our signal that we have found all of the divisors of 28. We list the divisors in order:

factors of 28: $\boxed{1, 2, 4, 7, 14, 28}$

Discussion: What do you think about the argument shown below?

3 is a divisor of 21 because there exists another integer, namely 7, so that $21 = 3 \cdot 7$. As we established that 3 is a divisor of 21, we also found that 7 is also a divisor of 21. In other words, divisors always come in pairs. For example, 28 has six divisors that we found in three pairs: 1 with 28, 2 with 14, and 4 with 7. Consequently, every positive integer has an even number of positive divisors.

Definition: An integer is a **prime number** if it has exactly two divisors: 1 and itself.

For example, 37 is a prime number.

We will later prove all of the following statements. They will cut down on the work as we look for divisors of a number.

Theorem: A number n is divisible by 2 if its last digit is 0, 2, 4, 6, or 8.
A number n is divisible by 5 if its last digit is 0, or 5.
A number n is divisible by 4 if the two-digit number formed of its last two digits is divisible by 4.
A number n is divisible by 3 if the sum of its all digits is divisible by 3.
A number n is divisible by 9 if the sum of its all digits is divisible by 9.

Practice Problems

1. List all the factors of 48.
2. Which of the following is NOT a prime number? $53, 73, 91, 101, 139$
3. Consider the following numbers. 128, 80, 75, 270, 64
   a) Find all numbers on the list that are divisible by 5.
   b) Find all numbers on the list that are divisible by 3.
   c) Find all numbers on the list that are divisible by 4.

Enrichment

1. Two mathematicians are having a conversation. Mathematician A asks B about his kids. B answers: "I have three children, the product of their ages is 36." A says: "I still don't know the ages of your children." Then B tells A the sum of his three kids' ages. A answers: "I still don't know how old they are. Then B adds: "The youngest one has red hair." Now A knows the ages of all three children. Do you?
2. A king has his birthday. So he decides to let go some of his prisoners. He actually has 100 prisoners at the moment. They are each in a separate cell, numbered from 1 to 100. Well, he is a high tech king. He can close or open any prison door by a single click on the cell's number on his royal laptop. When he clicks at a locked door, it opens. When he clicks at an open door, it locks. At the beginning, every door is locked. First the king clicks on every number from 1 to 100 (therefore opening every door). Then he clicks on every second number from 1 to 100, (i.e. 2, 4, 6, 8, 10,...). Then he clicks on every third number. And so on. Finally, he only clicks on the number 100. Then he orders that the prisoners who find their door open may go free. Who gets to go and who has to stay?

Chapter 4

Class 4

4.1 The Set of All Integers

Recall that the set of all natural numbers (also called counting numbers) is $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. This set was historically the first set that people used. In this course, it will be a recurring theme that a mathematical system or set would be enlarged. This was the case with the natural numbers. Why would mathematicians of past centuries feel the need to step beyond the natural numbers? One reason is closure.

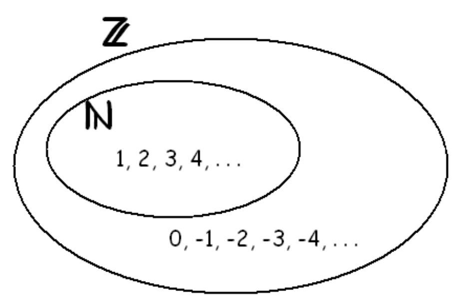
Recall the meaning of closure. The set $\mathbb{N}$ is closed under addition. In other words, the sum of *any* two natural numbers is also a natural number. $\mathbb{N}$ is also closed under multiplication. However, we do not have closure under subtraction and division. We *can* find a subtraction, say $3 - 12$, or a division, $10 \div 7$ that do not result in natural number. If we stay within the set of natural numbers, this means that the results for $3 - 12$ or $10 \div 7$ do not exist. It is a common theme in mathematics to work towards closure.

The set of all natural numbers, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ is closed under addition and multiplication, but not under subtraction and division.

Definition: The set of all integers, denoted by $\mathbb{Z}$, is the set

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The set of integers completely contains the set of natural numbers. In other words, the set of all natural numbers is a subset of the set of all integers, $\mathbb{N} \subseteq \mathbb{Z}$. We can also imagine that we started with the natural numbers and added zero and the opposite of each natural number to form the set of all integers.



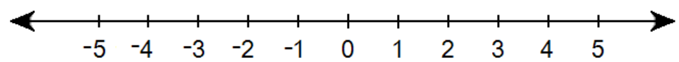
Definition: The **opposite** of 3 is written as -3. For any number, the sum of the number and its opposite is zero. Another expression for the opposite is the **additive inverse**.

The opposite of 3 is -3. The opposite of -3 is 3. The opposite of zero is zero itself.

The negative sign already has a meaning, that of subtraction. We now are facing an ambiguity that is often the source of confusion. Does a negative sign denote the opposite of a number, or does it denote subtraction? This is a question that we often need to ask ourselves. While the answer always clearly exists, it very much depends on the context. For example, the negative sign in -3 clearly denotes that we are talking about the opposite of 3 or negative 3. However, if we place a number in front of it, like in $8-3$, the same sign here denotes subtraction. And what about $8(-3)$? Now the parentheses tells us that the negative sign does not denote subtraction, rather it describes the number after it as negative.

-3	$8-3$	$8(-3)$
the opposite of 3	subtraction	the opposite of 3

We often depict integers using a number line.



Definition: (Ordering on $\mathbb{Z}$) Between two integers, the one on the right is greater.

Another way of envisioning this is to think of a positive number as money and a negative number as debt, and ask: who is richer? $2<10$ was obvious, and also that $-5<3$, but now we see that between -100 and -2, -2 is greater. After all, the person who has no money and only 2 dollars of debt is better off than another person who has no money and a 100 dollars of debt. And so $-100<-2$.

We can swap inequalites, as long as the smaller part of the inequality sign points to the smaller number.

$-100<-2$	read: -100 is less than -2	$-100\leq-2$	read: -100 is less than or equal to -2
$-2>-100$	read: -2 is greater than -100	$-2\geq-100$	read: -2 is greater than or equal to -100

Wether we plot them on a number line or think money and debt, we will agree that $-1000\,000$ (negative one million) is less than 5. But what if we wanted to compare the size of numbers, ignoring their signs? Suppose we want to say that a million dollar debt is a lot of debt. In this case, we use the concept of the absolute value of a number.

Definition: The **absolute value of a number** is its distance from zero on the number line. We denote the absolute value of a number x by $|x|$.

Distances can never be negative. -5 is 5 units away from zero on the number line. So is 5, only it is in the other direction. So, the absolute value of 5 and -5 are both 5.

Example 1. Compute each of the following.

a) $|-2|$ b) $|2|$ c) $|0|$ d) $-|-5|$

Solution: a) The number -2 is 2 units away from zero on the number line. Thus $|-2|=2$.

b) The number 2 is 2 units away from zero on the number line. Thus $|2|=2$.

c) The distance between zero and zero on the number line is zero. Thus $|0|=0$.

d) This is an example for two negatives not making a positive. The way we can read this as: *the opposite of the absolute value of negative five*. The absolute value of -5 is 5. The opposite of that is -5. Using notation, $-|-5|=-5$.

Addition of Integers: Again, think money and debt. Positive numbers represent money, negative numbers represent debt. Adding zero to anything will leave the other number unchanged.

Example 2. Compute each of the following sums.

a) $-4+7$ b) $-3+(-8)$ c) $3+(-14)$ d) $-7+2$ e) $-2+0$

Solution: a) We think of $-4+7$ as follows. We start with a person who has no money and is in debt by 4 dollars. To this, we add 7 dollars. So the person pays off all that 4 dollar debt and is still left with 3 dollars. So $-4+7=3$.

b) We think of $-3+(-8)$ as follows. We start with a person who has no money and is in debt by 3 dollars. To this, we add another debt of 8 dollars. So this person is still now in debt by 11 dollars. So $-3+(-8)=-11$.

c) We think of $3+(-14)$ as follows. We start with a person who has 3 dollars. To this, we add a debt of 14 dollars. So the person pays off all the debt he can - that is 3 dollars and is still in debt by 11 dollars. So $3+(-14)=-11$.

d) We think of $-7+2$ as follows. We start with a person who has no money and is in debt by 7 dollars. Then this person gets 2 dollars. So the person pays off all the debt she can. After she pays off 2 dollars of debt, she has no money and is still left with 5 dollars of debt. So $-7+2=-5$.

e) Adding zero to any number leaves the other number unchanged. Therefore, $-2+0=-2$.

We already know how to add two positive numbers, and we know that the sum is positive. Adding two negative numbers is similar, we are summing debts. So we know that the sum of two negative numbers is also negative. If we add a negative and a positive number, the result may be positive or negative, depending on which number's size (or absolute value) is greater.

Adding zero to any number leaves that number unchanged. In other words, for any integer x, $x+0=x$ and $0+x=x$. In the language of algebra, we refer to a number that has no effect in an operation as an identity or identity element.

> **Definition:** When added to any number, zero has no effect. Because of this behavior, we call zero an **additive identity**.

Discussion: Based on its behavior, can you find a multiplicative identity within the set of all integers?

Now that we can add integers, we need to return to absolute values. The absolute value sign is also a grouping symbol that overwrites order of operations. So if there is a sum (or any other expression) within the absolute value sign, we need to perform those until we are left with just a number within the absolute value sign. Then we take the absolute value of that number.

Example 3. Compute each of the following.

a) $|-9+4|$ b) $|-9|+|4|$ c) $|8|+|-7|$ d) $|8+(-7)|$

Solution: a) The absolute value sign is also a pair of parentheses. We perform the addition $-9+4$ inside, and get -5. Then we take the absolute value of -5.

$|-9+4|=|-5|=5$

b) In this example we take the absolute values and then add. $|-9|+|4|=9+4=13$

c) We take the absolute values and then add. $|8|+|-7|=8+7=15$

d) We first perform the addition inside and then take the absolute value. $|8+(-7)|=|1|=1$

Subtraction of Integers: the following statement is always true, and is often extremely useful.

To subtract is to add the opposite.

Of course, we don't always use this fact. In the subtraction $10-3$, we would only complicate things by applying this fact. It would still get us the right result. Instead of subtracting positive 3, we add its opposite, negative 3.

$$10-3=7 \quad \text{and also,} \quad 10+(-3)=7$$

Consider the subtraction $100-(-20)$. We are asked to subtract negative 20. To subtract is to add the opposite. So, instead of subtracting negative 20, we will add its opposite, positive 20.

$$100-(-20)=100+20=120$$

Example 4. Compute each of the following.

a) $-7-8$ b) $-9-(-5)$ c) $1-7$ d) $6-(-3)$

Solution: a) First, the negative sign in front of the 7 cannot denote subtraction. We are asked to subtract positive 8 from negative 7. To subtract is to add the opposite. Instead of subtracting positive 8, we will add its opposite, negative 8.
$-7-8=-7+(-8)=-15$

b) To subtract is to add the opposite. Instead of subtracting negative 5, we will add its opposite, positive 5.
$-9-(-5)=-9+5=-4$

c) To subtract is to add the opposite. Instead of subtracting positive 7, we will add its opposite, negative 7.
$1-7=1+(-7)=-6$

d) To subtract is to add the opposite. Instead of subtracting negative 3, we will add its opposite, positive 3.
$6-(-3)=6+3=9$

Why is $100-(-20)=100+20$? Even if we understand how to compute this, it would be nice to understand why this is correct. So here is one way to think about this.

Imagine that we have both a bank account a credit card with a bank. Suppose that at the moment, we have 150 dollars in the bank but we also owe 50 dollars to the bank on the credit card. So our net worth is 100 dollars.

Money in bank	Debt on credit card	Total Net worth
150	50	100

Suppose now that we have collected enough bonus points on the credit card to earn rewards. So the bank reduces our credit card debt by 20 dollars. (i.e. subtracts 20 debt, i.e. subtracts negative 20). We still have our 150 in cash, but now our debt is reduced to 30 dollars. So our net worth is now 120 dollars. That is 20 dollars more than before.

	Money in bank	Debt on credit card	Total Net worth
before	150	50	100
after	150	30	120

After all, reducing our debt by 20 dollars is almost the same as if someone gave us 20 dollars so that we can pay off some of our debts.

Multiplication of Integers: Multiply the absolute values. If two integers have the same sign, their product is positive. If two integers have different signs, their product is negative. If any of the factors is zero, the product is zero.

Example 5. Compute each of the following.

a) $-3 \cdot 5$ b) $-4(-5)$ c) $10(-2)$ d) $0(-3)$ e) $-1(8)$

Solution: a) The product of a negative and a positive number is negative.
$-3 \cdot 5 = -15$

b) The product of two negative numbers is positive.
$-4(-5) = 20$

c) The product of a positive and a negative number is negative.
$10(-2) = -20$

d) If any of the factors is zero, the product is zero.
$0(-3) = 0$

e) The product of a negative and a positive number is negative.
$-1(8) = -8$

Notice that if we multiply any integer by -1, the result is the opposite of that integer. This will be very useful later.

Why do these rules work this way? Here is one possible explanation. Multiplication is defined as repeated addition. For example, $4 \cdot 7$ means that we add 7 to itself, 4 times.

$$4 \cdot 7 = 7 + 7 + 7 + 7 = 28$$

Consider now $4 \cdot (-7)$. This means that we add -7 to itself, 4 times

$$4 \cdot (-7) = -7 + (-7) + (-7) + (-7) = -28$$

The logic becomes a bit tortured, but it also works with the first factor being negative. Consider now the product $-5 \cdot 8$. We can interpret the first negative sign as repeated subtraction. So, we are subtracting 8 repeatedly, 5 times. If we feel that we don't have anything to subtract the first 8 from, we can fix that by inserting a zero. We know that adding zero will not change any value.

$$\begin{aligned} -5 \cdot 8 &= -8 - 8 - 8 - 8 - 8 \\ &= 0 - 8 - 8 - 8 - 8 - 8 && \text{to subtract is to add the opposite} \\ &= 0 + (-8) + (-8) + (-8) + (-8) + (-8) \\ &= -40 \end{aligned}$$

The most interesting case is probably when we are multiplying two negative numbers. Consider the product $-4 \cdot (-10)$. The first negative sign is interpreted as repeated subtraction, the second one is that we are subtracting negative numbers. So we are subtracting negative 10 repeatedly, 4 times. If we need something to subtract the first negative 10 from, we will just insert a zero at the beginning.

$$\begin{aligned} -4 \cdot (-10) &= 0 - (-10) - (-10) - (-10) - (-10) && \text{to subtract is to add the opposite} \\ &= 0 + 10 + 10 + 10 + 10 = 40 \end{aligned}$$

Division of integers: We will deal with zero later. For the quotient of any two non-zero integers, the rules are very simple and similar to those of multiplication. Divide the absolute values. If the the integers have the same sign, the quotient is positive. If they have different signs, the quotient is negative.

Example 6. Compute each of the following.

a) $14 \div (-2)$ b) $-24 \div (-6)$ c) $-10 \div 5$

Solution: a) Divide the absolute values. The quotient of a positive and a negative number is negative.
$14 \div (-2) = -7$

b) Divide the absolute values. The quotient of two negative numbers is positive.
$-24 \div (-6) = 4$

c) Divide the absolute values. The quotient of a negative and a positive number is negative.
$-10 \div 5 = -2$

Division is often denoted with a horizontal bar. The same computations can also be written as

$$\frac{14}{-2} = -7 \text{ and } \frac{-24}{-6} = 4 \text{ and } \frac{-10}{5} = -2$$

Why do these rules work this way? Division is defined in terms of multiplication backward. In other words,

$$\frac{20}{4} = 5 \text{ is true because } 4 \cdot 5 = 20$$

Let us apply this idea. What is the result of $14 \div (-2)$?

$$\frac{14}{-2} = \boxed{?} \text{ would be true because } -2 \cdot \boxed{?} = 14$$

Since $-2 \cdot 7$ would result in -14, we can only choose -7 to make the multiplication backward work. $-2(-7) = 14$, therefore $\frac{14}{-2} = -7$.

$$\text{Similarly, } \frac{-24}{-6} = \boxed{?} \text{ would be true because } -6 \cdot \boxed{?} = -24$$

We need to multiply -6 by a positive number to get a negative product. Only positive 4 will work, and so $\frac{-24}{-6} = 4$.

Division by Zero: Now that we understand that division is defined in terms of multiplication backward, we can easily deal with zero. The expressions $\frac{0}{3}$ and $\frac{3}{0}$ look very similar, and yet they are very different.

$$\frac{0}{3} = \boxed{?} \text{ would be true because } 3 \cdot \boxed{?} = 0$$

In this case, we can only use zero to make the multiplication backward work. Let us investigate the other case.

$$\frac{3}{0} = \boxed{?} \text{ would be true because } 0 \cdot \boxed{?} = 3$$

Now we are in trouble. If we multiply any number by zero, the product is zero. Therefore, we can not meaningfully complete this division, and so we say that $\frac{3}{0}$ is undefined. In written notation, $\frac{3}{0} = \text{undefined}$.

We have established that we cannot divide a non-zero number by zero. What about $\frac{0}{0}$?

$$\frac{0}{0} = \boxed{?} \text{ would be true because } 0 \cdot \boxed{?} = 0$$

Now the problem is that *every* number would work, because any number times zero is zero. Mathematicians prefer one clean answer as a result of an operation. We do not like an operation that results in several numbers, let alone every number! So, one fundamental rule of mathematics is that division by zero is not allowed. Indeed, division by zero is not just an error: it is one of the worst errors.

The first commandment of mathemtics: *Thou shall not divide by zero. Ever...*

Changes in Notation

With the introduction of negative numbers, our notation will have to be modified. It is a widely accepted convention that if there are several signs (operations or negative) between two numbers, a pair of parentheses must separate them.

$-2+-6$	$-5--3$	$-3\cdot-4$	$-30\div-5$
$+-$ is not allowed	$--$ is not allowed	$\cdot-$ is not allowed	$\div-$ is not allowed

For this reason, until a few decades ago, we used to put a pair of parentheses around *every negative number*.

$(-2)+(-6)$	$(-5)-(-3)$	$(-3)\cdot(-4)$	$(-30)\div(-5)$
old style	old style	old style	old style

The development of mathematical notation is an ongoing process. The most important goal in notation is clarity. As long as clarity is not jeopardized, mathematicians are in the habit of omitting things. A few decades ago we stopped putting the parentheses around the first negative number in the line or inside a parentheses, because there was no risk that we would read the sign incorrectly as subtraction. Also, there is rarely an operation sign in front of the first number.

$-2+(-6)$	$-5-(-3)$	$-3\cdot(-4)$	$-30\div(-5)$
more modern	more modern	more modern	more modern

In the case of multiplication, we can omit one more thing. Recall that multiplication is the default operation; if we see two numbers with *nothing* between them, that indicates multiplication. For example, there is no operation sign or parentheses in $2x$ or ab and yet it is clear that the operation is multiplication. Now that most negative numbers must be placed in parentheses, we can often omit the dot indicating multiplication.

$-2+(-6)$	$-5-(-3)$	$-3(-4)$	$-30\div(-5)$
cannot omit anything	cannot omit anything	we can omit the dot	cannot omit anything

The most common modern style is minimalistic, ommitting as much as possible, as long as confusion is avoided. This can lead to apparent irregularities in notation. For example, our notation will be $2(-3)$, but when we swap the two factors, it will be $-3\cdot 2$.

Our Larger Number Sysytem

> **The set of all integers is closed under addition, subtraction, and multiplication. It is not closed under division.**

As a matter of fact, the set of all integers ($\mathbb{Z}$) is the smallest set that contains the set of all natural numbers ($\mathbb{N}$) and is closed under subtraction. Another way to state this is that $\mathbb{Z}$ is the closure of $\mathbb{N}$ under subtraction. In the future, we will further enlarge our number system to obtain closure under division.

Practice Problems

1. Label each of the following statements as true or false.

   a) $-3 \in \mathbb{Z}$ c) $\mathbb{Z} \subseteq \mathbb{N}$ e) $-|-2| = -2$

   b) $-3 \notin \mathbb{N}$ d) $-3 \geq -3$ f) For every integer x, $|x| \geq x$

2. Label each of the following statements as true or false.

   a) Every integer is a natural number.

   b) Every natural number is an integer.

   c) $3 < -5$ or $-2 > -8$

   d) $3 < -5$ and $-2 > -8$

   e) Zero is also called the additive identity.

   f) $5 \leq 5$ and $-|-8| = -8$

   g) $-2 < -2$ or $|-2| > |-10|$

3. Place an inequality sign between the given numbers to make the statement true.

   a) $5 \quad -7$ b) $-12 \quad -4$ c) $0 \quad -8$ d) $-1 \quad -4$ e) $-7 \quad -7$

4. Simplify each of the following.

   a) $|-5|$ b) $|5|$ c) $-|5|$ d) $-|-5|$ e) $|0|$ f) $|-12+9|$ g) $|-12|+|9|$

5. Perform the indicated operations.

   a) $-2+7$ e) $-8 \cdot 0$ i) $-4 \cdot 7$ m) $-3-0$ p) $0 \div (-1)$

   b) $-7-(-4)$ f) $-3-(-10)$ j) $-6-|-7|$ n) $0(-4)$ q) $9+|-1|$

   c) $12 \div (-2)$ g) $-20 \div 0$ k) $-3 \div (-3)$

   d) $5(-3)$ h) $-12 \div 3$ l) $|9|+(-1)$ o) $\dfrac{-5}{0}$ r) $|9+(-1)|$

4.2 Order of Operations on Integers

The good news is that order of operations on the set of all integers ($\mathbb{Z}$) is the same as on the set of all natural numbers ($\mathbb{N}$). This is not by chance: when mathematicians enlarge our number system, they make certain that the new numbers and operations defined are in harmony with properties of the smaller system. We will refer to this consistent effort as **the expansion principle**. The bad news is that notation became more complex with the introduction of zero and the negative numbers.

The Negative Sign

The negative sign has a dual role: it denotes subtraction or it describes the number after it. This ambiguity never leads to confusion, otherwise it wouldn't be allowed. It is extremely important that we correctly interpret these roles before starting a computation. Here is a neat trick: when facing a negative sign, we should ask ourselves if it could denote subtraction. If it can denote subtraction, then it *does* denote subtraction. If it cannot, then it is a sign to be read as '*negative*' or '*the opposite of*'. A negative sign denoting subtraction is one that is loose between two numbers.

-3	$-(-3)$	$8-3$	$8(-3)$	$8-(-3)$	$8(-(-3))$
negative 3	the opposite of negative 3	subtraction	negative 3	subtract negative 3	8 times the opposite of negative 3

The following situation is often difficult for students.

Example 1. Simplify the expression $20-3(-5)$.

Solution: The first negative sign indicates subtraction, the second one describes 5 as being negative. We start with the multiplication. However, we do not include the subtraction sign in the multiplication. $3(-5)=-15$, and that is what we will subtract.

$$\begin{aligned} 20-3(-5) &= 20-(-15) && \text{to subtract is to add the oposite} \\ &= 20+15 && \text{addition} \\ &= \boxed{35} \end{aligned}$$

Example 2. Consider the expression $20-3(-4-(-2)5)$.

a) Pair the opening and closing parentheses. Classify them as grouping symbols or clarifying prentheses.

b) How many operations are indicated?

c) Perform the indicated operations. For each step, write a separate line.

Solution: a) There are two pairs of parentheses, nested inside each other. The outside pair is a grouping symbol, the one enclosing -2 is a clarifying parentheses.

b) We scan the expression left to right and count the operations. At this point we must be clear on the role of each negative sign. The negative signs before 4 and 2 do not denote subtraction.

1. subtraction between 20 and 3 (although it is not going to be 3 what we are subtracting)
2. multiplication between 3 and the expression in parentheses
3. subtraction between -4 and (-2) (although it is not going to be -2 what we are subtracting)
4. multiplication between -2 and 5

So there are four operations.

c) We will write a separate line for each step. There are two advantages of this. First, at every point we just need to find the *one* operation to execute. Second, each line we write is a new and easier problem we need to solve. Within the parentheses, there is a subtraction and a multiplication. We start with the multiplication $-2 \cdot 5$.

$$\begin{aligned} 20-3(-4-(-2)5) &= 20-3(-4-(-10)) && \text{to subtract is to add the opposite} \\ &= 20-3(-4+10) && \text{addition (also, drop parentheses)} \\ &= 20-3\cdot 6 && \text{multiplication} \\ &= 20-18 && \text{subtraction} \\ &= \boxed{2} \end{aligned}$$

The Trouble with Exponents - It's an Order of Operations Thing

Recall that a negative sign in front of anything can be interpreted as '*the opposite of*', which can also be interpreted as mutliplication by -1. We can interpret -3 as $-1\cdot 3$, and so we can re-interpret the original question from comparing -3^2 and $(-3)^2$ to a question comparing $-1\cdot 3^2$ and $(-1\cdot 3)^2$. The rest is really just an order of operations problem.

Recall that in our order of operations agreement, exponentiation superseeds multiplication. So, when presented by multiplication and exponentiation, we first execute the exponentiation and then the multiplication.

If there is no parentheses, we have	If we have parentheses, then
$\begin{aligned} -3^2 &= -1\cdot 3^2 && \text{exponentiation first} \\ &= -1\cdot 9 && \text{multiplication} \\ &= -9 \end{aligned}$	$\begin{aligned} (-3)^2 &= (-1\cdot 3)^2 && \text{multiplication first} \\ &= (-3)^2 && \text{square the number } -3 \\ &= 9 \end{aligned}$

The difference between -3^2 and $(-3)^2$ is truly an order of operations thing: we are talking about taking the opposite and squaring, but in different orders.

-3^2 is the opposite of the square of 3 $(-3)^2$ is the square of the opposite of 3

Example 3. Simplify each of the given expressions.
a) -2^4 b) $(-2)^4$ c) -1^3 d) $(-1)^3$ e) $-(-2)^2$ f) $-\left(-2^2\right)$

Solution: a) $-2^4 = -1\cdot 2^4 = -1\cdot(2\cdot 2\cdot 2\cdot 2) = -1\cdot 16 = -16$
-2^4 can be read as the opposite of 2^4.

b) $(-2)^4 = (-1\cdot 2)^4 = (-1\cdot 2)(-1\cdot 2)(-1\cdot 2)(-1\cdot 2) = (-2)(-2)(-2)(-2) = 16$
$(-2)^4$ can be read as the fourth power of -2.

c) $-1^3 = -1\cdot 1^3 = -1\cdot 1\cdot 1\cdot 1 = -1$

d) $(-1)^3 = (-1)(-1)(-1) = -1$

e) $-(-2)^2 = -1\cdot\left((-1\cdot 2)^2\right) = -1\cdot\left((-2)^2\right) = -1\cdot 4 = -4$

f) Careful! The exponent is inside the parentheses. This is squaring 2 and then taking the opposite of the result twice.

$-\left(-2^2\right) = -1\cdot\left(-1\cdot 2^2\right) = -1\cdot(-1\cdot 4) = -1(-4) = 4$

Discussion: Explain why in the expression $-(-5)^2$, the two negatives do not cancel out to a positive.

Example 4. Evaluate each of the given expressions.

a) $5(-2)^2$ b) $5\left(-2^2\right)$

Solution: We need to pay attention to the subtle difference between the two expressions: the exponent is inside the parentheses in part b. In that second example, the parentheses is not necessary for the exponentiation, without it we would be looking at subtraction.

a) $5(-2)^2 = 5 \cdot 4 = 20$ b) $5\left(-2^2\right) = 5(-4) = -20$

Example 5. Simplify the given expression. Write a separate line for each step. $-3^2 - 2\left(3\left(10-(-2)^2\right)-4^2\right)^2$

Solution: We start with the exponentiation in the innermost parentheses.

$$-3^2 - 2\left(3\left(10-(-2)^2\right)-4^2\right)^2 =$$

$=$	$-3^2-2\left(3(10-4)-4^2\right)^2$	subtraction in innermost parentheses
$=$	$-3^2-2\left(3\cdot 6-4^2\right)^2$	exponentiation in parentheses
$=$	$-3^2-2(3\cdot 6-16)^2$	multiplication in parentheses
$=$	$-3^2-2(18-16)^2$	subtraction in parentheses
$=$	$-3^2-2\cdot 2^2$	first exponentiation left to right; $-3^2=-9$
$=$	$-9-2\cdot 2^2$	exponentiation
$=$	$-9-2\cdot 4$	multiplication
$=$	$-9-8$	subtraction
$=$	$\boxed{-17}$	

The Trouble with the Absolute Value Sign

Absolute value signs are more difficult to pair off because their shapes do not tell us whether they are opening or closing. The following order of operations problems illustrate that very similarly looking problems can actually be quite different. The key to solving these types of is to first pair off the absolute value signs.

Example 6. Evaluate each of the following expressions.

a) $|-5-3|-|7+2|$ b) $||-5-3|-7|+2$ c) $-5-||3-7|+2|$ d) $-5|-3-|7+2||$

Solution: a) Consider the expression $|-5-3|-|7+2|$.

We will not be able to get far in solving this problem without pairing the absolute values signs. Naturally, the first vertical bar can only be an opening sign. The second one can not be an another opening sign, becuase $|-|$ does not make any sense. Therefore, we have two pairs, one after the other. We should indicate the pairing using different colors or different sizes for the two pairs of absolute values signs. Keep in mind, they also serve as grouping symbols. After we paired up the signs, the problem becomes quite easy.

a) $|-5-3|-|7+2|=$

$$
\begin{aligned}
&= \Big|-5-3\Big| - |7+2| && \text{subtraction in first parentheses}\\
&= \Big|-8\Big| - |7+2| && \text{evaluate the absolute value of } -8\\
&= 8-|7+2| && \text{addition in parentheses}\\
&= 8-|9| && \text{evaluate the absolute value of 9}\\
&= 8-9=\boxed{-1} && \text{subtraction}
\end{aligned}
$$

b) Consider the expression $||-5-3|-7|+2$.

It is easy to see that the first and second absolute value signs can not be a pair, because || does not make sense. Therefore, since they are at the beginning, they are both opening. This means that we have one pair of absolute value signs nested inside the other. The one that opens last has to close first. After we have established that, the problem becomes much easier.

$$
\begin{aligned}
||-5-3|-7|+2 &= \Big||-5-3|-7\Big|+2 && \text{subtraction in innermost parentheses}\\
&= \Big||-8|-7\Big|+2 && \text{evaluate the absolute value of } -8\\
&= |8-7|+2 && \text{subtraction in parentheses}\\
&= |1|+2 && \text{evaluate the absolute value of 1}\\
&= 1+2=\boxed{3} && \text{addition}
\end{aligned}
$$

c) Consider the expression $-5-||3-7|+2|$.

We again see that || in front of 3 can not be a pair. Thus they are both opening, and so we have one pair inside the other.

$$
\begin{aligned}
-5-||3-7|+2| &= -5-\Big||3-7|+2\Big| && \text{subtraction in innermost parentheses}\\
&= -5-\Big||-4|+2\Big| && \text{evaluate the absolute value of } -4\\
&= -5-|4+2| && \text{addition in parentheses}\\
&= -5-|6| && \text{evaluate the absolute value of 6}\\
&= -5-6=\boxed{-11} && \text{subtraction}
\end{aligned}
$$

d) Consider the expression $-5|-3-|7+2||$. We again see that || after 2 can not be a pair. Thus they are both closing, and so we have one pair inside the other.

$$
\begin{aligned}
-5|-3-|7+2|| &= -5\Big|-3-|7+2|\Big| && \text{addition in innermost parentheses}\\
&= -5\Big|-3-|9|\Big| && \text{evaluate the absolute value of 9}\\
&= -5|-3-9| && \text{subtraction in parentheses}\\
&= -5|-12| && \text{evaluate the absolute value of } -12\\
&= -5\cdot 12=\boxed{-60} && \text{multiplication}
\end{aligned}
$$

Practice Problems

Simplify each of the follwing expressions by applying the order of operations agreement. Write a separate line for each step.

1. $-2-3\left(1-4^2\right)$
2. $-2-3(1-4)^2$
3. $(-2-3(1-4))^2$
4. $(-2-3)(1-4)^2$
5. $-2(-3(1-4))^2$
6. $\dfrac{-7-(-2)^3-(-1)^4}{-2(5-(-5))}$
7. $\dfrac{-24\div 2(-3)}{5-(-1)^2}$
8. $(2-5)\left(12-(-4)^2\right)$
9. $2-5\left(12-(-4)^2\right)$
10. $13-5(7-10)$
11. $13-5\,|7-10|$
12. $\dfrac{12-3(8-5)}{2^4-(-4)^2}$
13. $-3^2-\left((-2)^2-3(-1)^3\right)$
14. $(-3)^2-2\left(3-4\left(5-2^3\right)\right)$
15. $\left(\left(\left(5-(-2)^2\right)^2-3\right)^2+1\right)^2-1$
16. $(-2)^2-(-2)^3-(-2)^4$
17. $-(-5)^2-2\left(3-5^2\right)$
18. $-(-5)^2-2(3-5)^2$
19. $-\left((-5)^2-2(3-5)^2\right)$
20. $\dfrac{5(-2)-3\left|-5+(-1)^2\right|}{-2(3-(-2))-(-1)^2}$
21. $\dfrac{10-2\left(-3-(-1)^3\right)}{(-2)^3+(-1)^4}$
22. $\dfrac{12-2(3(4(5-7)+5)-3)}{-3+2\left(7-(-3)^2\right)-(-1)^3}$
23. $4-\left(6-\left(8-\left(10-\left(12-4^2\right)\right)\right)\right)$

Absolute value signs are more difficult to pair off because their shapes do not tell us whether they are opening or closing. The following order of operations problems can be solved by first pairing off the absolute value signs.

24. $5-2\,||-4+7|-10|$
25. $5\,|-2\,|-4+7||-10$
26. $5-2\,|-4+|7-10||$
27. $5\,|-2-4+|7-10||$
28. $5-|2-|4+7|-10|$
29. $|5-|2-4||+7-10$
30. $|5-2|-|4+7-10|$
31. $||5-2|-4|+7-10$
32. $||5-2|-4+7|-10$

Enrichment

1. We would hope that our order of operations agreement is void of ambiguities. This is not true. In case of the absolute value sign, we cannot tell whether it is opening or closing. Because of this, it is possible to find expressions that can be interpreted correctly in several ways, and the results are different! Evaluate each of the given expressions correctly in two different ways. What are your results?

   a) $|5-2|-4+7|-10|$ b) $2\,|-1-5|-3\,|-4|$

2. Enter pairs of parentheses or absolute value signs into the expression on the left-hand side to make the follwing statements true.

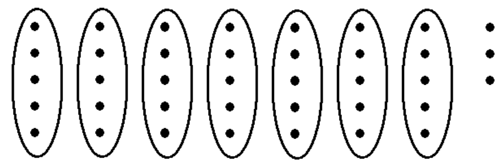
   a) $5-2-3\ ^2-4-7\ ^2+2=9$ b) $5-2-3\ ^2-4-7\ ^2+2=-9$

4.3 Division with Remainder

Theorem: (Division with Remainder) For every integer N and positive integer m, there exist unique integers q and r such that

$$N = mq + r \qquad \text{where } \ 0 \leq r \ \text{ and } \ r < m$$

For example, if $N = 38$ and $m = 5$, then the quotient is $q = 7$ and the remainder is $r = 3$. The picture illustrates the division $38 \div 5 = 7$ R 3, or, in other form: $38 = 5 \cdot 7 + 3$. Note that there are two different ways to represent this division:
$38 \div 5 = 7$ R 3 and $\dfrac{38}{5} = 7\dfrac{3}{5}$.



Example 1. Perform the given division with remainder. $6071 \div 17$

Solution: One way would be is to perform long division. Another is to utilize our calculator. First we enter the division into the calculator and see the result as a decimal: 357.1176470588. Note that this is NOT division with remainder. However, we can quickly adjust this. Since the result is between 357 and 358, we already know that the quotient is 357 and now we need to only find the remainder.

$$6071 \div 17 = 319 \text{ R } ?$$

If the quotient is 357, then we have already accounted for $17 \cdot 357 = 6069$. How much is missing from 6071? Clearly 2. Thus the answer is: $6071 \div 17 = 319$ R 2.

How can we check? We should look for two things: did we account for all 6071? Indeed, $17 \cdot 357 + 2 = 6071$. The other question is: Is the remainder less than the divisor? Indeed, $2 < 17$ and so our solution is correct.

Example 2. Perform the given division with remainder. $8271 \div 45$

Solution: First we enter the division into the calculator and see the result as a decimal: 183.8. Note that this is NOT division with remainder. However, we can quickly adjust this. Since the result is between 183 and 184, we already know that the quotient is 183 and now we need to only find the remainder. If the quotient is 183, then we have already accounted for $183 \cdot 45 = 8235$. How much is missing from 6071? $8271 - 8235 = 36$ Thus the answer is: $8271 \div 45 = 184$ R 36.

It is always a good idea to check: $183 \cdot 45 + 36 = 8271$, and $36 < 45$ and so the remainder is less than the divisor. So our solution is correct.

An Application: Orbits

Example 3. What is the last digit of 2^{99}?

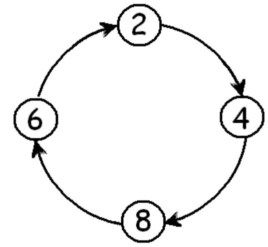
Solution: Let us first investigate what the last digits of smaller $2-$powers are.

$$\begin{array}{llllll}
2^1 = & 2 & 2^5 = & 3\boxed{2} & 2^9 = & 51\boxed{2} \\
2^2 = & 4 & 2^6 = & 6\boxed{4} & 2^{10} = & 102\boxed{4} \\
2^3 = & 8 & 2^7 = & 12\boxed{8} & 2^{11} = & 204\boxed{8} \\
2^4 = & 1\boxed{6} & 2^8 = & 25\boxed{6} & 2^{12} = & 409\boxed{6}
\end{array}$$

Before the 2-powers become uncomfortably large, we can observe a repeating pattern: the last digits being

$$2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6, \ldots.$$

There are only 4 numbers appearing as last digits, following each other in a repeating pattern, we also call an **orbit**. The question only remains: where does 99 land in this pattern?



One way to determine that is to apply division with remainder. Consider the last row:

$$2^4 = 1\boxed{6}, \quad 2^8 = 25\boxed{6}, \quad 2^{12} = 409\boxed{6},$$

We can see that if the exponent is divisible by 4, then the last digit is 6. Consider now the first row:

$$2^1 = \boxed{2}, \quad 2^5 = 3\boxed{2}, \quad 2^9 = 51\boxed{2},$$

These exponents, 1, 5, 9, 13, ... are right after a number divisible by 4. These are numbers that result in a remainder 1 when divided by 4. In the next row,

$$2^2 = \boxed{4}, \quad 2^6 = 6\boxed{4}, \quad 2^{10} = 102\boxed{4},$$

the exponents are even but not divisible by 4. These can also be expressed as numbers that result in a remainder 2 when divided by 4. In the last row,

$$2^3 = \boxed{8}, \quad 2^7 = 12\boxed{8}, \quad 2^{11} = 204\boxed{8},$$

the exponents have a remainder 3 when divided by 4. We amend our results with these labels.

$2^1 =$	2	$2^5 =$	$3\boxed{2}$	$2^9 =$	$51\boxed{2}$	remainder:	1	last digit: 2
$2^2 =$	4	$2^6 =$	$6\boxed{4}$	$2^{10} =$	$102\boxed{4}$	remainder:	2	last digit: 4
$2^3 =$	8	$2^7 =$	$12\boxed{8}$	$2^{11} =$	$204\boxed{8}$	remainder:	3	last digit: 8
$2^4 =$	$1\boxed{6}$	$2^8 =$	$25\boxed{6}$	$2^{12} =$	$409\boxed{6}$	remainder:	0	last digit: 6

We can find where 99 lands in this pattern if we divide it by 4 and focus on the remainder.

$$99 \div 4 = 24 \text{ R } 3$$

The remainder is 3. Therefore, 99 lands in the second to last row with exponents $3, 7, 11, 15, \ldots$, and so the last digit of 2^{99} is 8.

Practice Problems

1. Perform the indicated divisions with remainders.

   a) $132 \div 7$ b) $1145 \div 12$ c) $918 \div 8$ d) $201 \div 12$

2. Find the last digit of each of the following numbers.

   a) 7^{2017} b) 9^{120} c) 2^{1001} d) 3^{286}

4.4 Problem Set 2

1. List all factors of 24.

2. Suppose $A = \{2, 3, 4, 5, 7, 8\}$, and $B = \{1, 2, 4, 6, 8, 10\}$. Draw a Venn-diagram depicting A and B.

3. Suppose that $U = \{0, 1, 2, 3, \ldots, 19, 20\}$. Find each of the following sets.

   a) $A = \{x \in U : x \text{ is divisible by } 3\}$
   b) $B = \{x \in U : x \text{ is divisible by } 5 \text{ or } x < 8\}$
   c) $C = \{x \in U : x \text{ is divisible by } 5 \text{ and } x < 8\}$
   d) $D = \{x \in U : x < 12 \text{ or } x \geq 7\}$
   e) $E = \{x \in U : x < 12 \text{ and } x \geq 7\}$
   f) $F = \{x \in U : x < 4 \text{ or } x < 8\}$
   g) $G = \{x \in U : x < 4 \text{ and } x < 8\}$
   h) $H = \{x \in U : x \text{ is divisible by } 4\}$
   i) $I = \{x \in U : x \text{ is divisible by 3 or } x \text{ is divisible by } 4\}$
   j) $J = \{x \in U : x \text{ is divisible by 3 and } x \text{ is divisible by } 4\}$

4. Recall the following definitions. A **rectangle** is a four-sided polygon with four right angles. A **square** is a rectangle with four equal sides. Let R be the set of all rectangles and S the set of all squares.

   a) Label each of the following statements as true or false.
      i) Every square is a rectangle.
      ii) Every rectangle is a square.
      iii) $R \subseteq S$
      iv) $S \subseteq R$

   b) Describe x if we know that $x \in R$ and $x \notin S$.

5. Perform the divisions with remainder. Show both the quotient and the remainder. For example, $19 \div 7 = 2$ R 5.

   a) $99 \div 4$ b) $52 \div 7$ c) $2018 \div 7$

6. Perform the indicated operations. Show all steps.

   a) $12 - (3)(-5)$
   b) -3^2
   c) $(-3)^2$
   d) $-(-4)$
   e) $-|-4|$
   f) $-(-5)^2$
   g) $\dfrac{4^2 - 2^4}{2^5 - 5^2}$
   h) $\dfrac{10 - 2\left(3^2 - 8\right)}{(-1)^2 + (-1)^3}$
   i) $|3 + (-8)|$
   j) $|3| + |-8|$
   k) $|3| - |-8|$
   l) $12 - 3 + 4$
   m) $12 \div 2 \cdot 3$
   n) $(2+3)^2$
   o) $2^2 + 3^2$
   p) $(3-7)^2$
   q) $3^2 - 7^2$

7. Perform the indicated operations. Show all steps.

   a) $8 - 2\left(7 - 3^2 + 1\right) + 5$
   b) $\left(\left(3^2 - 13\right)^2 - 11\right)^2 - 1^3$
   c) $\dfrac{-20 - 4^2}{-2 - 3(-2)}$
   d) $-20 - \dfrac{(-4)^2}{-2}$
   e) $17 - 4\left(-20 - 2\left(-3^2\right)\right)$
   f) $\dfrac{60 \div (-3) \cdot 2 - 2}{|-12 - 3 + 1|}$
   g) $\dfrac{-4^2 + (-5)^2}{-1^2 + (-8) \div (-2)^3}$
   h) $\dfrac{(-3)^2 - (-2)^3 - (7 - (-6))}{20 - 2 \cdot 3^2 - 2(-1)^3}$
   i) $30 - 2\left(-3 + 2\left(7(-4) + (-5)^2\right)\right)$
   j) $\dfrac{-11 - \left|-3^2 + 4\right|}{-2^3 - 3(-4)}$
   k) $\dfrac{|-17| + \left|-2^3\right| + |4 - 9|}{|3^3 - (-4)| - 1}$
   l) $\left(\left((5-7)^2 - 3\right)^2 - 4\right)^2$

8. Find the last digit of 3^{2018}.

9. Compute each of the following.

   a) $-100 + (-99) + (-98) + \ldots + 98 + 99 + 100 + 101 + 102$
   b) $-100(-99)(-98) \ldots \cdot 98 \cdot 99 \cdot 100 \cdot 101 \cdot 102$

Chapter 5

Class 5

5.1 Set Operations

We have previously studied sets. At this point, we can define and compare sets. We will now start studying operations on sets.

Definition: If A and B are sets, then the **intersection** of A and B, denoted by $A \cap B$, is the set such that for all x,

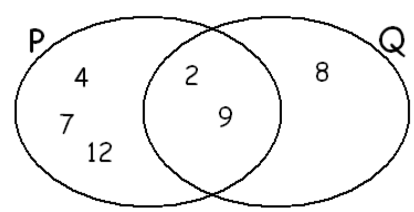
$$x \in A \cap B \text{ if and only if } x \in A \text{ and } x \in B.$$

The intersection of two sets is the set of all elements that belong to *both* sets.

Example 1. Suppose that $P = \{2, 4, 7, 9, 12\}$ and $Q = \{2, 8, 9\}$. Find $P \cap Q$.

Solution: The intersection of P and Q is the set containing those elements that are in **both** P and Q. Since P and Q are small sets, we check from element to element, and collect those that belong to both. We can see that $P \cap Q = \{2, 9\}$.

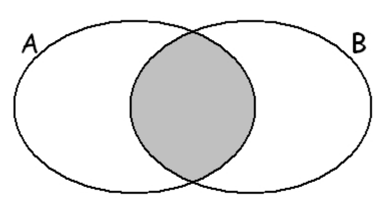
A picture such as this one is called a Venn diagram. Venn diagrams often provide useful visual tools to solve set theory problems. We can depict the intersection using Venn Diagrams.



Example 2. Suppose that $A = \{1, 3, 5, 7, 9\}$ and $B = \{5, 6, 7, 8, 9, 10\}$. Find $A \cap B$.

Solution: The intersection of A and B is the set containing those elements that are in both A and B. Since A and B are small sets, we check from element to element, and collect those that belong to both. We can see that $A \cap B = \{5, 7, 9\}$.

If we use a Venn diagram, the intersection of the two sets is the 'overlap' between the two sets as shown.



The shaded region is $A \cap B$

Example 3. Suppose that $T = \{3, 4, 7, 10\}$ and $Q = \{1, 6, 8\}$. Find $T \cap Q$.

Solution: As we look for elements in common, we find none. Thus $T \cap Q = \varnothing$. When this happens, we say that the two sets are **disjoint**.

We want the intersection of two sets to always be a set. In other words, we want the set of sets (!) to be closed under intersection. This is why it was important for us to define $\varnothing$, the empty set.

Example 4. Find $\mathbb{N} \cap \mathbb{Z}$.

Solution: If we start with the natural numbers, we notice that they are automatically in $\mathbb{Z}$. Indeed, $\mathbb{N}$ is a subset of $\mathbb{Z}$, and so every element in $\mathbb{N}$ is also in $\mathbb{Z}$ and thus in both sets. However, with the negative integers and zero we find that they are not in both sets because they are not in $\mathbb{N}$. Thus the intersection of the two sets is $\mathbb{N}$. In short, $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$.

Example 5. Let $S = \{3, 8, 14\}$. Find $S \cap \varnothing$.

Solution: The intersection of two sets is the set of elements in *both* sets. Since there is nothing in the empty set, there cannot be anything in the intersection. Thus $S \cap \varnothing = \varnothing$.

Definition: If A and B are sets, then the **union** of A and B, denoted by $A \cup B$, is the set such that for all x,

$$x \in A \cup B \text{ if and only if } x \in A \text{ or } x \in B$$

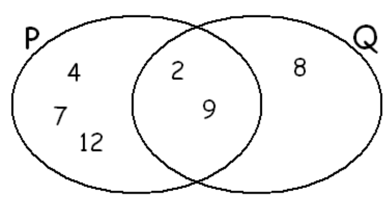
The word 'or' is used in the strict mathematical sense. $x \in A$ or $x \in B$ is true if either x is in A only, or x is in B only, or if x is in both. So, x is in the union of A and B if it is in A, in B, or in both A and B.

The union of two sets is the set of all elements from one set, put together with the set of all elements of the other. Imagine we throw the elements of both sets together and then we list them as elements of a single set, ignoring repetitions.

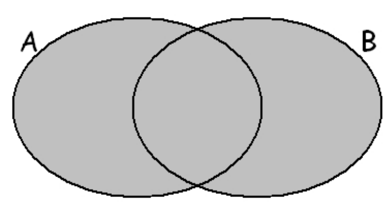
Example 6. Suppose that $P = \{2, 4, 7, 9, 12\}$ and $Q = \{2, 8, 9\}$. Find $P \cup Q$.

Solution: The union of P and Q is the set containing those elements that are in P or in Q. Since P and Q are small sets, we check from element to element, and collect those that belong to either sets or to both. Another way of visualizing the union is to throw together P and Q and list the resulting set without repetition. We can see that $P \cup Q = \{2, 4, 7, 8, 9, 12\}$.

A Venn diagram might help again. For the union, we collect every element from the three separate regions.



If we use a Venn diagram, the union of the two sets is the collection of those three regions as shown.



The shaded region is $A \cup B$

Example 7. Find $\mathbb{N} \cup \mathbb{Z}$.

Solution: Let us start with the integers this time. All integers are in the union since they are in $\mathbb{Z}$. Now we look at the other set, $\mathbb{N}$, and notice that all natural numbers are already listed in the union because they are automatically in $\mathbb{Z}$. Indeed, $\mathbb{N}$ is a subset of $\mathbb{Z}$, and so every element in $\mathbb{N}$ is also in $\mathbb{Z}$. Therefore, $\mathbb{N}$ does not bring anything new to the union, and so $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$.

Example 8. Suppose $T = \{2, 10, 12, 30\}$. Find $T \cup \varnothing$.

Solution: The union obviously contains all four elements of T. Now we need to add the elements that are not in T but are in the empty set. Since there isn't anything in the empty set, it does not bring anything new to the union, and so $T \cup \varnothing = \{2, 10, 12, 30\}$. We can also state the answer as $T \cup \varnothing = T$.

Practice Problems

1. Suppose that $P = \{1, 4, 6, 9\}$ and $Q = \{1, 2, 3, 4, 5\}$. Find each of the following.

   a) $P \cap Q$ b) $P \cup Q$ c) $P \cap \varnothing$ d) $Q \cup \varnothing$

2. Let $A = \{1, 2, 5, 8, 9\}$ and $B = \{2, 4, 6, 8\}$.

   a) Draw a Venn diagram depicting these sets.

   b) Find each of the following.

      i) $A \cap B$ ii) $A \cup B$ iii) $B \cup (A \cap B)$

   c) Label each of the following statements as true or false.

      i) $A \subseteq A \cap B$ ii) $B \subseteq A \cup B$ iii) $A \cap B \subseteq A \cup B$

3. Let P denote the set of all students taking physics at Truman College. Let M denote the set of all students taking mathematics at Truman College.

   a) describe the set $P \cap M$ b) describe the set $P \cup M$

4. Label each of the following statements as true or false.

   a) $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$ b) $\mathbb{N} \cap \mathbb{Z} = \mathbb{Z}$ c) $\mathbb{N} \cup \mathbb{Z} = \mathbb{N}$ d) $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$

5. Label each of the following statements as true or false. (Hint: make up suitable examples for yourself and then investigate!)

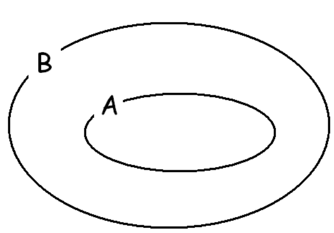
   a) If $A \subseteq B$, then $A \cap B = A$

   b) If $A \subseteq B$, then $A \cup B = B$

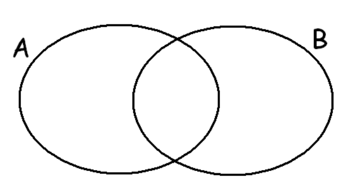
   c) For all sets A and B, $A \cap B \subseteq A$

   d) For all sets A and B, $B \subseteq A \cup B$

6. Recall that a visual representation of subset is to draw one set inside the other. However, this is not a Venn diagram. In case of a Venn diagram, we must have the three distinct regions.

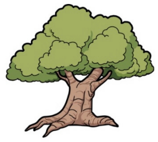


This is not a Venn diagram.



This is a Venn diagram.

Given a Venn diagram depicting sets A and B, how does it show up that A is a subset of B?



Enrichment

Suppose that A and B are sets such that $A \cap B = \{1, 2, 5\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$. How many different sets are possible for A?

5.2 Perimeter and Area of Rectangles

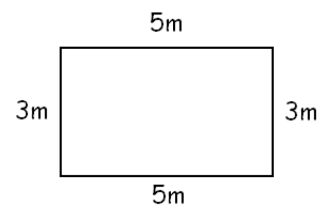
Part 1 - Perimeter

Definition: The **perimeter** of any geometric object is the length of its boundary.

In general, we can always think of the perimeter as a fencing problem. If we have a property, how long of a fence do we need to buy to completely fence around the property? We will denote perimeter by P. Perimeter is a *length*, we measure it in meters (m), centimeters (cm), inches (in), feet (ft), kilometers (km), or miles (mi).

Example 1. Compute the perimeter of a rectangle with sides 3 meters and 5 meters long.

Solution: If we think fencing, we mentally walk around a rectangle-shaped property to figure out how much fencing to buy. That is the same as simply adding the lengths of all four sides. The lengths of only two sides were given, but this should not be a problem. The opposite sides of a rectangle are equally long. Thus we can compute the perimeter as



$$P = 3\,\mathrm{m} + 5\,\mathrm{m} + 3\,\mathrm{m} + 5\,\mathrm{m} = 16\,\mathrm{m}$$

So the perimeter of this rectangle is $\boxed{P = 16\,\mathrm{m}}$.

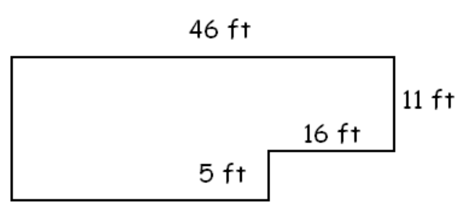
In general, the perimeter of a quadrilateral with sides a, b, c, and d is $P = a + b + c + d$. In case of a rectangle, the opposite sides are equally long, so $c = a$ and $d = b$ and this makes the primeter formula simpler.

Theorem: The perimeter of a rectangle with sides a and b is $P = 2a + 2b$.

This means that in our previous example, the perimeter of a $3\,\mathrm{m}$ by $5\,\mathrm{m}$ rectangle is

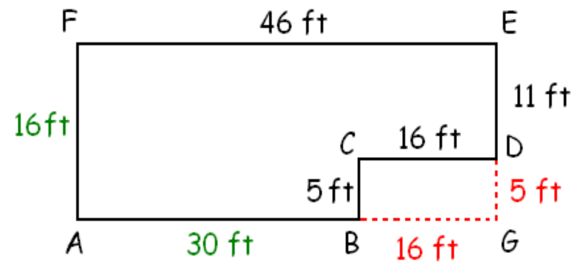
$$P = 2 \cdot 3\,\mathrm{m} + 2 \cdot 5\,\mathrm{m} = 6\,\mathrm{m} + 10\,\mathrm{m} = 16\,\mathrm{m}$$

Example 2. Find the perimeter of the object shown. Angles that look like right angles are right angles.



Solution: It is much easier to discuss geometry if we label points and sides. Consider the picture shown. We are clearly missing the lengths of some sides for the perimeter, so we need to figure out those lengths first.

We first draw line AB beyond point B and line ED beyond point D as shown. These lines intersect each other in point G. Since $BGDC$ is a rectangle, its opposite sides are equally long. Therefore, $BG = 16\,\text{ft}$ and $DG = 5\,\text{ft}$.



The quadrilateral $AGEF$ is also a rectangle, and so FA is the same length as EG. $FA = 16\,\text{ft}$, because $FA = EG = ED + DG = 11\,\text{ft} + 5\,\text{ft} = 16\,\text{ft}$. Also, AG is as long as FE.

$$\begin{aligned} AG &= FE \\ AB + BG &= 46\,\text{ft} && \text{we know } BG = 16\,\text{ft} \\ AB + 16\,\text{ft} &= 46\,\text{ft} && \text{subtract } 16\,\text{ft} \\ AB &= 30\,\text{ft} \end{aligned}$$

We can now compute the perimeter, starting from point A, and moving counterclockwise.

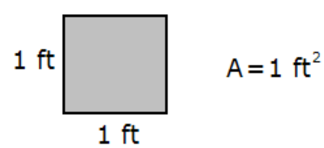
$$P = AB + BC + CD + DE + EF + FA = 30\,\text{ft} + 5\,\text{ft} + 16\,\text{ft} + 11\,\text{ft} + 46\,\text{ft} + 16\,\text{ft} = \boxed{124\,\text{ft}}$$

Part 2 - Area

The **area** of a geometric object is a measurement of its surface.

Understanding and remembering area formulas are easier if we know how they were derived. While we could think about perimeter as a fencing problem, area can be thought of as follows. Suppose a geometric object is a room. How many tiles do we need to buy to cover the entire room? Of course, we have to first agree on the size of the tiles we use to measure area.

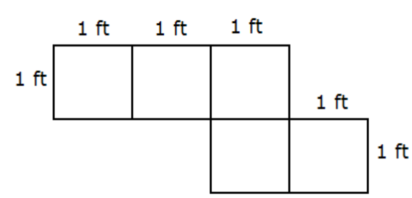
Definition: The **area** of a 1 ft by 1 ft square (shown on the picture) is defined to be $1\,\text{ft}^2$ (square-foot). Similar definitions can be formulated with mi^2, cm^2, in^2, etc. The area of any object, measured in ft^2, is the number of these 1 ft by 1 ft square tiles needed to cover the object, cutting and pasting allowed.



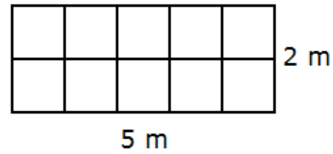
Area is not a length like perimeter. Area is measured in square-meters (m^2), square-centimeters (cm^2), square-inches (in^2), square-feet (ft^2), square-kilometers (km^2), or square-miles (mi^2). etc., and is usually denoted by A.

Example 3. Find the area of the figure shown on the picture.

Solution: We simply count the tiles we need to cover this object. Since the figure can be covered using five unit tiles, its area is $\boxed{A = 5\,\text{ft}^2}$.

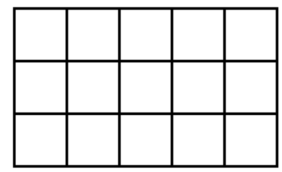


Example 4. Find the area of the figure shown on the picture.



Solution: Since the figure can be covered using ten $1\,\mathrm{m}$ by $1\,\mathrm{m}$ squares, its area is $\boxed{A = 10\,\mathrm{m}^2}$.

> **Theorem:** The area of a rectangle with sides x and y is $A = xy$.



Proof: We will not formally prove this theorem. Instead, we will just informally argue for this formula. The main idea should be clear from the previous example. Consider a rectangle with sides $3\,\mathrm{m}$ and $5\,\mathrm{m}$. The area of this rectangle will be as many square-meters as many $1\,\mathrm{m}$ by $1\,\mathrm{m}$ square tiles are needed to cover it.

Once we place a grid on the rectangle, it is easy to see how many such squares are needed. The rectangle is composed of five columns of squares, where each column consists of three squares. Thus we split the rectangle into fifteen unit tiles, and so the area is $15\,\mathrm{m}^2$. This shows that as long as the lengths of the sides are integers, we can place a grid on it, and the number of unit square tiles is the product of the length of the two sides.

In reality, this theorem is very difficult to prove, because not all side lengths happen to be integers. Mathematicians proved that this formula is true even if the sides of the rectangle are not integers.

Another interesting fact is that logically, we counted how many meter^2 we have. The computation for the area, however, is slightly different with regards to units. Instead of counting meter^2, we literally multiply meter by meter.

$$A = xy = 3\,\mathrm{m}\,(5\,\mathrm{m}) = 15\,\mathrm{m}^2 \quad \text{and not} \quad 15 \cdot 1\,\mathrm{m}^2$$

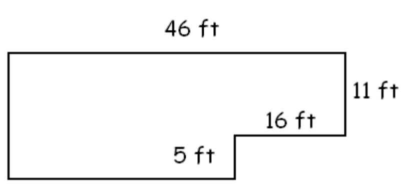
Area computation will always yield for the right unit.

Example 5. Find the area of a rectangle with sides 13 in and 7 in.

Solution: We apply the formula.

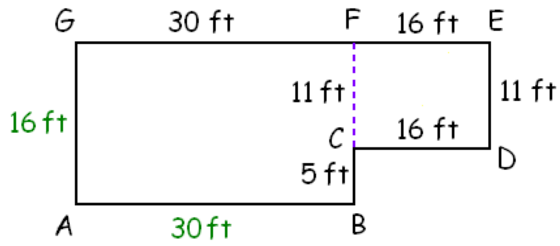
$$A = xy = 13\,\mathrm{in}\,(7\,\mathrm{in}) = \boxed{91\,\mathrm{in}^2}$$

Example 6. Find the area of the object shown. Angles that look like right angles are right angles.



Solution: To compute the area, we have two options.

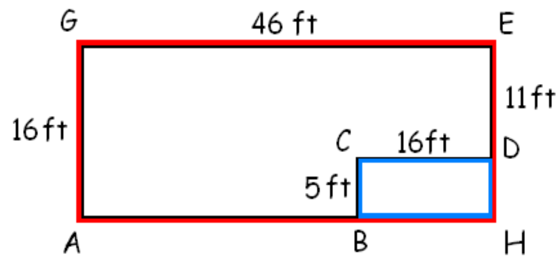
Method 1: We can think of the region as the sum of two rectangles: $CDEF$ is a rectangle with sides 11 ft by 16 ft, and $ABFG$ is a rectangle with sides 30 ft by 16 ft. So the area is



$$\begin{aligned} A &= A_{CDEF} + A_{ABFG} \\ &= 11\,\text{ft} \cdot 16\,\text{ft} + 16\,\text{ft} \cdot 30\,\text{ft} \\ &= 176\,\text{ft}^2 + 480\,\text{ft}^2 = \boxed{656\,\text{ft}^2} \end{aligned}$$

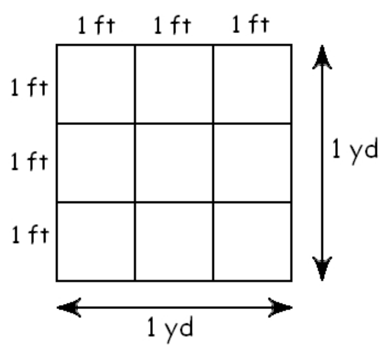
Method 2: We can also think of the region as a big rectangle from which a corner was removed.

The big (red) rectangle, $AHEG$ has sides 16 ft and 46 ft long. The smaller (blue) rectangle has sides 5 ft by 16 ft. So the area is the difference.



$$\begin{aligned} A &= A_{AHEG} - A_{BHDC} \\ &= 46\,\text{ft} \cdot 16\,\text{ft} - 16\,\text{ft} \cdot 5\,\text{ft} \\ &= 736\,\text{ft}^2 - 80\,\text{ft}^2 = \boxed{656\,\text{ft}^2} \end{aligned}$$

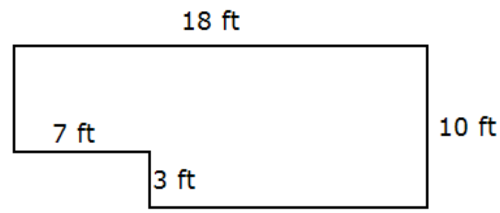
Discussion: One yard equals to three feet. Consider the picture below and discuss: how many square-feet is one square-yard? Can you show the same result algebraically?



Practice Problems

1. Compute the perimeter and area of a rectangle with sides 12 cm by 8 cm.

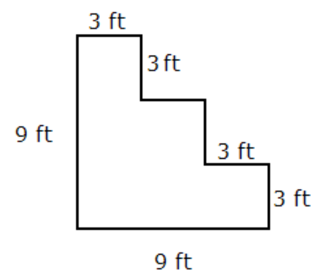
2. Consider the figure shown. Angles that look like right angles are right anges.



   a) Compute the perimeter of the figure. Include units in your computation and answer.

   b) Compute the area of the figure. Include units in your computation and answer.

3. Consider the figure shown. Angles that look like right angles are right anges.



   a) Compute the perimeter of the figure. Include units in your computation and answer.

   b) Compute the area of the figure. Include units in your computation and answer.

Chapter 6

Class 6

6.1 Algebraic Expressions and Statements

Part 1 – Algebraic Expressions

We will now start studying the expressions and statements frequently made in algebra.

> **Definition**: A **numerical expression** is an expression that combines numbers and operations. To **evaluate** a numerical expression is to compute its value.

For example, $3 \cdot 5^2$ is a numerical expression. So are $-\dfrac{12}{3+1}$ and $5^2 - 2^2$ and $-|-5|$. We can evaluate numerical expressions by performing the operations indicated in the expression. Naturally, we must correctly apply the order of operations agreement. For that, we must clearly understand notation. It is a fundamental principle that notation can be objectively understood in only *one* possible way.

Example 1. Evaluate each of the given numerical expressions.

a) $3 \cdot 5^2$ b) $-\dfrac{12}{3+1}$ c) 3^2+2^2 d) $(3+2)^2$ e) -3^2 f) $(-3)^2$ g) $-|-5|$

Solution: a) Between exponentiation and multiplication, we first perform the exponentiation. $3 \cdot 5^2 = 3 \cdot 25 = \boxed{75}$

b) The addition in the denominator must be performed before we divide. (Why?) $-\dfrac{12}{3+1} = -\dfrac{12}{4} = \boxed{-6}$

c) $3^2+2^2 = 9+4 = \boxed{13}$ and d) $(3+2)^2 = 5^2 = \boxed{25}$

Note: The error of confusing 3^2+2^2 with $(3+2)^2$ is called the "Freshman's Dream Error".

e) $-3^2 = \boxed{-9}$ and f) $(-3)^2 = \boxed{9}$

Note: In looking at -3^2 and $(-3)^2$, we can interpret the minus sign in front of 3 as 'the opposite of'. That is the same as multiplication by -1. Now we can apply order of operations, and exponentiation comes before multiplication.

$$-3^2 = -1 \cdot 3^2 = -1 \cdot 9 = -9 \quad \text{but} \quad (-3)^2 = (-3)(-3) = 9$$

In the case of -3^2, we take the opposite of the square of three. In the case of $(-3)^2$, we square the opposite of three.

g) $-|-5| = \boxed{-5}$

This is a perfect example that two minuses don't always make a plus. What happens here?

Definition: An **algebraic expression** is an expression that combines numbers, operations, and variables.

Variables are letters that represent numbers. They are subjects to the same rules as numbers. We use variables for different reasons. Often because the variable represents a number we do not know but would like to know. However, there are other reasons. Sometimes we use variables because we would like to discuss general statements. Consider the following statement as an example. "*For every numbers x and y*, $x+y=y+x$". This statement is not about the numbers x and y; it is rather about the operation addition. No matter what two numbers we add, the order of the numbers in the addition does not matter. We express this property of addition by saying that addition is commutative. In this case, we use variables because we would like to talk about all the numbers at the same time.

In our modern language, we prefer to use x as a variable, especially if it is the type of unknown we want to find. A fundamental principle is that within the context of a problem, a variable has one fixed meaning. Suppose that we are solving a word problem that involves the number of books and the number of pencils. In this case, we can not label both of them x, unless we are certain that the number of books is the same as the number of pencils. **To denote quantities that may be different, we must use different letters.** For example, we can denote the number of books by x and the number of pencils by y. We might find out later in the problem that the values of x and y might be equal. That is perfectly fine. We must use different variables for quantities that *could* be different.

For example, $3x^2-1$ is an algebraic expression. So are $-x+3$ and $2a-b$ and $5y+3$. If the values of all variables in an expression are given, we can evaluate it. To evaluate an algebraic expression, we substitute the given values of the variables into the expression, and evaluate the resulting numerical expression.

When subtituting a number into an algebraic expression, it is our responsibility to preserve the indicated operations and their order correctly. Consider the expression $2x$. If $x=5$, we can not write 25 instead of $2x$, because our notation would indicate a two-digit number with no operation. If $x=-5$, we can not write $2-5$ instead of $2x$, because we would incorrectly indicate subtraction instead of the multiplication.

$$2x \quad \text{when } x=5 \quad \Longrightarrow \quad 2\cdot 5 \quad \text{or} \quad 2(5) \qquad\qquad 2x \quad \text{when } x=-5 \quad \Longrightarrow \quad 2(-5) \quad \text{or} \quad 2\cdot(-5)$$

To evaluate an algebraic expression, we can perform the following steps.

Step 1. Copy the entire expression with one modification: replace each variable by an empty pair of parentheses.

Step 2. Insert the values into the parentheses. Now the problem became an order of operations problem.

Step 3. Drop the unnecessary parentheses and work out the order of operations problem. (It may appear awkward first to create all these parentheses but they are extremely helpful.)

Example 2. Evaluate the algebraic expression $3x^2-x+5$ given the values of x.

a) $x=-2$ b) $x=3$

Solution: a) We first copy the entire expression, replacing the letter x by little pairs of parentheses.

$$3x^2-x+5=3(\ \ )^2-(\ \ )+5$$

Then we insert the number -2 into each pair of parentheses.

$$3x^2-x+5=3(-2)^2-(-2)+5$$

Now the problem became an order of operations problem. We start with the exponent.

$$\begin{aligned} 3(-2)^2-(-2)+5 &= 3\cdot 4-(-2)+5 && \text{perform multiplication} \\ &= 12-(-2)+5 && \text{subtraction:}\quad 12-(-2)=12+2=14 \\ &= 14+5 && \text{addition} \\ &= \boxed{19} \end{aligned}$$

Notice that because we substituted a negative value for x, all little parentheses proved to be necessary.

Students' work should look like this:

$$\boxed{\begin{aligned} 3x^2-x+5 &= 3(-2)^2-(-2)+5 \\ &= 3\cdot 4+2+5 \\ &= 12+2+5=\boxed{19} \end{aligned}}$$

b) Evaluate $3x^2-x+5$ when $x=3$.

We first copy the entire expression, replacing the letter x by little pairs of paretheses.

$3x^2-x+5=3(\ \ )^2-(\ \ )+5$

Then we insert the number 3 into each pair of parentheses.

$3x^2-x+5=3(3)^2-(3)+5$

Because we substituted a positive number, most parentheses are unnecessary. We will drop them:

$3x^2-x+5=3(3)^2-(3)+5=3\cdot 3^2-3+5$

Then we solve the resulting order of operations problem. We start with the exponent.

$$\begin{aligned} 3\cdot 3^2-3+5 &= 3\cdot 9-3+5 && \text{perform multiplication} \\ &= 27-3+5 && \text{subtraction} \\ &= 24+5=\boxed{29} && \text{addition} \end{aligned}$$

Example 3. We ejected a small object upward from the top of a 720 ft tall building and started measuring time in seconds. We find that t seconds after launching, the vertical position of the object is $-16t^2+64t+720$ feet.

a) Evaluate the given expression with $t=0$. What does your result mean?

b) Where is the object 5 seconds after launch?

c) Where is the object 9 seconds after launch?

Solution: a) We evaluate the expression with $t=0$.

$$\begin{aligned} -16t^2+64t+720 &= -16\cdot 0^2+64\cdot 0+720 \\ &= -16\cdot 0+64\cdot 0+720=\boxed{720} \end{aligned}$$

This means that at the time of the launch of the object, it is at a height of 720 feet.

b) To find out where the object is after 5 seconds, we evaluate the expression with $t=5$.

$$\begin{aligned} -16t^2+64t+720 &= -16\cdot 5^2+64\cdot 5+720 \\ &= -16\cdot 25+64\cdot 5+720 \\ &= -400+320+720=-80+720=640 \end{aligned}$$

So the object is at a height of $\boxed{\text{640 feet}}$ 5 seconds after launch.

c) To find out where the object is after 9 seconds, we evaluate the expression with $t = 9$.

$$\begin{aligned} -16t^2 + 64t + 720 &= -16 \cdot 9^2 + 64 \cdot 9 + 720 \\ &= -16 \cdot 81 + 64 \cdot 9 + 720 \\ &= -1296 + 576 + 720 = -720 + 720 = 0 \end{aligned}$$

When we started at the top of the building, the location was 720. This means that a height of zero indicates that the object is at the ground. So the object is at a height of $\boxed{0 \text{ feet}}$ 9 seconds after launch.

Every time we enlarge our language, we have to return to old concepts and discuss them in the light of the new concepts. This is the case with the negative sign. When learning about the order of operations, we discussed that if a negative sign can denote subtraction, then it does denote subtraction. If not, then it is a sign describing the thing after it as *negative*, or *opposite of*. This might have seemed silly until now. Why would we say "*the opposite of three*" instead of "*negative three*"? With the introduction of variables, this is the moment when this interpretation becomes useful.

It is a common mistake to assume that the negative sign in $-x$ means that $-x$ is negative. This is incorrect. We are much better off thinking of $-x$ as "*the opposite of* x".

Example 4. Evaluate $-x$ with the given values of x. a) when $x = 5$ b) when $x = -4$

Solution: a) We can carefully use notation: $-x = -(5) = \boxed{-5}$.
We can also use language: $-x$ is the opposite of x. If x is 5, its opposite is -5.

b) We can carefully use notation: $-x = -(-4) = \boxed{4}$.
We can also use language: $-x$ is the opposite of x. If x is -4, its opposite is 4.

Example 5. Evaluate each of the given expressions with the given values of x.

a) $-x^2$ when $x = 5$ c) $(-x)^2$ when $x = 2$ e) $-(-x)^2$ when $x = 6$

b) $-x^2$ when $x = -5$ d) $(-x)^2$ when $x = -2$ f) $-(-x)^2$ when $x = -6$

Solution: a) $-x^2$ is the opposite of the square of x. We square 5 and then take the opposite.
$-x^2 = -(5)^2 = -1 \cdot 5^2 = \boxed{-25}$

b) We square -5 and then take the opposite.
$-x^2 = -(-5)^2 = -1 \cdot (-5)^2 = -1 \cdot 25 = \boxed{-25}$.

If we think about this, it is not so surprising that we got the same answer. The square of a number and its opposite is the same. Then if we take the opposite of both, we still have the same result.

c) $(-x)^2$ means the square of the opposite of x. When we carefully substitute $x = 2$, we will have two pairs of parentheses. One came with the expression, the other with the substitution.
$(-x)^2 = (-(2))^2 = (-2)^2 = \boxed{4}$.

d) $(-x)^2$ means the square of the opposite of x. When we carefully substitute $x = -2$, we will have two pairs of parentheses. One came with the expression, the other with the substitution.
$(-x)^2 = (-(-2))^2 = 2^2 = \boxed{4}$.

e) $-(-x)^2$ means that we take the opposite of x, we square it, and then take the opposite again. At this point, notation might be the safest way.
$-(-x)^2 = -(-(6))^2 = -(-6)^2 = -1 \cdot (-6)^2 = -1 \cdot 36 = \boxed{-36}$.

f) $-(-x)^2$ means that we take the opposite of x, we square it, and then take the opposite again.
$-(-x)^2 = -(-(-6))^2 = -(6)^2 = -1 \cdot 6^2 = -1 \cdot 36 = \boxed{-36}$.

Discussion: Evaluate each of the following algebraic expressions with $x=2$ and $x=-2$. How are these results similar to or different from the results in Example 5? Can you explain why?

a) $-x^3$ b) $(-x)^3$ c) $-(-x)^3$

Part 2 – Algebraic Statements

The most frequantly made statements in algebra are equations and inequalities.

Definition: An **equation** is a pair of numerical or algebraic expressions connected with an equal sign.

For example, $2+3=9$ and $1+2a=5b$ are equations. So are $x+y=y+x$, and $x^3+4=x^2+4x$. Some equations have variables in them, some don't.

Definition: A **solution of an equation** is a value (or ordered pair of values) of the unknown(s) that, when substituted into both sides of the equation, makes the statement of equality true.

Example 6. Consider the equation $x^3+4=x^2+4x$. Evaluate both sides of the equation with the given value of x to determine whether it is a solution of the equation or not.

a) $x=3$ b) $x=-2$ c) $x=-1$ d) $x=2$

Solution: a) We substitute $x=3$ into both sides of the equation $x^3+4=x^2+4x$ and compare the values. We will denote the left-hand side by LHS and the right-hand side by RHS.

If $x=3$, then

$\text{LHS} = 3^3+4$ $= 27+4 = 31$	$\text{RHS} = 3^2+4\cdot 3$ $= 9+12 = 21$

The left-hand side is 31, and the right-hand side is 21. Since these are not equal, $x=3$ is not a solution of the given equation. In short, $\text{LHS} = 31 \neq 21 = \text{RHS}$, so 3 is not a solution.

b) We substitute $x=-2$ into both sides of the equation $x^3+4=x^2+4x$ and compare the values.

If $x=-2$, then

$\text{LHS} = (-2)^3+4$ $= -8+4 = -4$	$\text{RHS} = (-2)^2+4(-2)$ $= 4+(-8) = -4$

Since the two sides are equal, -2 is a solution of the equation.

c) We substitute $x=-1$ into both sides of the equation $x^3+4=x^2+4x$.

If $x=-1$, then

$\text{LHS} = (-1)^3+4$ $= -1+4 = 3$	$\text{RHS} = (-1)^2+4(-1)$ $= 1+(-4) = -3$

Since the two sides are not equal, -1 is not a solution of the equation.

d) We substitute $x = 2$ into both sides of the equation $x^3 + 4 = x^2 + 4x$.

If $x = 2$, then

$\text{LHS} = 2^3 + 4$ $= 8 + 4 = 12$	$\text{RHS} = 2^2 + 4 \cdot 2$ $= 4 + 8 = 12$

Since the two sides are equal, 21 is a solution of the equation.

We have seen that both 2 and -2 are solutions of the equation $x^3 + 4 = x^2 + 4x$. We will leave it for the reader to verify that $x = 1$ is also a solution. So, it is not true that equations can have only one solution! To solve an equation means to find *all* solutions of it.

Example 7. Consider the equation $1 + 2a = 5b$. Evaluate both sides of the equation with the given ordered pairs of values to determine whether they are a solution of the equation or not.

a) $a = 7$ and $b = 3$ b) $a = 3$ and $b = 7$

Solution: a) The pair $a = 7$ and $b = 3$ is often denoted by $(7,3)$. We substitute these numbers into both sides of the equation $1 + 2a = 5b$.

$$\text{LHS} = 1 + 2 \cdot 7 = 1 + 14 = 15 \quad \text{and} \quad \text{RHS} = 5 \cdot 3 = 15$$

The left-hand side and the right-hand side are both 15. Thus the ordered pair $(7,3)$ is a solution of the equation.

b) We substitute $a = 3$ and $b = 7$ into both sides of the equation.

$$\text{LHS} = 1 + 2 \cdot 3 = 1 + 6 = 7 \quad \text{and} \quad \text{RHS} = 5 \cdot 7 = 35$$

The values of the right-hand side and the left-hand side are different, and so the ordered pair $(3,7)$ is not a solution of the equation. Notice that $(7,3)$ is a solution, but $(3,7)$ is not. This is why we call such pairs *ordered* pairs.

Definition: An **inequality** is a pair of numerical or algebraic expressions connected with an inequality sign.

For example, $3x - 1 \geq 5x - 7$, and $2x < 4y - 5$, and $2x^2 \leq 7x - 4$ are all inequalities.

Definition: A **solution of an inequality** is a value (or ordered pair of values) of the unknown(s) that, when substututed into both sides of the inequality, makes the statement of inequality true.

Example 8. Consider the equation $3x-1 \geq 5x-7$. Evaluate both sides of the equation with the given value of x to determine whether it is a solution of the inequality or not.

a) $x=5$ b) $x=-5$ c) $x=3$ d) $x=-3$

Solution: a) We evaluate both sides of the inequality $3x-1 \geq 5x-7$ with $x=5$ to see whether the inequality statement is true.

If $x=5$, then

$\text{LHS} = 3 \cdot 5 - 1 = 15 - 1 = 14$	$\text{RHS} = 5 \cdot 5 - 7 = 25 - 7 = 18$

This means that if $x=5$, then $3x-1 \geq 5x-7$ becomes

$$14 \geq 18$$

This statement is false, therefore 5 is not a solution of the inequality.

b) We evaluate both sides of the inequality with $x=-5$.

If $x=-5$, then

$\text{LHS} = 3(-5) - 1 = -15 - 1 = -16$	$\text{RHS} = 5(-5) - 7 = -25 - 7 = -32$

This means that if $x=-5$, then $3x-1 \geq 5x-7$ becomes

$$-16 \geq -32$$

This statement is true, therefore -5 is a solution of the inequality.

c) We evaluate both sides of the inequality with $x=3$.

If $x=3$, then

$\text{LHS} = 3 \cdot 3 - 1 = 9 - 1 = 8$	$\text{RHS} = 5 \cdot 3 - 7 = 15 - 7 = 8$

This means that if $x=3$, then $3x-1 \geq 5x-7$ becomes $8 \geq 8$. This statement is true, therefore 3 is a solution of the inequality.

d) We evaluate both sides of the inequality with $x=-3$.

If $x=-3$, then

$\text{LHS} = 3(-3) - 1 = -9 - 1 = -10$	$\text{RHS} = 5(-3) - 7 = -15 - 7 = -22$

This means that if $x=-3$, then $3x-1 \geq 5x-7$ becomes $-10 \geq -22$. This statement is true, therefore -3 is a solution of the inequality.

Part 3 – Translations

We will often need to translate English sentences into algebraic statements. While this might seem intimidating first, we just need to practice this skill. Here are a few basic statements for a start. Suppose our number is x.

y is twice as great as x. $\Longrightarrow$	$y=2x$	y is two greater than x. $\Longrightarrow$	$y=x+2$
y is three times as great as x.	$y=3x$	y is three more than x.	$y=x+3$
y is half of x.	$y=\frac{x}{2}$	y is five less than x.	$y=x-5$
y is one-fifth of x.	$y=\frac{x}{5}$		

Recall a few additional expressions we might also need.

the sum of a and b $\Longrightarrow$	$a+b$	the order of a and b does not matter
the product of a and b	ab	the order of a and b does not matter
the differrence of a and b	$a-b$	the order of a and b does matter (first mentioned, first written)
the quotient of a and b	$\frac{a}{b}$	the order of a and b does matter (first mentioned is upstairs)
the opposite of a	$-a$	

Notice that the statements "*y is the sum of x and three*" and "*y is three greater than x*" will result in the same equation: $y=x+3$. Similarly, "*P is four less than Q*" and "*P is the difference of Q and four*" will both result in $P=Q-4$. We will see all of these expressions frequently.

Just like in English, we can create many statements by combining these basic expressions and basic statements.

Example 9. Translate each of the following statements into an equation.

a) M is the sum of A and three times B.

b) M is three times the sum of A and B.

c) y is three less than twice x.

d) y is twice as much as the sum of three and x.

e) A is three more than the product of B and C.

f) The opposite of m is five less than one-third of n.

Solution: a) *M is the sum of A and three times B.*

We can first translate "*three times B*" to $3B$. Then we have: "*M is the sum of A and* $3B$". Then we can translate "*the sum of A and* $3B$" to $A+3B$. Thus the translation is $\boxed{M=A+3B}$.

b) *M is three times the sum of A and B.*

We can first translate the sum of A and B into $A+B$. So now we have: *M is three times* $A+B$. However, the translation $M=3\cdot A+B$ would be incorrect. This way we would multiply only A by 3 and not the entire sum. $3\cdot A+B$ is the sum of three times A and B. If we want the sum to be multiplied by 3, we would have to force the multiplication to be performed after the addition. We can easily do that with a pair of parentheses; the correct answer is $\boxed{M=3(A+B)}$.

c) Given: *y is three less than twice x.*

We can translate "*twice x*" as $2x$. Then we have: y is three less than $2x$. This is one of the basic statements: $\boxed{y=2x-3}$.

d) *y is twice as much as the sum of three and x.*
We can first translate "*the sum of three and x*" into $3+x$. Then we have: *y is twice as much as $3+x$*. This means that we get y if we muttiply $3+x$ by 2. However, $y=2\cdot3+x$ would be incorrect. According to order of operations, we would not multiply the sum by 2, only three. So the correct way to express this is $y=2(x+3)$. The parentheses overwrites the usual order of operations, so we really multiply the entire sum and not just parts of it. So the translation is $\boxed{y=2(3+x)}$.

e) *A is three more than the product of B and C.*
The product of B and C is simply BC. So now we have that *A is three more than BC*. This is again one of the basic ones: $\boxed{A=BC+3}$.

f) *The opposite of m is five less than one-third of n.*
We can translate "*the opposite of m*" to $-m$, and "*one-third of n*" to $\dfrac{n}{3}$. Then we have: " *$-m$ is five less than $\dfrac{n}{3}$*", which can be translated to $\boxed{-m=\dfrac{n}{3}-5}$.

Sometimes the same expressions show up "disguised". One such frequently occuring expression is c*onsecutive integers*. How can we translate three consecutive integers?

Consecutive means that they come right after the other, like 3, 4, and 5. If we denote the smallest number by x, then the other two would be $x+1$ and $x+2$. If we denote the largest number by x, then the three numbers would be expressed as x, $x-1$, and $x-2$. In word problems we often have the freedom of selecting which of the three numbers should we denote by x, and the best choice often depends on the particular problem.

What about four consecutive even numbers? First let us look at an example, say 6, 8, 10, and 12. We see that each one is two greater than the one before. So if we denote the smallest even number by x, then these numbers can be labeled as x, $x+2$, $x+4$, and $x+6$.

Sample Problems

1. Evaluate each of the following numerical expressions.
   a) $2-5(3-7)$ b) $24-10+2$ c) -4^2 d) $(-4)^2$ e) $|3|-|8|$ f) $|3-8|$

2. Evaluate each of the algebraic expressions when $p=-7$ and $q=3$.
   a) $15-p$
   b) $pq-|p-2|$
   c) $4p-q^3$
   d) $\dfrac{q^2-p}{2q+p+1}$
   e) p^2-q^2
   f) $(p-q)^2$
   g) $2q^2$
   h) $(2q)^2$
   i) $15-\dfrac{p+q}{|1-p|}$
   j) $(p+q)^2-(5q+2p)^4$
   k) $-p^2-p+8$

3. Evaluate the expression $3x^2-x+5$ with the given values of x.
   a) $x=0$ b) $x=-1$

4. We ejected a small object upward from the top of a 720 ft tall building and started measuring time in seconds. We find that t seconds after launching, the vertical position of the object is $-16t^2+64t+720$ feet.
   a) Where is the object 2 seconds after launch?
   b) Where is the object 8 seconds after launch?

5. Let $a=-4$, $b=2$, and $x=-3$. Evaluate each of the following expressions.

   a) a^2-b^2 b) $(a-b)^2$ c) $a^b-2bx-x^2-2x$ d) $\dfrac{-x^2+(x+2)^2}{(x-1)}$ e) $\dfrac{x-1}{x+3}$

6. Consider the equation $2x^2+x+34=21x-8$. In case of each number given, determine whether it is a solution of the equation or not.

   a) $x=1$ b) $x=3$ c) $x=4$ d) $x=7$

7. Consider the equation $x^2-10x+x^3-4=4(x+5)$. In case of each number given, determine whether it is a solution of the equation or not.

   a) $x=0$ b) $x=-2$ c) $x=-3$ d) $x=2$

8. Consider the equation $3a-2b-1=(a-b)^2+4$. In case of each pair of numbers given, determine whether it is a solution of the equation or not.

   a) $a=8$ and $b=5$ or, as an ordered pair, $(8,5)$

   b) $a=10$ and $b=7$ or, as an ordered pair, $(10,7)$

9. Consider the inequality $3(2y-1)+1\leq 5y-7$. In case of each number given, determine whether it is a solution of the inequality or not.

   a) $y=-10$ b) $y=3$ c) $y=-5$ d) $y=0$

10. Consider the inequality $\dfrac{2x+1}{3}+5<\dfrac{3x-1}{2}$. In case of each number given, determine whether it is a solution of the inequality or not.

    a) $x=1$ b) $x=13$ c) $x=7$ d) $x=-5$

11. Translate each of the following statements to an algebraic statement.

    a) y is three less than four times x

    b) The opposite of A is one greater than the difference of B and three times C.

    c) Twice M is five less than the product of N and the opposite of M.

12. The longer side of a rectangle is three units shorter than five times the shorter side. If we label the shorter side by x, how can we express the longer side in terms of x?

13. Suppose we have three consecutive integers.

    a) Express them in terms of x if x denotes the smallest number.

    b) Express them in terms of y if y denotes the number in the middle.

    c) Express them in terms of L if L denotes the greatest number.

Practice Problems

1. Evaluate each of the following numerical expressions.
   a) $24-5+1$ c) -1^2 e) $-|4|-|7|$ g) 6^2-4^2
   b) $24 \div 3 \cdot 2$ d) $(-1)^2$ f) $-|4-7|$ h) $(6-4)^2$

2. Evaluate each of the algebraic expressions when $x=6$ and $y=8$.
   a) $19-y+x$ d) x^2+y^2 g) $3(y-x)$ i) $5x-\dfrac{y}{2}$
   b) $19-(y+x)$ e) $(x+y)^2$
   c) $2x^2-5y+3$ f) $3y-x$ h) $\dfrac{5x-y}{2}$ j) $\dfrac{x^2-5x+4}{y-3}$

3. Consider the expression $\dfrac{6x-3y-xy+2x^2}{2x-y}-3$. Evaluate this expression if
   a) $x=-1$ and $y=2$ or the ordered pair $(-1,2)$ c) $x=3$ and $y=-2$ or the ordered pair $(3,-2)$
   b) $x=-3$ and $y=-6$ or the ordered pair $(-3,-6)$ d) $x=-7$ and $y=4$ or the ordered pair, $(-7,4)$

4. Evaluate $-m^2-m$ if
   a) $m=2$ b) $m=-2$ c) $m=0$ d) $m=5$ e) $m=-5$

5. Evaluate $\dfrac{8x+x^2-33}{x+11}$ if
   a) $x=0$ b) $x=7$ c) $x=-4$ d) $x=-11$ e) $x=-1$

6. a) It is a common mistake to think that the expressions $2x-3$ and $2x+3$ are opposites. They are not. Evaluate these expressions for the values given below to fill out the table below.

	$x=2$	$x=5$	$x=6$	$x=10$	$x=-1$	$x=-5$	$x=-8$
$2x-3$	1						
$2x+3$	7						

   b) the opposite of $2x-3$ is actually $-2x+3$. Evaluate these expressions for the values given below to fill out the table below.

	$x=2$	$x=5$	$x=6$	$x=10$	$x=-1$	$x=-5$	$x=-8$
$2x-3$							
$-2x+3$							

7. Evaluate $\dfrac{x-2}{2-x}$ if
   a) $x=0$ b) $x=10$ c) $x=2$ d) $x=-13$

8. Evaluate each of the following algebrac expressions with the value(s) given.
   a) $3x^2-x+7$ if $x=-1$ b) $-a+5b$ if $a=3$ and $b=-2$ c) $\dfrac{x^x-1}{x-1}$ if $x=2$

9. The absolute value of a number is its distance from zero on the number line. (Recall that distances can never be negative.) If we wanted to know the distance between two numbers on the number line, the absolute value is very helpful. Consider the expression $|a-b|$. Evaluate this expression for each of the pairs of values given, and see whether $|a-b|$ really results in the distance between a and b on the number line.
   a) $a=8$ and $b=3$ b) $a=2$ and $b=10$ c) $a=-2$ and $b=-9$ d) $a=2$ and $b=-1$

10. Consider the equation $\dfrac{2x^2-11x-21}{2x+3}=3x-(2x+7)$. In case of each number given, determine whether it is a solution of the equation or not.

    a) $x=8$ b) $x=13$ c) $x=10$

11. Consider the equation $-x^2-2x\left(3-x^2\right)=-x+2$. In case of each number given, determine whether it is a solution of the equation or not.

    a) $x=0$ b) $x=1$ c) $x=-1$ d) $x=2$ e) $x=-2$

12. Consider the equation $y=\dfrac{5x-3}{2}$. In case of each pair of numbers given, determine whether it is a solution of the equation or not.

    a) $x=1$ and $y=1$ or, as an ordered pair, $(1,1)$
    b) $x=9$ and $y=4$ or, as an ordered pair, $(9,4)$
    c) $x=3$ and $y=6$ or, as an ordered pair, $(3,6)$
    d) $x=17$ and $y=41$ or, as an ordered pair, $(17,41)$

13. Consider the equation $(p-q)^2+\dfrac{3p-1}{6-q}=4(p+1)$. In case of each pair of numbers given, determine whether it is a solution of the equation or not.

    a) $p=8$ and $q=5$ b) $p=7$ and $q=1$

14. Consider the inequality $-x+2<-x^2+2(x+6)$. In case of each number given, determine whether it is a solution of the inequality or not.

    a) $x=-5$ b) $x=-2$ c) $x=0$ d) $x=3$ e) $x=7$

15. Consider the inequality $\dfrac{x}{3}+1\geq\dfrac{x+1}{2}-1$. In case of each number given, determine whether it is a solution of the inequality or not.

    a) $x=-9$ b) $x=-3$ c) $x=27$ d) $x=15$ e) $x=-15$

16. Translate each of the given statements to algebraic statements.

    a) The difference of A and B is four less than the product of A and the opposite of B.

    b) If we square x, the result is eight less than five times the opposite of x.

    c) If we subtract ten from P, we get a number that is five less than the sum of Q and twice R.

    d) Four times a number y is one greater than twice the sum of y and seven.

    e) The number x is ten greater than its own opposite.

    f) The square of the sum of x and y is ten greater than the difference of the square of x and square of y.

17. The longer side of a rectangle is seven units longer than three times its shorter side. If we label the shorter side by x, express the longer side in terms of x.

18. Express four consecutive even numbers if

    a) we denote the smallest number by x

    b) we denote the greatest number by x

6.2 Problem Set 3

1. Label each of the following statements as true or false.

   a) 3 is a multiple of 3.

   b) $4 < 4$

   c) $5 \geq 5$

   d) 1 is a prime number.

   e) 2 is a prime number.

   f) 14 is a multiple of 4 or 7 is a prime number.

   g) 14 is a multiple of 4 and 7 is a prime number.

   h) If a number n is divisible by 2 and 3, then it is also divisible by 6.

   i) If a number n is divisible by 4 and 6, then it is also divisible by 24.

   j) Every number divisible by 12 is also divisible by 6.

   k) Every number divisible by 6 is also divisible by 12.

2. List all factors of each of 90.

3. Consider the given numbers. 101010, 1189188, 35530, 1234321, 20172017. List all numbers from the list that are divisible by: a) 4 b) 6 c) 9 d) 11

4. List the first five prime numbers.

5. Which of the given numbers are primes? 501, 737, 91, 101, 2017, 407

6. Perform the given division with remainders.

   a) $2017 \div 13$ b) $12091 \div 27$ c) $1234 \div 18$ d) $5624 \div 37$

7. Let $A = \{1,4,6,8,9\}$ and $B = \{2,3,4,8,10\}$.

   a) Find $A \cup B$ and $A \cap B$.

   Label the following statements as true or false.

   b) $A \cap B \subseteq A$ c) $B \subseteq A \cup B$ d) $A \cap \varnothing = A$ e) $B \cup \varnothing = B$

8. Find each of the following sets and if possible, present them by listing their elements.

   a) $A = \{a \in \mathbb{N} \,|\, a < 6\}$

   b) $B = \{b \in \mathbb{N} \,|\, b < 7 \text{ and } b > 3\}$

   c) $C = \{c \in \mathbb{N} \,|\, c < 7 \text{ or } c > 3\}$

   d) $D = \{x \in \mathbb{N} \,|\, x \leq 10 \text{ and } x \text{ is even}\}$

9. Let $S = \{x \in \mathbb{N} : x \geq 2\}$ and $T = \{x \in \mathbb{N} : x < 10\}$. Find $S \cap T$ and $S \cup T$.

10. Let $A = \{n \in \mathbb{N} : n \text{ is divisible by } 2\}$, $B = \{n \in \mathbb{N} : n \text{ is divisible by } 6\}$. Which (if any) of the following is true?

    $A \subseteq B$ $B \subseteq A$

11. Let E be the set of all even integers and O be the set of all odd integers. Find $E \cap O$ and $E \cup O$.

12. Let R be the set of all rectangles and S be the set of all squares. Label each of the following as true or false.

    a) $R \subseteq S$ b) $S \subseteq R$ c) $R \cap S = R$ d) $R \cup S = R$

13. Let $P = \{n \in \mathbb{N} : n \text{ is divisible by } 2\}$, $Q = \{n \in \mathbb{N} : n \text{ is divisible by } 5\}$. What is $P \cap Q$?

14. Label each of the following statements as true or false.

a) For all sets A, $\varnothing \subseteq A$.

b) For all sets A, $A \cap \varnothing = A$.

c) For all sets A and B, $A \cap B \subseteq A$.

d) For all sets A and B, $A \subseteq A \cup B$.

e) For all sets A and B, if $A \subseteq B$, then $A \cup B = B$.

f) $\mathbb{N} \cup \mathbb{Z} = \mathbb{N}$

g) $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$

15. Evaluate each of the given numerical expressions.

a) $-2-3$ b) $-2(-3)$ c) $-(2-3)$ d) $(-2)-3$

e) $(-2)^2$ f) $(-2)^3$ g) $(-2)^4$ h) $(-2)^5$

i) -2^2 j) -2^3 k) -2^4 l) -2^5

m) 3^2-7^2 n) $(3-7)^2$ o) 3^2+7^2 p) $(3+7)^2$

q) $10-3(-8)$ r) $10-(3-8)$ s) $10(-3-8)$

t) $(10-3)(-8)$ u) $10(-(3-8))$

16. Evaluate each of the given numerical expressions.

a) $(-2)^2$ b) -2^2 c) $\left(-2^2\right)$ d) $-(2)^2$

e) $5(-2)^2$ f) $5-2^2$ g) $5\left(-2^2\right)$ h) $5-(2)^2$

i) 2^2+5^2 j) $(2+5)^2$ k) $\left(2+5^2\right)$ l) 2^2-5^2

m) $(2-5)^2$ n) $\left(2-5^2\right)$ o) $2^2(-5)^2$ p) $2^2\left(-5^2\right)$

q) $\left(2^2-5\right)^2$ r) $2^2-(-5)^2$ s) $2^2-\left(-5^2\right)$ t) $(2-(-5))^2$

u) $\left(2-(-5)^2\right)$ v) $2^2\left(-\left(-5^2\right)\right)$ w) $2^2\left(-(-5)^2\right)$ x) $2^2(-(-5))^2$

17. Simplify each of the following expressions by applying the order of operations agreement. **Show all steps. Perform only one operation in each step.**

a) $7\cdot 3^2-\left(3-2^2\cdot 5-1\right)\div 2$

b) $\dfrac{5-1+2}{-1^2+(-1)^2}$

c) $\dfrac{(-2)^3-5(-3)-(-1)^4+(-3)^2}{-2^2-(-1)}$

d) $|3-8|-(|3|-|8|)$

e) $2^3-2\left(5-(-3)^2\right)^2$

f) $\left(\left((8-5)^2-7\right)^2-2\right)^2-1$

g) $\dfrac{4^2+5^2-6\div 2\cdot 3}{4^2-8\cdot 2}$

h) $3+2\left(5+3\left(15-2^3\right)-2^2-1\right)$

i) $4\left(3\left(2(2^2-1)-1\right)-1\right)+5$

j) $-3^2-(-24)\div\left(5-(-1)^3\right)\cdot 2$

k) $-2^2-3\left(5-(-2)^2\right)-(-1)^3$

l) $-2-5\left(-3^2-2(-7)\right)$

m) $|-10-7|-|1-4|$

n) $|-10-7-|1-4||$

o) $|-10-7|1-4||$

p) $|-10-|7-1-4||$

q) $|-10|-7-1-4||$

18. Let $p=4$, $q=-3$, and $s=1$. Evaluate each of the following expressions.

a) $-q^2-pq$ b) $\dfrac{2p-q}{p-(s-q)}$ c) p^2-2s^2 d) $p^2-(2s)^2$ e) $|p-q|-3s$ f) $p-|q-3s|$

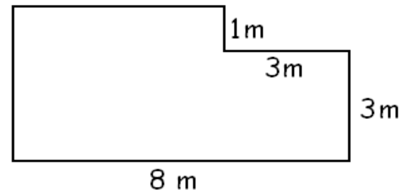
19. Suppose that $x=4$ and $y=-3$. Evaluate each of the algebraic expressions.

a) $2x-y+1$ b) $-y^2-3x^2y$ c) $(-x)^2-5y$ d) $5x-2y+2x+y$ e) $\left|\dfrac{x^2-y^2}{y^2-x^2}\right|$

20. Consider the equation $x^3-x^2+7=x^2+5x+1$. Which of the given numbers are solutions of the equation? The given numbers: $0, 1, -1, 2$, and -2

21. Consider the inequality $x^2+3x \leq x+24$. Which of the given numbers are solutions of the inequality? The given numbers: 5, 6, 0, -10, 3, and 4

22. Compute the perimeter and area of the figure shown. Include units in your computation and answer.



23. Translate each of the following sentences into algebraic statements.

    a) X is three less than twice Y.

    b) The opposite of x is five greater than the sum of y and half of z.

    c) The product of a and b is fourteen less than three times the sum of a and b.

    d) The sum of m and twice n is one greater than the quotient of m and n.

    e) Three times the difference of x and y is one less than the product of x and the opposite of y.

    f) Suppose that Peter's age is ten less than twice the age of his younger brother, David. If Peter's age is denoted by P and David's age is denoted by D, write an algebraic statement expressing P in terms of D.

    g) Suppose that a cab-fare for one person in Chicago is 2.50 dollars for the first mile and 1.5 dollars for each additional mile. Write an algebraic expression for the price of a cab ride in Chicago of a distance of x miles.

    h) We are thinking of three consecutive numbers. Express them in terms of x if x represents the smallest number.

    i) The longer side of a rectangle is three feet shorter than four times the shorter side. If the shorter side of the rectangle is denoted by x, express the area of the rectangle in terms of x.

    j) Suppose that Ann has A dollars and Beatrix has B dollars. The girls made a bet that Ann lost, so she must pay Beatrix 30 dollars. Express how much money do the girls have after Ann paid.

24*. (Enrichment) There are thirty students in our Math 99 class. Twenty of them also takes English 101, fifteen of them also takes Speech 101, and ten of them takes both English 101 and Speech 101? How many of the students in Math 99 are taking neither English 101 nor Speech 101? (Hint: draw a Venn Diagram!)

25*. (Enrichment) Two mathematicians are having a conversation. Mathematician A asks B about his kids. B answers: "I have three children, the product of their ages is 36." A says: "I still don't know how old your children are." Then B tells A the sum of his three kids' ages. A answers: "I still don't know how old they are. Then B adds: "The youngest one has red hair." Now A knows how old the kids are. Do you?

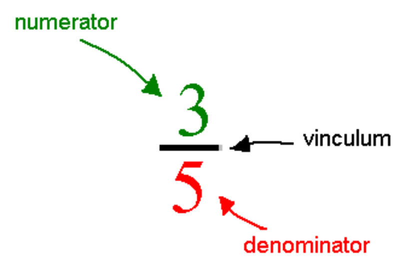
Chapter 7

Class 7

7.1 Definition of a Fraction

A fraction has three components as shown on the picture. The important parts are the **numerator** above and the **denominator** below the little line that is called the vinculum.

At first let us not even consider a fraction alone. We will just define a **fraction of something**.



> **Definition:** $\frac{3}{5}$ of a quantity can be obtained as follows.
>
> Step 1. We first divide the quantity into 5 equal shares.
>
> Step 2. Let us take 3 such shares. That is $\frac{3}{5}$ of our quantity.

So the numerator tells us how many shares we have. The denominator tells us how big each share is.

Example 1. Find $\frac{3}{5}$ of 100 dollars.

Solution: Step 1. Divide 100 dollars into five equal shares. This means that we exchange a 100 dollar bill into five twenty-dollar bills. In other words, $\frac{1}{5}$ of 100 dollars is 20 dollars.

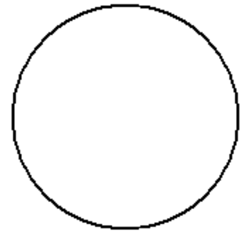
Step 2. To obtain $\frac{3}{5}$, we take three such shares. In this case this means taking three twenty-dollar bills. That is 60 dollars. In other words, $\frac{3}{5}$ of 100 dollars is $\boxed{\text{60 dollars}}$.

Example 2. Compute $\frac{4}{7}$ of 42.

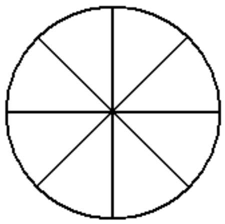
Solution: Step 1. We divide 42 into seven equal shares. $\frac{1}{7}$ of 42 is 6.

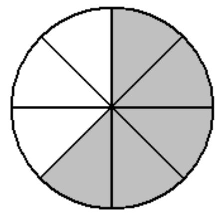
Step 2. We take four such shares. Thus $\frac{4}{7}$ of 42 is $4 \cdot 6 = 24$. The answer is $\boxed{24}$.

Example 3. Shade the region on the picture that corresponds to $\frac{5}{8}$ of the circle.



Solution: Step 1. Divide the circle into 8 equal parts. Step 2. We shade five such parts.





Example 4. A cake was sliced into equal slices. Amy ate 2 slices and Betsy ate 3 slices. If 2 slices were remaining, what fraction of the cake was eaten?

Solution: We need to first figure out how many slices made up the cake. If 2 were eaten by Amy and 3 by Betsy and 2 more were left, then there were all together $2+3+2=7$ slices. 5 slices were eaten which were $\frac{5}{7}$ of the cake. So the answer is that $\boxed{\frac{5}{7}\text{ of the cake was eaten}}$.

Example 5. Compute $\frac{8}{100}$ of 2000 dollars.

Solution: We divide 2000 dollars into 100 equal shares. Then each share is 20 dollars. (We divide 2000 by 100). Then we take 8 such shares, that is $8 \cdot 20$ dollars $= 160$ dollars. Thus $\frac{8}{100}$ of 2000 dollars is $\boxed{\text{160 dollars}}$.

Definition: We often use fractions with 100 in the denominator. Such a fraction is also called a **percent** and is denoted by %.

For example, computing 8% of a quantity is exactly the same as computing $\frac{8}{100}$ of it.

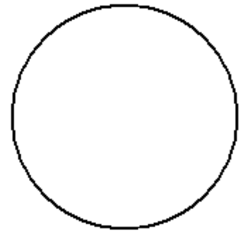
Example 6. In 2012, there were approximately 235 000 000 people eligible to vote in the USA. If 59% of them voted in 2012, how many people voted?

Solution: We need to compute 59% of 235 000 000. That is the same as computing $\frac{59}{100}$ of 235 000 000.

We first divide 235 000 000 by 100. Division by 100 is very easy in this case: just cut off the last two zeroes. So, 1% or $\frac{1}{100}$ of the number is 2 350 000. Now we take 59 of that, we multiply 2 350 000 by 59. The result is 138 650 000. Thus, $\boxed{\text{approximately 138 650 000 people voted}}$ in 2012.

Practice Problems

1. Compute $\frac{4}{9}$ of 63.
2. Compute $\frac{5}{6}$ of 24.
3. Shade $\frac{5}{9}$ of the circle given.



4. The price of a TV is \$400. We want to raise the price by 5%. What is the new price?
5. The price of a couch is \$700. Next week, it will go on a 15% off sale. What is the new price? (A 15% sale means that the price of the item is lowered by 15%.)
6. Find $\frac{3}{4}$ of 56.
7. We placed \$2000 into a bank account with 6% yearly interest rate. How much money do we have in the bank after one year?
8. This problem is about a method of comparing fractions.

   a) Compute $\frac{3}{7}$ of 420.

   b) Compute $\frac{4}{10}$ of 420.

   c) Based on the results of parts a) and b), which fraction is larger, $\frac{3}{7}$ or $\frac{4}{10}$?
9. Bert has made \$54 000 last year. If he has to pay 32% of his income in taxes, how much taxes does he owe and how much of his income will he keep?
10. Sally used to make \$2400 per month, but now she got a 3% raise. How much is her monthly salary now?
11. Mr. X won \$600 000 in the lottery two yers ago. By now he has spent some of the money. When he was asked what happened, he said the following. "*I didn't spend it all. I spent $\frac{1}{3}$ of the money by taking a luxury yacht trip around the world. Then I put the rest in the bank. Later I decided to buy a house I really liked. So I took $\frac{3}{4}$ of the money out of the bank and baught the house. For half of what was left, I purchased stocks that completely lost their value. Finally, I gave $\frac{2}{5}$ of what's left to my niece for her college education.*"

    How much money is left from the winnings?

7.2 One- and Two-Step Equations

Equations are a fundamental concept and tool in mathematics.

> **Definition**: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign.

For example, $3x^2 - x = 4x + 28$ is an equation. So is $x^2 + 5y = -y^2 + x + 2$.

> **Definition**: A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality true.

Example 1. a) Verify that -2 is not a solution of the equation $3x^2 - x = 4x + 28$.
b) Verify that 4 is a solution of the equation $3x^2 - x = 4x + 28$.

Solution: a) Consider the equation $3x^2 - x = 4x + 28$ with $x = -2$. We substitute $x = -2$ into both sides of the equation and evaluate the expressions.

If $x = -2$, the left-hand side of the equation is $\begin{aligned} \text{LHS} &= 3x^2 - x \\ &= 3(-2)^2 - (-2) \\ &= 3 \cdot 4 + 2 = 12 + 2 = 14 \end{aligned}$	If $x = -2$, the right-hand side of the equation is $\begin{aligned} \text{RHS} &= 4x + 28 \\ &= 4(-2) + 28 \\ &= -8 + 28 = 20 \end{aligned}$

Since the two sides are not equal, $14 \neq 20$, the number -2 is not a solution of this equation.

b) Consider the equation $3x^2 - x = 4x + 28$ with $x = 4$. We evaluate both sides of the equation after substituting 4 into x.

If $x = 4$, the left-hand side of the equation is $\begin{aligned} \text{LHS} &= 3x^2 - x \\ &= 3 \cdot 4^2 - 4 \\ &= 3 \cdot 16 - 4 = 48 - 4 = 44 \end{aligned}$	If $x = 4$, the right-hand side of the equation is $\begin{aligned} \text{RHS} &= 4x + 28 \\ &= 4 \cdot 4 + 28 \\ &= 16 + 28 = 44 \end{aligned}$

Since the two sides are equal, $x = 4$ is a solution of this equation.

> **Definition**: To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Caution! Finding one solution for an equation is not the same as solving it. For example, the number 2 is a solution of the equation $x^3 = 4x$. However, -2 is also a solution of this equation.

If we think about it a little, trial and error is never a legitimate method because there is no way for us to gurantee that there are no other solutions are there. It is impossible for us to try all real numbers because there are infinitely many of them, and we have finite lives.

So we will need to develop systematic methods to solve equations. We will start with the easiest group of equations, linear equations. There are several types of linear equations, and we will start with the easiest type that is called one-step equations.

To solve a linear equation, we isolate the unknown by applying the same operation(s) to both sides. Consider, for example, Ann and Dewitt who has the same monthly salary. This month they both get a 40 dollar raise. Who is making ore money now? It is clear that if we start with two equal quantities and we add the same amount to them, they will still stay equal. This is the underlying principle of solving equations. We always apply the same operations to both sides in an effort to bring the equation in a simple form such as $x=-2$. The following equations are **one-step equations** because there is only one operation that separates us from the desired form.

Example 2. Solve each of the given equations. Make sure to check your solutions.

a) $x-8=10$ b) $3y=-12$ c) $\dfrac{x}{-3}=8$ d) $m+10=-5$

Equations like these are called **one-step equations** because they can be solved in only one step. We need to isolate the unknown on one side. In order to do that, we perform the inverse operation. (The inverse operation of additon is subtraction and vica versa. The inverse operation of multiplication is division and vica versa.)

Solution: a) In order to isolate the unknown, we add 8 to both sides.

$$\begin{aligned} x-8 &= 10 \qquad \text{add } 8 \\ x &= 18 \end{aligned}$$

So the only solution of this equation is 18. We can also say that the solution set is $\{18\}$. We should check; if $x=18$, the left-hand side is

$$\text{LHS} = x-8 = 18-8 = 10 = \text{RHS} \checkmark$$

So our solution, $\boxed{x=18}$ is correct.

b) In order to isolate the unknown, we divide both sides by 3.

$$\begin{aligned} 3y &= -12 \qquad \text{dividde by } 3 \\ y &= -4 \end{aligned}$$

So the only solution of this equation is -4. We check; if $y=-4$, then

$$\text{LHS} = 3y = 3(-4) = -12 = \text{RHS} \checkmark$$

So our solution, $\boxed{y=-4}$ is correct.

c) In order to isolate the unknown, we multiply both sides by -3.

$$\begin{aligned} \frac{x}{-3} &= 8 \qquad \text{multiply by } -3 \\ x &= -24 \end{aligned}$$

So the only solution of this equation is -24. We check; if $x=-24$, then

$$\text{LHS} = \frac{-24}{-3} = 8 = \text{RHS} \checkmark$$

So our solution, $\boxed{x=-24}$ is correct.

d) In order to isolate the unknown, we we subtract 10 from both sides.

$$\begin{aligned} m+10 &= -5 \qquad \text{subtract } 10 \\ m &= -15 \end{aligned}$$

So the only solution of this equation is -15. We check; if $m=-15$, then

$$\text{LHS} = m+10 = -15+10 = -5 = \text{RHS}$$

So our solution, $\boxed{m=-15}$ is correct.

Discussion: Solve each of the following equations. How are these unusual?

a) $5x=5$ b) $5x=0$ c) $x-4=-4$ d) $\dfrac{x}{3}=0$

Example 3. One side of a rectangle is 12 feet long. Find the length of the other side if the area of the rectangle is 60 square-feet.

Solution: Let us denote the missing side by x. We will write and solve an equation expressing the area of the rectangle.

$$\begin{aligned} 12x &= 60 \qquad \text{divide by } 12 \\ x &= 5 \end{aligned}$$

Thus the other side is $\boxed{5 \text{ feet}}$ long. Note that if we carry the units in the computation, they will work out perfectly.

$$\begin{aligned} (12\,\text{ft})\,x &= 60\,\text{ft}^2 \qquad \text{divide by } 12\,\text{ft} \qquad \text{margin work: } \frac{60\,\text{ft}^2}{12\,\text{ft}} = 5\frac{\text{ft}\cdot\text{ft}}{\text{ft}} = 5\,\text{ft} \\ x &= 5\,\text{ft} \end{aligned}$$

Part 2 – Two-Step Equations

Suppose we decide to hide a small object, say a coin. We put the coin on the table, then place an envelope over it, and then, just to be sure, we place a hat on top of the envelope. Let us find the coin! To do that, what do we need to remove, and in what order? We would first remove the hat and then the envelope, right?

This is the basis of solving two-step equations. To isolate the unknown, we will perform the inverse operations, in the reverse order. What happened last can be undone first.

Example 4. Solve each of the given equations. Make sure to check your solutions.

a) $10=3x-11$ b) $3x+8=-7$ c) $\dfrac{t-7}{2}=-8$ d) $\dfrac{x}{-3}+4=15$

Solution: a) The equation $10 = 3x - 11$ looks unusual in the sense that two-step equations often contain the unknown on the left-hand side. We are always allowed to swap two sides of an equation. If $A = B$, then clearly, also $B = A$. We will do this first. This is an optional step that is always available.

$$\begin{aligned} 10 &= 3x - 11 && \text{we swap the two sides} \\ 3x - 11 &= 10 \end{aligned}$$

We now look at the side that contains x and ask: *What happened to the unknown*? The answer is: *Multiplication by* 3 *and then subtraction of* 11. We need to apply the inverse operations, in reverse order. In this case, this means that we will add 11 to both sides and then divide both sides by 3.

$$\begin{aligned} 3x - 11 &= 10 && \text{add 11} \\ 3x &= 21 && \text{divide by 3} \\ x &= 7 \end{aligned}$$

So the only solution of this equation is 7. We check: if $x = 7$, then

$$\text{LHS} = 3x - 11 = 3 \cdot 7 - 11 = 21 - 11 = 10 = \text{RHS} \checkmark$$

So our solution, $\boxed{x = 7}$ is correct.

b) As we look at the equation $3x + 8 = -7$, we ask: *What happened to the unknown*? The answer is: *Multiplication by* 3 *and then addition of* 8. We need to apply the inverse operations, in a reverse order. In this case, this means that we will subtract 8 from both sides and then divide both sides by 3.

$$\begin{aligned} 3x + 8 &= -7 && \text{subtract 8} \\ 3x &= -15 && \text{divide by 3} \\ x &= -5 \end{aligned}$$

So the only solution of this equation is -5. We check: if $x = -5$, then

$$\text{LHS} = 3x + 8 = 3(-5) + 8 = -15 + 8 = -7 = \text{RHS} \checkmark$$

So our solution, $\boxed{x = -5}$ is correct.

c) What happened to the unknown? On the left-hand side, there was a subtraction of 7 and then a division by 2. To reverse that, we will multiply both sides by 2 and then add 7 to both sides.

$$\begin{aligned} \frac{t-7}{2} &= -8 && \text{multiply by 2} \\ t - 7 &= -16 && \text{add 7} \\ t &= -9 \end{aligned}$$

So the only solution of this equation is -9. We check: if $t = -9$, then

$$\text{LHS} = \frac{t-7}{2} = \frac{-9-7}{2} = \frac{-16}{2} = -8 = \text{RHS} \checkmark$$

So our solution, $\boxed{t = -9}$ is correct.

d) What happened to the unknown? On the left-hand side, there was a division by -3 and then an addition of 4. To reverse that, we will subtract 4 from both sides by and then multiply both sides by -3.

$$\begin{aligned} \frac{x}{-3}+4 &= 15 && \text{subtract } 4 \\ \frac{x}{-3} &= 11 && \text{multiply by } -3 \\ x &= -33 \end{aligned}$$

So the only solution of this equation is -33. We check: if $x=-33$, then

$$\text{LHS} = \frac{x}{-3}+4 = \frac{-33}{-3}+4 = 11+4 = 15 = \text{RHS} \checkmark$$

So our solution, $\boxed{x=-33}$ is correct.

Example 5. The sum of three times a number and seven is -5. Find this number.

Solution: Let us denote our mystery number by x. The equation will be just the first sentence, translated to algebra. The sum of three times the number and seven is $3x+7$. So our equation is $3x+7=-5$. We will know the number if we solve this equation.

$$\begin{aligned} 3x+7 &= -5 && \text{subtract } 7 \\ 3x &= -12 && \text{divide by } 3 \\ x &= -4 \end{aligned}$$

Good news! We do not need to check if -4 is indeed the solution of the equation. What if we correctly solved the *wrong* equation? Recall that *we* came up with the equation, it was not given. Instead of checking the number against the equation, we should check if our solution satisfies the conditions stated in the problem. Is it true the sum of three times -4 and seven is -5? Indeed, $3(-4)+7=-12+7=-5$. Thus our solution, $\boxed{-4}$, is correct.

Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x-5=17$
2. $\dfrac{a-10}{5}=-3$
3. $\dfrac{t}{4}-10=-4$
4. $\dfrac{t-5}{12}=4$
5. $2x-7=-3$
6. $\dfrac{x+8}{3}=-2$
7. $\dfrac{x}{3}+8=-2$
8. $-2x+3=3$
9. $3(x+7)=36$
10. $3x-10=-10$
11. $-4x+6=-18$

Solve each of the following application problems.

12. Paul invested his money on the stock market. First he bet on a risky stock and lost half of his money. Then he became a bit more careful and invested money in more conservative stocks that involved less risk but also less profit. His investments made him 80 dollars. If he has 250 dollars in the stock market today, with how much money did he start investing?
13. In a hotel, the first night costs 45 dollars, and all additional nights cost 35 dollars. How long did Mr. Williams stay in the hotel if his bill was 325 dollars?

Practice Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x-3=-11$
2. $-2x-3=7$
3. $5x-3=17$
4. $\dfrac{x-3}{7}=-2$
5. $\dfrac{x}{7}-3=-1$
6. $-4x-3=13$
7. $\dfrac{a+1}{4}=-9$
8. $5x-6=-6$
9. $\dfrac{x}{7}-1=-3$
10. $-x+5=-7$
11. $\dfrac{2x-1}{7}=-3$
12. $5(x-2)=-20$
13. $\dfrac{x-8}{7}=-2$
14. $3b+13=-5$
15. $\dfrac{x}{3}-7=7$

Solve each of the following application problems.

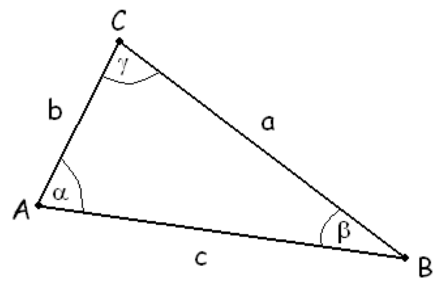
16. Three times the difference of x and 7 is -15. Find x.
17. Ann and Bonnie are discussing their financial situation. Ann said: *If you take* 50 *bucks from me and then doubled what is left, I would have* \$300. Bonnie answers: *That's funny. If you doubled my money first and then took* \$50, *then I would have* \$300! How much money do they each have?
18. Susan was asked about her age. She answered as follows: My big brother's age is six less than three times my age. How old is Susan if her big brother is 21 years old?

7.3 Perimeter and Area of Right Triangles

Part 1 – Standard Labeling

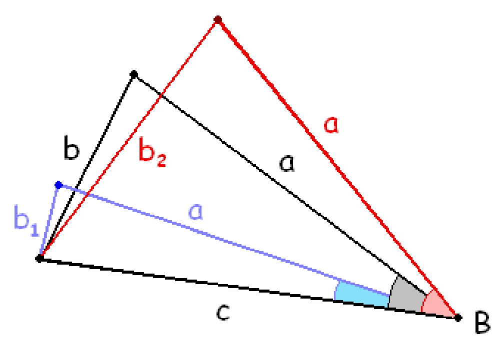
The perimeter of any triangle is simply the sum of the lengths of its three sides. We will have to learn much more for a discussion of the area of right triangles. But before we do that, let us agree first on a method of notation that avoids confusion and lengthy explanations in geometry problems. This agreement is called *standard labeling*, and it establishes a connection between the labels of sides, vertices, and angles in triangles. Every triangle has three of the following three components.

1. **vertices** (singular: vertex)
   Points are usually denoted by uppercase letters. In case of triangles, we often use A, B, and C.
2. **angles**
   Angles are usually denoted by lowercase Greek letters. In case of triangles, we often use α (alpha), β (beta), and γ (gamma).
3. **sides**
   Lines and line segments are usually denoted by lowercase letters. In case of triangles, we often use a, b, and c.



In case of standard labeling, we automatically associate sides, vertices, and angles. A vertex is associated with the angle located at that vertex. These two are associated with the side opposite these. For example, angle α is always assumed to be located at point A, and side a is always assumed to be the side oppsite to point A and angle α. Point B, angle β, and side b are similarly grouped. Unless otherwise indicated, we should always assume standard labeling when presented with data that uses these letters.

Standard labeling is a smart approach to triangles, because there is a natural connection between an angle in a triangle and the side opposite that angle. Consider, for example, the triangle shown above with standard labeling. What if we fixed sides a and c and only modified angle β? Imagine that we have two rods in the lengths of a and c attached to each other at one end and we can freely change the angle between them. If we increase the angle between sides a and c (see the red lines), the side opposite will also increase. If we decrease the angle between sides a and c (see the blue lines), the side opposite will also decrease. So, there seems to be a natural correspondance between side b and angle β.



> **Theorem:** In any triangle ABC, there is a correspondance between the length of a side and the measure of the angle opposite that side:
> The longest side is opposite the greatest angle, and vica versa: the greatest angle is opposite the longest side. The shortest side is opposite the smallest angle, and vica versa: the smallest angle is opposite the shortest side.

So, the order between the three sides is the same as the order between the corresponding angles, and vica versa. We recommend that sides in triangles are tracked by their corresponding sides. This is because we can perceive the difference in angles much better than in side lengths.

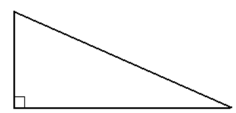
Example 1. Suppose that ABC is a triangle with $\alpha = 82°$ and $\gamma = 39°$. List the length of the sides of the triangle in an increasing order.

Solution: Recall that the three angles in a triangle add up to $180°$. This means that if two angles are given, we can compute the third one. $\beta = 180° - (82° + 39°) = 180° - 121° = 59°$. Now we can see the order between the angles. γ is the smallest angle, β is in the middle, and α is the greatest angle. In short: $\gamma < \beta < \alpha$. The order between the lengths of the sides is the same: c is the shortest side, b is in the middle, and a is the longest side. In short: $c < b < a$.

There is an easy but important consequence of this property.

Theorem: In any triangle ABC, if two angles have equal measure, then the sides opposite them have equal length. If two sides are equally long, then the angles opposite those sides have equal mesaures. Such a triangle is called **isosceles**.

Definition: A **right triangle** is a triangle with a right angle. A right angle measures $90°$.



We should never assume that a triangle is right, unless it is formally stated either in the problem, or on a picture. Just because an angle appears to be a right angle, it could have a measure of $89.5°$, and that is not right. The measure must be exactly $90°$.

Theorem: In a right triangle, there can only be one right angle and it is the greatest angle in the triangle.

Discussion

1. Find an algebraic and geometric argument for the theorem stated above. Why is it true?
2. What does that mean for the sides of a right triangle?

Definition: In any right triangle, there is a single longest side, and it is opposite of the right angle. This side is called the **hypotenuse** of the right triangle. The shorter sides are called **leg**s.

The hypotenuse of a triangle is often denoted by c, but this is just a convenient habit and not a rule! We should never assume that the rigth angle in a triangle is γ. For all we now, it could be any of α, β, or γ.

We took two pages to establish that there is a longest side in a right triangle. The perimeter and area formulas will be very easy.

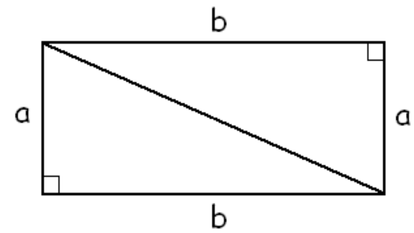
Part 2 - Perimeter and Area Formulas

Recall that the perimeter is just the length of the boundary. This means that the perimeter of any triangle can be computed by adding the lengths of its three sides.

Theorem: The perimeter of a triangle ABC can be computed as $P = a + b + c$.

When we reviewed the area formula for rectangles, we have mentioned that that was a very difficult formula to prove. It is often the case in mathematics that, once we worked very hard for a formula, we use that result over and over. When it comes to area, *all* we know are rectangles. In other words, every area formula was derived from the area formula of the rectangle.

Every right triangle is half of a rectangle. In other words, given any right triangle, we can use two identical copies of it to form a rectangle. We know how to find the area of the rectangle: $A = ab$.
Because the rectangle consists of two identical (also called congruent) right triangles, it naturally follows that each takes up half of the area of the rectangle. Thus the area of the right triangle is $A = \dfrac{ab}{2}$.



Theorem: The area of a right triangle ABC, where c denotes the hypotenuse, is $A = \dfrac{ab}{2}$.

What is unusual about this formula is that we don't need the length of the hypotenuse, only the lengths of the other two sides. This means that given the three sides of a right triangle, we need to know to only use the lengths of the two shorter sides.

Example 2. Compute the perimeter and area of a right triangle with sides 10 ft, 24 ft, and 26 ft.

Solution: The perimeter is just the sum of all three sides.

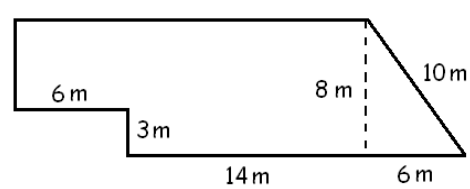
$$P = a + b + c = 10\,\text{ft} + 24\,\text{ft} + 26\,\text{ft} = \boxed{60\,\text{ft}}$$

For the area, we only need the two shorter sides.

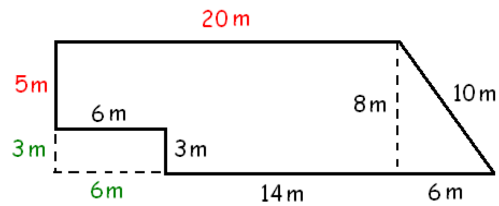
$$A = \frac{ab}{2} = \frac{10\,\text{ft} \cdot 24\,\text{ft}}{2} = \frac{240\,\text{ft}^2}{2} = \boxed{120\,\text{ft}^2}.$$

Although we will derive formulas for the area of more complicated shapes, we can often avoid those by cutting our objects into rectangles and right triangles.

Example 3. Compute the perimeter and area of the figure shown on the picture. Angles that look like right angles are right angles. Units are in meters.



Solution: For the perimeter, we need to find two missing sides. The vertical side is 5 m long since the opposite sides of rectangles are equally long. Similarly, the missing horizontal side is 20 m long. We can now compute the perimeter. Notice that the vertical line in the inside is not part of the boundary. We will add all the sides, starting with the long horizontal side.



$$P = 20\,\text{m} + 10\,\text{m} + 6\,\text{m} + 14\,\text{m} + 3\,\text{m} + 6\,\text{m} + 5\,\text{m} = \boxed{64\,\text{m}}$$

Now for the area: We will compute the area of the big rectangle and from it we will subtract the area of the smaller rectangle.

$$A_1 = 8\,\text{m} \cdot 20\,\text{m} - 3\,\text{m} \cdot 6\,\text{m} = 160\,\text{m}^2 - 18\,\text{m}^2 = 142\,\text{m}^2$$

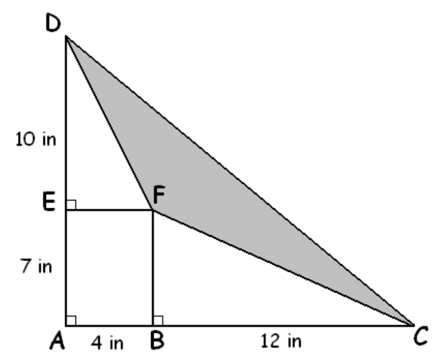
To this we add the area of the right triangle.

$$A_2 = \frac{6\,\text{m} \cdot 8\,\text{m}}{2} = \frac{48\,\text{m}^2}{2} = 24\,\text{m}^2$$

The area of the entire figure is the sum of the two areas:

$$A = A_1 + A_2 = 142\,\text{m}^2 + 24\,\text{m}^2 = \boxed{166\,\text{m}^2}$$

Example 4. Compute area of the shaded region shown on the picture. Units are in inches.



Solution: First we will compute the area of right triangle ACD. $AC = 16\,\text{in}$ and $AD = 17\,\text{in}$.

$$A_{\text{ACD}} = \frac{16\,\text{in} \cdot 17\,\text{in}}{2} = \frac{272\,\text{in}^2}{2} = 136\,\text{in}^2$$

To get the area of the shaded region, we will subtract the areas of the rectangle $ABFE$ and right triangles DEF and BCF.

$$\begin{aligned} A_{\text{ABFE}} &= 7\,\text{in} \cdot 4\,\text{in} = 28\,\text{in}^2 \\ A_{\text{DEF}} &= \frac{10\,\text{in} \cdot 4\,\text{in}}{2} = 20\,\text{in}^2 \\ A_{\text{BCF}} &= \frac{7\,\text{in} \cdot 12\,\text{in}}{2} = \frac{84\,\text{in}^2}{2} = 42\,\text{in}^2 \end{aligned}$$

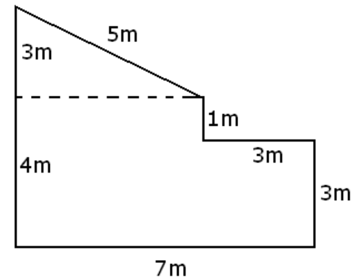
So the shaded area is when we subtract the white areas from the big right triangle.

$$\begin{aligned} A &= A_{\text{ACD}} - (A_{\text{ABFE}} + A_{\text{DEF}} + A_{\text{BCF}}) \\ &= 136\,\text{in}^2 - \left(28\,\text{in}^2 + 20\,\text{in}^2 + 42\,\text{in}^2\right) = 136\,\text{in}^2 - 90\,\text{in}^2 = \boxed{46\,\text{in}^2} \end{aligned}$$

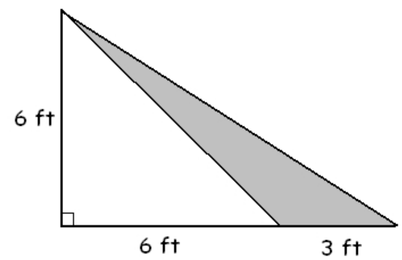
Practice Problems

1. Compute the perimeter and area of a right triangle with sides 7 ft, 25 ft, and 24 ft. Include units in your computation and answer.

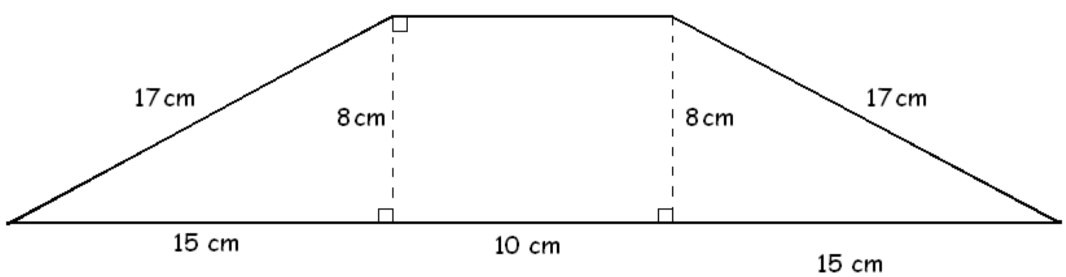
2. Compute the perimeter and area of the figure shown on the picture. Angles that look like right angles are right angles. Units are in meters. Include units in your computation and answer.



3. Compute the area of the shaded region shown on the picture. Include units in your computation and answer.

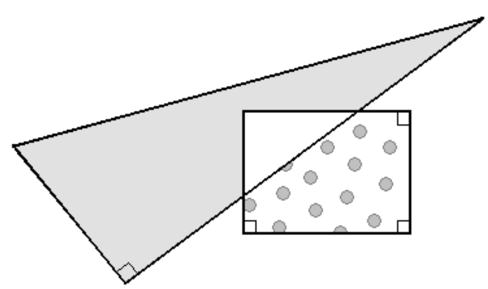


4. Compute the perimeter and area of the figure shown on the picture below. Units are in meters. Include units in your computation and answer.



Enrichment

Consider the figure shown on the picture. The shorter sides of the right triangle are $6\,\mathrm{cm}$ and $10\,\mathrm{cm}$ long. The sides of the rectangle are $5\,\mathrm{cm}$ and $6\,\mathrm{cm}$ long. Which region has the greater area, the shaded or the dotted?



7.4 Problem Set 4

1. Let T be the set of all integers divisible by 2 and S the set of all integers divisible by 6. Label each of the following as true or false.

   a) $-8 \in T$ b) $0 \in S$ c) $T \subseteq S$ d) $S \subseteq T$ e) $T \cup S = S$

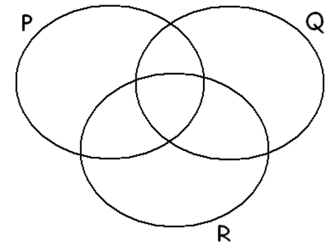
2. Let $A = \{2,3,5,6,8,9,10\}$ and $B = \{1,3,4,8,10\}$. Label the following statements as true or false.

   a) $2 \in A$ b) $2 \in B$ c) $3 \in A \cap B$ d) $9 \notin A \cup B$

3. Let $P = \{1,4,5,9,10\}$, $Q = \{1,2,3,5,8,10\}$, and $R = \{1,2,4,6,9,10\}$. Compute each of the following.

   a) $P \cap Q$ b) $(P \cap Q) \cup R$ c) $P \cap (Q \cup R)$ d) $(P \cap Q) \cap R$ e) $(P \cap R) \cup (Q \cap R)$

4. Shade the region(s) corresponding to the sets identified in the previous problem on a Venn diagram shown with the three sets.



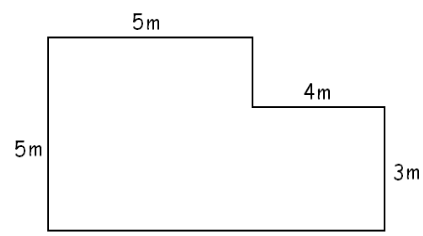
5. List all factors of 60.

6. Compute the perimeter and area of the right triangle with sides 7 in, 25 in, and 24 in long.

7. Consider the object shown on the picture below. Angles that look like right angles are right angles. Dimensions are in meters.

   a) Compute the perimeter of the object. Include units in your computation and answer.

   b) Compute the area of the object. Include units in your computation and answer.



8. Perform the division with reminder. $2018 \div 12$.

9. a) Compute $\frac{3}{7}$ of 420.

   b) Compute $\frac{4}{100}$ of 3500. (This is the same as 4% of 3500.)

10. Saturday was a good day for the bookstore. The store opened with 1200 books in its inventory. Until noon, the store sold $\frac{1}{6}$ of its books. In the afternoon, the store sold $\frac{2}{5}$ of what's left. How many books were in the store at closing time?

11. Simplify each of the following. Show all steps.

    a) $-2^2 - 3\left(5 - (-3)^2\right) - 2\,|3-8|$

    b) $3^2(-2) + |2 - 3\,|-5||$

    c) $-5^2 - 24 \div 2(-3)$

    d) $\dfrac{3 - 2^2(-5)}{(-1)^3 + 6 \div 6}$

    e) $(-3)^2 + \dfrac{-20}{4-9}$

12. Consider the equation $-x^2 + x^3 - x = -2x^2 + 5x$. In each case, determine whether the number given is a solution of the equation or not.

    a) $x = 3$ b) $x = -3$ c) $x = 2$ d) $x = -2$

13. Consider the inequality $\dfrac{2x-1}{3} \leq \dfrac{3x+1}{2} - 5$. In each case, determine whether the number given is a solution of the equation or not.

    a) $x = -7$ b) $x = 5$ c) $x = 17$

14. Solve each of the following equations. Make sure to check your solutions.

a) $x - 17 = -20$

b) $\dfrac{m}{5} = -8$

c) $2x = 0$

d) $x + 8 = -2$

e) $3x - 5 = 22$

f) $\dfrac{y}{-5} - 1 = -4$

g) $\dfrac{y-1}{-5} = -4$

h) $3(a+1) = -15$

i) $4x - 10 = 10$

j) $4x - 10 = -10$

15. The sum of three times a number x and 10 is -8. Find x.

16. Peter and Robert are discussing their financial situation. Peter said: *If you give* 50 *dollars to me and then doubled the total, I would have* 500 dollars. Robert answers: *That's funny. If you doubled my money first and then gave me* 50 dollars, *then I would have* 500! How much money do they each have?

17. Ayesha was asked about her age. She answered as follows: My older brother's age is ten less than twice my age. How old is Ayesha if her older brother is 16 years old?

18*. (Enrichment) Suppose that $X \cap Y = \{1,2,3\}$, and $X \cup Y = \{1,2,3,4,5,6\}$. How many different sets are possible for X?

Appendix A

Answers

Answers for 1.2 – The Words And and Or

Practice Problems

1. true 2. false 3. true 4. true 5. false 6. true 7. false 8. true 9. false
10. false 11. true 12. false 13. true 14. false 15. true 16. false 17. true
18. true 19. true 20. false

Enrichment

1. there are 8 cases

A	B	C	A or B or C
true	true	true	true
true	true	false	true
true	false	true	true
false	true	true	true
true	false	false	true
false	true	false	true
false	false	true	true
false	false	false	false

2. there are 8 cases

A	B	C	A and B and C
true	true	true	true
true	true	false	false
true	false	true	false
false	true	true	false
true	false	false	false
false	true	false	false
false	false	true	false
false	false	false	false

3. there are 8 cases

A	B	C	(A and B) or C
true	true	true	true
true	true	false	true
true	false	true	true
false	true	true	true
true	false	false	false
false	true	false	false
false	false	true	true
false	false	false	false

4. there are 8 cases

A	B	C	A and (B or C)
true	true	true	true
true	true	false	true
true	false	true	true
false	true	true	false
true	false	false	false
false	true	false	false
false	false	true	false
false	false	false	false

Answers for 2.2 – Order of Operations on Integers

Sample Problems

1. 5 2. 8 3. 13 4. 55 5. 25 6. 81 7. 7

Practice Problems

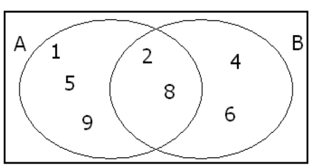
1. 43 2. 51 3. 9 4. 12 5. 4 6. 4 7. 1 8. 40 9. 15 10. 5 11. 3
12. 20 13. 85 14. 2 15. 1

Answers for 2.3 – Problem Set 1

1. a) true b) false c) false d) true
2. a) $1,2,3,4,5$ b) $4,5,6$ c) all natural numbers d) $2,4,6,8,10$
3. a) 9 b) 10 c) 12 d) 5 e) 4 f) 2 g) 8 h) 49 i) 2 j) 12 k) 5 l) 27 m) 1
4. $36-2\cdot\left((5-2)^2+4\right)=10$

Answers for 3.1 - Introduction to Set Theory

Practice Problems

1. a) true b) false c) true d) false e) true
2. 

3. a) true b) false c) true d) false e) true
4. a) $\{1, 2\}$ b) $\{1, 2, 3, 4, 5, 6\}$ c) $\{3, 4, 5, 6, 7\}$ d) $\mathbb{Z}$ (the set of all integers)
5. Y is not an element of set X. Instead, Y is a subset of X, denoted by $Y \subseteq X$.
6. This is naturally true if $A = B$ as every set is a subset of itself. However, if A and B are different sets, at least one of $A \subseteq B$ and $B \subseteq A$ will be false.
7. We will list the subsets of A by organizing by the number of their elements.

0-element subsets:	$\varnothing$	1 subset
1-element subsets:	$\{1\}, \{2\}, \{3\}, \{4\}$	4 subsets
2-element subsets:	$\{1, 2\}$ $\{1, 3\}$ $\{2, 3\}$ $\{1, 4\}$ $\{2, 4\}$ $\{3, 4\}$	6 subsets
3-element subsets:	$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$	4 subsets
4-element subsets:	$\{1, 2, 3, 4\}$	1 subset

So there are 16 subsets.

Answers for 3.2 – Introduction to Number Theory

Practice Problems

1. $1,2,3,4,6,8,12,16,24,48$ 2. 91   3. a) 80, 75, 270   b) 75, 270   c) $128, 80, 64$

Answers for 4.1 – The Set of All Integers

Practice Problems

1. a) true b) true c) false d) true e) true f) true
2. a) false b) true c) true d) false e) true f) true g) false
3. a) $5 > -7$ or $5 \geq -7$ b) $-12 < -4$ or $-12 \leq -4$ c) $0 > -8$ or $0 \geq -8$ d) $-1 > -4$ or $-1 \geq -4$
   e) $-7 \geq -7$ or $-7 \leq -7$
4. a) 5 b) 5 c) -5 d) -5 e) 0 f) 3 g) 21
5. a) 5 b) -3 c) -6 d) -15 e) 0 f) 7 g) undefined h) -4 i) -28 j) -13
   k) 1 l) 8 m) -3 n) 0 o) undefined p) 0 q) 10 r) 8

Answers for 4.2 – Order of Operations on Integers

Practice Problems

1. 43 2. -29 3. 49 4. -45 5. -162 6. 2 7. 0 8. 9 9. 12 10. 22 11. 28
12. -2 13. undefined 14. -16 15. -21 16. 24 17. -4 18. 19 19. -33 20. -17
21. -2 22. -6 23. -8 24. -9 25. 20 26. 3 27. 15 28. -14 29. 0 30. 2
31. -2 32. -4

Enrichment

1. a) 69 and 11   b) 0 and 40

   The pairs are colored differently. If your printout is black and white, look at the document at your computer.

   a) $|5-2|-4+7|-10| = 69$ and $|5-2|-4+7|-10| = 11$

   b) $2|-1-5|-3|-4| = 0$ and $2|-1-5|-3|-4| = 40$

2. a) $5-2\left((-3)^2-\left((4-7)^2+2\right)\right) = 9$ b) $5-\left((2-3)^2-(4-7)\right)^2+2 = -9$

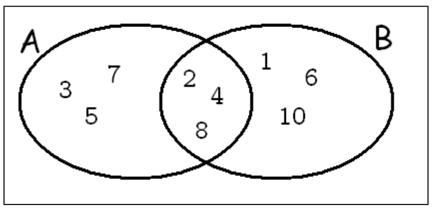
Answers for 4.3 – Division with Remainder

Practice Problems

1. a) 18 R 6 b) 95 R 5 c) 114 R 6 d) 16 R 9 2. a) 7 b) 1 c) 2 d) 9

Answers for 4.4 – Problem Set 2

1. 1, 2, 3, 4, 6, 8, 12, 24
2.



3. a) $\{0, 3, 6, 9, 12, 15, 18\}$ b) $\{0, 1, 2, 3, 4, 5, 6, 7, 10, 15, 20\}$
   c) $\{0, 5\}$ d) U or $\{0, 1, 2, 3, \ldots, 19, 20\}$ e) $\{7, 8, 9, 10, 11\}$
   f) $\{0, 1, 2, 3, 4, 5, 6, 7\}$ g) $\{0, 1, 2, 3\}$ h) $\{0, 4, 8, 12, 16, 20\}$
   i) $\{0, 3, 4, 6, 8, 9, 12, 15, 16, 18, 20\}$ j) $\{0, 12\}$

4. a) i) true ii) false iii) false iv) true
   b) x is a rectangle that is NOT a square, i.e. a rectangle that has two sides with different lengths.

5. a) 24 R 3 b) 7 R 3 c) 288 R 2

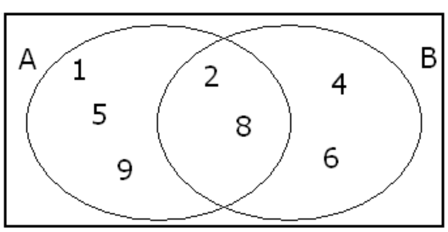
6. a) 27 b) -9 c) 9 d) 4 e) -4 f) -25 g) 0 h) undefined i) 5 j) 11 k) -5 l) 13
   m) 18 n) 25 o) 13 p) 16 q) -40

7. a) 15 b) 24 c) -9 d) -12 e) 25 f) -3 g) undefined h) 1 i) 48 j) -4 k) 1 l) 9

8. 9 9. a) 203 b) 0

Answers for 5.1

1. a) $\{1, 4\}$ b) $\{1, 2, 3, 4, 5, 6, 9\}$ c) $\varnothing$ d) $\{1, 2, 3, 4, 5\}$ or Q
2. a)

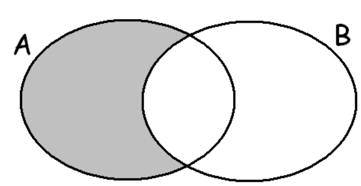


   b) i) $\{2,8\}$ ii) $\{1,2,4,5,6,8,9\}$ iii) $\{2,4,6,8\}$
   c) i) false ii) true iii) true

3. a) $P \cap M$ - the set of all students taking mathematics and physics at Truman College.
   b) $P \cup M$ - the set of all students taking mathematics or physics or both at Truman College.

4. a) true b) false c) false d) true 5. a) true b) true c) true d) true

6. If A is a subset of B, then there is no element that belongs to A but not to B. This will show up on the Venn diagram by the shaded region shown containing no elements.



Answers for 5.2

Discussion

$1\,\text{yd}^2 = 9\,\text{ft}^2$ Algebraically: if $1\,\text{yd} = 3\,\text{ft}$, then $1\,\text{yd}^2 = 1\,\text{yd}\cdot 1\,\text{yd} = 3\,\text{ft}\cdot 3\,\text{ft} = 9\,\text{ft}^2$

Practice Problems

1. $P = 40\,\text{cm}$, $A = 96\,\text{cm}^2$ 2. $P = 56\,\text{ft}$, $A = 159\,\text{ft}^2$ 3. $P = 36\,\text{ft}$, $A = 54\,\text{ft}^2$

Answers for 6.1

Sample Problems

1. a) 22 b) 16 c) -16 d) 16 e) -5 f) 5
2. a) 22 b) -30 c) 1 d) undefined e) 40 f) 100 g) 18 h) 36 i) 17 j) 15 k) -34
3. a) 5 b) 9 4. a) 784 ft b) 208 ft 5. a) 12 b) 36 c) 25 d) 2 e) undefined
6. a) $37 \neq 13$ no b) $55 = 55$ yes c) $70 \neq 76$ no d) $139 = 139$ yes
7. a) $-4 \neq 20$ no b) $12 = 12$ yes c) $8 = 8$ yes d) $-12 \neq 28$ no
8. a) $13 = 13$ yes b) $15 \neq 13$ no
9. a) $-62 \leq -57$ yes b) $16 \not\leq 8$ no c) $-32 \leq -32$ yes d) $-2 \not\leq -7$ no
10. a) $6 \not< 1$ no b) $14 < 19$ yes c) $10 < 10$ no d) $2 < -8$ no
11. a) $y = 4x - 3$ b) $-A = (B - 3C) + 1$ c) $2M = N(-M) - 5$ 12. $5x - 3$
13. a) $x, x+1$, and $x+2$ b) $y-1, y$, and $y+1$ c) $L-2, L-1$, and L

Practice Problems

1. a) 20 b) 16 c) -1 d) 1 e) -11 f) -3 g) 20 h) 4 i) 13 j) 17
2. a) 17 b) 5 c) 35 d) 100 e) 196 f) 18 g) 6 h) 11 i) 26 j) 2
3. a) -1 b) undefined c) 3 d) -7 4. a) -6 b) -2 c) 0 d) -30 e) -20
5. a) -3 b) 4 c) -7 d) undefined e) -4
6. a)

	$x=2$	$x=5$	$x=6$	$x=10$	$x=-1$	$x=-5$	$x=-8$
$2x-3$	1	7	9	17	-5	-13	13
$2x+3$	7	13	15	23	1	-7	-19

b)

	$x=2$	$x=5$	$x=6$	$x=10$	$x=-1$	$x=-5$	$x=-8$
$2x-3$	1	7	9	17	-5	-13	13
$-2x+3$	-1	-7	-9	-17	5	13	-13

7. a) -1 b) -1 c) undefined d) -1 8. a) 11 b) -13 c) 3
9. a) 5 b) 6 c) 7 d) 3 $|a-b|$ always gives us the distance between a and b on the number line
10. a) $1 = 1$ yes b) $6 = 6$ yes c) $3 = 3$ yes

11. a) $0 \neq 2$ no b) $-5 \neq 1$ no c) $3 = 3$ yes d) $0 = 0$ yes e) $-8 \neq 4$ no

12. a) $1 = 1$ yes b) $4 \neq 21$ no c) $6 = 6$ yes d) $41 = 41$ no

13. a) $32 \neq 36$ no b) $40 \neq 32$ no

14. a) $7 \not< -23$ no b) $4 \not< 4$ no c) $2 < 12$ yes d) $-1 < 9$ yes e) $-5 \not< -23$ no

15. a) $-2 \geq -5$ yes b) $0 \geq -2$ yes c) $10 \not\geq 13$ no d) $6 \not\geq 7$ no e) $-4 \geq -8$ yes

16. a) $A - B = A(-B) - 4$ b) $x^2 = 5(-x) - 8$ c) $P - 10 = (Q + 2R) - 5$ d) $4y = 2(y+7) + 1$
e) $x = -x + 10$ f) $(x+y) = x^2 - y^2 + 10$ 17. $3x + 7$

18. a) $x, x+2, x+4$, and $x+6$ b) $x-6, x-4, x-2$, and x

Answers for 6.2 – Problem Set 3

1. a) true b) false c) true d) false e) true f) true g) false h) true i) false j) true k) false

2. $1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90$

3. a) 1189 188 b) 101010, 1189 188, c) 1189 188 d) 1189 188, 35530, 1234 321

4. $2, 3, 5, 7, 11$ 5. 101 and 2017 6. a) 155 R 2 b) 447 R 22 c) 68 R 10 d) 152

7. a) $A \cup B = \{1, 2, 3, 4, 6, 8, 9, 10\}$ $A \cap B = \{4, 8\}$ b) true c) true d) false e) true

8. a) $\{1, 2, 3, 4, 5\}$ b) $\{4, 5, 6\}$ c) $\mathbb{N}$ (all natural numbers) d) $\{2, 4, 6, 8, 10\}$

9. $S \cap T = \{2, 3, 4, 5, 6, 7, 8, 9\}$ and $S \cup T = \mathbb{N}$ (all natural numbers) 10. $B \subseteq A$ 11. $E \cap O = \varnothing$ and $E \cup O = \mathbb{Z}$

12. a) false b) true c) false d) true 13. $\{n \in \mathbb{N} : n \text{ is divisible by } 10\} = \{10, 20, 30, 40, \ldots\}$

14. a) true b) false c) true d) true e) true f) false g) true

15. a) -5 b) 6 c) 1 d) -5 e) 4 f) -8 g) 16 h) -32 i) -4 j) -8 k) -16
l) -32 m) -40 n) 16 o) 58 p) 100 q) 34 r) 15 s) -110 t) -56 u) 50

16. a) 4 b) -4 c) -4 d) -4 e) 20 f) 1 g) -20 h) 1 i) 29 j) 49 k) 27 l) -21 m) 9
n) -23 o) 100 p) -100 q) 1 r) -21 s) 29 t) 49 u) -23 v) 100 w) -100 x) 100

17. a) 72 b) undefined c) 5 d) 10 e) -24 f) 3 g) undefined h) 45 i) 61 j) -1 k) -6
l) -27 m) 14 n) 20 o) 31 p) 12 q) 120

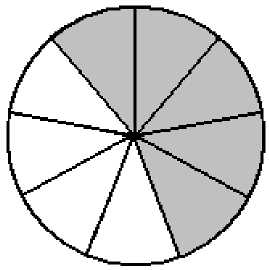
18. a) 3 b) undefined c) 14 d) 12 e) 4 f) -2 19. a) 12 b) 135 c) 5 d) 31 e) 1

20. -2 and 1 21. 0, 3, and 4 22. $P = 24\,\text{m}$, $A = 29\,\text{m}^2$

23. a) $X = 2Y - 3$ b) $-x = \left(y + \frac{z}{2}\right) + 5$ c) $ab = 3(a+b) - 14$ d) $m + 2n = \frac{m}{n} + 1$
e) $3(x-y) = x(-y) - 1$ f) $P = 2D - 10$ g) $2.5 + (x-1)1.5$ h) $x, x+1$, and $x+2$ i) $x(4x-3)$
j) Ann has $A - 30$ dollars and Beatrix has $B + 30$. 24. 5

Answers for 7.1 – Fractions 1 – The definition

1. 16 2. 20 3. See below 4. \$420 5. \$595 6. 42 7. \$2120 8. a) 180 b) 168 c) $\frac{3}{7}$ is larger



9. \$17 280 in taxes and will keep \$36 720. 10. \$2472 11. \$30 000

Answers for 7.2 – One- and Two-Step Equations

Discussion

a) 1 b) 0 c) 0 d) 0

One thing that is unusual in this problem is the idea of cancellation. Cancellation results in 0 or 1, depending on the operation.

Sample Problems

1. 11 2. -5 3. 24 4. 53 5. 2 6. -14 7. -30 8. 0 9. 5 10. 0 11. -4

12. \$340 13. 9 nights

Practice Problems

1. -4 2. -5 3. 4 4. -11 5. 14 6. -4 7. -37 8. 0 9. -14 10. 12

11. -10 12. -2 13. -6 14. -6 15. 42 16. 2 17. Ann has \$200 and Bonnie has \$175

Answers for 7.3 – Perimeter and Area of Right Triangles

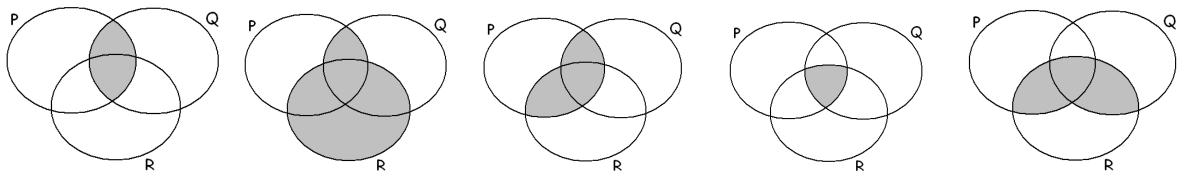
1. $P = 56\,\text{ft}$, $A = 84\,\text{ft}^2$ 2. $P = 26\,\text{m}$, $A = 31\,\text{m}^2$ 3. $A = 9\,\text{ft}^2$ 4. $P = 84\,\text{cm}$, $A = 200\,\text{cm}^2$

Answers for 7.4 – Problem Set 4

1. a) true b) true c) false d) true e) false 2. a) true b) false c) true d) false

3. a) $\{1,5,10\}$ b) $\{1,2,4,5,6,9,10\}$ c) $\{1,4,5,9,10\}$ d) $\{1,10\}$ e) $\{1,2,4,9,10\}$

4. a) b) c) d) e)



5. 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 6. $P = 56\,\text{in}, A = 84\,\text{in}^2$ 7. a) $P = 28\,\text{m}$ b) $A = 37\,\text{m}^2$

8. 168 R 2 9. a) 180 b) 140 10. 600 books

11. a) -2 b) -5 c) 11 d) undefined e) 13

12. a) no, $15 \neq -3$ b) yes, $-33 = -33$ c) yes, $2 = 2$ d) no, $-10 \neq -18$

13. a) no, $-5 \not< -15$ b) yes, $3 \leq 3$ c) yes, $11 < 21$

14. a) -3 b) -40 c) 0 d) -10 e) 9 f) 15 g) 21 h) -6 i) 5 j) 0

15. -6 16. Peter has \$250 and Robert has \$225 17. 13 18. 8

Appendix B

Solutions

B.1 Solutions for 2.2 - The Order of Operations Agreement

Simplify each of the following expressions by applying the order of operations agreement.

1. $2\cdot 3^2 - \left(6^2 - 2\cdot 5\right) \div 2$

   Solution: We start with the parentheses. We will work within the parentheses until the entire expression within it becomes one number. In the parentheses, there is an exponentiation, a subtraction, and a multiplication. Since it is stronger, we start with the exponent. 6^2 in the parentheses.

$$\begin{aligned} 2\cdot 3^2 - \left(6^2 - 2\cdot 5\right) \div 2 &= 2\cdot 3^2 - (36 - 2\cdot 5) \div 2 && \text{next, multiplication within parentheses} \\ &= 2\cdot 3^2 - (36 - 10) \div 2 && \text{subtraction within parentheses} \\ &= 2\cdot 3^2 - (26) \div 2 && \text{we may drop parentheses now} \\ &= 2\cdot 3^2 - 26 \div 2 \end{aligned}$$

   Now that there is no parentheses, we perform all exponents, left to right. There is only one, so we have

$$2\cdot 3^2 - 26 \div 2 = 2\cdot 9 - 26 \div 2$$

   Now we execute all multiplications, divisions, left to right.

$$\begin{aligned} 2\cdot 9 - 26 \div 2 &= 18 - 26 \div 2 && \text{next: division} \\ &= 18 - 13 && \text{subtraction} \\ &= \boxed{5} \end{aligned}$$

2. $18 - 7 - 3$

   Solution: It is a common mistake to subtract 4 from 18. This is not what order of operations tell us to do. The two subtractions have to be performed left to right. The first subtraction is $18 - 7$

$$\begin{aligned} 18 - 7 - 3 &= 11 - 3 && \text{the other subtraction} \\ &= \boxed{8} \end{aligned}$$

3. $5^2 - 2\left(10 - 2^2\right)$

   Solution: We start with the parentheses. Within the parentheses, exponents are the strongest, so we start there

$$\begin{aligned} 5^2 - 2\left(10 - 2^2\right) &= 5^2 - 2(10-4) && \text{subtraction in parentheses} \\ &= 5^2 - 2(6) && \text{drop parentheses} \\ &= 5^2 - 2 \cdot 6 && \text{exponent} \\ &= 25 - 2 \cdot 6 && \text{multiplication} \\ &= 25 - 12 && \text{subtraction} \\ &= \boxed{13} \end{aligned}$$

4. $8^2 - 3^2$

   Solution: There are three operations, two exponents and a subtraction. We start with the exponents, left to right.

$$\begin{aligned} 8^2 - 3^2 &= 64 - 3^2 && \text{exponent} \\ &= 64 - 9 && \text{subtraction} \\ &= \boxed{55} \end{aligned}$$

5. $(8-3)^2$

   Solution: We start with the parentheses. There is only a subtraction there.

$$\begin{aligned} (8-3)^2 &= (5)^2 && \text{drop parentheses} \\ &= 5^2 && \text{exponent} \\ &= \boxed{25} \end{aligned}$$

   This problem and the previous one tells us a very important thing: $a^2 - b^2$ and $(a-b)^2$ are different expressions! In $a^2 - b^2$ we first square a and b and the subtract. In $(a-b)^2$ we first subtract b from a and then square the difference.

6. $\left(3^3 - 4 \cdot 5 + 2\right)^2$

   Solution: We will work within the parentheses until it becomes a number. Within the parentheses, we start with the exponents.

$$\begin{aligned} \left(3^3 - 4 \cdot 5 + 2\right)^2 &= (27 - 4 \cdot 5 + 2)^2 && \text{multiplication within parentheses} \\ &= (27 - 20 + 2)^2 \end{aligned}$$

   There is an addition and a subtraction in the parentheses. **It is not true that addition comes before subtraction!** Addition and subtraction are equally strong; we execute them left to right. So, we start with the subtraction.

$$\begin{aligned} (27 - 20 + 2)^2 &= (7+2)^2 && \text{addition within parentheses} \\ &= (9)^2 && \text{drop parentheses} \\ &= 9^2 && \text{exponents} \\ &= \boxed{81} \end{aligned}$$

7. $\dfrac{3+2\left(20-3^2-5\right)}{3^2-2^2}+4^1$

Solution: The division bar stretching over entire expressions is a case of the **invisible parentheses**. It instruct us to work out the top until we obtain a number, the bottom until we obtain a number, and finally divide. The invisible parentheses here means

$$\frac{3+2\left(20-3^2-5\right)}{3^2-2^2}=\left[3+2\left(20-3^2-5\right)\right]\div\left[3^2-2^2\right]$$

And now we see that the invisible parentheses was developed to simplify notation. We will start with the top. Naturally, we stay within the parentheses until they disappear. We will start with the exponent in the parentheses.

$$\begin{aligned}
\frac{3+2\left(20-3^2-5\right)}{3^2-2^2}+4^1 &= \frac{3+2\left(20-9-5\right)}{3^2-2^2}+4^1 && \text{now subtraction in parentheses, left to right}\\
&= \frac{3+2\left(11-5\right)}{3^2-2^2}+4^1 && \text{other subtraction in parentheses}\\
&= \frac{3+2\left(6\right)}{3^2-2^2}+4^1 && \text{drop parentheses}\\
&= \frac{3+2\cdot 6}{3^2-2^2}+4^1 && \text{multiplication on top}\\
&= \frac{3+12}{3^2-2^2}+4^1 && \text{addition on top}\\
&= \frac{15}{3^2-2^2}+4^1
\end{aligned}$$

Now we work out the bottom, applying order of operations. We start with the exponents, left to right.

$$\begin{aligned}
\frac{15}{3^2-2^2}+4^1 &= \frac{15}{9-2^2}+4^1 && \text{other exponent}\\
&= \frac{15}{9-4}+4^1 && \text{subtraction}\\
&= \frac{15}{5}+4^1 && \text{exponent, } 4^1=4\\
&= \frac{15}{5}+4 && \text{division}\\
&= 3+4 && \text{addition}\\
&= \boxed{7}
\end{aligned}$$

B.2 Solutions for 6.1 – Algebraic Expressions and Statements

1. Evaluate each of the following numerical expressions.

a) $2-5(3-7)$

Solution: We will apply order of operations. First we perform the subtraction in the parentheses.

$$\begin{aligned} 2-5(3-7) &= && \text{subtraction in parentheses} \\ 2-5(-4) &= && \text{multiplication} \\ 2-(-20) &= && \text{subtraction} \\ 2+20 &= \boxed{22} \end{aligned}$$

b) $24-10+2$

Solution: It is NOT true that addition comes before subtraction. Addition and subtraction are equally strong, so between those two, we perform them left to right. First come, first served.

$$24-10+2 = 14+2 = \boxed{16}$$

c) -4^2

Solution: as it was discussed before, -4^2 is quite different from $(-4)^2$. This is $-1\cdot 4^2 = \boxed{-16}$.

d) $(-4)^2$

This is when -4 is squared. So $(-4)^2 = -4(-4) = \boxed{16}$

e) $|3|-|8|$

Solution: We subtract the absolute value of 8 from the absolute value of 3. So $|3|-|8| = 3-8 = \boxed{-5}$

f) $|3-8|$

Solution: This is the absolute value of the difference. Absolute value signs also function of grouping symbols (i.e. parentheses) to overwrite the usual order of operations. So $|3-8| = |-5| = \boxed{5}$

2. Evaluate each of the algebraic expressions when $p=-7$ and $q=3$.

a) $15-p$

Solution:

Step 1. We re-write the expression with one modification: we replace each variable by an empty pair of parentheses.

Step 2. We insert the values into the parentheses. Now the problem becomes an order of operations problem.

Step 3. We drop the unnecessary parentheses and work out the order of operations problem. (It may appear awkward to create these parentheses but they will later become extremely helpful.)

$$\begin{array}{lrcl} \text{Step 1.} & 15-p & = & 15-(\ ) \\ \text{Step 2.} & & = & 15-(-7) \\ \text{Step 3.} & & = & 15+7 \\ & & = & \boxed{22} \end{array}$$

b) $pq-|p-2|$

Solution:

$$\begin{array}{lrcl} \text{Step 1.} & pq-|p-2| & = & (\ )(\ )-|(\ )-2| \\ \text{Step 2.} & & = & (-7)(3)-|-7-2| \\ \text{Step 3.} & & = & -7\cdot 3-|-9| = -7\cdot 3-9 = -21-9 = \boxed{-30} \end{array}$$

c) $4p-q^3$
Solution:

$$\begin{array}{lrcll}
\text{Step 1.} & 4p-q^3 & = & 4(\ \ )-(\ \ )^3 & \\
\text{Step 2.} & & = & 4(-7)-(3)^3 & \\
\text{Step 3.} & & = & 4\cdot 7-3^3 & \text{exponentiation} \\
 & & = & 4\cdot 7-27 & \text{multiplication} \\
 & & = & 28-27 & \text{subtraction} \\
 & & = & \boxed{1} &
\end{array}$$

d) $\dfrac{q^2-p}{2q+p+1}$

Solution:

$$\begin{array}{lrcll}
\text{Step 1.} & \dfrac{q^2-p}{2q+p+1} & = & \dfrac{(\ \ )^2-(\ \ )}{2(\ \ )+(\ \ )+1} & \\
\text{Step 2.} & & = & \dfrac{(3)^2-(-7)}{2(3)+(-7)+1} & \text{drop extra parentheses} \\
 & & = & \dfrac{3^2-(-7)}{2\cdot 3+(-7)+1} & \text{exponent upstairs}
\end{array}$$

$$\begin{array}{lcll}
\text{Step 3.} & = & \dfrac{9-(-7)}{2\cdot 3+(-7)+1} & \text{subtraction upstairs; } 9-(-7)=9+7 \\
 & = & \dfrac{16}{2\cdot 3+(-7)+1} & \text{multiplication} \\
 & = & \dfrac{16}{6+(-7)+1} & \text{additions, left to right} \\
 & = & \dfrac{16}{-1+1}=\dfrac{16}{0} & \text{Division by zero is not allowed!} \\
 & = & \boxed{\text{undefined}} &
\end{array}$$

e) p^2-q^2

Solution:

$$\begin{array}{rcll}
p^2-q^2 & = & (\ \ )^2-(\ \ )^2 & \\
 & = & (-7)^2-(3)^2=(-7)^2-3^2 & \text{exponents,} \\
 & = & 49-3^2 & \text{left to right} \\
 & = & 49-9 & \text{subtraction} \\
 & = & \boxed{40} &
\end{array}$$

f) $(p-q)^2$

Solution:

$$\begin{array}{rcll}
(p-q)^2 & = & [(\ \ )-(\ \ )]^2 & \\
 & = & [(-7)-(3)]^2=[-7-3]^2 & \text{subtraction in parentheses} \\
 & = & (-10)^2 & \text{exponentiation} \\
 & = & \boxed{100} &
\end{array}$$

g) $2q^2$

Solution:

$$\begin{aligned} 2q^2 &= 2(\ \ )^2 \\ &= 2(3)^2 \\ &= 2\cdot 3^2 && \text{exponentiation} \\ &= 2\cdot 9 && \text{multiplication} \\ &= \boxed{18} \end{aligned}$$

h) $(2q)^2$

Solution:

$$\begin{aligned} (2q)^2 &= [2(\ \ )]^2 \\ &= [2(3)]^2 \\ &= (2\cdot 3)^2 && \text{multiplication in parentheses} \\ &= 6^2 && \text{exponents} \\ &= \boxed{36} \end{aligned}$$

i) $15-\dfrac{p+q}{|1-p|}$

Solution: From here on, we show computations **in the form they should appear**. Once you wrote down the expression with little parentheses instead of the letters, you can insert the values into it.

$$\begin{aligned} 15-\frac{p+q}{|1-p|} &= 15-\frac{(\ \ )+(\ \ )}{|1-(\ \ )|} \\ &= 15-\frac{(-7)+(3)}{|1-3|} \\ &= 15-\frac{-7+3}{|1-3|} && \text{invisible parentheses! addition on top} \\ &= 15-\frac{-4}{|1-3|} && \text{subtraction in absolute value sign} \\ &= 15-\frac{-4}{|-2|} && \text{evaluate the absolute value of 2} \\ &= 15-\frac{-4}{2} && \text{division} \\ &= 15-(-2) && \text{subtraction} \\ &= \boxed{17} \end{aligned}$$

j) $(p+q)^2-(5q+2p)^4$
Solution:

$$\begin{aligned}
(p+q)^2-(5q+2p)^4 &= [(-7)+(3)]^2-[5(3)+2(-7)]^4 \\
&= (-7+3)^2-(5\cdot 3+2\cdot(-7))^4 && \text{addition in first parentheses} \\
&= (-4)^2-(5\cdot 3+2\cdot(-7))^4 && \text{multiplications in parentheses} \\
&= (-4)^2-(15+2\cdot(-7))^4 && \text{left to right} \\
&= (-4)^2-(15+(-14))^4 && \text{addition in parentheses} \\
&= (-4)^2-1^4 && \text{exponents, left to right} \\
&= 16-1^4 && \text{careful! } 1^4\neq 4 \\
&= 16-1 && \text{subtraction} \\
&= \boxed{15}
\end{aligned}$$

k) $-p^2-p+8$

Solution:

$$\begin{aligned}
-p^2-p+8 &= -(\ \ )^2-(\ \ )+8 \\
&= -(-7)^2-(-7)+8 \\
&= -49+7+8 \\
&= -42+8=\boxed{-34}
\end{aligned}$$

3. Evaluate the expression $3x^2-x+5$ with the given values of x.

a) $x=0$

Solution: We first copy the entire expression, replacing the letter x by little pairs of paretheses.

$$3x^2-x+5=3(\ \ )^2-(\ \ )+5$$

Then we insert the number 0 into each pair of parentheses.

$$3x^2-x+5=3(0)^2-(0)+5$$

Because we substituted zero, most parentheses are unnecessary. We will drop them:

$$3x^2-x+5=3(0)^2-(0)+5=3\cdot 0^2-0+5$$

Then we solve the resulting order of operations problem. We start with the exponent.

$$\begin{aligned}
3\cdot 0^2-0+5 &= 3\cdot 0-0+5 && \text{perform multiplication} \\
&= 0-0+5 && \text{subtraction} \\
&= 0+5 && \text{addition} \\
&= \boxed{5}
\end{aligned}$$

b) Evaluate $3x^2-x+5$ when $x=-1$.

Solution: We evaluate the expression with $x=-1$.

$$\begin{aligned}
3x^2-x+5 &= 3(-1)^2-(-1)+5 \\
&= 3\cdot 1+1+5=3+1+5=\boxed{9}
\end{aligned}$$

4. We ejected a small object upward from the top of a 720 ft tall building and started measuring time in seconds. We find that t seconds after launching, the vertical position of the object is $-16t^2+64t+720$ feet.

   a) Where is the object 2 seconds after launch?

   b) Where is the object 8 seconds after launch?

   a) Solution: To find out where the object is after 2 seconds, we evaluate the expression with $t=2$.

$$\begin{aligned} -16t^2+64t+720 &= -16\cdot 2^2+64\cdot 2+720 \\ &= -16\cdot 4+64\cdot 2+720 \\ &= -64+128+720=64+720=784 \end{aligned}$$

   So the object is at a height of $\boxed{\text{784 feet}}$ 2 seconds after launch.

   b) Solution: To find out where the object is after 8 seconds, we evaluate the expression with $t=8$.

$$\begin{aligned} -16t^2+64t+720 &= -16\cdot 8^2+64\cdot 8+720 \\ &= -16\cdot 64+512+720 \\ &= -1024+512+720=-512+720=208 \end{aligned}$$

   So the object is at a height of $\boxed{\text{208 feet}}$ exactly 8 seconds after launch.

5. Let $a=-4$, $b=2$, and $x=-3$. Evaluate each of the following expressions.

   a) a^2-b^2
   Solution: First we re-write the expression with one change, we write little pairs of parentheses instead of the letters.

$$a^2-b^2=(\ )^2-(\ )^2$$

   We now write the values inside the parentheses. From here on this is an order of operations problem.

$$\begin{aligned} a^2-b^2 &= (-4)^2-(2)^2 && \text{drop extra parentheses} \\ &= (-4)^2-2^2 && \text{exponents} \\ &= 16-4 && \text{subtraction} \\ &= \boxed{12} \end{aligned}$$

   b) $(a-b)^2$
   Solution: First we re-write the expression with one modification: we write little pairs of parentheses instead of the letters.

$$(a-b)^2=((\ )-(\ ))^2$$

   We now write the values inside the parentheses. From here on this is an order of operations problem.

$$\begin{aligned} (a-b)^2 &= ((-4)-(2))^2 && \text{drop extra parentheses} \\ &= (-4-2)^2 && \text{subtraction in parentheses} \\ &= (-6)^2 && \text{exponent} \\ &= \boxed{36} \end{aligned}$$

   This and the previous problem are here to remind you that $(a-b)^2$ and a^2-b^2 are two different expressions.

c) $a^b - 2bx - x^2 - 2x$

Solution: First we re-write the expression with one modification only: we write little pairs of parentheses instead of the letters.

$$a^b - 2bx - x^2 - 2x = (\ \ )^{(\ \ )} - 2(\ \ )(\ \ ) - (\ \ )^2 - 2(\ \ )$$

We now write the values inside the parentheses. From here on this is an order of operations problem.

$a^b - 2bx - x^2 - 2x =$

$$\begin{aligned}
&= (\ \ )^{(\ \ )} - 2(\ \ )(\ \ ) - (\ \ )^2 - 2(\ \ ) \\
&= (-4)^{(2)} - 2(2)(-3) - (-3)^2 - 2(-3) && \text{drop extra parentheses} \\
&= (-4)^2 - 2\cdot 2(-3) - (-3)^2 - 2(-3) && \text{exponents, left to right} \\
&= 16 - 2\cdot 2(-3) - (-3)^2 - 2(-3) \\
&= 16 - 2\cdot 2(-3) - 9 - 2(-3) && \text{multiplications, left to right} \\
&= 16 - 4(-3) - 9 - 2(-3) \\
&= 16 - (-12) - 9 - 2(-3) \\
&= 16 - (-12) - 9 - (-6) && \text{additions, subtractions, left to right} \\
&= 16 + 12 - 9 - (-6) \\
&= 28 - 9 - (-6) \\
&= 19 - (-6) = 19 + 6 = \boxed{25}
\end{aligned}$$

d) $\dfrac{-x^2 + (x+2)^2}{(x-1)}$

Solution: First we re-write the expression with only one modification: we write little pairs of parentheses instead of the letters.

$$\frac{-x^2 + (x+2)^2}{(x-1)} = \frac{-(\ \ )^2 + ((\ \ ) + 2)^2}{((\ \ ) - 1)}$$

We now write the values inside the parentheses. From here on this is an order of operations problem.

$$\begin{aligned}
\frac{-x^2 + (x+2)^2}{(x-1)} &= \frac{-(-3)^2 + ((-3)+2)^2}{((-3)-1)} && \text{drop parentheses} \\
&= \frac{-(-3)^2 + (-3+2)^2}{(-3-1)} && \text{addition in parentheses upstairs} \\
&= \frac{-(-3)^2 + (-1)^2}{(-3-1)} && \text{subtraction downstairs in parentheses} \\
&= \frac{-(-3)^2 + (-1)^2}{(-4)} && \text{drop parentheses} \\
&= \frac{-(-3)^2 + (-1)^2}{-4} && \text{exponents upstairs} \\
&= \frac{-9+1}{-4} && \text{addition} \\
&= \frac{-8}{-4} && \text{division} \\
&= \boxed{2}
\end{aligned}$$

e) $\dfrac{x-1}{x+3}$

Solution: First we re-write the expression with only one modification: we write little pairs of parentheses instead of the letters.

$$\frac{x-1}{x+3}=\frac{(\ )-1}{(\ )+3}$$

We write the values inside the parentheses and evaluate the expression.

$$\frac{x-1}{x+3}=\frac{(-3)-1}{(-3)+3}=\frac{-4}{0}=\boxed{\text{undefined}}$$

6. Consider the equation $2x^2+x+34=21x-8$. In case of each number given, determine whether it is a solution of the equation or not.

a) $x=1$

Solution: We need to substitute 1 for x into both sides of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

If $x=1$, then			
	$\text{LHS}=2(1)^2+(1)+34$ $=2\cdot 1+1+34$ $=2+1+34=37$	$\text{RHS}=21(1)-8$ $=21\cdot 1-8$ $=13$	$2x^2+x+34=21x-8$ becomes $37=13$ This is false.

So 1 is not a solution of the equation.

b) $x=3$

Solution: We need to substitute 3 for x into both the left-hand side and right-hand side of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

If $x=3$, then			
	$\text{LHS}=2\cdot 3^2+3+34$ $=2\cdot 9+3+34$ $=18+3+34$ $=21+34=55$	$\text{RHS}=21\cdot 3-8$ $=63-8$ $=5$	$2x^2+x+34=21x-8$ becomes $55=55$ This is true!

So 3 is a solution of the equation.

c) $x=4$

Solution: We will substitute 4 for x in both sides of the equation and compare the values.

If $x=4$, then			
	$\text{LHS}=2\cdot 4^2+4+34$ $=2\cdot 16+4+34$ $=32+4+34$ $=36+34=70$	$\text{RHS}=21\cdot 4-8$ $=84-8$ $=76$	$2x^2+x+34=21x-8$ becomes $70=76$ This is false.

Therefore 4 is not a solution of the equation.

d) $x = 7$

Solution: We will substitute 7 for x into both sides of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

If $x = 7$, then	LHS	RHS	
	$\text{LHS} = 2 \cdot 7^2 + 7 + 34$ $= 2 \cdot 49 + 7 + 34$ $= 98 + 7 + 34$ $= 105 + 34 = 139$	$\text{RHS} = 21 \cdot 7 - 8$ $= 147 - 8$ $= 139$	$2x^2 + x + 34 = 21x - 8$ becomes $139 = 139$ This is true!

Therefore 7 is a solution of the equation.

7. Consider the equation $x^2 - 10x + x^3 - 4 = 4(x+5)$. In each case, determine whether the number given is a solution of the equation or not.

a) $x = 0$

Solution: We simply evaluate both sides of the equation when $x = 0$.

If $x = 0$, then	LHS	RHS	
	$\text{LHS} = 0^2 - 10 \cdot 0 + 0^3 - 4$ $= 0 - 0 + 0 - 4$ $= -4$	$\text{RHS} = 4(0+5)$ $= 4 \cdot 5$ $= 20$	$x^2 - 10x + x^3 - 4 = 4(x+5)$ becomes $-4 = 20$ This is false.

Therefore 0 is not a solution of the equation.

b) $x = -2$

Solution: We simply evaluate both sides of the equation when $x = -2$.

If $x = -2$, then	LHS	RHS	
	$\text{LHS} = (-2)^2 - 10(-2) + (-2)^3 - 4$ $= 4 - 10(-2) + (-8) - 4$ $= 4 + 20 - 8 - 4$ $= 24 - 8 - 4 = 12$	$\text{RHS} = 4(-2+5)$ $= 4 \cdot 3$ $= 12$	$x^2 - 10x + x^3 - 4 = 4(x+5)$ becomes $12 = 12$ This is true.

Therefore -2 is a solution of the equation.

c) $x = -3$

Solution: We simply evaluate both sides of the equation when $x = -3$.

If $x = -3$, then	LHS	RHS	
	$\text{LHS} = (-3)^2 - 10(-3) + (-3)^3 - 4$ $= 9 - 10(-3) + (-27) - 4$ $= 9 + 30 - 27 - 4$ $= 39 - 27 - 4$ $= 12 - 4 = 8$	$\text{RHS} = 4(-3+5)$ $= 4 \cdot 2$ $= 8$	$x^2 - 10x + x^3 - 4 = 4(x+5)$ becomes $8 = 8$ This is true.

Therefore -3 is a solution of the equation.

8. Consider the equation $3a-2b-1=(a-b)^2+4$. In case of each pair of numbers given, determine whether it is a solution of the equation or not.

   a) $a=8$ and $b=5$ or, as an ordered pair, $(8,5)$

   Solution: We will substitute $a=8$ and $b=5$ into both sides of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

If $a=8$ and $b=5$, then	$\text{LHS}=3\cdot 8-2\cdot 5-1$ $=24-2\cdot 5-1$ $=24-10-1$ $=14-1=13$	$\text{RHS}=(8-5)^2+4$ $=3^2+4$ $=9+4$ $=13$	$3a-2b-1=(a-b)^2+4$ becomes $13=13$ This is true!

   Therefore $(8,5)$ is a solution of the equation.

   b) $a=10$ and $b=7$

   Solution: We need to substitute $a=10$ and $b=7$ into both sides of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

If $a=10$ and $b=7$, then	$\text{LHS}=3\cdot 10-2\cdot 7-1$ $=30-2\cdot 7-1$ $=30-14-1$ $=16-1=15$	$\text{RHS}=(10-7)^2+4$ $=3^2+4$ $=9+4$ $=13$	$3a-2b-1=(a-b)^2+4$ becomes $15=13$ This is false.

   Therefore $(10,7)$ is not a solution of the equation.

9. Consider the inequality $3(2y-1)+1\leq 5y-7$. In case of each number given, determine whether it is a solution of the inequality or not.

   a) $y=-10$

   Solution: We need to substitute $y=-10$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than or equal to the right-hand side.

If $y=-10$, then	$\text{LHS}=3(2(-10)-1)+1$ $=3(-20-1)+1$ $=3(-21)+1$ $=-63+1=-62$	$\text{RHS}=5(-10)-7$ $=-50-7$ $=-57$	$3(2y-1)+1\leq 5y-7$ becomes $-62\leq -57$ This is true.

   Thus -10 is a solution of the inequality.

   b) $y=3$

   Solution: We need to substitute $y=3$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than or equal to the right-hand side.

If $y=3$, then	$\text{LHS}=3(2\cdot 3-1)+1$ $=3(6-1)+1$ $=3\cdot 5+1$ $=15+1=16$	$\text{RHS}=5\cdot 3-7$ $=15-7$ $=8$	$3(2y-1)+1\leq 5y-7$ becomes $16\leq 8$ This is false.

Thus 3 is not a solution of the inequality.

c) $y = -5$

Solution: We need to substitute $y = -5$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than or equal to the right-hand side.

If $y = -5$, then	$\begin{aligned} \text{LHS} &= 3(2(-5)-1)+1 \\ &= 3(-10-1)+1 \\ &= 3(-11)+1 \\ &= -33+1 = -32 \end{aligned}$	$\begin{aligned} \text{RHS} &= 5(-5)-7 \\ &= -25-7 \\ &= -32 \end{aligned}$	$3(2y-1)+1 \le 5y-7$ becomes $-32 \le -32$ This is true.

Thus -5 is a solution of the inequality.

d) $y = 0$

Solution: We need to substitute $y = 0$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than or equal to the right-hand side.

If $y = 0$, then	$\begin{aligned} \text{LHS} &= 3(2\cdot 0-1)+1 \\ &= 3(0-1)+1 \\ &= 3(-1)+1 \\ &= -3+1 = -2 \end{aligned}$	$\begin{aligned} \text{RHS} &= 5\cdot 0-7 \\ &= 0-7 \\ &= -7 \end{aligned}$	$3(2y-1)+1 \le 5y-7$ becomes $-2 \le -7$ This is false.

Thus 0 is not a solution of the inequality.

10. Consider the inequality $\dfrac{2x+1}{3} + 5 < \dfrac{3x-1}{2}$. In case of each number given, determine whether it is a solution of the inequality or not.

a) $x = 1$

Solution: We need to substitute $x = 1$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than the right-hand side. The left-hand side:

$$\text{LHS} = \frac{2\cdot 1+1}{3} + 5 = \frac{2+1}{3} + 5 = \frac{3}{3} + 5 = 1 + 5 = 6$$

The right-hand side:

$$\text{RHS} = \frac{3\cdot 1-1}{2} = \frac{3-1}{2} = \frac{2}{2} = 1$$

So the statement $\dfrac{2x+1}{3} + 5 < \dfrac{3x-1}{2}$ becomes $6 < 1$. Since this is a false statement, $x = 1$ is not a solution of the inequality.

b) $x = 13$

Solution: We need to substitute $x = 13$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than the right-hand side. The left-hand side:

$$\text{LHS} = \frac{2(13)+1}{3} + 5 = \frac{26+1}{3} + 5 = \frac{27}{3} + 5 = 9 + 5 = 14$$

The right-hand side:

$$\text{RHS} = \frac{3(13)-1}{2} = \frac{39-1}{2} = \frac{38}{2} = 19$$

So the statement $\frac{2x+1}{3} + 5 < \frac{3x-1}{2}$ becomes $14 < 19$. Since this is a true statement, $x = 13$ is a solution of the inequality.

c) $x = 7$

Solution: We need to substitute $x = 7$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than the right-hand side. The left-hand side:

$$\text{LHS} = \frac{2(7)+1}{3} + 5 = \frac{14+1}{3} + 5 = \frac{15}{3} + 5 = 5 + 5 = 10$$

The right-hand side:

$$\text{RHS} = \frac{3(7)-1}{2} = \frac{21-1}{2} = \frac{20}{2} = 10$$

So the statement $\frac{2x+1}{3} + 5 < \frac{3x-1}{2}$ becomes $10 < 10$. Since this is a false statement, $x = 7$ is not a solution of the inequality.

d) $x = -5$

Solution: We need to substitute $x = -5$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than the right-hand side. The left-hand side:

$$\text{LHS} = \frac{2(-5)+1}{3} + 5 = \frac{-10+1}{3} + 5 = \frac{-9}{3} + 5 = -3 + 5 = 2$$

The right-hand side:

$$\text{RHS} = \frac{3(-5)-1}{2} = \frac{-15-1}{2} = \frac{-16}{2} = -8$$

So the statement $\frac{2x+1}{3} + 5 < \frac{3x-1}{2}$ becomes $2 < -8$. Since this is a false statement, $x = -5$ is not a solution of the inequality.

11. Translate each of the following statements to an algebraic statement.

a) y is three less than four times x

Solution: "*four times* x" can be translated as $4x$.

Then we have: "y *is three less than* $4x$", which can be translated as $\boxed{y = 4x - 3}$.

b) The opposite of A is one greater than the difference of B and three times C.

Solution: "*The opposite of* A" can be translated as $-A$. "*three times* C" can be translated as $3C$.

Now we have: "$-A$ *is one greater than the difference of* B *and* $3C$".

"*The difference of* B *and* $3C$" can be translated as $B - 3C$. It is important to write them in the order they are mentioned; first come first served.

Now we have: "$-A$ *is one greater than* $B - 3C$". This can be translated as $\boxed{-A = (B - 3C) + 1}$.

The parentheses turns out to be unnecessary, but it does make sense here. We are adding 1 to the entire difference $B - 3C$, not just $-3C$.

c) Twice M is five less than the product of N and the opposite of M.

Solution: "*Twice* M" can be translated to $2M$ and "*the opposite of* M" can be translated as $-M$.

Now we have: "$2M$ *is five less than the product of* N *and* $-M$".

"*The product of* N *and* $-M$" can be translated as $N(-M)$ or $N \cdot (-M)$.

So now we have: "$2M$ *is five less than* $N(-M)$". This can be translated as $\boxed{2M = N(-M) - 5}$.

12. The longer side of a rectangle is three units shorter than five times the shorter side. If we label the shorter side by x, how can we express the longer side in terms of x?

Solution: The longer side is three less than five times the shorter side. (The shorter side is x.)

So, the longer side is three less than five times x.

Or, the longer side is three less than $5x$.

Or, the longer side is $\boxed{5x - 3}$.

13. Suppose we have three consecutive integers.

a) Express them in terms of x if x denotes the smallest number.

Solution: To get from one consecutive integer to the next one, we simply need to add 1. So the middle number is $x + 1$ and the largest number we obtain by adding 1 again, so we get $x + 2$. The answer is $\boxed{x, x+1, x+2}$.

b) Express them in terms of y if y denotes the number in the middle.

Solution: then the smallest number is one less than y, which is $y - 1$, and the largest number is one greater than y, which is $y + 1$. So the answer is $\boxed{y - 1, y, y + 1}$.

c) Express them in terms of L if L denotes the greatest number.

Solution: Then we have to subtract one to get to the middle number, and two to get to the smallest number. So the answer is $\boxed{L - 2, L - 1, L}$.

B.3 Solutions for 7.2– One- and Two-Step Equations

Solve each of the following equations. Make sure to check your solutions.

1. $2x-5=17$

   Solution:

$$\begin{aligned} 2x-5 &= 17 && \text{add 5 to both sides} \\ 2x &= 22 && \text{divide by 2} \\ x &= 11 \end{aligned}$$

   We check: if $x=11$, then

$$\text{RHS} = 2(11)-5 = 22-5 = 17 = \text{LHS}$$

   Thus our solution, $\boxed{x=11}$ is correct.

2. $\dfrac{a-10}{5} = -3$

   Solution:

$$\begin{aligned} \frac{a-10}{5} &= -3 && \text{multiply both sides by 5} \\ a-10 &= -15 && \text{add 10 to both sides} \\ a &= -5 \end{aligned}$$

   We check: if $a=-5$, then

$$\text{LHS} = \frac{-5-10}{5} = \frac{-15}{5} = -3 = \text{RHS}$$

   Thus our solution, $\boxed{a=-5}$ is correct.

3. $\dfrac{t}{4} - 10 = -4$

   Solution:

$$\begin{aligned} \frac{t}{4} - 10 &= -4 && \text{add 10 to both sides} \\ \frac{t}{4} &= 6 && \text{multiply both sides by 4} \\ t &= 24 \end{aligned}$$

   We check: if $t=24$, then

$$\text{RHS} = \frac{t}{4} - 10 = \frac{24}{4} - 10 = 6-10 = -4 = \text{LHS}$$

   Thus our solution, $\boxed{t=24}$ is correct.

4. $\dfrac{t-5}{12}=4$

Solution:

$$\begin{aligned} \frac{t-5}{12} &= 4 && \text{multiply both sides by 12} \\ t-5 &= 48 && \text{add 5 to both sides} \\ t &= 53 \end{aligned}$$

We check: if $t=53$, then $\text{LHS}=\dfrac{53-5}{12}=\dfrac{48}{12}=4=\text{RHS}.$

Thus our solution, $\boxed{t=53}$ is correct.

5. $2x-7=-3$

Solution: We apply all operations to both sides.

$$\begin{aligned} 2x-7 &= -3 && \text{add } 7 \\ 2x &= 4 && \text{divide by 2} \\ x &= 2 \end{aligned}$$

We check: if $x=2$, then

$$\text{LHS}=2(2)-7=4-7=-3=\text{RHS}$$

Thus our solution, $\boxed{x=2}$ is correct.

6. $\dfrac{x+8}{3}=-2$

Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x+8}{3} &= -2 && \text{multiply by } 3 \\ x+8 &= -6 && \text{subtract } 8 \\ x &= -14 \end{aligned}$$

We check: $\text{LHS}=\dfrac{-14+8}{3}=\dfrac{-6}{3}=-2=\text{RHS}$

Thus our solution, $\boxed{x=-14}$ is correct.

7. $\dfrac{x}{3}+8=-2$

Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x}{3}+8 &= -2 && \text{subtract } 8 \\ \frac{x}{3} &= -10 && \text{multiply by } 3 \\ x &= -30 \end{aligned}$$

We check:

$$\text{LHS}=\frac{-30}{3}+8=-10+8=-2=\text{RHS}$$

Thus our solution, $\boxed{x=-30}$ is correct.

8. $-2x+3=3$

Solution: We apply all operations to both sides.

$$\begin{aligned} -2x+3 &= 3 && \text{subtract } 3 \\ -2x &= 0 && \text{divide by } -2 \\ x &= 0 \end{aligned}$$

We check: if $x=0$, then

$$\text{LHS} = -2\cdot 0+3 = 0+3 = 3 = \text{RHS}$$

Thus our solution, $\boxed{x=0}$ is correct.

9. $3(x+7)=36$

Solution: We apply all operation to both sides,

$$\begin{aligned} 3(x+7) &= 36 && \text{divide by } 3 \\ x+7 &= 12 && \text{subtract } 7 \\ x &= 5 \end{aligned}$$

We check: if $x=5$, then

$$\text{LHS} = 3(5+7) = 3\cdot 12 = 36 = \text{RHS}$$

Thus our solution, $\boxed{x=5}$ is correct.

10. $3x-10=-10$

Solution:

$$\begin{aligned} 3x-10 &= -10 && \text{add } 10 \text{ to both sides} \\ 3x &= 0 && \text{divide by } 3 \\ x &= 0 \end{aligned}$$

We check: if $x=0$, then

$$\text{LHS} = 3\cdot 0-10 = 0-10 = -10 = \text{RHS}$$

Thus our solution, $\boxed{x=0}$ is correct.

11. $-4x+6=-18$

Solution:

$$\begin{aligned} -4x+6 &= -18 && \text{subtract } 6 \\ -4x &= -24 && \text{divide by } -4 \\ x &= 6 \end{aligned}$$

We check:

$$\text{RHS} = -4x+6 = -4\cdot 6+6 = -24+6 = -18 = \text{LHS}$$

Thus our solution, $\boxed{x=6}$ is correct.

12. Paul invested his money on the stock market. First he bet on a risky stock and lost half of his money. Then he became a bit more careful and invested money in more conservative stocks that involved less risk but also less profit. His investments made him 80 dollars. If he has 250 dollars in the stock market today, with how much money did he start investing?

    Solution: Let us denote the amount of money with which Paul started to invest by x. First he lost half of his money, so he had $\frac{x}{2}$. Then he gained 80 dollars end ended up with 250 dollars. So, we can write the equation $\frac{x}{2}+80=250$. We will solve this two-step equation for x. What happened to the unknown was first division by 2 and then addition of 80. To reverse these operations, we will first subtract 80 and then multiply by 2.

$$\begin{aligned} \frac{x}{2}+80 &= 250 && \text{subtract 80} \\ \frac{x}{2} &= 170 && \text{multiply by 2} \\ x &= 340 \end{aligned}$$

    So Paul started with 340 dollars. We check: If we lose half of 340,dollars we have 170 dollars left. Then when we add 80,dollars we indeed end up with 250 dollars. So our solution is correct, $\boxed{\text{Paul started with 340 dollars}}$.

13. In a hotel, the first night costs 45 dollars, and all additional nights cost 35 dollars. How long did Mr. Williams stay in the hotel if his bill was 325 dollars?

    Solution: Suppose that Mr. Williams stayed for the first night and the an additional x many nights. Then the bill would be $45+x\cdot 35$ or $35x+45$. So we write and then solve the equation $35x+45=325$.

$$\begin{aligned} 35x+45 &= 325 && \text{subtract 45} \\ 35x &= 280 && \text{divide by 35} \\ x &= 8 \end{aligned}$$

    Thus Mr. Williams stayed in the hotel for 9 nights. Why not 8 if we got $x=8$? Remember, the first night was counted separately; there was the first night and then $x=8$ additional nights. This is why it is a good idea to read the text of the problem one more time before we state our final answer. So $\boxed{\text{Mr. Williams stayed 9 nights in the hotel}}$. We check: the bill for 9 nights would be $45+8(35)=325$, and so our solution is correct.