

- A 1. Rewriting the equation as $y = (-2/3)x + 2$ gives $m = -2/3$, $b = 2$, so the desired equation is $y = 2x - 2/3$ or $6x - 3y = 2$.
- B 2. $g(f(-4)) - f(g(-4)) = g(20) - f(-2) = 22 - 6 = 16$.
- A 3. $\triangle BCD$ is isosceles with vertex angle 50° , so $m\angle BDC = m\angle DBC = (180^\circ - 50^\circ)/2 = 65^\circ$.
- C 4. Solving the equations $1A + 2B = 16$ (for fiber) and $4A + 3B = 39$ (for protein) simultaneously yields $A = 6$ slices, $B = 5$ slices, for a total of 11 slices of bread.
- D 5. Letting $\log_8 128 = x$ means $8^x = (2^3)^x = 2^{3x} = 128 = 2^7$, so $3x = 7$ and $x = 7/3$.
- B 6. Diagonal JN separates the vertices into 2 disjoint sets, $\{K, L, M\}$ and $\{O, P, Q, R, S\}$. The first set produces only one diagonal, KM, and the second set produces 6, OQ, OR, OS, PR, PS, and QS.
- A 7. Solving the equations simultaneously produces $x = 5$, $y = -3$. The equation of a line through the origin passing through $(5, -3)$ is $3x + 5y = 0$.
- D 8. The stamps not on the border form an 8×8 square, so the required probability is $64/100$ or 0.64 .
- LALHMJ 9. The original substitution changes AMATYC into 1,13,1,20,25,3, which the function transforms into 12,1,12,8,13,10, which becomes LALHMJ.
- C 10. Answer A is eliminated because it doesn't have an amplitude of 2, and of the remaining choices, only C passes through the point $(0, 2)$ as the original function does.
- D 11. The requirement is satisfied by all multiples of 10 (10, 20, 30, 40, ..., 90), all multiples of 9 (18, 27, 36, ..., 81, 99), and also 12, 21, 42, 48, and 84, for a total of $9 + 9 + 5 = 23$.
- E 12. Let L and W be each rectangle's length and width, so that $2L + 2W = 20$ and $L + W = 10$. The given square's side is $L + W$, so its area must be 10^2 or 100.
- D 13. $(2^{2002})(5^{2005}) = (2^{2002})(5^{2002})(5^3) = (10^{2002})(5^3) = 125000...000$, whose digits sum to $1 + 2 + 5$.
- E 14. Let R be the number of red cards in the first pile and $7R$ the number of black cards in that pile.
- Then there are $26 - R$ red and $26 - 7R$ black cards in the 2nd pile, with $26 - R$ an integer multiple of $26 - 7R$. Checking $R = 1, 2, 3$ (the only possible values) yields $R = 2$, so the 2nd pile has $26 - 2 = 24$ red cards.
- E 15. Suppose each person eats an amount s of food daily, so the food supply is $640s$. If n is the number of people sent, they need $5sn$ amount of food, leaving $640s - 5sn$ amount of food behind to last $80 - n$ people for 10 days. Then $640s - 5sn = 10s(80 - n)$, $640 - 5n = 10(80 - n)$, and $n = 160/5 = 32$ people.
- D 16. Let $x = 100a + 10b + c$ ($1 \leq a \leq 9$, $0 \leq b \leq 9$, $1 \leq c \leq 9$), so $y = 100c + 10b + a$ and $x + y = 101a + 20b + 101c = 101(a + c) + 20b$. Now $a + c$ can take on any integer value from 2 to 18, and b can take any integer value from 0 to 9, so there are $17(10) + 10 = 170$ possible values of z .
- C 17. First note that m and n must both be positive, because otherwise $f(x)$ would have a horizontal, not an oblique, asymptote. Also, for $f(x)$ to have an oblique asymptote, $m = n + 1$. Finally, the denominator must be 0 for $x = 1$ and $x = -1$, while the numerator must be 0 only at $x = 1$. Then the denominator must be $x^2 - 1$ and the numerator $x^3 - 1$, so $mn = 6$.
- D 18. Let D be the origin, A the point $(0, 30)$, and C the point $(30, 0)$. Then the line through D and M has equation $y = 2x$, the line through A and C has equation $x + y = 30$, and the two lines intersect at $(10, 20)$, which is point E. The required area is the area of $\triangle ABC$ (with altitude and base both 30) minus the area of $\triangle AEM$ (with base 15 and altitude $30 - 20$), which equals $30(30)/2 - 10(15)/2$, or 375.
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- B 19. To be tangent, the parabolas can only intersect in one point. Thus $x(r - x) = x^2 + 10$ (or equivalently,

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- B 1. Since $b^{2.5} = 12$, $b^5 = 144$. Of the choices, $2.5^5 = 3125/32 < 100$ and $3^5 = 243$, so checking 2.7^5 yields 143.48907, which is the answer.
- A 2. The given information implies $200a + 3b = c$, so $3b - (-200)a = c$, and $(3, -200)$ is a solution.
- E 3. $i^{-321} = 1/i^{321} = i^3/i^{324} = i^3/(i^4)^{81} = -i/1^{81} = -i$.
- D 4. $m\angle SLM = 180^\circ - 30^\circ - 40^\circ = 110^\circ$, so $m\angle SLP = 55^\circ$ and $m\angle SPL = 180^\circ - 55^\circ - 30^\circ = 95^\circ$.
- C 5. Let P be the midpoint of side MH and let s be the length of a side of the square. Then $PA = PT = 10$ (since they are both radii), and $s^2 + s^2/4 = 10^2$. Thus $5s^2/4 = 100$ and $s^2 = 80$, the square's area.
- A 6. Three 7 g weights is 21 g, while two 11 g weights is 22 g, which will balance with the 7 g weights plus the 1 g mass. These five weights are the minimum number.
- A 7. If $P(x+2) = 3x+1 = 3(x+2)-5$, then $P(x) = 3x-5$ and $P(x+3) = 3(x+3)-5 = 3x+4$. The equation $3x+4 = 13$ is satisfied by $x = 3$.
- B 8. Squaring both sides of the equation and subtracting x^2 from both sides yields $x^3 - 2x^2 - x + 2 = 0$. But $x^3 - 2x^2 - x + 2 = x^2(x-2) - (x-2) = (x^2-1)(x-2)$, which has the zeroes $\pm 1, 2$. Only 1 and 2 check into the original equation.
- E 9. The consecutive integers from 5 to 13 sum to 81, but these 9 numbers fill the rows of the square, so each row (and hence each column and long diagonal) sums to 27. Only 9 is involved in four sums of 27, so 9 is the center number. Then 12 is in the upper left, 10 is in the lower right, and $*$ = 11.
- 48 10. If my age is $10t + u$, its reverse is $10u + t$. Then $(10u + t)/3 + 20 = 10t + u$, $10u + t + 60 = 30t + 3u$, and $29t = 7u + 60$. The only single digit values of t and u satisfying this equation are $t = 4$ and $u = 8$.
- D 11. The given series $= -1 + 2 - 3 + 4 - 5 + \dots + 2002 - 2003$. This $= 1 + 1 + \dots + 1$ (1001 times) $- 2003$, which equals -1002.
- A 12. Since the new parabola passes through (1,3), the original parabola passes through (0,2), so its y-intercept is 2.
- D 13. Let C be the center of the larger circle, A one point of tangency of the circle and the angle, B one point of tangency of the smaller circle and the angle, and P the angle's vertex. Then $m\angle CPA = 30^\circ$, $m\angle CAP = 90^\circ$, and $CP = 8$ (since $\triangle CAP$ is a 30-60-90 triangle). If O is the smaller circle's center and r its radius, then $\triangle OBP$ a 30-60-90 triangle means $OP = 2r$, and $CP = 4 + r + 2r$. But $CP = 8$, so $3r = 4$ and $r = 4/3$.
- A 14. $f(f(x)) = a(ax+b) + b = a^2x + ab + b$. Thus $a^2 = 4$ and $ab + b = 5$. If $a = 2$, then $b = 5/3$, which is not allowed, since a and b are integers, so $a = -2$ and $b = -5$. Then $f(2) = -2(2) - 5 = -9$.
- C 15. In Quadrant I, $x \geq y$ implies $x \leq 25$, and $x \leq y$ implies $y \leq 25$. In Quadrant II, $x < 0$ and $x - y < 0$, so $2y - 2x \leq 50$. By similar reasoning, $x \geq -25$ and $y \geq -25$ in Quadrant III, and $2x - 2y \leq 50$ in Quadrant IV. The resulting region consists of two squares of side 25 in Quadrants I and III, and two isosceles triangles with legs of length 25 in Quadrants II and IV. The total area is $(2)25^2 + 2(1/2)25^2$ or 1875.
- C 16. Let the sides be a, a+d, a+2d, the semiperimeter a+3d, and the area a+4d. Then $(a + a+d + a+2d)/2 = 1.5a + 1.5d = a+3d$, so $a/2 = 1.5d$, $a = 3d$, and the sides are 3d, 4d, and 5d. But $(3d)^2 + (4d)^2 = (5d)^2$, so the triangle is right with area $(1/2)(3d)(4d) = 6d^2 = 7d$. Then $d = 7/6$ and the perimeter is 12d or 14.
- B 17. First, A cannot be even, for then A+2 would be even and each would be twice an odd prime. But odd primes always differ by at least two, so A and A+2 would differ by 4, not 2. Thus A+1 is twice a prime ≥ 11 , and A and A+2 are odd. It is easy to eliminate 22 (23 prime), 26 ($25 = (5)5$), 34 ($35 = (7)5$), 38 (37

prime), 46 (47 prime), 58 (59 prime), 62 (61 prime), 74 (73 prime), and 82 (83 prime). If $A+1 = 86$, then $A = 85 = 5(17)$ and $A+2 = 87 = 3(29)$ are all strange.

- C 18. The required numbers satisfy $a + ar + ar^2 = 42$, so $a(1 + r + r^2) = 42$. Since $1 + r + r^2$ must be a factor of 42, then $r^2 + r + 1 = 42/a$ must factor. The only values of a which work are 2 (with $r = 4$ or -5), 6 (with $r = 2$ or -3), and 14 (with $r = 1$ or -2). The corresponding sequences are 2, 8, 32; 2, -10, 50; 6, 12, 24; 6, -18, 54; 14, 14, 14 (not distinct integers); and 14, -28, 56 for a total of five sequences.

- E 19. The conditions $\cos x = \tan y = \frac{\sin y}{\cos y}$ and $\cos y = \tan x = \frac{\sin x}{\cos x}$ imply that $\cos x \cos y = \sin y$ and $\cos y \cos x = \sin x$. But then $\sin y = \sin x$, and since x and y are acute, $x = y$. Thus $\cos^2 x = \sin x$ and so $1 - \sin^2 x = \sin x$. This means $1 = (\sin x)^2 + \sin x$.

- E 20. The hypotenuse of the right triangle formed by the two cuts has length $2\sqrt{10}$ inches and forms a chord of the circular pie. The area of the region formed by the chord and the pie's edge can be found by finding the area of the sector formed by the radii to the ends of this chord and subtracting the triangle formed by the three segments. The altitude of the triangle is $\sqrt{5^2 - \sqrt{10}^2} = \sqrt{15}$, so the vertex angle of the

triangle is $2 \arctan \frac{\sqrt{10}}{\sqrt{15}} = 1.36944$. The desired area is $1.36944(25/2) - (2\sqrt{10})(\sqrt{15})/2 = 17.11798 -$

$\sqrt{150} = 4.87$. The total area of the slice is the area formed by this chord plus the area of the 2 by 6 right triangle, which is $(1/2)(2)(6) = 6$. Thus the total area of the slice is $4.87 + 6$, or 10.9 to the nearest tenth.