

Test #2 AMATYC Student Mathematics League February/March
2013

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Solutions

Contributors: Kenneth Syharath indicated that the initial solution of problem 14 was incorrect.

1. Ms. Pham writes 2 final exams, each with 25 problems. If the exams have 12 problems in common, how many problems does she write?

- A. 24 B. 26 C. 37 D. 38 E. 49

Solution. Denote by $|C|$ the number of elements of a finite set C . The inclusion-exclusion principle applied to any finite sets A, B , states that $|A \cup B| = |A| + |B| - |A \cap B|$.

Let A, B be the sets of Ms. Pham's problems for the first and the second exams. Then by the inclusion-exclusion principle, the number of problems Ms. Pham is to write is $|A \cup B| = 25 + 25 - 12 = 38$

Answer - D

2. A triangle has two sides of length 8.1 and 1.4. If the length of the third side is an even integer, its length must be

- A. 2 B. 4 C. 6 D. 8 E. 10

Solution. Denote the sides of the triangle by $a = 8.1, b = 1.4$ and c . By the triangle inequalities, $a - b \leq c \leq a + b$, that is $6.7 \leq c \leq 9.5$. Since c is an integer, then $7 \leq c \leq 9$. Since c is an even integer, then $c = 8$

Answer - D

3. If (a, b) is the solution to the system of equations
$$\begin{cases} \pi x + (\pi + e)y = \pi + 2e \\ (\pi + 3e)x + (\pi + 4e)y = \pi + 5e \end{cases}$$
 find $b - a$.

- A. -3 B. -1 C. 0 D. 1 E. 3

Solution. Rewrite the system in the form
$$\begin{cases} \pi(x + y) + ey = \pi + 2e \\ (\pi + 3e)(x + y) + ey = \pi + 5e \end{cases}$$

Subtracting the first equation from the second one, we will obtain $3e(x + y) = 3e$, or $x + y = 1$. Then from the first equation in the last system we will get $y = 2$, and thus $x = 1 - y = -1$. The solution of the system is $(a, b) = (-1, 2)$, and $b - a = 3$

Answer - E

4. The year 2013 has the property that when its distinct prime factors 3, 11, and 61 are each reduced by 1 and written in increasing order (that is, 2, 10, 60) each number is a factor of the next. Find the next year with this property.

- A. 2014 B. 2015 C. 2016 D. 2017 E. 2018

Solution. Prime factor the integers following 2013.

2014 = $2 \cdot 19 \cdot 53$. Not with the property

2015 = $5 \cdot 13 \cdot 31$. Not with the property

$2016 = 2^5 \cdot 3^2 \cdot 7$. With the property

Answer - C

5. If the lines with equations $y = 2x + b$ and $y = mx - 6$ intersect at a point on the x axis, then

- A. $mb = 12$ B. $mb + 12 = 0$ C. $m = 3b$ D. $m + 3b = 0$ E. $3m = b$

Solution. Denote by $(a, 0)$ the point of intersection of the lines. Then

$$\begin{cases} 2a + b = 0 \\ ma - 6 = 0 \end{cases} \quad \text{Eliminating } a \text{ from this system, we get } mb + 12 = 0$$

Answer - B

6. Find the smallest positive integer value of n for which $\frac{1}{a} + \frac{1}{b} = \frac{1}{n}$ has at least three solutions (a, b) in integers with $a \geq b > 0$.

- A. 3 B. 4 C. 5 D. 6 E. 8

Solution. Since $\frac{1}{b} < \frac{1}{a} + \frac{1}{b} \leq \frac{2}{b}$ for $0 < b \leq a$, then $\frac{1}{b} < \frac{1}{n} \leq \frac{2}{b}$, from which $n < b \leq 2n$. We are looking for the smallest n with at least three (a, b) solutions. For a fixed n , denote by B_n the set of potential values of b . From the inequality $n < b \leq 2n$ we have

$$n = 1, B_1 = \{2\}$$

$$n = 2, B_2 = \{3, 4\}$$

$$n = 3, B_3 = \{4, 5, 6\}. \text{ But for } b = 5, \frac{1}{n} - \frac{1}{b} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \text{ is not of the form } \frac{1}{a}$$

$n = 4, B_4 = \{5, 6, 7, 8\}$. Here for $b = 5, 6, 8$, the expression $\frac{1}{n} - \frac{1}{b}$ is of the form $\frac{1}{a}$. Value $n = 4$ is the requested value.

Answer - B

7. The equation $a^3 + b^2 + c^2 = 2013$ has a solution in positive integers for which b is a multiple of 5. Find $a + b + c$ for this solution.

- A. 55 B. 57 C. 59 D. 61 E. 63

Solution. We have $a^3 < 2013$, so that $a < \sqrt[3]{2013}, 1 \leq a \leq 12$. We have $b^2 < 2013$, so that $1 \leq b \leq 44$. Since b is a multiple of 5, then $b = 5, 10, 15, 20, 25, 30, 35, 40$. For selected a, b we find $c = \sqrt{2013 - a^3 - b^2}$ which should be a positive integer. Calculations performed using TI-83/84 graphing calculator show that the only possible values are $a = 4, b = 10, c = 43$, and then $a + b + c = 57$

Answer - B

8. Each letter A through Z of the alphabet is assigned a unique integer from 2 to 27. If $A \cdot M \cdot A \cdot T \cdot Y \cdot C = 3^2 \cdot 5^2 \cdot 7 \cdot 11^2$, find $M + T + Y + C$.

- A. 30 B. 34 C. 36 D. 38 E. 42

Solution. Number 11 cannot be a factor of two of the numbers among M, T, Y, C since the product of 11 and any other prime from $\{3, 5, 7\}$ is greater than 27. Thus $11 = A$ and $M \cdot T \cdot Y \cdot C = 3^2 \cdot 5^2 \cdot 7 = 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 = 3(3 \cdot 5)(5)(7)$. Let $M = 3, T = 15, Y = 5, C = 7$. Calculate $M + T + Y + C = 3 + 15 + 5 + 7 = 30$

Answer - A

9. The third-degree polynomial $P(x)$ has only nonnegative integer coefficients. If $P(0) \cdot P(3) = 139$ and $P(1) \cdot P(2) = 689$, find $P(-1)$.

- A. -2 B. -1 C. 0 D. 1 E. 2

Solution. Numbers $P(0), P(1), P(2), P(3)$ are nonnegative integers and $P(0) < P(1) < P(2) < P(3)$. Since number 139 is a prime, then $P(0) \cdot P(3) =$

$1 \cdot 139$, and $P(0) = 1, P(3) = 139$. Since $P(1) \cdot P(2) = 689 \equiv 13 \cdot 53$, then $P(1) = 13, P(2) = 53$. Denoting $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and selecting $x = 0, 1, 2, 3$ we will form the system of equations.

$$\begin{cases} a_0 & & & & = & 1 \\ a_0 + a_1 + a_2 + a_3 & = & 13 \\ a_0 + 2a_1 + 4a_2 + 8a_3 & = & 53 \\ a_0 + 3a_1 + 9a_2 + 27a_3 & = & 139 \end{cases}$$

Solving this system, I simply used function *rref* of my *TI-84* calculator, we obtain $a_0 = 1, a_1 = 4, a_2 = 5, a_3 = 3$, so that $P(x) = 1 + 4x + 5x^2 + 3x^3$. Finally, $P(-1) = -1$

Answer - B

10. Find the smallest positive value of t such that *cost* is the same whether t is in radians or in degrees. Write your answer (rounded to 3 decimal places) in the corresponding blank on the answer sheet.

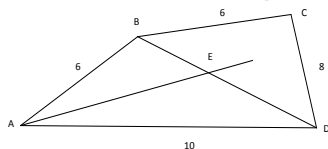
Solution. We are looking for the smallest positive value of t such that $\cos t_{rad} = \cos t^\circ$. For solving the problem I applied my *TI-84* graphing calculator. I switched the calculator to *radian* mode. Since $t^\circ = (\frac{\pi}{180}t^\circ)_{rad}$, then $\cos(t_{rad}) = \cos(\frac{\pi}{180}t^\circ)_{rad}$. I graphed functions $Y_1 = \cos(X)$, $Y_2 = \cos(\frac{\pi}{180}X)$ and used button *CALC* to find the intersection (x, y) of the graphs with the smallest $x > 0$. Calculations produced positive $x_{min} \approx 6.175$

Answer - $t \approx 6.175$

11. In quadrilateral $ABCD$, $AB = 6, BC = 6, CD = 8, AD = 10$, and $\angle C = 90^\circ$. If the angle bisector of $\angle A$ meets diagonal BD at point E , find BE .

- A. $\frac{15}{4}$ B. 4 C. 5 D. 6 E. $\frac{25}{4}$

Solution. See the picture



$\triangle BCD$ is a right triangle. Then $BD = \sqrt{|BC|^2 + |CD|^2} = \sqrt{6^2 + 8^2} = 10$. Denote $x = |BE|$, then $|ED| = 10 - x$. Let $\sin(\angle BAE) = \sin(\angle EAD) = a$. Let $\sin(\angle AEB) = b$, then $\sin(\angle AED) = \sin(\pi - \angle AED) = \sin(\angle AED) = b$

Apply the law of sines to $\triangle ABE$ and $\triangle AED$. We'll have $\frac{x}{a} = \frac{6}{b}$ and $\frac{10-x}{a} = \frac{10}{b}$. Solving these for x we get $x = \frac{15}{4}$

Answer - A

12. Line L has intercepts 2 and 4, while line M has intercepts 4 and 6. If L and M intersect at (a, b) , which of the following could NOT be $3a + b$?

- A. 0 B. 4 C. 8 D. 12 E. 32

Solution. Rename the point of intersection of two lines from (a, b) to (c, d) and keep the usual notations of a and b for x and y intercepts of lines. We have to through two possible assignments for x and y intercepts for both L and M lines

Intercepts form of a line is $\frac{x}{a} + \frac{y}{b} = 1$. Denote $A = \frac{1}{a}, B = \frac{1}{b}$. Then the previous equation of a line becomes $Ax + By = 1$. If values A, B are related to L

line, and values C, D are related to M line, then the lines point of intersection (c, d) is the solution of the following system of equations

$$\begin{cases} Ax + By = 1 \\ Cx + Dy = 1 \end{cases}$$

Solving this system, we find $c = \frac{D-B}{AD-BC}, d = \frac{A-C_2}{AD-BC}$, and $3c + d = \frac{(A-3B)-(C-3D)}{AD-BC}$

$$c = \frac{B_2-B_1}{A_1B_2-A_2B_1}, d = \frac{A_1-A_2}{A_1B_2-A_2B_1}, \text{ and } 3c + d = \frac{(A_1-3B_1)-(A_2-3B_2)}{A_1B_2-A_2B_1}$$

Case $a_1 = 2, b_1 = 4; a_2 = 4, b_2 = 6$. Then $A = \frac{1}{2}, B = \frac{1}{4}, C = \frac{1}{4}, D = \frac{1}{6}$, and $3c + d = 0$

Case $a_1 = 4, b_1 = 2; a_2 = 4, b_2 = 6$. Then $A = \frac{1}{4}, B = \frac{1}{2}, C = \frac{1}{4}, D = \frac{1}{6}$, and $3c + d = 12$

Case $a_1 = 2, b_1 = 4; a_2 = 6, b_2 = 4$. Then $A = \frac{1}{2}, B = \frac{1}{4}, C = \frac{1}{6}, D = \frac{1}{4}$, and $3c + d = 4$

Case $a_1 = 4, b_1 = 2; a_2 = 6, b_2 = 4$. Then $A = \frac{1}{4}, B = \frac{1}{2}, C = \frac{1}{6}, D = \frac{1}{4}$, and $3c + d = 32$

Answer - C

Notes. Computations are done using TI-84 calculator. For each case the values of A, B, C, D are stored in the calculator variables with the same names. Then the screen line $\frac{(A-3B)-(C-3D)}{AD-BC}$ is repeatedly called and executed.

13. Sue traveled continuously starting on 1/1/2012. Her first trip was less than 3 months, and each successive trip was 2 days longer than the previous trip. If her last trip ended on 12/31/2012, which of these was the length in days of one of her trips?

- A. 54 B. 58 C. 65 D. 72 E. 77

Solution. The durations of Sue's trips form an arithmetic series with the difference $d = 2$ and unknown length a of the first trip and the number of trips n . The total duration S of Sue's trips is 366 days, which is the number of day in 2012 year. The formula for the sum of an arithmetic series is $S = \frac{n}{2}(2a + (n-1)d)$. In our case this reduces to $n(a + (n-1)) = 366$, or $n^2 + (a-1)n - 366 = 0$. By the quadratic formula, the solution n of this equation is $n = \frac{-(a-1) + \sqrt{(a-1)^2 + 1464}}{2}$. The feasible values a are positive integers under condition $a < 91$ such that the value of n calculated by the last formula is a positive integer.

To calculate n , I typed into my TI-84 graphing calculator function $Y_1 = (-(X-1) + \sqrt{((X-1)^2 + 1464)})/2$ (Y_1 represents n and X represents a), and browsed through the table of Y_1 values in the range $1 \leq X \leq 90$. The only solution value is $(a, n) = (56, 6)$.

For $(a, n) = (56, 6)$, the complete set of trip lengths is $\{56, 58, 60, 62, 64, 66\}$

Answer - B

14. A binary string is a sequence of 1's and 0's, such as 10011 or 11101010. How many different binary strings of length 6 are there such that no two are reversals of each other or add up to 111111?

- A. 22 B. 23 C. 24 D. 25 E. 26

Solution. Rewrite the conditions of the problem to make it more visual. Replace digit 0 in the problem conditions with number -1 . Consider the strings

$f = f_{-3}, f_{-2}, f_{-1}, f_1, f_2, f_3$, where all elements are one of the $-1, 1$.

We will interpret such a string as a function f , call it b-function, with the discrete domain $D = \{-3, -2, -1, 1, 2, 3\}$ and with values $f(i) = f_i, i \in D$. We might wish to set $f(0) = 0$. Let U be the set of all b-functions. By the Fundamental Counting Principle, the cardinality $|U| = 2^6 = 64$

We introduce two unary operations x and y on b-functions.

For a b-function f , the reversal function f^y is defined as $f^y(x) = f(-x)$. The graph of f^y in the xy-plane is a reflection of graph of f across the y-axis.

For a b-function f , denote the b-function f^x by $f^x(x) = -f(x)$. The graph of f^x in the xy-plane is a reflection of graph of f across the x-axis. It is clear that $f + f^x = 0$.

In further reading, please keep the pictures of graphs in mind.

It is evident that $f^{xx} = f, f^{yy} = f, f^{xy} = f^{yx}, f^x \neq f$.

Denote $E = \{f \in U | f^y = f\}$ to be the set of even b-functions. The equality $f^y = f$ is identical to $f^x = f^{xy}$.

Denote $O = \{f \in U | f^{xy} = f\}$ to be the set of odd b-functions. The equality $f^{xy} = f$ is identical to $f^x = f^y$.

By the Fundamental Counting Principle, $|E| = |O| = 2^3 = 8$. $E \cap O = \emptyset$.

Denote $N = U \setminus (E \cup O)$ to be the set of neither even, not odd functions. $|N| = |U| - |E| - |O| = 64 - 8 - 8 = 48$

For any $f \in U$ we introduce f -set $\{f, f^y, f^x, f^{yx} \equiv f^{xy}\}$. It is easy to see that if two f -sets have a common element, then they coincide.

Call $F = \{f \in U | f^x \notin F \text{ or } f^y \notin F\}$ a feasible set. We have to find $\max |F|$ over all feasible sets.

Consider some fixed feasible set F .

Let $f \in E$. The f -set is $\{f = f^y, f^x = f^{xy}\}$, as it is also easy to see geometrically. So if $f = f^y \in F$, then $f^x = f^{xy} \notin F$. Thus there may be maximum 4 even functions in F .

Let $f \in O$. The f -set is $\{f = f^{xy}, f^x = f^y\}$, as it is also easy to see geometrically. So if $f = f^{xy} \in F$, then $f^x = f^y \notin F$. Thus there may be maximum 4 odd functions in F .

Let $f \in N$. The b-functions in f -set $\{f, f^y, f^x, f^{yx} (\equiv f^{xy})\}$ are from N and are different. Neither any three of them, say $f^y = (f^{yx})^x, f^x = (f^{yx})^y, f^{yx}$, are in F , which is also clear geometrically. Then of four functions f, f^x, f^y, f^{yx} maximum two of them may be in F . Thus $|F \cap N| \leq \frac{|N|}{2} = 24$.

Totally $|F| = |F \cap E| + |F \cap O| + |F \cap N| \leq 4 + 4 + 24 = 32$.

On the other hand, the set $\tilde{F} = \{f \in U | f(-3) = -1\}$ is feasible since $f^x \notin \tilde{F}$ which follows from $f^x(-3) = -f(-3) = 1$. Cardinality $|\tilde{F}| = \frac{|U|}{2} = 32$

Answer - 32 . This answer is not among suggested answers.

14 Alternative. In the frame of problem **14**, let us change the question of the problem onto 'How many different binary strings of length 6 are there such that no two are reversals of each other **AND** add up to 111111?'

Solution. The following changes in the above solution should be made.

Call $F = \{f \in U | f^x \notin F \text{ AND } f^y \notin F\}$ a feasible set. We have to find $\max |F|$ over all feasible sets.

Consider some fixed feasible set F .

Let $f \in E$. Since $f = f^y$, then $f \notin F$. No even b-function may be included into F .

Let $f \in O$. Since $f \neq f^x$, and $f^x = f^y$, then f may be a member of F with $f^x = f^y$ being outside F . Thus, as before, there may be maximum 4 odd functions in F .

Let $f \in N$. As above, no three b-functions out of f -set $\{f, f^y, f^x, f^{yx}\}$ may be in F . But now not any two functions from f -set are allowed in F , but two specific pairs. Namely, permissible are b-functions f, f^{xy} with f^x, f^y being outside F , or functions f^x, f^y with f, f^{xy} being outside F . This is clear geometrically. Thus, as before, $|F \cap N| \leq \frac{|N|}{2} = 24$

Totally, $|F| = |F \cap E| + |F \cap O| + |F \cap N| \leq 0 + 4 + 24 = 28$.

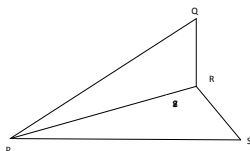
A concrete feasible set F with $|F| = 28$ can be constructed following the steps of the proof.

Answer - 28 .

15. In quadrilateral $PQRS$, $\angle P = \angle Q = \angle S = 45^\circ$, $\angle QPR = \angle RPS$, and $PR = 8\sqrt{2}$ Find the area of quadrilateral $PQRS$ to the nearest integer.

A. 60 B. 61 C. 62 D. 63 E. 64

Solution. See the picture



In $\triangle PQR$, by the law of sines $|QR| = \frac{|PR|}{\sin(\angle Q)} \sin(\angle QPR) = \frac{8\sqrt{2}}{\sin(\pi/4)} \sin(\frac{\pi}{8})$. Then $Area(\triangle PQR) = \frac{1}{2}|PR||QR| \sin(\angle PRQ) = \frac{1}{2}|PR||QR| \sin(\pi - \frac{\pi}{4} - \frac{\pi}{8}) = \frac{1}{2}8\sqrt{2} \cdot \frac{8\sqrt{2}}{\sin(\pi/4)} \sin(\frac{\pi}{8}) \cdot \sin(\frac{5\pi}{8}) = 64 \frac{\sin(\frac{\pi}{8}) \sin(\frac{5\pi}{8})}{\sin(\pi/4)}$.

$\triangle PQR$ and $\triangle PRS$ are congruent since they have equal corresponding angles and share side PR .

Thus $Area(PQRS) = 2Area(\triangle PQR) = 128 \frac{\sin(\frac{\pi}{8}) \sin(\frac{5\pi}{8})}{\sin(\pi/4)} = 64$

Answer - E

16. The numbers 2 and 1 are the smallest positive integers for which the square of the first is 2 more than twice the square of the second. If a and b are the smallest such pair with $a > 10$, find $a - b$.

A. 13 B. 15 C. 17 D. 19 E. 21

Solution. We have $a^2 = 2 + 2b^2$. Thus $b = \sqrt{\frac{a^2-2}{2}}$. Use TI-83/84 calculator, TABLE facility to find the smallest solution of the last formula. The results are $a = 58, b = 41, a - b = 17$

Answer - C

17. A number is chosen at random from among all 5-digit numbers containing exactly one each of the digits 1, 2, 3, 4, and 5. Find the probability that no two adjacent digits in the number are consecutive integers.

A. $\frac{1}{10}$ B. $\frac{7}{60}$ C. $\frac{2}{15}$ D. $\frac{3}{20}$ E. $\frac{1}{6}$

Solution. The total number N of 5-digit numbers containing exactly one each of the digits 1, 2, 3, 4, is $N = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

We will use tree diagram method to calculate the count of 5-digit numbers where no two adjacent digits in the number are consecutive integers.

First digit of numbers: 1,2,3,4,5

First two digits of numbers: 13,14,15;24,25;31,35;41,42,51,52,53

First three digits of numbers: 135,142,152,153,241,251,253,314,315,351,352,413,415,425,513,514,524,531

First four digits of numbers: 1352,1425,1524,2413,2415,2513,2514,2531,3142,3152,3514,3524,4135,4152,4153,4251,4253,5142,5241,5314

All five digits of numbers: 13524,14253,24135,24153,25314,31425,31524,35142,35241,41352,42513,42531,52413,53142

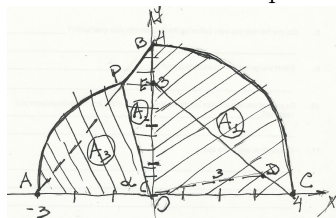
The required probability is $\frac{14}{120} = \frac{7}{60}$

Answer - B

18. The triangular region with vertices $(0,0)$, $(4,0)$, and $(0,3)$ is rotated 90° counterclockwise around the origin. Find the area of the figure formed by this rotation to the nearest hundredth.

- A. 19.96 B. 20.04 C. 20.12 D. 20.20 E. 20.28

Solution. See the picture



We have to find the area of the region below the bold curve and above the x-axis. The region is the union of three subregions: R_1 - quarter of circle OB of radius $r_1 = 4$; R_2 - triangle OPB with $|OP| = 3$, $|OB| = 4$, $\angle POB = \pi/2 - \alpha$; R_3 - circular sector OAP of radius $r_2 = 3$ and central angle α .

Region R_1 is swept in rotating line segment OC

Region R_2 is swept in rotating all line segments starting at O and ending at points of segment DC . Length $|OD| = 3$

Region R_3 is swept in rotating line segment OE (of length 3)

I used my TI-84 graphing calculator to do the computations below.

Area of region R_1 is $A_1 = \pi r_1^2 / 4 = \pi 4^2 / 4 = 4\pi$

The equation of the line AB is $y = \frac{4}{3}x + 4$, the equation of the circular arc AP is $y = -\sqrt{9 - x^2}$. Find the intersecting point P for the line and the circle: $P = (-0.84, 2.88)$. Find the angle $\alpha = \tan^{-1}(\frac{2.88}{0.84})$.

Area of region R_2 is $A_2 = \frac{1}{2}|OP||OB|\sin(\frac{\pi}{2} - \alpha) = \frac{1}{2}3 \cdot 4 \cos(\tan^{-1}(\frac{2.88}{0.84})) = 6 \cos(\tan^{-1}(\frac{2.88}{0.84}))$

Area of region R_3 is $A_3 = \frac{1}{2}r_2^2\alpha = \frac{1}{2}3^2 \tan^{-1}(\frac{2.88}{-0.84}) = \frac{9}{2} \tan^{-1}(\frac{2.88}{0.84})$

The total area is $A_1 + A_2 + A_3 = 4\pi + 6 \cos(\tan^{-1}(\frac{2.88}{0.84})) + \frac{9}{2} \tan^{-1}(\frac{2.88}{0.84}) = 20.03788059 \approx 20.04$

Answer - B

19. For how many pairs of positive integers (n, m) with $n, m < 100$ are both of the polynomials $x^2 + mx + n$ and $x^2 + mx - n$ factorable over the integers?

- A. 4 B. 5 C. 6 D. 7 E. 8

Solution. By the quadratic formula, the required values of m, n are such that both expressions $\sqrt{m^2 - 4n}, \sqrt{m^2 + 4n}$ are integers.

Computerized solution. The selection of appropriate m, n be resolved by creating and running a simple computer program in TI-84 graphing calculator. However here I wrote and run a short program in a modern computer language Ruby

```
include Math
(1..99).each do |n|
  (1..99).each do |m|
    discr1 = m * m - 4 * n
    discr2 = m * m + 4 * n
    puts "n = #{n}, m = #{m}" if discr1 >= 0 and sqrt(discr1) == sqrt(discr1).floor
    and sqrt(discr2) == sqrt(discr2).floor
  end
end

The printout is
n=6, m = 5
n=24, m = 10
n=30, m = 13
n=54, m = 15
n=60, m = 17
n=84, m = 25
n=96, m = 20
```

Analytical solution. Denote $\sqrt{m^2 - 4n} = p, \sqrt{m^2 + 4n} = q, m < p < q$. By squaring both parts here we will get $m^2 - 4n = p^2, m^2 + 4n = q^2$. Summing together the corresponding sides of these equations, we will obtain $p^2 + q^2 = 2m^2$. We derive the solution of this equation in the way similar to getting formulas for Pythagorean triples. Denote $\frac{p}{m} = x, \frac{q}{m} = y, 0 < x < 1 < y$, and rewrite the last equation as $x^2 + y^2 = 2$. Transform this equation to the forms $y^2 - 1 = 1 - x^2, (y - 1)(y + 1) = (1 - x)(x + 1)$, to obtain $\frac{1-x}{y-1} = \frac{y+1}{x+1} \equiv t$, for some rational value $t > 1$. Rewrite the last double equation to obtain $\frac{1-x}{y-1} = t, \frac{y+1}{x+1} \equiv t$, and get $x + ty = t + 1, tx - y = -t + 1$. Solving this system, we obtain $x = \frac{-t^2 + 2t + 1}{t^2 + 1}, y = \frac{t^2 + 2t - 1}{t^2 + 1}$. Writing $t = \frac{u}{v}, v, u$ are integers, $v < u$, we will get $\frac{p}{m} \equiv x = \frac{-u^2 + 2uv + v^2}{u^2 + v^2}, \frac{q}{m} \equiv y = \frac{u^2 + 2uv - v^2}{u^2 + v^2}$, or $p = \frac{-u^2 + 2uv + v^2}{u^2 + v^2}m, q = \frac{u^2 + 2uv - v^2}{u^2 + v^2}m$. Denote $\frac{m}{u^2 + v^2} \equiv r$ and obtain $p = (-u^2 + 2uv + v^2)r, q = (u^2 + 2uv - v^2)r, m = (u^2 + v^2)r$. Solving $\sqrt{m^2 - 4n} \equiv p$ for n , we receive $n = \frac{m^2 - p^2}{4} = \frac{1}{4}([(u^2 + v^2)r]^2 - [(-u^2 + 2uv + v^2)r]^2) = \dots = uv(u^2 - v^2)r^2$. We can consider u, v mutually prime because otherwise the common factor can be factored out and combined with coefficient r .

In summary, required values m, n can be found from formulas $m = (u^2 + v^2)r, n = uv(u^2 - v^2)r^2$, where $0 < v < u$ are mutually prime integers and r is a

positive integer. The possible values of u, v, r are restricted by the conditions $1 \leq m, n \leq 99$

The calculations are summarized into the following table.

(u, v)	r	$n = r^2 uv(u^2 - v^2)$	$m = r(u^2 + v^2)$
$(2, 1)$	1	6	5
$(2, 1)$	2	24	10
$(2, 1)$	3	54	15
$(2, 1)$	4	96	20
$(4, 1)$	1	60	17
$(3, 2)$	1	30	13
$(4, 3)$	1	84	25

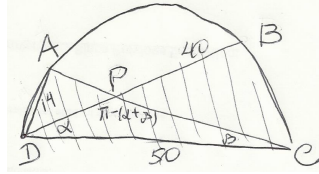
Other values of (u, v) yield either already obtained results or violate limitations on m, n

Answer - D

20. Triangles ACD and BCD ($AD = 14, BD = 40$) are inscribed in a semicircle with diameter $CD = 50$. If $AB > 25$, find the area of their union.

A. 625 B. 637.5 C. 652.5 D. 673.5 E. 675

Solution. See the picture



$$Area(\triangle DAC) = \frac{1}{2}|DA||AC| = \frac{1}{2}|DA|\sqrt{|DC|^2 - |DA|^2} = \frac{1}{2}14\sqrt{50^2 - 14^2} = 336$$

$$Area(\triangle DBC) = \frac{1}{2}|DB||BC| = \frac{1}{2}|DB|\sqrt{|DC|^2 - |DB|^2} = \frac{1}{2}40\sqrt{50^2 - 40^2} = 600$$

From $\triangle DAC$, $\beta = \sin^{-1}(\frac{|DA|}{|DC|}) = \sin^{-1}(\frac{7}{25})$. From $\triangle DBC$, $\alpha = \cos^{-1}(\frac{|DB|}{|DC|}) = \cos^{-1}(\frac{4}{5})$. From $\triangle DPC$, by the law of sines $|DP| = |DC|\frac{\sin \beta}{\sin(\pi - (\alpha + \beta))} = 50\frac{\sin \beta}{\sin(\alpha + \beta)}$

$$Area(\triangle DPC) = \frac{1}{2}|DC||DP|\sin \alpha = \frac{1}{2}50 \cdot 50 \frac{\sin \beta}{\sin(\alpha + \beta)} \sin \alpha = 1250 \frac{\sin \alpha \cdot \sin \beta}{\sin(\alpha + \beta)} = 1250 \frac{\sin(\cos^{-1}(\frac{4}{5})) \sin(\sin^{-1}(\frac{7}{25}))}{\sin(\cos^{-1}(\frac{4}{5}) + \sin^{-1}(\frac{7}{25}))} = 262.5$$

The requested area is (see reasoning for *Problem#1*) $Area(\triangle DAC) + Area(\triangle DBC) - Area(\triangle DPC) = 336 + 600 - 262.5 = 673.5$

Answer - D