

Test #2. AMATYC Student Mathematics League, Feb/Mar 2014

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## Solutions

Note. For problem 20, my answer differs from the listed answer. Please let me know if you find errors in my reasoning.

1. A store advertises, "We pay the sales tax!" If sales tax is 8%, what discount to the buyer to the nearest tenth of a percent does this represent?

- A. 7.4%    B. 7.5%    C. 7.6%    D. 7.7%    E. 7.8%

*Solution.* Denote by  $A$  the initial prize of an item and by  $D$  the store discount on the initial item prize. Then  $A = (A - D)1.08 \Rightarrow \frac{D}{A} = \frac{0.08}{1.08} \Rightarrow 7.4\%$

Answer - A

2. The lines with equations  $ax + 2y = c$  and  $bx - 3y = d$  are perpendicular. Find  $a \cdot b$ .

- A. -6    B. -1.5    C. -1    D. 1.5    E. 6

*Solution.* The slopes of the equations are  $m_1 = -\frac{a}{2}, m_2 = \frac{b}{3}$ . Since the lines are perpendicular, then  $-1 = m_1 \cdot m_2 = (-\frac{a}{2})(\frac{b}{3}) = -\frac{ab}{6}$ . Thus  $ab = 6$ .

Answer - E

3. Sue owes \$12,000 on a loan. She makes monthly payments of \$200, and \$10 interest is added each month to her balance. In how many months is the loan paid off?

- A. 60    B. 61    C. 62    D. 63    E. 64

*Solution.* Denote by  $n$  the number of months the loan be paid off. Each month, except the last month, the outstanding balance on Sue's loan decreases by  $\$200 - \$10 = \$190$ . At the the last month Sue pays the difference  $0 \leq 12000 - (n - 1)190 \leq 200$ . Solving this double inequality for  $n$ , we get  $\frac{11800}{190} + 1 \leq n \leq \frac{12000}{190} + 1$ , or  $63 < n < 65$ . Since  $n$  is integer, then  $n = 64$

Answer - E

4. The polynomial  $3x^2 + 4xy - 4y^2$  can be factored as the product of two first-degree polynomials. The sum of the two factors is

- A.  $4x$     B.  $4y$     C.  $2x$     D.  $2x + 2y$     E.  $4x + 4y$

*Solution.* We have  $3x^2 + 4xy - 4y^2 = 3x^2 + (6xy - 2xy) - 4y^2 = (3x^2 + 6xy) - (2xy + 4y) = 3x(x + 2y) - 2y(x + 2y) = (x + 2y)(3x - 2y)$ . The sum of the factors is  $(x + 2y) + (3x - 2y) = 4x$

Answer - A

5. The lines with equations  $2x + 3y = 6$  and  $x + 2y = 5$  intersect at the point  $(a, b)$ . The sum  $a + b$  equals

- A. -2    B. -1    C. 0    D. 1    E. 2

*Solution.* Subtracting from the first equation the second equation, we obtain  $x + y = 1$

Answer - D

6. A domino is a  $1 \times 2$  rectangle. When 8 dominos are formed into all possible rectangles with no spaces or gaps, let  $P$  be the greatest possible perimeter and  $p$  the least possible perimeter. Find  $P/p$ .

- A. 1.25    B. 1.75    C. 2    D. 2.125    E. 2.375

*Solution.* The area of a possible rectangle is  $8(1 \cdot 2) = 16$ . The dimensions of possible rectangles, being positive factorings of 16, are  $16 = 16 \cdot 1 = 8 \cdot 2 = 4 \cdot 4$ . The corresponding perimeters of possible rectangles are  $2(16 + 1) = 34, 2(8 + 2) = 20, 2(4 + 4) = 16$ . The required ratio is  $\frac{34}{16} = 2.125$

Answer - D

7. The 5-digit number  $217xy$  has 5 different digits and a factor of 45. Find  $x + y$ .

- A. 8    B. 9    C. 10    D. 11    E. 12

*Solution.* Since  $217xy$  is divided by 9, then  $2 + 1 + 7 + x + y$  is divided by 9, or  $1 + x + y$  is divided by 9. Thus,  $x + y = 8$ , or  $x + y = 17$ . Since  $217xy$  is divided by 5, then  $y = 0$  or 5. Thus  $x + y = 8$

Answer - A

8. Ed and Em order sodas at the 8-12 store. After Ed drinks half of his and Em drinks  $\frac{1}{3}$  of hers, they have the same number of ounces of soda left. If the two sodas totaled 28 oz. originally, how many ounces of soda total do the two of them have left?

- A. 12    B. 15    C. 16    D. 18    E. 20

*Solution.* Denote by  $x, y$  the amounts ordered by Ed and Em. Ed drank  $\frac{1}{2}x$ , Em drank  $\frac{1}{3}y$ . Since they had the same amounts left, then  $\frac{1}{2}x = \frac{2}{3}y$ . As  $y = 28 - x$ , then  $\frac{1}{2}x = \frac{2}{3}(28 - x)$ , and  $x = 16$ . The total amount left is  $2(\frac{1}{2}x) = 16$

Answer - C

9. Let  $S = \{3, 5, 7, 11, 13, 17\}$ . How many elements of  $S$  are factors of  $2^{60} - 1$ ?

- A. 2    B. 3    C. 4    D. 5    E. 6

*Solution.* We have  $2^{60} - 1 = (2^{30} - 1)(2^{30} + 1) = 1073741823 \cdot 1073741825$ . Denote  $x = 1073741823$ ,  $y = 1073741825$ . We check that 3 divides  $X$ , 5 divides  $Y$ , 7 divides  $X$ , 11 divides  $X$ , 13 divides  $Y$ , 17 divides neither  $X$ , nor  $Y$ .

Answer - D

10. On Jan. 27, postal rates rose from  $46\text{¢}$  to  $49\text{¢}$  an ounce. Vi buys some new  $49\text{¢}$  stamps and some  $3\text{¢}$  stamps to use with her leftover  $46\text{¢}$  stamps. If she spends \$4.10 and buys more  $49\text{¢}$  stamps than  $3\text{¢}$  stamps, how many stamps does she buy?

- A. 12    B. 14    C. 16    D. 18    E. it cannot be determined

*Solution.* Denote by  $x, y$  the corresponding number of  $46\text{¢}$  and  $3\text{¢}$  stamps bought. Then  $49x + 3y = 410$ . We have to find nonnegative integer solutions  $x, y$  of this equations, provided that  $x > y$ . Estimate  $49x \leq 49x + 3y = 410 \Rightarrow x \leq \frac{410}{49} < 9, 410 = 49x + 3y \leq 52x \Rightarrow x \geq \frac{410}{52} > 7$ . The only possible value of  $x$  is  $x = 8$ . From the equation  $49x + 3y = 410$  it follows that  $y = \frac{410 - 49 \cdot 8}{3} = 6$ . The total number of stamps is  $8 + 6 = 14$

Answer - B

11. The equation  $a^4 + b^2 + c^2 = 2014$  has a unique solution in positive integers. For this solution, find  $a + b + c$ .

A. 56      B. 58      C. 60      D. 62      E. 64

*Solution.* From the equation it follows that  $a^4 \leq 2014$ , that is  $a < \sqrt[4]{2014} < 7$ . Since  $a$  is a positive integer, then  $a \in \{1, 2, 3, 4, 5, 6\}$ . For each of the feasible values of  $a$  we will numerically find the positive integer solutions  $(b, c)$  of the equation  $b^2 + c^2 = 2014 - a^4$  rewritten in the form  $c(b) = \sqrt{2014 - a^4 - b^2}$ . The values of  $b$  are limited by  $b \leq \sqrt{2014 - a^4 - c^2} < \sqrt{2014} < 45$ . Since  $b$  is a positive integer, then  $1 \leq b \leq 45$ .

We will employ graphing calculator model  $TI-83/84$ . For a fixed numerical value  $a \in \{1, 2, 3, 4, 5, 6\}$ , type the function  $Y_1 = \sqrt{(2014 - A^4 - X^2)}$ . In the *TABLE* facility look for the integer value of  $Y_1$  in the set of integer  $1 \leq X \leq 45$ .

Case  $A = 1$ . No integer solution.

Case  $A = 2$ . No integer solution.

Case  $A = 3$ .  $(X = 13, Y_1 = 42), (X = 42, Y_1 = 13)$

Case  $A = 4$ . No integer solution.

Case  $A = 5$ . No integer solution.

Case  $A = 6$ . No integer solution.

The integer solutions  $(a, b, c) = (3, 13, 42), (3, 42, 13)$ . The value  $a + b + c = 58$

Answer - B

**12.** Different letters are placed on the 18 faces of 3 standard 6-sided dice, one per face. Choosing 1 letter from each die, the following words can be formed: bow, boy, cot, dry, gas, hat, oat, old, one, pay, pie, red, six. Which of the following could also be spelled?

A. eat      B. rap      C. top      D. wad      E. won

*Solution.* Arbitrary, one of the dice we denote by (A), another by (B), and the last by (C). We have to find the letters on each dice.

We consider letter cases over the given words. For each case, we create distributions of letters over the dices.

1) word **bow**. All three letters belong to different dice. Let they have distribution A)b; B)o; C)w

2) word **boy**. From letter distribution 1) we have A)b; B)o; C)w,y

3) words **oat**, **pay**. From **oat** and the above, **a** is in (A) or (C). From **pay** and the above, **a** is in (A), or (B). Then **a** is in (A). Letter distribution is A)b,a B)o C)w,y

3) words **oat**, **pay**. From the distribution above, **t** is in (C), and **p** is in (B). We have A)b,a B)o,p C)w,y,t

4) words **cot**, **hat**. From the distribution above, **c** is (A), **h** is in (B). We have A)b,a,c B)o,p,h C)w,y,t

5) words **dry**, **old**. From **dry** and the above, **d** is in (A) or (B). From **old** and the above, **d** is in (A) or (C). Then **d** is in (A). Letter distribution is A)b,a,c,d B)o,p,h C)w,y,t

6) words **dry**, **old**. From the distribution above, **r** is in (B), and **l** is in (C). We have A)b,a,c,d B)o,p,h,r C)w,y,t,l

7) word **red**. From above, **e** is in (C). We have A)b,a,c,d B)o,p,h,r C)w,y,t,l,e

8) words **one**, **pie**. From above, **n** is in (A), **i** is in (A). We have A)b,a,c,d,n,i B)o,p,h,r C)w,y,t,l,e

9) words **gas**, **six**. From **gas** and the above, **g,s** should be in different (B),(C). From **six** and the above, **s,x** should be in different (B),(C). Then it follows that **s** is in (C) and **g,x** are in (B)

The final distributions of letters is A)b,a,c,d,n,i B)o,p,h,r,g,x C)w,y,t,l,e,s

Words **eat**, **rap**, **top**, **wad** cannot be spelled using different dices, but the word **won** can

Answer - E.

**13.** The fraction  $\frac{a}{b}$  is 0.455 when rounded to 3 decimal places. If  $\frac{a+1}{b+1}$  is 0.467 when rounded to 3 decimal places, find  $a + b$ .

A. 63 B. 64 C. 65 D. 66 E. 67

*Solution.* By the conditions of the problem,  $0.4545 \leq \frac{a}{b} \leq 0.4554 \Rightarrow 0 \leq \frac{a}{b} - 0.4545 \leq 0.0009$ . Denote  $\frac{a}{b} - 0.4545 = \delta, 0 \leq \delta \leq 0.0009$ . We have  $\frac{a}{b} = 0.4545 + \delta \Rightarrow a = (0.4545 + \delta)b$ . Let  $A = 0.4545 + \delta$ , so that  $a = Ab$ .

By the conditions of the problem,  $0.4665 \leq \frac{a+1}{b+1} \leq 0.4674 \Rightarrow 0 \leq \frac{a+1}{b+1} - 0.4665 \leq 0.0009$ . Denote  $\frac{a+1}{b+1} - 0.4665 = \varepsilon, 0 \leq \varepsilon \leq 0.0009$ . We have  $\frac{a+1}{b+1} = 0.4665 + \varepsilon \Rightarrow a + 1 = (0.4665 + \varepsilon)(b + 1)$ . Let  $B = 0.4665 + \varepsilon$ , so that  $a + 1 = B(b + 1)$ .

Substituting here the expression  $a = Ab$ , we get  $Ab + 1 = B(b + 1)$ . Solving this equation for  $b$ , we obtain  $b = \frac{1-B}{B-A} = \frac{1-(0.4665+\varepsilon)}{(0.4665+\varepsilon)-(0.4545+\delta)} = \frac{0.5335-\varepsilon}{0.012+(\varepsilon-\delta)}$ . Using the last equality, estimate the value of  $b$ . We have  $\frac{0.5335-0.0009}{0.012+(0.0009-0)} \leq b \leq \frac{0.5335-0}{0.012+(0-0.0009)} \Rightarrow 41 < b < 49$ . Since  $b$  is an integer, then  $42 \leq b \leq 48$ .

We have  $a = Ab = (0.4545 + \delta)b$ , so that  $0.4545b \leq a \leq (0.4545 + 0.0009)b$ , or  $0.4545b \leq a \leq 0.4554b$ . Find possible integer solutions  $a$  of this double inequality using the above possible values of  $b$ .

Case  $b = 42$ . We have  $0.4545 \cdot 42 \leq a \leq 0.4554 \cdot 42 \Rightarrow 19.089 \leq a \leq 19.1268$ . No integer solution for  $a$ .

Case  $b = 43$ . We have  $0.4545 \cdot 43 \leq a \leq 0.4554 \cdot 43 \Rightarrow 19.5435 \leq a \leq 19.5822$ . No integer solution for  $a$ .

Case  $b = 44$ . We have  $0.4545 \cdot 44 \leq a \leq 0.4554 \cdot 44 \Rightarrow 19.998 \leq a \leq 20.0375$ . Solution  $a = 20$ .

Case  $b = 45$ . We have  $0.4545 \cdot 45 \leq a \leq 0.4554 \cdot 45 \Rightarrow 20.4525 \leq a \leq 20.493$ . No integer solution for  $a$ .

Case  $b = 46$ . We have  $0.4545 \cdot 46 \leq a \leq 0.4554 \cdot 46 \Rightarrow 20.907 \leq a \leq 20.9484$ . No integer solution for  $a$ .

Case  $b = 47$ . We have  $0.4545 \cdot 47 \leq a \leq 0.4554 \cdot 47 \Rightarrow 21.3615 \leq a \leq 21.4038$ . No integer solution for  $a$ .

Case  $b = 48$ . We have  $0.4545 \cdot 48 \leq a \leq 0.4554 \cdot 48 \Rightarrow 21.816 \leq a \leq 21.8592$ . No integer solution for  $a$ .

The only solution  $(a, b) = (20, 44)$ , and  $a + b = 64$

Answer -B

The case calculations can be simplified by using graphing calculator *TI-83/84*. Type  $\backslash Y_1 = 0.4545X$ ,  $\backslash Y_2 = 0.4554X$  and use *Table* facility of the calculator.

**14.** If  $ax + b = 15$  and  $15x + a = b$  have the same unique solution, where  $a$  and  $b$  are positive integers both less than or equal to 30, find the sum of all

possible values of  $a$ .

- A. 28      B. 43      C. 58      D. 78      E. 93

*Solution.* From the above equations eliminate  $x$  to obtain  $b = \frac{a^2+15^2}{a+15} \equiv a - 15 + \frac{450}{a+15} \geq 0$  for  $a \geq 0$ . To simplify calculations to integer solutions  $a$ , employ graphing calculator  $TI-83/84$ . Type  $\backslash Y_1 = \frac{450}{X+15}$  and use *TABLE* facility for  $1 \leq X \leq 30$  to obtain positive integer values of  $X$  for which  $Y$  is an integer. Calculations show that possible values of  $X$  are 3, 10, 15, 30. Their sum is 58

Answer - C

**15.** If  $(r, s, t, u, v)$  satisfies the system

$$3r + 10s + 16t + 30u + 25v = 10$$

$$4r + 15s + 20t + 36u + 36v = 11$$

$$5r + 20s + 24t + 42u + 49v = 20,$$

then the value of  $6r + 25s + 28t + 48u + 64v$  is

- A. 33      B. 34      C. 35      D. 36      E. 37

*Solution.* From two times the second equation subtract the sum of the first and the third equation to obtain  $-2v = -8$ , or  $v = 4$ . The system is simplified to

$$3r + 10s + 16t + 30u = -90$$

$$4r + 15s + 20t + 36u = -133$$

$$5r + 20s + 24t + 42u = -176$$

In the last system, the third equation is two times the second equation minus the first equation and can be removed. We will have

$$3r + 10s + 16t + 30u = -90$$

$$4r + 15s + 20t + 36u = -133$$

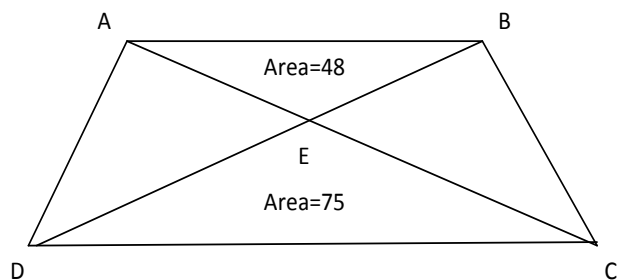
Multiply the first equation above by  $-2$ , multiply the second equation above by 3, and add the products to obtain  $-2(3r + 10s + 16t + 30u) + 3(4r + 15s + 20t + 36u) = (-2)(-90) + 3(-133) \Rightarrow 6r + 25s + 28t + 48u = -219$ . Since  $v = 4$ , then  $6r + 25s + 28t + 48u + 64v = -219 + 64(4) \equiv 37$

Answer - E.

**16.** In trapezoid  $ABCD$ ,  $\overline{AB} \parallel \overline{CD}$  and  $E$  is the point of intersection of  $\overline{AC}$  and  $\overline{BD}$ . If the area of  $\triangle CDE$  is 75 and the area of  $\triangle ABE$  is 48, find the area of the trapezoid.

- A. 216      B. 225      C. 240      D. 243      E. 246

*Solution.* Triangles  $\triangle DEC$ ,  $\triangle ABE$  are congruent as having equal corresponding angles. So  $\frac{|CE|}{|AE|} = \frac{|DE|}{|BE|} = k$ , where  $k$  is a constant. Comparing the given areas of the triangles,



$$\frac{75}{48} = \frac{\frac{1}{2}|CE||DE|\sin(\angle DEC)}{\frac{1}{2}|AE||BE|\sin(\angle AEB)} = k^2, \text{ and } k = \sqrt{\frac{75}{48}} = \frac{5}{4}.$$

$$\text{Then } \text{area}(\triangle ADE) = \frac{1}{2}|DE||AE|\sin(\angle AED) = \frac{1}{2}(k|BE||AE|\sin(180^\circ - \angle AEB)) = k(\frac{1}{2}|BE||AE|\sin \angle AEB) = k(\text{area}(\triangle ABE)) = \frac{5}{4}48 = 60$$

$$\text{and } \text{area}(\triangle BCE) = \frac{1}{2}|BE||CE|\sin(\angle BEC) = \frac{1}{2}|BE|(k|AE|)\sin(180^\circ - \angle AEB) = k(\frac{1}{2}|BE||AE|\sin \angle AEB) = k(\text{area}(\triangle ABE)) = \frac{5}{4}48 = 60$$

$$\text{The total area is } 75 + 48 + 60 + 60 = 243$$

Answer - D

**17.** There is a unique integer  $N$  with the property that  $N$  has the 4-digit representation  $pqrs$  in base 7 and the 4-digit representation  $qrst$  in base 9 ( $p \neq 0, q \neq 0$ ). Write the base-10 representation of  $N$  in the corresponding blank on the answer sheet.

*Solution.* We have  $N = pqrs_7 =qrst_9$ , or  $N = (p7^3 + q7^2 + r7 + s)_{10} = (q9^3 + r9^2 + s9 + p)_{10}$ , where  $1 \leq p, q \leq 6; 0 \leq r, s \leq 6$ . Rewrite the last equality as  $(9^3 - 7^3)q + (9^2 - 7^2)r + (9 - 1)s = (7^3 - 1)p$ , or  $680q + 74r + 8s = 342p$ . We have  $680q \leq 342p \leq 342 \cdot 6 = 2052$ , or  $q \leq \frac{2052}{680} < 4$ . Thus the possible values of  $q$  are 1, 2, 3

Case  $q = 3$ . We have  $680 \cdot 3 + 74r + 8s = 342p$ , or  $74r + 8s + 2014 = 342p$ . We have  $342p \geq 2014$ , or  $p \geq \frac{2014}{342} > 5$ . Thus  $p = 6$ . We have  $74r + 8s + 2014 = 342 \cdot 6$ , or  $74r + 8s = 38$ . Then it follows that  $r = 0$  and  $s$  is not integer. Contradiction.

Case  $q = 2$ . We have  $680 \cdot 2 + 74r + 8s = 342p$ , or  $74r + 8s + 1360 = 342p$ . Then  $342p \geq 1360$ , or  $p \geq \frac{1360}{342} > 3$ . Thus  $p = 4, 5, 6$

For numerical solution we will use graphing calculator  $TI-83/84$ . Rename variables  $r = X, s = Y, p = A$ , and rewrite the previous equation in the form  $Y = \frac{342A - 1360 - 74X}{8}$ . Type this function into the calculator, sequentially assign to  $A$  values 4, 5, 6 and use the *TABLE* facility of the calculator. For  $0 \leq X \leq 6$  select only integer solutions  $0 \leq Y \leq 6$ . Calculations show that for  $q = 2$  the only solution  $(p, q, r, s) = (A, 2, X, Y)$  is  $(4, 2, 0, 1)$ . Thus  $N = p7^3 + q7^2 + r7 + s = 4 \cdot 7^3 + 2 \cdot 7^2 + 0 \cdot 7 + 1 = 1471$ .

Case  $q = 1$ . We have  $680 \cdot 1 + 74r + 8s = 342p$ , or  $74r + 8s + 680 = 342p$ . As in the previous case, form function  $Y = \frac{342A - 680 - 74X}{8}$ . Type this function into your calculator, sequentially assign to  $A$  values 1, 2, 3, 4, 5, 6 and use the *TABLE* facility of the calculator. For  $0 \leq X \leq 6$  select only integer solutions  $0 \leq Y \leq 6$ . Calculations show that in this case there are no solutions.

Answer -  $N = 1471$

**18.** In *approval voting*, each voter can distribute up to 5 votes among 6 candidates. For example, you could cast 3 votes for one candidate and 2 for

another, or you could cast 1 vote for each of 4 candidates (and not cast your fifth vote). In how many ways can you distribute your votes?

- A. 252    B. 256    C. 462    D. 480    E. 720

*Solution.* Let  $(A, B, C, D, E, F, G)$  be any valid distribution of votes, where  $A, B, C, D, E, F$  are corresponding numbers of votes for 1,2,3,4,5,6 candidates, and  $G$  is a number of unused votes. Then  $A + B + C + D + E + F + G = 5$ , so that  $A, B, C, D, E, F, G$  satisfy the conditions  $0 \leq A \leq 5$ ,  $0 \leq B \leq 5 - A$ ,  $0 \leq C \leq 5 - A - B$ ,  $0 \leq D \leq 5 - A - B - C$ ,  $0 \leq E \leq 5 - A - B - C - D$ ,  $0 \leq F \leq 5 - A - B - C - D - E$ ,  $G = 5 - A - B - C - D - E - F$ .

To find the total number of valid distributions I wrote and run the following simple program *S2014P18* in graphing calculator *TI-83/84*

```
:ClrHome
:Disp "——START——"
:0->S
:For (A,0,5)
:For (B,0,5-A)
:For (C,0,5-(A+B))
:For (D,0,5-(A+B+C))
:For (E,0,5-(A+B+C+D))
:For (F,0,5-(A+B+C+D+E))
S+1->S
:END
:END
:END
:END
:END
:END
:Disp "S:", S
:Disp "——END——"
```

—  
The displayed answer - 462, run time - 10 sec

Answer - C

**19.** The polynomial  $P(x) = x^4 + mx^3 + nx^2 - 24x + 144$  has exactly 2 distinct integer roots, and no other roots, real or complex. Find  $m + n$ .

- A. -27    B. -25    C. -23    D. -21    E. -19

*Solution.* The polynomial should have a form  $P(x) = (x - a)(x - b)^3$ , or  $P(x) = (x - a)^2(x - b)^2$ , where  $a, b$  are integers

1. Case  $P(x) = (x - a)(x - b)^3$ . Simplifying, we obtain  $P(x) = x^4 - (a + 3b)x^3 + 3b(a + b)x^2 - b^2(3a + b)x + ab^3$ . Since  $ab^3 = 144 = (2^4)(3^2) = (18)(2^3) = (144)(1)^3$ , then  $(a, b)$  should be one of the  $(18, 2), (-18, -2), (144, 1), (-144, -1)$ .

For each of these subcases calculations show that  $-b^2(3a + b)x \neq -24x$

2. Case  $P(x) = (x - a)^2(x - b)^2$ . Simplifying, we obtain  $P(x) = x^4 - 2(a + b)x^3 + (a^2 + 4ab + b^2)x^2 - 2ab(a + b)x + a^2b^2$ . Since  $a^2b^2 = 144 = (\pm 2)^4(\pm 3)^2 = (\pm 4)^2(\pm 3)^2 = (\pm 2)^2(\pm 6)^2 = (\pm 1)^2(\pm 12)^2$ , then, up to the symmetry,  $(a, b)$  is one of the  $(\pm 4, \pm 3), (\pm 2, \pm 6), (\pm 1, \pm 12)$

For each of these subcases only  $(a, b) = (-4, 3)$  satisfies condition  $-b^2(3a + b)x = -24x$ . Taking  $a = -4, b = 3$ , calculate  $m = -2(a + b) = 2, n = a^2 + 4ab + b^2 = -23, m + n = 2 + (-23) = -21$

Answer - D

**20.** A subset  $S$  of  $\{1, 2, 3, \dots, n\}$  is called *odd-neighbored* if for each even number  $k$  in  $S$ , if  $k < n$  then  $S$  contains both  $k - 1$  and  $k + 1$ , and if  $k = n$  then  $S$  contains  $k - 1$ . For example,  $\emptyset, \{1, 3, 5, 7\}, \{1, 2, 3, 5\}$ , and  $\{3, 4, 5, 7, 8\}$  are all odd-neighbored subsets of  $\{1, 2, 3, \dots, 8\}$ . Find the number of nonempty odd-neighbored subsets of  $\{1, 2, 3, \dots, 12\}$ .

A. 232    B. 264    C. 324    D. 376    E. 432

*Solution.* For any set  $C$  denote by  $|C|$  the number of elements in  $C$ . We assume that all sets below are arranged in ascending order of their elements.

Denote by  $J_n$  the set of all nonempty odd-neighbored subsets of  $\{1, 2, 3, \dots, n\}$ . Denote by  $K_n$  the set of all nonempty odd-neighbored subsets of  $\{1, 2, 3, \dots, n\}$  whose last element is  $n$ . Denote  $A_n = |J_n|, B_n = |K_n|$ . It is clear that  $J_n = \cup_{k=1}^n K_k = J_{n-1} + K_n$  and  $A_n = A_{n-1} + B_n$ . We have to find the expression for  $B_n$ .

Case  $n$  is even. If  $S \in K_n$ , then  $S = G \cup \{n\}$ , where  $G \in K_{n-1}$ . We have  $B_n = B_{n-1}$ .

Case  $n$  is odd. If  $S \in K_n$ , then  $S = G \cup \{n\}$ , where  $G \in K_m$  with  $(m \leq n - 2$  and  $m$  is odd) or  $(m = n - 1)$ . Thus we have  $B_3 = B_1 + B_2; B_n = B_1 + B_3 + \dots + B_{n-2} + B_{n-1} = B_1 + B_3 + \dots + B_{n-4} + B_{n-3} + (B_{n-1} + B_{n-2} - B_{n-3}) = B_{n-2} + (B_{n-1} + B_{n-2} - B_{n-3}) = B_{n-1} + 2B_{n-2} - B_{n-3}, n \geq 5$

Now starting with  $J_1 = \{\{1\}\}, K_1 = \{\{1\}\}, A_1 = 1, B_1 = 1$ , we calculate

$$B_2 = B_1 = 1, A_2 = A_1 + B_2 = 1 + 1 = 2$$

$$B_3 = B_1 + B_2 = 1 + 1 = 2, A_3 = A_2 + B_3 = 2 + 2 = 4$$

$$B_4 = B_3 = 2, A_4 = A_3 + B_4 = 4 + 2 = 6$$

$$B_5 = B_4 + 2B_3 - B_2 = 2 + 2 \cdot 2 - 1 = 5, A_5 = A_4 + B_5 = 6 + 5 = 11$$

$$B_6 = B_5 = 5, A_6 = A_5 + B_6 = 11 + 5 = 16$$

$$B_7 = B_6 + 2B_5 - B_4 = 5 + 2 \cdot 5 - 2 = 13, A_7 = A_6 + B_7 = 16 + 13 = 29$$

$$B_8 = B_7 = 13, A_8 = A_7 + B_8 = 29 + 13 = 42$$

$$B_9 = B_8 + 2B_7 - B_6 = 13 + 2 \cdot 13 - 5 = 34, A_9 = A_8 + B_9 = 42 + 34 = 76$$

$$B_{10} = B_9 = 34, A_{10} = A_9 + B_{10} = 76 + 34 = 110$$

$$B_{11} = B_{10} + 2B_9 - B_8 = 34 + 2 \cdot 34 - 13 = 89, A_{11} = A_{10} + B_{11} = 110 + 89 = 199$$

$$B_{12} = B_{11} = 89, A_{12} = A_{11} + B_{12} = 199 + 89 = 288$$

My answer - 288,

The listed answer - D(376)