

1. If $y = 2x$ and $z = 2y$, then $x + y + z$ equals:

A) x B) $3x$ C) $5x$ D) $7x$ E) $9x$

Answer: $x + y + z = x + 2x + 2(2x) = 7x$

2. The diagonal of square I is $a + b$. The perimeter of square II with twice the area of square I is:

A) $(a + b)^2$ B) $\sqrt{2}(a + b)^2$ C) $2(a + b)$ D) $\sqrt{8}(a + b)$ E) $4(a + b)$

Answer: Square I has sides $\frac{a + b}{\sqrt{2}}$. Square II then has sides $a + b$, and so its perimeter is $4(a + b)$.

3. What is the y -intercept of the line passing through $(\sqrt{2}, 1)$ and $(-2, 2)$?

A) 1 B) 1.5 C) $\sqrt{2}$ D) $\sqrt{3}$ E) $5 - 1$

Answer: Let m denote the slope and b the y -intercept. We obtain a system substituting the points given into $y = mx + b$.

$$\begin{cases} 1 = \sqrt{2}m + b \\ 2 = -2m + b \end{cases} \quad \text{multiply by } \sqrt{2} \quad \Rightarrow \quad \begin{cases} \sqrt{2} = 2m + \sqrt{2}b \\ 2 = -2m + b \end{cases}$$

We add the two equations and obtain $\sqrt{2} + 2 = b(1 + \sqrt{2})$. Thus $b = \frac{\sqrt{2} + 2}{1 + \sqrt{2}} = \frac{\sqrt{2}(1 + \sqrt{2})}{1 + \sqrt{2}} = \sqrt{2}$

4. For $x > 0$, $y > 0$, and $b > 0$ where $b \neq 1$ and $y \neq 1$, let $\log_b x = m$ and $\log_b y = n$. Three of the following statements are true. Which one is, in general, not true?

A) $\log_b xy = nm$ B) $\log_b x^p = pm$ C) $\log_y x = \frac{m}{n}$ D) $\log_b \frac{x}{y} = m - n$

Answer: $\log_b xy = \log_b x + \log_b y = n + m$ and so A is clearly false. For example, $b = x = y = 5$ will result in $n = m = \log_5 5 = 1$ and $\log_b xy = \log_5 25 = 2 \neq 1 = nm$.

5. How many real solutions has the equation $|x^2 - 6x| = 9$?

A) 0 B) 1 C) 2 D) 3 E) 4

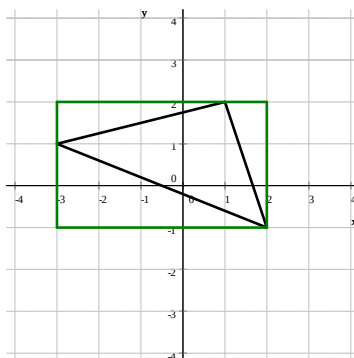
Answer: the original equation translates to the pair $x^2 - 6x = 9$ and $x^2 - 6x = -9$. The first equation has two real solutions, $3 \pm 3\sqrt{2}$ the second one has one, 3. These are clearly different.

6. Find the area of the triangle with vertices $(-3, 1)$, $(1, 2)$, and $(2, -1)$.

A) 6 B) $\frac{13}{2}$ C) 7 D) $\frac{15}{2}$ E) 8

Answer: The area of the rectangle is $3(5) = 15$, from which we subtract the 'corners':

$$15 - \left(\frac{2(5)}{2} + \frac{3(1)}{2} + \frac{1(5)}{2} \right) = 6$$



7. Find the positive number x for which $\sqrt{x} = \sqrt[3]{y}$ and $\sqrt{y} = 8$.

A) 2 B) $2\sqrt{2}$ C) 4 D) 16 E) 64

Answer: $\sqrt{y} = 8 \implies y = 64 \implies \sqrt[3]{y} = 4 \implies \sqrt{x} = 4 \implies x = 16$

8. When the three-digit numbers $6a3$ and $2b5$ are added together, the answer is a number divisible by 9. The largest value of $a + b$ is:

A) 2 B) 9 C) 11 D) 17 E) 20

Answer:

$6a3 + 2b5$ is divisible by 9 $\implies 6 + a + 3 + 2 + b + 5 = a + b + 16$ is divisible by 9 $\implies a + b + 7$ is divisible by 9. Since a and b are digits, their sum is at most 18. The possible values are 2 and 11.

9. Find the real part of $\frac{i}{1 + \frac{i}{1 + \frac{i}{1 + i}}}$

A) $-\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{4}{5}$ D) $\frac{2}{3}$ E) none of these

Answer:

$$\begin{aligned} \frac{i}{1 + \frac{i}{1 + \frac{i}{1 + i}}} &= \frac{i}{1 + \frac{i}{\frac{1+2i}{1+i}}} = \frac{i}{1 + i \cdot \frac{1+i}{1+2i}} = \frac{i}{1 + \frac{i-1}{1+2i}} = \frac{i}{\frac{1+2i+i-1}{1+2i}} \\ &= \frac{i}{\frac{3i}{1+2i}} = i \cdot \frac{1+2i}{3i} = \frac{1+2i}{3} = \frac{1}{3} + \frac{2}{3}i \end{aligned}$$

10. The solution of the equation $\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} = 3$ is:

A) 3 B) $\frac{3}{5}$ C) $\frac{4}{5}$ D) $\frac{5}{4}$ E) $\frac{5}{3}$

Answer:

$$\begin{aligned} \frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} &= 3 \\ \sqrt{x+1} + \sqrt{x-1} &= 3\sqrt{x+1} - 3\sqrt{x-1} \\ 4\sqrt{x-1} &= 2\sqrt{x+1} \\ 2 &= \sqrt{\frac{x+1}{x-1}} \\ 4 &= \frac{x+1}{x-1} \\ 4x - 4 &= x + 1 \\ 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$$

11. Which of the following describes the asymptotes for the hyperbola $\frac{(x-8)^2}{16} - \frac{(y-3)^2}{4} = 1$
- A) $y - 3 = \pm \frac{1}{2}(x - 8)$ B) $y - 1 = \pm \frac{3}{8}(x - 2)$ C) $y + 2 = \pm \frac{3}{8}(x - 4)$
- D) $y - 1 = \pm \frac{1}{2}(x - 2)$ E) $y + 2 = \pm \frac{1}{4}(x - 4)$

Answer: We solve for y and get $y - 3 = \pm \frac{1}{2}\sqrt{x^2 - 16x + 48}$ which, for large values of x and y , is close to $y - 3 = \pm \frac{1}{2}(x - 8)$

12. $\sin y + \sin(x - y) = \sin x$ for all y provided that x is:

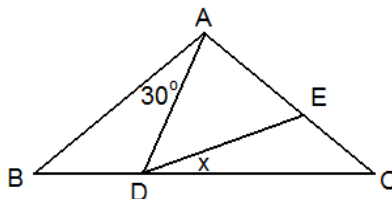
A) 60° B) 90° C) 180° D) 270° E) 360°

Answer:

$$\begin{aligned}\sin y + \sin(x - y) &= \sin x \\ \sin(x - y) &= \sin x - \sin y\end{aligned}$$

The statement is clearly true for $x = 360^\circ$.

13. In the figure $\overline{AB} = \overline{AC}$, angle $BAD = 30^\circ$, and $\overline{AE} = \overline{AD}$. Then x equals:



A) $7\frac{1}{2}^\circ$ B) 10° C) $12\frac{1}{2}^\circ$ D) 15° E) 20°

14. Of the following, which fraction is an integer multiple of each of the fractions $\frac{6}{7}$, $\frac{5}{14}$, $\frac{10}{21}$?

A) $\frac{7}{30}$ B) $\frac{7}{15}$ C) $\frac{15}{7}$ D) $\frac{30}{7}$ E) $\frac{80}{21}$

Answer: an integer multiple of a fraction can only 'lose' factors from its denominator, never gain. Thus we only need to consider C and D. Since 6 is not a factor of 15, we also rule out C.

15. Fifteen billiard balls are lying on a table in such a way that they are just squeezed inside an equilateral triangular frame whose inside perimeter is 876. The radius of a billiard ball is:

A) $\frac{73}{2}$ B) $\frac{146}{4 + \sqrt{3}}$ C) $\frac{146}{2 + \sqrt{3}}$ D) $\frac{146}{3 + \sqrt{3}}$ E) none of these

16. The least positive integer which has remainders 1, 1, and 5 when divided by 3, 5, and 7 respectively, is:

A) 166 B) 151 C) 145 D) 131 E) none of these

Answer: remainders 1, 1 when divided by 3 and 5 translate to remainder 1 when divided by 15. Among the first seven such positive numbers, we will find the one with remainder 5 when divided by 7. We consider $k \cdot 15 + 1$, where $k = 0, 1, 2, 3, 4, 5, 6$. We obtain 1, 16, 31, 46, 61, 76. The remainders when divided by 7 are 1, 2, 3, 4, 5, 6 and so 61 is the smallest such number.

17. Each valve A , B , and C , when open, releases water into a tank at its own constant rate. With all three valves open, the tank fills in one hour, with only valves A and C open it takes 1 hour and 20 minutes, and with only valves B and C open it takes 2 hours. The time it takes to fill the tank with only valves A and B open is:

A) $\frac{2}{3}$ hr B) $\frac{4}{3}$ hr C) $\frac{3}{2}$ hr D) 2 hr E) $\frac{9}{4}$ hr

Answer: Let a denote the time it takes to fill the tank with valve A open only. Similarly, let b and c denote the time valve B and C fills the tank. Then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

$$\frac{4}{3} \left(\frac{1}{a} + \frac{1}{c} \right) = 1$$

$$2 \left(\frac{1}{b} + \frac{1}{c} \right) = 1$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

$$\frac{1}{a} + \frac{1}{c} = \frac{3}{4} \implies \frac{1}{b} = \frac{1}{4}$$

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{2} \implies \frac{1}{a} = \frac{1}{2}$$

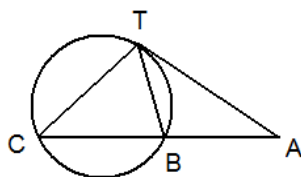
$$\frac{1}{a} + \frac{1}{b} = \frac{3}{4} \quad \text{and so} \quad \frac{4}{3} \left(\frac{1}{a} + \frac{1}{b} \right) = 1$$

18. In how many different arrangements can a careless office boy place 5 letters into 5 mailboxes so that no one gets the right letter?

A) 32 B) 44 C) 60 D) 120 E) 225

Answer: 5-cycles: $4! = 24$ 3+2: $\binom{5}{3} (2) = 20$

19. In the given diagram, points B , C , and T are on the circle, and AT is tangent to the circle at T . If $AB = 3$ and $BC = 4$, find $\frac{AB + AT}{AT + AC}$.



Answer: $AT = \sqrt{12}$ and so $\frac{3 + \sqrt{12}}{\sqrt{12} + 7} = \frac{2\sqrt{3} + 3}{2\sqrt{3} + 7} = \frac{8\sqrt{3} + 9}{37}$

20. Find the limiting value of $\frac{1}{9} + \frac{3}{27} + \frac{5}{81} + \frac{7}{243} + \dots + \frac{2k-1}{3^{k+1}}$

$$\frac{1}{3}$$

$$\left(\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots \right) + ()$$