

1. Ms. Pham writes 2 final exams, each with 25 problems. If the exams have 12 problems in common, how many problems does she write?
- A. 24 B. 26 C. 37 D. 38 E. 49

Solution: Suppose she writes the 12 question first that are common to the final exam. Then she will have to write 13 more for the first exam, to have 25 questions. She will also have to write 13 more for the second exam, to have 25 questions. Thus she writes $12 + 13 + 13 = 38$ questions. The correct answer is D.

2. A triangle has two sides of length 8.1 and 1.4. If the length of the third side is an even integer, its length must be
- A. 2 B. 4 C. 6 D. 8 E. 10

Solution: Recall the triangle inequality. If a , b , and c are sides of a triangle, then $a + b > c$. Let us denote the missing sides by x . The triangle inequality gives us that

$$8 + 1.4 > x \quad \text{and} \quad 1.4 + x > 8$$

We solve these inequalities and get $6.6 < x < 9.4$. The only even integer in this interval is 8. The correct answer is D.

3. If (a, b) is the solution to the system of equations $\begin{cases} \pi x + (\pi + e)y = \pi + 2e \\ (\pi + 3e)x + (\pi + 4e)y = \pi + 5e \end{cases}$ find $b - a$.
- A. -3 B. -1 C. 0 D. 1 E. 3

Solution: This is just a fairly intimidating, however linear system in two variables. Keep in mind, the problem is asking for the value of $y - x$. We have two options: we can solve the system for x and y and then add, or perhaps, there is a shortcut to the value of $y - x$ without actually finding x or y . We first investigate what would happen if we added or subtracted the equations in the system. It looks very promising if we subtract the first equation from the second. (We will just add the opposite.)

$$\begin{aligned} -\pi x - (\pi + e)y &= -\pi - 2e \\ (\pi + 3e)x + (\pi + 4e)y &= \pi + 5e \end{aligned}$$

$$\begin{aligned} 3ex + 3ey &= 3e && \text{divide by } 3e \\ x + y &= 1 \end{aligned}$$

Too bad we are not asked for the value of $x + y$. This is still useful information. Let us look at the first equation again.

$$\begin{aligned} \pi x + (\pi + e)y &= \pi + 2e \\ \pi x + \pi y + ey &= \pi + 2e \\ \pi(x + y) + ey &= \pi + 2e && \text{we know that } x + y = 1 \\ \pi + ey &= \pi + 2e \\ ey &= 2e && \text{divide by } e \\ y &= 2 \end{aligned}$$

If $y = 2$ and $x + y = 1$, then $x = -1$ and so $y - x = 2 - (-1) = 3$. The correct answer is E.

4. The year 2013 has the property that when its distinct prime factors 3, 11, and 61 are each reduced by 1 and written in increasing order (that is, 2, 10, 60) each number is a factor of the next. Find the next year with this property.

A. 2014

B. 2015

C. 2016

D. 2017

E. 2018

Solution: Let us look at the prime factorization of the potential answers. $2014 = 2 \cdot 19 \cdot 53$ and after subtracting 1, we obtain 1, 18, and 52. Since 18 is not a factor of 52, this number is ruled out as the correct answer.

$2015 = 5 \cdot 13 \cdot 31$ and after subtracting 1, we obtain 4, 12, and 52. Since 12 is not a factor of 52, this number is ruled out as the correct answer.

$2016 = 2^5 \cdot 3^2 \cdot 7$ and the prime factors are 2, 3, and 7. so it can not be the product of three prime number. After subtracting 1, we obtain 1, 2, and 6. This is clearly the solution so we could stop looking. The answer is C.

2017 is actually a prime number. (We just have to check for prime divisors until 45 since $\sqrt{2017} \approx 44.911$)

$2018 = 2 \cdot 1009$ and it does not have three distinct prime factors.

5. If the lines with equations $y = 2x + b$ and $y = mx - 6$ intersect at a point on the x -axis, then

A. $mb = 12$ B. $mb + 12 = 0$ C. $m = 3b$ D. $m + 3b = 0$ E. $3m = b$

Solution: When we solve the system $\begin{cases} y = 2x + b \\ y = mx - 6 \end{cases}$ the solution is a point on the x -axis, and so $y = 0$

$$\begin{aligned} 0 &= 2x + b \quad \text{and} \quad 0 = mx - 6 \\ -\frac{b}{2} &= x \quad \implies \quad 0 = m \left(-\frac{b}{2} \right) - 6 \end{aligned}$$

$$\begin{aligned} 0 &= m \left(-\frac{b}{2} \right) - 6 \\ 6 &= -\frac{mb}{2} \\ -12 &= mb \\ 0 &= mb + 12 \end{aligned}$$

So the correct answer is B.

6. Find the smallest positive integer value of n for which $\frac{1}{a} + \frac{1}{b} = \frac{1}{n}$ has at least three solutions (a, b) with integers $a \geq b > 0$

A. 3

B. 4

C. 5

D. 6

E. 7

Solution:

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} &= \frac{1}{n} && \text{multiply by } abn \\ bn + an &= ab && \text{subtract } (an + bn) \\ 0 &= ab - an - bn \end{aligned}$$

We will factor the right-hand side. Based on the three terms, we try $(a - n)(b - n) = ab - an - bn + n^2$. So, we can factor it if we just add n^2 to both sides.

$$\begin{aligned} n^2 &= ab - an - bn + n^2 \\ n^2 &= (a - n)(b - n) \end{aligned}$$

Before we proceed, let us notice that $a \geq b$ implies that $a - n \geq b - n$

Because $n = 1$ and 2 are not listed among the solutions offered, we don't need to consider them.

Case 1. $n = 3$

$$\begin{aligned}
 9 &= (a-3)(b-3) \\
 9 &= \begin{matrix} (a-3) & (b-3) \\ 9 & 1 \\ 3 & 3 \end{matrix} \implies \begin{matrix} a=12, b=4 \\ a=6, b=6 \end{matrix}
 \end{aligned}$$

This only gives us two solutions.

Case 2. $n = 4$

$$\begin{aligned}
 16 &= (a-4)(b-4) \\
 16 &= \begin{matrix} (a-4) & (b-4) \\ 16 & 1 \\ 8 & 2 \\ 4 & 4 \end{matrix} \implies \begin{matrix} a=20, b=5 \\ a=12, b=6 \\ a=8, b=8 \end{matrix}
 \end{aligned}$$

So this is the first number among the ones listed that has at least three solutions. The correct answer is B.

7. The equation $a^3 + b^2 + c^2 = 2013$ has a solution in positive integers for which b is a multiple of 5. Find $a + b + c$ for this solution.

A. 55

B. 57

C. 59

D. 61

E. 63

Solution:

$$a^3 + b^2 + c^2 = 2013 \quad \text{and} \quad b = 5k$$

First note that $\sqrt[3]{2013} \approx 12.62645$ and so $a \leq 12$. (Indeed, $12^3 = 1728$ and so a could be 12 but $13^3 = 2197$ that is too large.)

$$\begin{array}{rcccc}
 x & 0 & 1 & 2 & 3 & 4 \\
 x^2 & 0 & 1 & 4 & 4 & 1 \\
 x^3 & 0 & 1 & 3 & 2 & 4
 \end{array}$$

Re-write modulo 5 (That means to replace every number by the remainder we get after division by 5. For example, $7 \equiv 2$ and $10 \equiv 0$)

$$\begin{aligned}
 a^3 + 0^2 + c^2 &\equiv 3 \\
 a^3 + c^2 &\equiv 3
 \end{aligned}$$

$$\text{Case 1. } a^3 \equiv 3 \text{ and } c^2 \equiv 0 \implies a \equiv 2 \text{ and } c \equiv 0$$

$$\text{Case 2. } a^3 \equiv 2 \text{ and } c^2 \equiv 1 \implies a \equiv 3 \text{ and } c \equiv 1 \text{ or } 4$$

$$\text{Case 3. } a^3 \equiv 4 \text{ and } c^2 \equiv 4 \implies a \equiv 4 \text{ and } c \equiv 2 \text{ or } 3$$

$$\text{Case 1. } a \equiv 2 \text{ and } b \equiv c \equiv 0$$

$$a = 2, 7, 12$$

$$a = 2$$

$$\begin{aligned}
 2^3 + b^2 + c^2 &= 2013 \\
 b^2 + c^2 &= 2005
 \end{aligned}$$

This is impossible because if both b and c are divisible by 5 then their squares are divisible by 25, then and so is their sum. 2005 is not divisible by 25.

$$a = 7$$

$$\begin{aligned} 7^3 + b^2 + c^2 &= 2013 \\ b^2 + c^2 &= 1670 \end{aligned}$$

This is impossible because if both b and c are divisible by 5 then their squares are divisible by 25, then and so is their sum. But 1670 is not divisible by 25.

$$a = 12$$

$$\begin{aligned} 12^3 + b^2 + c^2 &= 2013 \\ b^2 + c^2 &= 285 \end{aligned}$$

This is impossible because if both b and c are divisible by 5 then their squares are divisible by 25, and so is their sum. But 285 is not divisible by 25.

Case 2. $a \equiv 3$ and $c \equiv 1$ or 4

$a \equiv 3$ means that $a = 3$ or 8

$$a = 3$$

$$\begin{aligned} 3^3 + b^2 + c^2 &= 2013 \\ b^2 + c^2 &= 1986 \end{aligned}$$

This is impossible for divisibility by 9.

$$\begin{array}{rcccccccc} x & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ x^2 & 0 & 1 & 4 & 0 & 7 & 7 & 0 & 4 & 1 \end{array}$$

1986 gives a remainder of 6 and there are no two squares modulo 9 that add up to 6.

$$a = 8$$

$$\begin{aligned} 8^3 + b^2 + c^2 &= 2013 \\ b^2 + c^2 &= 1501 \end{aligned}$$

Now b could be 5, 10, 15, 20, 25, 30, or 35

$b = 5$	$c^2 = 1988$	no solution
$b = 10$	$c^2 = 1913$	no solution
$b = 15$	$c^2 = 1788$	no solution
$b = 20$	$c^2 = 1613$	no solution
$b = 25$	$c^2 = 1388$	no solution
$b = 30$	$c^2 = 1113$	no solution

Case 3. $a \equiv 4$ and $c \equiv 2$ or 3

$a \equiv 4$ means that $a = 4$ or 9

$$a = 4$$

$$\begin{aligned} 4^3 + b^2 + c^2 &= 2013 \\ b^2 + c^2 &= 1949 \end{aligned}$$

Now b could be 5, 10, 15, 20, 25, 30, or 35

$b = 5$	$c^2 = 1924$	no solution
$b = 10$	$c^2 = 1849$	$c = 43$
$b = 15$	$c^2 = 1724$	no solution
$b = 20$	$c^2 = 1549$	no solution
$b = 25$	$c^2 = 1324$	no solution
$b = 30$	$c^2 = 1049$	no solution

Thus $a = 4$ and $b = 10$ and $c = 43$. We check: $4^3 + 10^2 + 43^2 = 2013$. So the answer is $4 + 10 + 43 = 57$ which is B.

There should be a better way.

8. Each letter A through Z of the alphabet is assigned a unique integer from 2 to 27.

If $A \cdot M \cdot A \cdot T \cdot Y \cdot C = 3^2 \cdot 5^2 \cdot 7 \cdot 11^2$, find $M + T + Y + C$.

- A. 30 B. 34 C. 36 D. 38 E. 42

Solution: There are 5 variables, A, M, T, Y, and C. They are all different, and based on the product, all odd. In fact, based on the product, the possible values are for these variables are 3, 5, 7, 9, 11, 15, 21, and 25.

$$A^2 \cdot M \cdot T \cdot Y \cdot C = 3^2 \cdot 5^2 \cdot 7 \cdot 11^2$$

Let us also notice that M, T, Y, and C are symmetrical. A is the only variable with a different role than the others.

First we will establish that only $A = 11$ is possible.

Suppose that A is divisible by 3. Then A^2 has two 3-factors in its prime factorization so the rest, M, Y, T, C all must have none. This leaves us only the values 5, 7, 11, and 25 for these but then the product of these numbers has one too many 5-factors. So, A is not divisible by 3.

Suppose now that A is divisible by 5. Then A^2 has two 5-factors in its prime factorization so the rest, M, Y, T, C all must have none. This leaves us only the values 3, 7, 9, 11, and 21 for these. 9 can not be one of them because if 9 is one of M, Y, T, C then the other three can not be divisible by 3 and so we only have 7 and 11 for the other three values. So M, Y, T, C must be 3, 7, 11, and 21. This is impossible too, because now the product has one too many 7-factors. So, A is not divisible by 5.

A can also not be divisible by 7 because then the product would be divisible by 49.

This leaves only one value for A, that is 11.

This leaves us 3, 5, 7, 9, 15, 21, and 25 for M, Y, T, C where

$$M \cdot T \cdot Y \cdot C = 3^2 \cdot 5^2 \cdot 7$$

Next we claim that none of these can be 9 or 25. Suppose one is 9. Then the rest of the numbers can have no 3-factor in them, leaving us only 5, 7, and 25 for the other three. But their product has one too many 5-factor. The argument goes similarly for 25. So now we have the values of 3, 5, 7, 15 and 21. The product of these numbers is $3^3 \cdot 5^2 \cdot 7^2$. We need to get the product $3^2 \cdot 5^2 \cdot 7$ by removing one of the numbers. So we remove 21. This gives us $A = 11$ and $M, T, Y, C = 3, 5, 7, 15$ and so the sum is $3 + 5 + 7 + 15 = 30$. The correct answer is A.

9. The third-degree polynomial $P(x)$ has only nonnegative integer coefficients. If $P(0) \cdot P(3) = 139$ and $P(1) \cdot P(2) = 689$, find $P(-1)$.

- A. -2 B. -1 C. 0 D. 1 E. 2

Solution: On the interval $[0, \infty)$, the polynomial $P(x)$ is increasing and has only integer values $P(n)$ for integer n . We first look at the prime-factorization of 139 and 689.

Fortunately, 139 is a prime. Since $P(x)$ is increasing, this means that $P(0) = 1$ and $P(3) = 139$. On the other hand, $689 = 13 \cdot 53$. This means that either $P(1) = 1$ and $P(2) = 689$ or $P(1) = 13$ and $P(2) = 53$. Because $P(x)$ is increasing, $P(1) = 1$ is impossible because $P(0) = 1$ already. So we have the following:

$$P(0) = 1 \quad P(1) = 13 \quad P(2) = 53 \quad \text{and} \quad P(3) = 139$$

$$P(x) = ax^3 + bx^2 + cx + d \quad a, b, c, d \in \mathbb{Z} \quad a, b, c, d \geq 0$$

$$\begin{aligned} P(0) &= 1 &\implies a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d &= 1 &\implies d = 1 \\ P(1) &= 13 &\implies a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d &= 13 &\implies a + b + c + d = 13 \\ P(2) &= 53 &\implies a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d &= 53 &\implies 8a + 4b + 2c + d = 53 \\ P(3) &= 139 &\implies a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d &= 139 &\implies 27a + 9b + 3c + d = 139 \end{aligned}$$

$$\begin{aligned} a + b + c + 1 &= 13 &\implies a + b + c &= 12 \\ 8a + 4b + 2c + 1 &= 53 &\implies 8a + 4b + 2c = 52 &\implies 4a + 2b + c = 26 \\ 27a + 9b + 3c + 1 &= 139 &\implies 27a + 9b + 3c = 138 &\implies 9a + 3b + c = 46 \end{aligned}$$

So we have the following system of equations: (we already know $d = 1$)

$$\begin{aligned} a + b + c &= 12 \\ 4a + 2b + c &= 26 \\ 9a + 3b + c &= 46 \end{aligned}$$

We eliminate c by subtracting the first equation from the other two

$$\begin{aligned} 3a + b &= 14 \\ 8a + 2b &= 34 &\implies 4a + b = 17 \end{aligned}$$

Now we just have

$$\begin{aligned} 3a + b &= 14 \\ 4a + b &= 17 \end{aligned}$$

After subtraction, we have $a = 3$. Then $4a + b = 17$ and $a = 3$ gives us $b = 5$. Then $a + b + c = 12$ gives us $c = 4$. Thus

$$a = 3 \quad b = 5 \quad c = 4 \quad d = 1$$

We check: if $P(x) = 3x^3 + 5x^2 + 4x + 1$, then $P(0) = 1$, $P(1) = 13$, $P(2) = 53$ and $P(3) = 139$. And $P(-1) = 3(-1)^3 + 5(-1)^2 + 4(-1) + 1 = -1$. The correct answer is B.

10. Find the smallest positive value of t such that $\cos t$ is the same whether t is in radians or in degrees. Write your answer (rounded to 3 decimal places) in the corresponding blank on the answer sheet.

Solution: Let x be our number. Then x° and x_{rad} denotes the two angles with $\cos x^\circ = \cos x_{\text{rad}}$. First, let us notice that x_{rad} denotes the larger angle.

There are two ways the cosine of two angles are the same: either one is co-terminal with the other, or one is co-terminal with the opposite of the other. The smallest positive co-terminal angle is obtained when we add 360° (or 2π) to an angle. If we are looking for the smallest positive angle like that, the second case is more promising because then x_{rad} is still less than 2π while in the first case, x_{rad} is greater than 2π .

So we will state that x_{rad} is 2π more than the opposite of x° . We need to decide first in how to write the equation: in degrees or in radians. It doesn't matter, the answer should be the same number. We will go with degrees.

$$\begin{aligned} x_{\text{rad}} \text{ in degrees} &= -x^\circ + 360^\circ \\ x_{\text{rad}} \cdot \left(\frac{180^\circ}{\pi}\right) &= -x^\circ + 360^\circ \end{aligned}$$

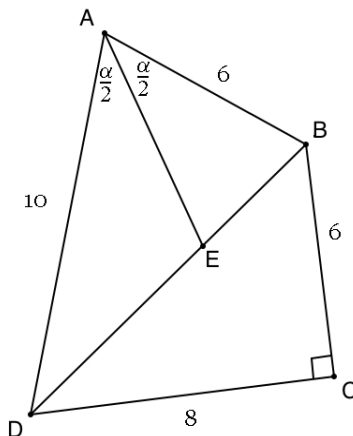
So our equation is

$$\begin{aligned}
 x \cdot \frac{180}{\pi} &= -x + 360 && \text{add } x \\
 x \left(\frac{180}{\pi} \right) + x &= 360 && \text{factor our } x \\
 x \left(\frac{180}{\pi} + 1 \right) &= 360 && \text{divide by } \frac{180}{\pi} + 1 \\
 x &= \frac{360}{\frac{180}{\pi} + 1} \approx 6.175
 \end{aligned}$$

11. In quadrilateral ABCD, $AB = 6$, $BC = 6$, $CD = 8$, $AD = 10$, and $\angle C = 90^\circ$. If the angle bisector of $\angle A$ meets diagonal BD at point E, find BE.

- A. $\frac{15}{4}$ B. 4 C. 5 D. 6 E. $\frac{25}{4}$

Solution: Consider the picture shown below.



First, diagonal BD is 10 units long by the Pythagorean theorem. Second, recall a theorem: in any triangle, an angle bisector splits the opposite sides into two parts in the same ratio as the ratio of the two sides around the angle. If we know this theorem, the solution is immediate: the other two sides are in a ratio of 3 to 5, so BE must be $\frac{3}{8}$ of BD (the other part is $\frac{5}{8}$) and so the answer is $\frac{3}{8} \cdot 10 = \frac{15}{4}$. The correct answer is A.

In case someone does not know this theorem, we will sort of derive it here: we will prove that $\frac{BE}{DE} = \frac{3}{5}$.

The trick is to compare the areas of triangles AEB and ADE. (Recall that the area of a triangle can be computed as $A = \frac{1}{2}ab \sin \gamma$)

$$\frac{A_{AEB}}{A_{ADE}} = \frac{\frac{1}{2} \cdot 6 \cdot AE \cdot \sin \frac{\alpha}{2}}{\frac{1}{2} \cdot 10 \cdot AE \cdot \sin \frac{\alpha}{2}} = \frac{3}{5}$$

The area of the same triangles can also be computed as $\frac{1}{2}ah$ and also notice that these triangles have the same height. So,

$$\frac{A_{AEB}}{A_{ADE}} = \frac{3}{5} = \frac{\frac{1}{2} \cdot EB \cdot h}{\frac{1}{2} \cdot ED \cdot h} = \frac{EB}{ED}$$

So now we know that the ratio of EB to ED is 3 to 5 and they add up to 10. Then EB is $\frac{3}{8}$ of 10 which is $\frac{15}{4}$.

12. Line L has intercepts 2 and 4, while line M has intercepts 4 and 6. If L and M intersect at (a, b) , which of the following could NOT be $3a + b$?

A. 0 B. 4 C. 8 D. 12 E. 32

Solution: There are four cases to consider.

Case 1. Intercepts of L are $(2, 0)$ and $(0, 4) \implies y = -2x + 4$

Intercepts of M are $(4, 0)$ and $(0, 6) \implies y = -\frac{3}{2}x + 6$

$$\begin{aligned} -2x + 4 &= -\frac{3}{2}x + 6 \\ -2x &= -\frac{3}{2}x + 2 && \text{multiply by 2} \\ -4x &= -3x + 4 \\ -x &= 4 \\ x &= -4 \implies y = 12 \implies 3a + b = 0 \end{aligned}$$

Case 2. Intercepts of L are $(2, 0)$ and $(0, 4)$

Intercepts of M are $(6, 0)$ and $(0, 4)$. The two lines clearly intersect at $(0, 4)$ and so $3a + b = 4$.

Case 3. Intercepts of L are $(4, 0)$ and $(0, 2)$

Intercepts of M are $(4, 0)$ and $(0, 6)$. The two lines clearly intersect at $(4, 0)$ and so $3a + b = 12$.

Case 4. Intercepts of L are $(4, 0)$ and $(0, 2) \implies y = -\frac{1}{2}x + 2$

Intercepts of M are $(6, 0)$ and $(0, 4) \implies y = -\frac{2}{3}x + 4$

$$\begin{aligned} -\frac{1}{2}x + 2 &= -\frac{2}{3}x + 4 \\ -\frac{1}{2}x &= -\frac{2}{3}x + 2 && \text{multiply by 6} \\ -3x &= -4x + 12 \\ x &= 12 \implies y = -4 \implies 3a + b = 32 \end{aligned}$$

The only number we didn't get is 8. The correct answer is C.

13. Sue traveled continuously starting on 1/1/2012. Her first trip was less than 3 months, and each successive trip was 2 days longer than the previous trip. If her last trip ended on 12/31/2012, which of these was the length in days of one of her trips?

A. 54 B. 58 C. 65 D. 72 E. 77

Solution: 2012 was a leap year with 365 days. Recall that the sum of the first n terms in an arithmetic sequence with first element a and common difference d is $s_n = \frac{2a + (n-1)d}{2}n$. In this case, $d = 2$ and $s_n = 365$

$$365 = \frac{2a + (n-1)2}{2}n = (a + (n-1))n = (a + n - 1)n$$

It clear that $n \geq 2$ and so $a + n - 1$ is greater than n . These two are also divisors of 365. The prime-factorization of 365 is $365 = 5 \cdot 73$

$$\begin{array}{rcl} 365 & = & (a + n - 1) \cdot n \\ & & \begin{array}{cc} 365 & 1 \\ 73 & 5 \end{array} \end{array} \quad \begin{array}{l} \text{impossible} \\ \text{since } n \geq 2 \end{array}$$

If $n = 5$ and $a + n - 1 = 73$, then $a + 4 = 73$ so $a = 69$ and so the lengths of the five trips are

$$69, 71, 73, 75, 77$$

and so the correct answer is E.

Solution: 2012 was a leap year with 365 days. Recall that the sum of the first n terms in an arithmetic sequence with first element a and common difference d is $s_n = \frac{2a + (n-1)d}{2}n$. In this case, $d = 2$ and $s_n = 365 + n - 1$

$$\begin{aligned} 364 + n &= \frac{2a + (n-1)2}{2}n = (a + n - 1)n \\ 364 &= (a + n - 1)n - n \\ 364 &= (a + n - 2)n \quad \text{where } n \geq 2 \end{aligned}$$

The prime factorization of 364 is $364 = 2^2 \cdot 7 \cdot 13$.dd $a + 52 - 2 = 7$, Solution is: -43

$$\begin{array}{rcl} 364 & = & (a + n - 2) \cdot n \\ & & \begin{array}{cc} 364 & 1 \\ 182 & 2 \\ 91 & 4 \\ 52 & 7 \\ 7 & 52 \\ 4 & 91 \end{array} \end{array} \quad \begin{array}{l} \text{impossible} \\ \text{since } n \geq 2 \\ \implies n = 2 \implies a = 182 \\ \implies n = 4 \implies a = 89 \\ \implies n = 7 \implies a = 47 \\ \implies n = 52 \implies a = -43 \\ \implies n = 91 \implies \end{array}$$

The first three months are $31 + 29 + 31 = 91$ days long and so $a = 89$ and $a = 47$ are both possible.

2 trips - first is too long

3 trips: $x + x + 2 + x + 4 = 367$ x is not an integer

4 trips: $x + x + 2 + x + 4 + x + 6 = 368$, Solution is: 89

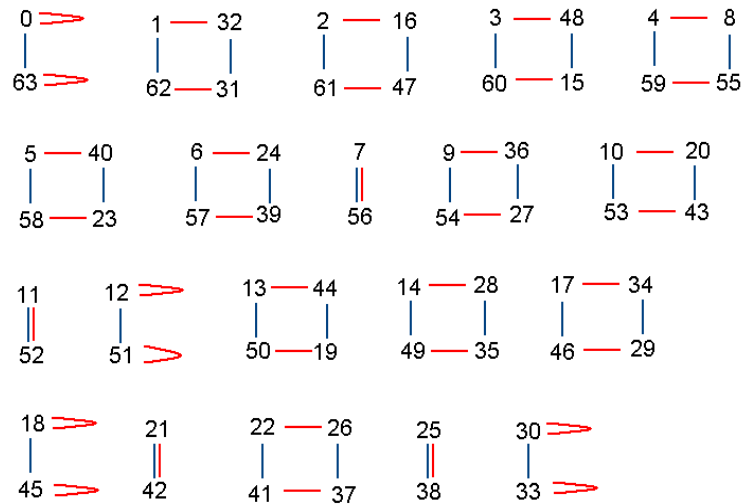
14. A binary string is a sequence of 1's and 0's, such as 10011 or 11101010. How many different binary strings of length 6 are there such that no two are reversals of each other or add up to 111111?

A. 22 B. 23 C. 24 D. 25 E. 26

Solution: There are $2^6 = 64$ six-long binary strings. We will count the complement and subtract it from 64. If we interpret these numbers as written in base 2, they are the numbers from 0 to 63. We can then pair them up so that the pairs add up to 111111 - which is the same as two numbers adding up to 63. So the pairs are 0 with 63, 1 with 62, and so on, 30 with 33, and finally 31 with 32. So if we select one from the pair into our collection, then we cannot select the other. How about the reversal? Each number has a unique number as their reversal. This is another number UNLESS the number is symmetrical and is therefore its own reversal. How many such symmetrical numbers are there? We claim 8. This is because we have complete freedom to select the first three digits - giving us 8 choice but then there is no choice but to duplicate the triple backwards to get a symmetrical string. For example 110 will give us the string 110011. So, out of the 64 strings 8 are symmetrical and so the other 56 can be paired into 28 pairs where

they are each other's reversal. The question is: can we pick just one from each of the 28 pairs so that no two add up to 63?

Solution: The blue lines connect two numbers that add to 111111 and the red lines connect strings that are reversals of each other.



If we wanted to collect numbers in a set such that no two are connected, then

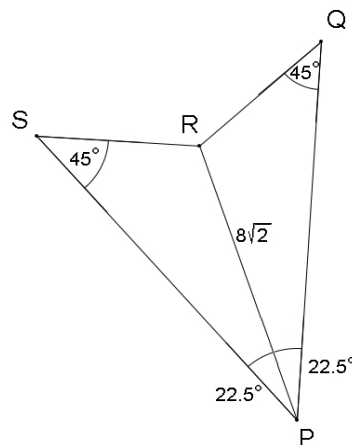
- we can pick one from each pairs such as 0-63 or 7-56
- we can pick exactly two from each of the squares. For example, the first square, containing 1-32-31-62, we can either select the pair 1 and 31 or the pair 32 and 62.

This means that a maximum of 32 such numbers can be collected.

15. In quadrilateral PQRS, $\angle P = \angle Q = \angle S = 45^\circ$, $\angle QPR = \angle RPS$, and $PR = 8\sqrt{2}$. Find the area of quadrilateral PQRS to the nearest integer.

- A. 60 B. 61 C. 62 D. 63 E. 64

Solution: Triangles PQR and RSP are congruent because they have two angles in common and they share side PR. Angle QRS is $360^\circ - 3 \cdot 45^\circ = 225^\circ$. Angle QRP is half of that, 112.5° .



We compute the length of RS using the Law of Sines

$$\frac{RS}{8\sqrt{2}} = \frac{\sin 22.5^\circ}{\sin 45^\circ} \implies RS = 8\sqrt{2} \left(\frac{\sin 22.5^\circ}{\frac{1}{\sqrt{2}}} \right) = 16 \sin 22.5^\circ$$

Recall that the area of triangle ABC is $\frac{1}{2}ab \sin \gamma$.

So triangle RSP has area

$$\begin{aligned} A &= \frac{1}{2} \cdot RS \cdot 8\sqrt{2} \sin 112.5^\circ = \frac{1}{2} \cdot (16 \sin 22.5^\circ) \cdot 8\sqrt{2} \sin 112.5^\circ \\ &= 64\sqrt{2} \sin 22.5^\circ \sin 112.5^\circ \end{aligned}$$

The area of the quadrilateral is twice that,

$$A_{PQRS} = 128\sqrt{2} \sin 22.5^\circ \sin 112.5^\circ \approx 64$$

So the correct answer is E.

16. The numbers 2 and 1 are the smallest positive integers for which the square of the first is 2 more than twice the square of the second. If a and b are the smallest such pair with $a > 10$, find $a - b$.

A. 13 B. 15 C. 17 D. 19 E. 21

Solution: We need to find integers a, b with $a > 10$ and $a^2 = 2b^2 + 2$.

Claim 1. b must be an odd number.

Proof: This is because of rules of divisibility by 4. If we square an odd number, the result will give a remainder 1 after divided by 4. If we square an even number, it will be divisible by 4. (In other words, if x is odd, then $x^2 \equiv 1 \pmod{4}$ and if x is even, then $x^2 \equiv 0 \pmod{4}$) Let n be an integer.

$$\begin{aligned} (2n+1)^2 &= 4n^2 + 4n + 1 = 4(n^2 + n) + 1 \quad \text{has a remainder 1 after division by 4} \\ (2n)^2 &= 4n^2 \quad \text{divisible by 4} \end{aligned}$$

So if b is an even number, then clearly $2b^2$ is divisible by 4 but then $2b^2 + 2$ is even but NOT divisible by 4. But then a^2 even but not divisible by 4 is impossible. There is a shorter way to discuss all this:

remainder of b after division by 4	0	1	2	3
b^2	0	1	0	1
$2b^2 + 2$	2	0	2	0

remainder of a after division by 4	0	1	2	3
a^2	0	1	0	1

The table above shows the same information. It is also pretty clear that a must be even.

Let us now follow what the last digit would be of b , b^2 , and $a^2 = 2b^2 + 2$.

last digit of b	1	3	5	7	9
last digit of b^2	1	9	5	9	1
last digit of $2b^2$	2	8	0	8	2
last digit of $2b^2 + 2$	4	0	2	0	4

last digit of a	0	2	4	6	8
last digit of a^2	0	4	6	6	4

Only 0 and 4 are possible for the last digit of both a^2 and $2b^2 + 2$. This means that either

Case 1. The last digit of b is 1 or 9 and the last digit of a is 4 or 8 or

Case 2. The last digit of b is 3 or 7 and the last digit of a is 0

So the last digit of b is either 1 or 3 or 7 or 9

Let's see.

If $b = 3$ then $2b^2 + 2 = 20$ - not a square

If $b = 7$ then $2b^2 + 2 = 100 = 10^2$ a square but $a > 10$ is not true

If $b = 9$ then $2b^2 + 2 = 164$ - not a square

If $b = 11$ then $2b^2 + 2 = 244$ - not a square

If $b = 13$ then $2b^2 + 2 = 340$ - not a square

If $b = 17$ then $2b^2 + 2 = 580$ - not a square

If $b = 19$ then $2b^2 + 2 = 724$ - not a square

If $b = 21$ then $2b^2 + 2 = 884$ - not a square

If $b = 23$ then $2b^2 + 2 = 1060$ - not a square

If $b = 27$ then $2b^2 + 2 = 1460$ - not a square

If $b = 29$ then $2b^2 + 2 = 1684$ - not a square

If $b = 31$ then $2b^2 + 2 = 1924$ - not a square

If $b = 33$ then $2b^2 + 2 = 2180$ - not a square

If $b = 37$ then $2b^2 + 2 = 2740$ - not a square

If $b = 39$ then $2b^2 + 2 = 3044$ - not a square

If $b = 41$ then $2b^2 + 2 = 3364 = 58^2$ - so $a = 58$

Then $a - b = 58 - 41 = 17$ and so the correct answer is C.

17. A number is chosen at random from among all 5-digit numbers containing exactly one each of the digits 1, 2, 3, 4, and 5. Find the probability that no two adjacent digits in the number are consecutive integers.

- A. $\frac{1}{10}$ B. $\frac{7}{60}$ C. $\frac{2}{15}$ D. $\frac{3}{20}$ E. $\frac{1}{6}$

Solution: If all digits are to appear only once, the total number of such 5-digit numbers is $5! = 120$. That is the denominator. For the numerator, we need to count all permutations of 1, 2, 3, 4, and 5 in which no two adjacent digits in the number are consecutive integers. Notice that 1 and 5 have only one "neighbor" while 2, 3, and 4 have two.

We will count these numbers, preferably in an increasing order.

Case 1. Suppose the number starts with 1. After 1 we have 3 choices: 3, 4, or 5

13524 after 3 we can only pick 5 and then we can only pick 2 and so on

14253

15 - now we are stuck with 3 consecutive integers - so, no number

3) 24135

4) 24153

5) 25314

6) 31425

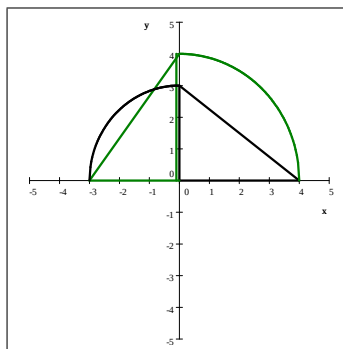
- 7) 31524
- 8) 35142
- 9) 35241
- 10) 41352
- 11) 42513
- 12) 42531
- 51 - nope
- 13) 52413
- 14) 53142

and so the probability is $\frac{14}{120} = \frac{7}{60}$ which is choice B.

18. The triangular region with vertices $(0, 0)$, $(4, 0)$, and $(0, 3)$ is rotated 90° counterclockwise around the origin. Find the area of the figure formed by this rotation to the nearest hundredth.

- A. 19.96 B. 20.04 C. 20.12 D. 20.20 E. 20.28

Solution:

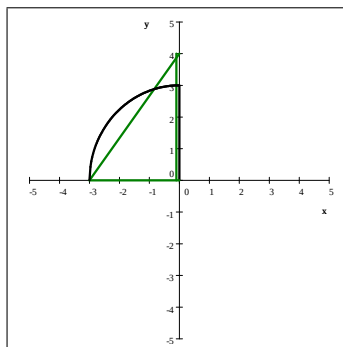


The area in the first quadrant is just a quarter of a circle with radius 4 and so the area is $A_1 = \frac{1}{4}(4)^2\pi = 4\pi$. The area in the second quadrant, denoted by A_2 is more complex:

Let A_T denote the area of the triangle defined by $(0, 0)$, $(0, 4)$, and $(-3, 0)$. Let A_C denote the area of the quarter of the circle in the second quadrant and let A_{int} denote the area of the intersection of these two. The area we are looking for is

$$A_2 = A_T + A_C - A_{\text{int}}$$

Clearly $A_T = \frac{1}{2}(3)(4) = 6$ and $A_C = \frac{1}{4}(3)^2\pi = \frac{9}{4}\pi$. The real challenge is to compute the area of the intersection



First we will compute the coordinates of the point where the circle and the line intersect each other. This means finding the other solution (other than $(-3, 0)$) of the system
$$\begin{cases} x^2 + y^2 = 9 \\ y = \frac{4}{3}x + 4 \end{cases}$$

We will use substitution. Because we know one of the solutions of this system is $(-3, 0)$, we will try to solve for y at the end. Quadratic equations are easy to factor if one of the roots is zero. So we first solve for x in terms of y and substitute that into the quadratic equation.

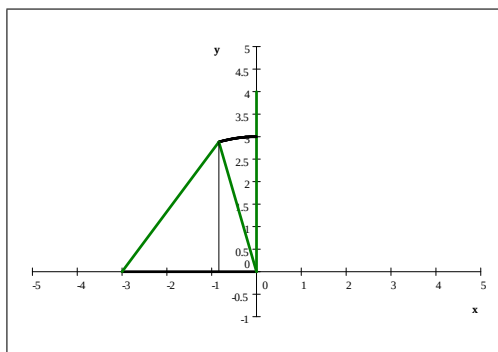
$$y = \frac{4}{3}x + 4 \quad \implies \quad x = \frac{3}{4}(y - 4) = \frac{3}{4}y - 3$$

$$\begin{aligned} x^2 + y^2 &= 9 \\ \left(\frac{3}{4}y - 3\right)^2 + y^2 &= 9 \\ \frac{9}{16}y^2 - \frac{9}{2}y + 9 + y^2 &= 9 \\ \frac{9}{16}y^2 - \frac{9}{2}y + 9 + y^2 &= 9 \\ \frac{25}{16}y^2 - \frac{9}{2}y &= 0 \\ \frac{25}{16}y \left(y - \frac{\frac{9}{2}}{\frac{25}{16}}\right) &= 0 \qquad \frac{\frac{9}{2}}{\frac{25}{16}} = \frac{9}{2} \cdot \frac{16}{25} = \frac{72}{25} \end{aligned}$$

$$y_1 = 0 \quad \text{and} \quad y_2 = \frac{72}{25} = 2.88$$

Then $x_2 = \frac{3}{4}y_2 - 3 = \frac{3}{4}(2.88) - 3 = -0.84$. So the other intersection point is $(-0.84, 2.88)$.

We separate the two areas as shown below. The area of the triangle is $\frac{1}{2}(3)(2.88) = 4.32$ as the height belonging to the 3 unit long side is 2.88. For the area of the sector, we need to find first the central angle.



Using right triangle trigonometry, we write

$$\tan \alpha = \frac{0.84}{2.88} = \frac{7}{24} \quad \implies \quad \alpha = \tan^{-1}\left(\frac{7}{24}\right)$$

So the area of the sector is

$$A_{\text{sector}} = \frac{r^2\theta}{2} = \frac{(3)^2}{2} \tan^{-1}\left(\frac{7}{24}\right) = \frac{9}{2} \tan^{-1}\left(\frac{7}{24}\right) \approx 3.42966$$

So the area of the intersection is

$$A_{\text{int}} = 4.32 + \frac{9}{2} \tan^{-1} \left(\frac{7}{24} \right)$$

We are ready to compute the area:

$$A = A_1 + A_2 = A_1 + A_T + A_C - A_{\text{int}} = 4\pi + 6 + \frac{9}{4}\pi - \left(4.32 + \frac{9}{2} \tan^{-1} \left(\frac{7}{24} \right) \right) \approx 20.0378806$$

After rounding to the nearest hundredth, we get 20.04 which is choice B.

19. For how many pairs of positive integers (n, m) with $n, m < 100$ are both of the polynomials $x^2 + mx + n$ and $x^2 + mx - n$ factorable over the integers?

A. 4 B. 5 C. 6 D. 7 E. 8

Solution: Let s_1, s_2 and t_1, t_2 be the integers in the factored form, i.e.

$$x^2 + mx + n = (x - s_1)(x - s_2) \quad \text{and} \quad x^2 + mx - n = (x - t_1)(x - t_2)$$

Then

$$x^2 + mx + n = x^2 - (s_1 + s_2)x + s_1s_2 \quad \text{and} \quad x^2 + mx - n = x^2 - (t_1 + t_2)x + t_1t_2$$

$$\begin{aligned} -(s_1 + s_2) &= m \\ s_1s_2 &= n \\ -(t_1 + t_2) &= m \\ t_1t_2 &= -n \end{aligned}$$

$m^2 - 4n$ a square and $m^2 + 4n$ a square

$$m^2 - 4n = A^2$$

$$m^2 + 4n = B^2$$

$$B^2 - A^2 = 8n$$

$$(B + A)(B - A) = 8n$$

20. Triangles ACD and BCD ($AD = 14$, $BD = 40$) are inscribed in a semicircle with diameter $CD = 50$. If $AB > 25$, find the area of their union.
21. A. 625 B. 637.5 C. 652.5 D. 673.5 E. 675