

- C 1. By rotating the line counterclockwise, it is clear that the maximum slope occurs when the line passes through the origin, producing a slope of $(-2 - 0)/(-3 - 0) = 2/3$.
- B 2. $4(1 + i)^{-1} = 4(1 - i)/[(1 + i)(1 - i)] = 4(1 - i)/(1 - i^2) = 2(1 - i)$
- B 3. The quadrilateral with the three vertices plus $(9, -2)$ is a right triangle with area 14 plus a trapezoid with area 22. The original triangle's area is this sum minus the area of a right triangle with area 16.
- D 4. Since $400 \div 13$ (12 diners plus 1 server) $= 30 \frac{10}{13}$, we can have 30 full groups of 12 diners. There is room for 10 more people, which makes 9 diners and a server, for a total of $(30)(12) + 9 = 369$ diners.
- E 5. $C = 1, T = 2, A = Y = 3, M = 4$. The A's can't be adjacent, so the only possible order is C, T, A, Y, A, M.
- D 6. The equation is equivalent to $2x - 1 = \pm 7$, which has the solutions 4 and -3.
- B 7. The runners first meet when the faster one has run 360 m. Each time they meet, they have run one lap between them, so they meet for the 8th time when the faster one has run 2880 m, 120 m short of the start.
- E 8. $R = kL/D^2$, so $R_{\text{new}} = k(1.5)L/(2.5D)^2 = (6/25)(kL/D^2) = (6/25)(600) = 144$ ohms.
- D 9. $\cot C = 2$ means $\cos C = 2 \sin C$, so $\sin 2C = 2 \sin C \cos C = 4 \sin^2 C$. But $\cot^2 C + 1 = 5 = \csc^2 C = 1/\sin^2 C$, so $4 \sin^2 C = 4/5$.
- E 10. There are 3 one-scoop cones, 3 two-scoop cones, and 1 three-scoop cone. Of these, four contain chocolate (chocolate, choc-vanilla, choc-strawberry, choc-van-straw), so the answer is $4/7$.
- A 11. Place the triangles at $A(8,0), B(0,0), C(0,6), D(0,5), E(12,0)$. The hypotenuses have the equations $6x + 8y = 48$ and $5x + 12y = 60$ and intersect at $F(3, 15/4)$. DF is $\sqrt{3^2 + (-5/4)^2} = 13/4$.
- A 12. Since $(x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$, the sum of all of the solutions is
- 3. The graph indicates one real solution of about 1.1, so the nonreal solutions must sum to about -4.1.
- D 13. The equation is equivalent to $(\cos^2 t - \sin^2 t) - (\cos t + \sin t) = 0$ or $(\cos t + \sin t)(\cos t - \sin t - 1) = 0$.
- The factor $\cos t + \sin t$ is 0 at $3\pi/4, 7\pi/4, -\pi/4$, and $-5\pi/4$, and $\cos t - \sin t = 1$ when $\cos t = 1$ and $\sin t = 0$ $(-2\pi, 0, 2\pi)$ or $\cos t = 0$ and $\sin t = -1$ $(3\pi/2, -\pi/2)$, for a total of nine solutions.
- D 14. Since $\log r + \log s = \log(rs)$, $rs = r + s$ with $r, s > 0$ (because of the domain of the log). Solving for s gives $s = r/(r - 1)$. For $r > 1$, this yields $s > 0$, while $r \leq 1$ yields either $s < 0$ or an undefined value.
- C 15. The area inside R_1 is 80. The region within one unit of R_2 is bounded by four segments each parallel to a side of R_2 outside the rectangle connected by four quarter circles at the corners, and an inner 4 by 2 rectangle. The area is $20 + \pi + [(6)(4) - (4)(2)] = 36 + \pi$, and the probability is $(36 + \pi)/80$.
- E 16. Let a_n = the number of ways of reaching n . You can reach n either by moving one unit from $n - 1$ or two units from $n - 2$, so $a_n = a_{n-1} + a_{n-2}$. Since $a_1 = 1$ and $a_2 = 2$, the formula yields $a_{10} = 89$.
- B 17. Linear programming theory shows that the minimum of a linear function on a polygonal region occurs at a vertex. The vertices of the given region are $(6, 10), (4, 6), (12, 6)$, and $(6, 3)$, and the function $f = 2x + y$ assumes its minimum value of 14 at $(4, 6)$.
- X 18. The first car's distance at time t is $30t$, the second's is $40t$. Their distances are two sides of a triangle with included angle 135° . By the Law of Cosines, the distance between them is $t\sqrt{2500 + 1200\sqrt{2}}$, and the speed at which they are moving apart is about $\sqrt{4196.8}$ or 65 mph. Due to a misprint, this was not one of the choices, so this question was not graded.

- E 19. Since $5^{2000} - 1 = (5^{125} - 1)(5^{125} + 1)(5^{250} + 1)(5^{500} + 1)(5^{1000} + 1)$, a factor of $5^{125} + 1$ or $5^{125} - 1$ is a factor of $5^{2000} - 1$. Since $x^n \pm 1$ is divisible by $x \pm 1$, 6 and 4 (and thus 3 and 2) are factors.
- C or D 20. If $\mu = 1$, then since each $x \geq 1$, all values must be 1, so $\mu = 1$ implies $M = m = 1$ as well. This eliminates A, B, and E. For C, a collection with 999 1's, 5 2's, 995 3's, and one 6 satisfies the requirements, and for D, a collection with 1200 1's and 800 6's satisfies them (other choices are possible).

Test #2

Student Mathematics League Short Solutions

February 2001

- B 1. The line M has slope $3/2$ and x-intercept 3, so the line L has equation $y = (-2/3)x + 2$.
- D 2. The squares of the distances are 233, 234, 233, 232, and 234 respectively.
- A 3. Replacing x with 2 yields $6 - b = 3b - 6$, so b must be 3.
- E 4. The given values are 7, 14, -1, 3, and 4 respectively.
- D 5. By the Difference Formula, $\cos(3\pi/2 - t) = \cos(3\pi/2)\cos t + \sin(3\pi/2)\sin t = 0 + (-1)(\sin t)$.
- D 6. Since $4^{3/2} = 8$ and $4^{5/2} = 32$, the required sum is 4.
- E 7. Neither of the fractions in the complex fraction can have 0 denominators, eliminating 2 and 1, but $x = -1$ would make the denominator of the complex fraction 0 as well, so it is also eliminated.
- B 8. The area of a region made up of two overlapping regions is the sum of the regions less the overlap (which was counted twice), so the answer is $(10)(6) + (4)(15) - (4)(6) = 96$.
- C 9. Subtracting the third equation from the first, and k times the third from the second, yields $(k - 1)y = 2$ and $(2 - k)y = 2 - k$. If $k = 2$, the last equation becomes $0 = 0$ and the first equation gives $y = 2$ (so $x = -1$), while if $k \neq 2$, the last equation yields $y = 1$ (and $x = 0$), making $k = 3$.
- D 10. The months from January to December have 31, 14, 10, 7, 6, 5, 4, 3, 3, 3, 2, and 2 lucky dates, respectively, for a total of 90 (leap years don't add lucky dates, since 2/29 isn't lucky). The answer is $90/365$ or about 0.25. The answer does not change if we count leap year ($90/366$), four years with one leap year ($360/1461$), or a full 400-year cycle of the Gregorian calendar ($36000/146097$).
- C 11. The numerator $= x^4 - 2x^3 + x^2 + x^2 - 2x + 1 = (x^2 + 1)(x - 1)^2 = (x + i)(x - i)(x - 1)^2$. Thus the numerator is zero for $x = \pm i$ or 1, but 1 is not a solution, since it is not in the domain of the fraction.
- B 12. The condition produces the equation $\sqrt{(x - 3)^2 + (y + 1)^2} = 2\sqrt{(x - 4)^2 + (y - 2)^2}$, which simplifies to $3x^2 + 3y^2 - 26x - 18y + 70 = 0$, the equation of a circle.
- E 13. A knight could never say, "I am an alternator" (it would be a lie), so Al is a knave or an alternator. If Al is a knave, "Bo is not an alternator" is false, so Bo is an alternator. If Al is an alternator, "I am an alternator" is true, and his next statement must be false, again making Bo an alternator. Since Bo must be an alternator, his statement "Al is a knave" could be true or false, so E is the only certain choice.
- C 14. T must be 1 less than A and C must be 2 less than A, so $T - C = (A - 1) - (A - 2) = 1$.
- B 15. If the terms were all different, there would be $8!$ or 40,320 arrangements. But the two A's can be rearranged in $2! = 2$ ways, and the b's and c's can each be rearranged in $3! = 6$ ways, all without affecting the actual expression, so the total number of arrangements is $8!/(2!3!3!) = 40320/72 = 560$ ways.
- A 16. By the Law of Cosines, $8^2 = 11^2 + 9^2 - 2(99)\cos A$, and $CM^2 = 11^2 + 4.5^2 + 99\cos A$. Subtracting twice the second equation from the first yields $64 - 2CM^2 = -80.5$, so $CM^2 = 289/4$ and $CM = 8.5$.
- B 17. $a_1 = 1$, $a_3 = 3$, $a_5 = 7$, $a_7 = 15$, and it is easy to show that $a_{2n-1} = 2^n - 1$, so $a_{2001} = 2^{1001} - 1$. Then $\sqrt[2001]{a_{2001}} = (2^{1001} - 1)^{1/2001}$ or approximately $2^{1001/2001}$, which is closest to $\sqrt{2}$.
- B 18. The original collection has the form 3 , with either two or three 4's above the 3. If there

were only two 4's, 4 couldn't be the unique mode, since the three blanks before the 3 could only be filled with one 1 and two 2's, or two 1's and one 2. Thus the collection must be $___ 3 4 4 4$, with the blanks filled with 1 2 2 or 2 2 3 (1 1 1, 2 3 3, 1 3 3, or 2 2 2 means 4 is not the unique mode, and 1 2 3 means 2 isn't the new mode). But 2 2 3 means 3 is the median of the new collection, so the original must have been 1 2 2 3 4 4 4, making the mean of the new collection $(1+2+2+2+2+3+4+4+4)/9 = 8/3$.

- E 19. Since $2001 = 3(23)(29)$ and $r^2 - s^2 = (r - s)(r + s)$, $r - s$ must be 1, 3, 23, or 29, making $r + s$ 2001, 667, 87, or 69, giving four solutions: (1001, 1000), (335, 332), (55, 32), and (49, 20).
- D 20. The number 8 cannot be a sum, since it can only be written as $1 + 2 + 5$ or $1 + 3 + 4$, so there is no third side for the triangle; 13 cannot be a sum, since it is only $2 + 5 + 6$ or $3 + 4 + 6$. For the numbers between 8 and 13, the following solutions exist (reading clockwise from the top of the triangle): 9: 1, 5, 3, 4, 2, 6; 10: the given example; 11: 2, 3, 6, 1, 4, 5; 12: 5, 1, 6, 2, 4, 3.

- Test #3** **Student Mathematics League Short Solutions** **March-April 2001**
- B 1. $r = -1$, $s = -7$, and $t = 44$, so the desired order is $s < r < t$.
- B 2. For $1 \leq x \leq 2$, $|x - 3| = 3 - x$, $|1 - x| = x - 1$, and $|4 - 2x| = 4 - 2x$.
- A 3. The sum of the intercepts of $y = 2x + b$ is $b/2$. The sum of the intercepts of $y = 2x + b + k$ is $(b + 4) + (-b/2 - 2) = b/2 + 2 = k + 2$.
- D 4. Carlos can make 4(5) meat, 4(3) cheese, and 4(5)(3) meat and cheese sandwiches. The total = 92.
- C 5. Let a = George's investment. Then $0.9(1.1a - 150) + 150 = 1.04a$, and $a = 300$, so his gain is \$12.
- C 6. The months from January to December have 31, 14, 10, 7, 6, 5, 4, 3, 3, 3, 2, and 2 lucky dates, respectively, so the answer is 3 months (September, October, and November).
- C 7. $A < 5$, so we have $4M + 4T = YC$. Since we want $Y = 9$, M and T must be 8 and 7, and $YC = 95$.
- E 8. $4 - x \geq 0$ implies $x \leq 4$, and $4 - \sqrt{4 - x} \geq 0$ implies $\sqrt{4 - x} \leq 4$, $4 - x \leq 16$, and $x \geq -12$.
- B 9. The sequence has $a_{n+6} = a_n$, so $a_{2001} = a_{3+6(333)} = a_3 = -4$.
- A 10. The region consists of 4 right triangular regions with right angles at (0,0) and areas 6, 9, 9, and 13.5.
- D 11. The region is a trapezoid, and the distance is its altitude. Since the two bases of the trapezoid have lengths 5 and 7.5, the altitude = $2(37.5)/(5 + 7.5) = 6$.
- D 12. Of the a_i 's, 1001 increase by 1 and 1000 decrease by 1, so μ increases by $1/2001$, eliminating A, C, and E. If a_{1001} (the median) = $a_{1002} - 1$, then these two elements switch values, so the median is unchanged. If $a_{1001} < a_{1002} - 1$, then the median increases by 1.
- E 13. In $\triangle ABC$, $\sec \alpha = AC/1 = AC$. In $\triangle ACD$, $\cot \alpha = CD/\sec \alpha$, so $CD = (\cot \alpha)(\sec \alpha) = \csc \alpha$.
- C 14. Since 17 isn't the sum of three different numbers ≤ 8 including a 1, 17 is impossible. The only such sum adding to 16 is $1 + 7 + 8$, so if such a square exists, one side is 7, 1, 8. The only sum adding to 16 including a 2 is $2 + 6 + 8$, so an adjacent side must be 8, 2, 6. But the only remaining sum including a 6 is $3 + 6 + 7$, and 6 and 7 are diagonally opposite. The square with values reading clockwise from a corner of 3, 4, 8, 1, 6, 2, 7, 5 has common sum 15.
- A 15. A function and its inverse are symmetric across $y = x$, their x - and y -intercepts are interchanged, and the sum is the same.
- C 16. The first series' sum = $s_1/(1 - r) = 8/3$, and the second series' sum = $s_1^2/(1 - r^2) = 2s_1$, so $s_1/[(1 - r)(1 + r)] = 2$. Thus $8/3 = 2(1 + r)$, and $r = 1/3$.
- D 17. $\triangle BCF$ is a right triangle with $BC = 13$ and $BF = 5$, so $CF = 12$. Then $CG = 1$, and by similar tri-

angles, CM (where M is the intersection of $\overline{CD}$ and $\overline{GH}$) $= 13/5$. By the same argument, $HM = 13/5$, and the overlap has area $2(1/2)(13)(13/5) = 33.8$ (two right triangles with legs 13 and $13/5$).

D 18. Every power of 2 reduces to 1; 3 cycles back to 3 (but no smaller); 5 cycles back to 5 (but no smaller); 6, 9, and 12 all cycle to 3; 7 cycles to 7 (but no smaller); 10 cycles to 5; 11, 13, and 14 all cycle to 7; 15 cycles to 15 (but no smaller); all higher numbers cycle to one of these for a total of 5: $\{1, 3, 5, 7, 15\}$.

B 19. The product of the LCM and GCF of x and y is xy , so the last two digits of $(2001! - 1)(2^{2001} + 1)$ are what we seek. $2001!$ ends in many zeros, so $2001! - 1$ ends in 99. Looking at the last two digits of powers of 2 shows that $2^{21}, 2^{41}, 2^{61}, \dots, 2^{2001}$ all end in 52, so $2^{2001} + 1$ ends in 53, and the required product has the same last two digits as $(99)(53) = (100 - 1)(53) = 5247$, so the answer is 47.

E 20. By the division algorithm, $P(x) = Q(x)(x^2 - 1) + R(x)$, where the degree of $R(x)$ must be ≤ 1 , so $R(x) = mx + b$. But $P(1) = -3 = Q(1)(0) + R(1)$, and $P(-1) = 5 = Q(-1)(0) + R(-1)$, so $-3 = m + b$ and $5 = -m + b$, which means $m = -4$ and $b = 1$.