Pre-Algebra

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Contents

Chapter 1

1.1 Place Value in the Decimal System

This topic was handled wonderfully by Professor Herbert Gross.

Video Lecture (link)

Textbook (Link)

Study Guide (link)

Numbers are adjectives, describing nouns. This means that we should never think of numbers in isolation. There are three houses, three cups of tea, three cats, etc. So numbers are always describe objects, namely, how many we have.

Even our notation of numbers is more easily understood as adjectives describing nouns. The numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 are called **digits**. We use digits in a place value system. Let us consider the number 278, two hundred seventy-eight. The three digits describe place values: eight units, seven tens, and two hundreds.

hundreds	tens	units
2	7	8

Each digit describes a different noun: 2 describes how many hundreds we have, 7 describes how many tens we have, and 8 describes the number of units. (Unit is just a fancy word for the number one.) And the entire number, 278 describes a noun: 278 dollars, 278 people, and so on. Each decimal place represent ten times the previous decimal place. A hundred is ten tens, a thousand is ten hundreds, etc.

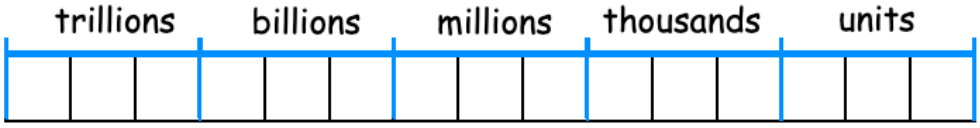
Another example would be 3051. This number illustrates the role of zero. Zero helps us distinguishing between numbers like 351 and 3051.

hundreds	tens	units
3	5	1

This number, 351 is three hundred fifty-one. On the other hand, the number with zero included is 3051:

thousands	hundreds	tens	units
3	0	5	1

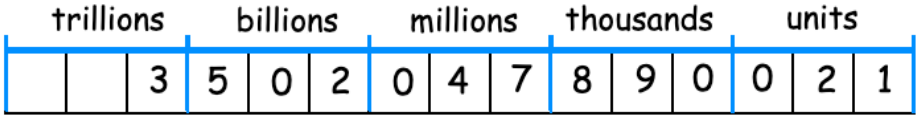
This number is 3051, three thousand, fifty-one. The place value system allows us to write vey large numbers. We group place values in groups of three, and use three-digit numbers to describe how many of each of the place value grouping.



Consider the number 3502047890021.

We start at the right end of the number and group the digits into groups of three. This can be done using commas or spacing: $3,502,047,890,021$ or 3 502 047 890 021

The first group is the units, the second group is the thousands, the third group is the millions, the fourth group is the billions, and the fifth group is the trillions.



We describe each group with a three-digit (or two or one-digit) number:

three trillion, five hundred two billion, forty-seven million, eight hundred ninety thousand, twenty-one

When we write numbers using digits, the commas separating the three-digit groups are optional. This is not the case when we write a number using letters. Also notice the hyphen in the two-digit numbers such as thirty-two, fifty-seven, etc.

Practice Problems

1. Consider the number 805 431 297. In this number, name the place value of the digits given

   a) 2 b) 5 c) 3 d) 9

2. Write each of the given numbers using digits.

   a) thirty-seven thousand, twenty-three

   b) two hundred eleven million, six hundred two thousand, seven hundred eighty

   c) five hundred thousand, five

   d) twenty billion, thirty-five million, ninety thosand, one hundred sixty-five

1.2 The Set of All Natural Numbers

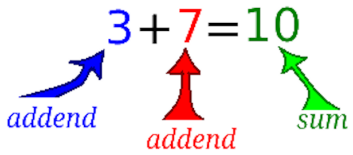
The **set of all natural numbers**, (sometimes also called the set of all counting numbers), denoted by $\mathbb{N}$, is the set

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

It is easy to imagine that the set of all natural numbers was the first set of numbers at which human beings looked. The four basic operations can be defined and performed on natural numbers as follows.

Addition, denoted by $+$, is defined as we usually think of addition: the addition of two natural numbers is obtaining the total amount of those quantities combined.

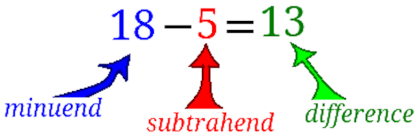
In the statement $3+7=10$, we say that 3 and 7 are **addend**s and 10 is called the **sum** of 3 and 7.



If we add two natural numbers, the sum is always a natural number. This is called **closure**. The set of all natural numbers is closed under addition: If n and m are natural numbers, then $n+m$ is also a natural number.

Subtraction, denoted by $-$, is a mathematical operation that represents the operation of removing objects from a collection.

In the statement $18-5=13$, we say that 18 is the **minuend** and 5 is the **subtrahend**, and 13 is called the **difference** of 18 and 5.



If we subtract a natural number from another natural number, the difference may or may not exits within $\mathbb{N}$. For example, the subtraction $8-5$ results in a natural number but the subtraction $3-11$ does not. In other words, the set of all natural numbers is NOT closed under subtraction.

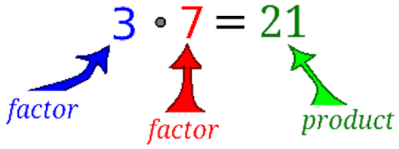
Addition and subtraction have interesting properties we did not discussed here. HIghly Recommeded! Check out what the great Professor Herbert Gross has to say about these operations. Please note that he refers to the set of whole numbers. Whole numbers are all the natural numbers and zero. Addition and Subtraction

Multiplication, denoted by $\cdot$, or by $\times$, or by nothing at all between two objects, is defined as we usually think of multiplication

$$3 \cdot 7 = 21 \quad \text{or} \quad 3 \times 7 = 21 \quad \text{or} \quad (3)7 = 21 \quad \text{or} \quad 3(7) = 21$$

Please note that in the last two equations, the **parentheses do not indicate multiplication**. The parentheses helps us interpret 3 and 7 as two separate numbers and not the number 37. Once we see two numbers with no operation sign between them, that NOTHING indicates multiplication.

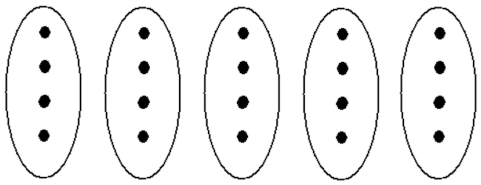
In the statement $3 \cdot 7 = 21$, we say that 21 is the **product** of 3 and 7. We also say that 3 and 7 are **divisors** or factors of 21.



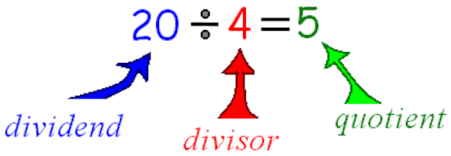
If we multiply two natural numbers, the product is always a natural number. In other words, the set of all natural numbers is closed under multiplication: if n and m are natural numbers, then nm is also a natural number.

Division, denoted by $\div$ or by $/$ is defined as we usually think of division. One example is: if we have 20 dots and we circle together every four dots, how many packages of four do we obtain? The answer is clearly five, because five packages of four will account for 20 dots.

$20 \div 4 = 5$ or $\frac{20}{4} = 5$



In the statement $20 \div 4 = 5$, we say that 20 is the **dividend**, 4 is the **divisor**, and 5 is the **quotient** of 20 and 4.



If we divide a natural number by another natural number, the quotient may or may not exist within $\mathbb{N}$. For example, the division $12 \div 3$ results in a natural number, but the division $20 \div 7$ does not. In other words, the set of all natural numbers is NOT closed under division.

Multiplication and division have interesting properties we did not discussed here. HIghly Recommeded! Check out what the great Professor Herbert Gross has to say about these operations.

Multiplication and Division

The video links are part of Professor Herbert Gross's online self-study arithmetics course. Check it out!

Discussion:

1. Consider the multiplication $4 \cdot 9 = 36$ and the division $30 \div 6 = 5$. While in the division all of 30, 6, and 5 have different names, both 4 and 9 are simply called factors in the multiplication. Can you explain why this will not cause any problems?

1.3 The Order of Operations Agreement

Lecture Video

The order of operations rule is an agreement among readers and writers of mathematics texts, and it simplifies notation. P stands for parentheses, E for exponents, M and D for multiplication and division, A and S for addition and subtraction. Notice that M and D are written next to each other. This is to suggest that multiplication and division are equally strong. Similarly, A and S are positioned to suggest that addition and subtraction are equally strong.

This is the hierarchy, and there are two basic rules.

1. **Between two operations that are on different levels of the hierarchy, we start with the operation that is higher.** For example, between a division and a subtraction, we start with the division since it is higher in the hierarchy than subtraction.
2. **Between two operations that are on the same level of the hierarchy, we start with the operation that comes first from reading left to right.**

Example 1. Perform the indicated operations. $20-3\cdot 4$

Solution: We observe two operations, a subtraction and a multiplication. Multiplication is higher in the hierarchy than subtraction, so we start there.

$$\begin{aligned} 20-3\cdot 4 &= && \text{multiplication} \\ 20-12 &= && \text{subtraction} \\ &= & \boxed{8} & \end{aligned}$$

Example 2. Perform the indicated operations. $36\div 3\cdot 2$

Solution: It is a common mistake to start with the multiplication. The letters M and D are in the same line because they are equally strong; among them we proceed left to right. From left to right, the division comes first.

$$\begin{aligned} 36\div 3\cdot 2 &= && \text{division} \\ 12\cdot 2 &= && \text{multiplication} \\ &= & \boxed{24} & \end{aligned}$$

Example 2v. Perform the indicated operations.

a) $40\div(5\cdot 2)$ b) $(40\div 5)\cdot 2$ c) $40\div 5\cdot 2$

Solution - Youtube link

Example 3. Perform the indicated operations. $24 \div 4 \div 2$

Solution: It is essential to perform these two divisions left to right. If we proceeded differently, we would get a different result.

$$\begin{aligned} 24 \div 4 \div 2 &= && \text{first division from left} \\ 6 \div 2 &= && \text{division} \\ &= & \boxed{3} & \end{aligned}$$

Example 4. Perform the indicated operations. $(8-5)^2 - 5 + 2$

Solution: We start with the subtraction within the parentheses. Then we can drop the parentheses.

$$\begin{aligned} (8-5)^2 - 5 + 2 &= 3^2 - 5 + 2 && \text{exponentiation} \\ &= 9 - 5 + 2 && \text{subtraction} \\ &= 4 + 2 && \text{addition} \\ &= \boxed{6} & & \end{aligned}$$

Example 4v. Simplify each of the given expressions.

a) $14 - 4\left(3^2 - 7\right)$ b) $(14-4)\left(3^2-7\right)$

Solution - Youtube link

Notations for Multiplication

There are different ways we denote multiplication. Suppose we wanted to express the multiplication 2 times 3, our options are $\cdot$ (dot), or $\times$ (cross), or *nothing*.

$2 \cdot 3$	or	2×3	or	$(2)(3)$ or $2(3)$ or $(2)3$
This is the preferred notation		We will almost never use this. It's a dinosaur, it is going away....		We will almost never use this with positive numbers

It is a common misconception to believe that parentheses mean multiplication. This is not true. **Parentheses never denote multiplication.** In the expression $2(3)$, the parentheses tell us that we are not looking at the two-digit number 23, but rather at the separate numbers 2 and 3, with *nothing* between them. In written notation, multiplication is the default operation. In other words, if we see two numbers with nothing between them, the *nothing* indicates multiplication. This becomes more clear if we consider expressions such as $2x$ or ab. There is no operation sign (or parentheses) in these expressions and yet we know that the operation is mutiplication. It is the *nothing* between 2 and x and between a and b means multiplication.

Understanding Parentheses

Parentheses have no meaning on their own. They merely help us to understand the context in the written language, like punctuation does in written English. Some parentheses are **grouping symbols** overwriting order of operations. Others clarify boundaries of numbers. For example, it is possible to write $2(3)$ instead of $2 \cdot 3$. The parentheses in $2(3)$ is not a grouping symbol. Rather, it helps us understand that we are not looking at the two-digit number 23. We will refer to this as **clarifying parentheses**. One easy way to tell the difference between the two types of parentheses is that if there is no operation inside a pair of parentheses, then it cannot be a grouping symbol.

We will often see expressions with several pairs of parentheses. Luckily, there are only two possibilities.

$4(3-1)-(7\cdot 2-9)$
one pair of parentheses follows the other

$20-\left(12-2\left(5\cdot 8-6^2\right)+1\right)$
one pair of parentheses is nested inside the other

When faced with an order of operations problem, *we must pair the parentheses before starting the computation*. Students are encouraged to use colored pencils to make the pairing more visual such as in $20-\left(12-2\left(5\cdot 8-6^2\right)+1\right)$. Sometimes authors aim to make notation easier to read by using pairs of parentheses with different shapes, such as [ ] or { }. These different looking parentheses serve as regular grouping symbols, they only made to look differently. For example,
$20-\left(12-2\left[5\cdot 8-6^2\right]+1\right)$ might be easier to read than $20-\left(12-2\left(5\cdot 8-6^2\right)+1\right)$.

When we have several pairs of parentheses following each other, we perform them left to right. When we are dealing with parentheses nested inside others, we start with the innermost one.

Sometimes we have a case of an **invisible parentheses**. When an operation sign encloses entire expressions, it serves as a grouping symbol. For example, there is no parentheses in sight in the expression $\frac{32-4}{9-2}$, and division comes before substraction. However, the division bar stretching over the entire expression indicates that we must perform the subtraction on the top and on the bottom, and only then divide. This becomes obvious if we switch the notation from the division bar to the symbol $\div$.

$$\frac{32-4}{9-2}=(32-4)\div(9-2)$$

In this exmple, the division bar is easier to read, so it will be the notation of choice. Note however, that students often make a mistake when entering things like this in the calculator. The correct answer is 7 but the calculator will give us the wrong answer if we enter it incorrectly as $32-4\div 9-2$. In these cases, the parentheses must become visible.

Example 5. Consider the expression $4(3-1)-(7\cdot 2-9)$

a) How many operations are there in the expression?

b) Simplify the expression by applying the order of operations agreement. For each step, write a separate line.

Solution: a) We "scan" the expression left to right, and count the operations.

1. multiplication between 4 and $(3-1)$
2. subtraction $3-1$
3. subtraction between the two expressions within the parentheses
4. multiplication between 7 and 2
5. subtracting 9

So there are five operations.

b) We have two pairs of parentheses. We will work them out, left to right. We start with the subtraction $3-1$. Once this subtraction is performed, we no longer need the parentheses. Instead, we will denote the multiplication using the dot notation.

$$\begin{aligned}
4(3-1)-(7\cdot 2-9) &= 4\cdot 2-(7\cdot 2-9) && \text{Now we work on the other parentheses.}\\
&&& \text{Multiplication before subtraction}\\
&= 4\cdot 2-(14-9) && \text{Subtraction within parentheses}\\
&&& \text{Drop parentheses}\\
&= 4\cdot 2-5 && \text{Multiplication}\\
&= 8-5 && \text{Subtraction}\\
&= \boxed{3}
\end{aligned}$$

Example 6v. Simplify each of the given expressions.

a) $67-4\left(5+3\left(4\cdot 7-5^2\right)\right)$

Solution - Youtube link

b) $\dfrac{24\div 4\cdot 3}{9-4+1}$

Solution - Youtube link

c) $2\left(7+3\left(2\cdot 8-5\right)\right)$

Solution - Youtube link

Example 7. Evaluate the expression $\left(7-\left(5-\left(11-(7-4)^2\right)^2\right)^2\right)^2$

Solution: This problem is a good example for why it is a good thing to make only one step in each line. Instead of sorting out all the things to do, we just need to find the one operation to do. Once we write down our next line, we have the same goal, but this time we are looking at a simpler problem. The first operation to perform is the subtraction $7-4$ in the innermost parentheses. In this example, we will immediately drop parentheses that are no longer necessary. This is the recommended practice.

$$\left(7-\left(5-\left(11-(7-4)^2\right)^2\right)^2\right)^2 =$$

$$\begin{aligned}
&= \left(7-\left(5-\left(11-3^2\right)^2\right)^2\right)^2 && \text{exponentiation in the innermost parentheses}\\
&= \left(7-\left(5-(11-9)^2\right)^2\right)^2 && \text{subtraction in the innermost parentheses}\\
&= \left(7-\left(5-2^2\right)^2\right)^2 && \text{exponentiation in the innermost parentheses}\\
&= \left(7-(5-4)^2\right)^2 && \text{subtraction in the innermost parentheses}\\
&= \left(7-1^2\right)^2 && \text{exponentiation in parentheses}\\
&= (7-1)^2 && \text{subtraction in parentheses}\\
&= 6^2 && \text{exponentiation}\\
&= \boxed{36}
\end{aligned}$$

Example 7v. Simplify $\left(\left(3^2-5\right)^2-14\right)^2-1^3$

Solution - Youtube link

Sample Problems

Simplify each of the following expressions by applying the order of operations agreement.

1. $2 \cdot 3^2 - \left(6^2 - 2 \cdot 5\right) \div 2$
2. $18 - 7 - 3$
3. $5^2 - 2\left(10 - 2^2\right)$
4. $8^2 - 3^2$
5. $(8-3)^2$
6. $\left(3^3 - 4 \cdot 5 + 2\right)^2$
7. $\dfrac{3 + 2\left(20 - 3^2 - 5\right)}{3^2 - 2^2} + 4^1$

Practice Problems

Simplify each of the following expressions by applying the order of operations agreement.

1. $2 \cdot 5^2 - \left(6 \cdot 5 - 3^2\right) \div 3$
2. $10^2 - 7^2$
3. $(10-7)^2$
4. $20 - 7 - 1$
5. $2^3 - \left(11 - 3^2\right)^2$
6. $\dfrac{5^2 - 3^2}{2^2}$
7. $\dfrac{(5-3)^2}{2^2}$
8. $120 \div 6 \cdot 2$
9. $\left((7-4)^2 - 5\right)^2 - 1$
10. $\dfrac{22 - 3^2 + 2\left(20 - 3^2 - 5\right)}{3^2 - 2^2}$
11. $\dfrac{5 + (5^2 - 3^2)}{3^2 - 2 \cdot 1^8}$
12. $30 - \left(2\left(15 - 2^3\right) - 2^2\right)$
13. $4\left(3\left(2(2^2 - 1) + 1\right) - 1\right) + 5$
14. $\dfrac{2\left(3^3 - 4 \cdot 5\right) - 2^2}{4^2 - (3^2 + 2)}$
15. $\left(\left(\left(19 - \left(8 - 2^2\right)^2\right)^2 - 7\right)^2 - 3\right)^2$

Enrichment

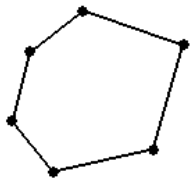
1. Place one or more pairs of parentheses into the expression on the left-hand side to make the equation true.

$$12 - 2 \cdot 3 - 1^2 + 2 - 3 + 4 = 20$$

1.4 Perimeter and Area of a Rectangle

Part 1 - Definition

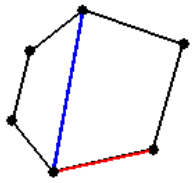
Definition: An $n-$sided **polygon** is n many points on a plane, connected with straight lines, as shown.



In every polygon, the number of points (also called vertices) and the number of sides are equal. Polygons with four sides are also called **quadrilaterals**.

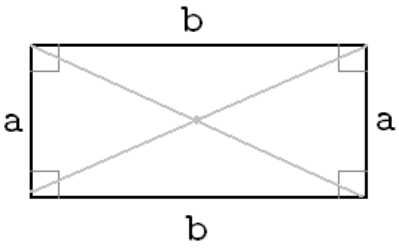
Definition: A line segment connecting two 'neighboring' points is a **side** of the polygon. A line segment conncecting two points, if it is not a side, then we call it a **diagonal**.

On the picture, the red line segment is a side, and the blue line segment is a diagonal of the polygon.



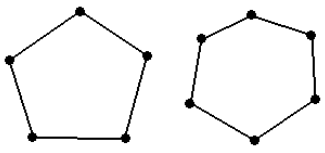
Definition: A **rectangle** is a four-sided polygon with four right angles.

We also refer to four-sided polygons as **quadrilateral**s. Although they are not part of the definition of a rectangle, there are a few important properties of rectangles that we should keep in mind. **In every rectangle, the opposite sides have equal lengths. The two diagonals of any rectangle are equally long and they bisect each other.** (To bisect something means to cut it into two equal parts.)



Enrichment

Draw and count the diagonals in each of the polygons shown.



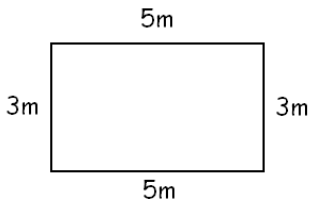
Part 2 - Perimeter

Definition: The **perimeter** of any geometric object is the length of its boundary.

In general, we can always think of the perimeter as a fencing problem. If we have a property, how long of a fence do we need to buy to completely fence around the property? We will denote perimeter by P. Perimeter is a *length*, we measure it in meters (m), centimeters (cm), inches (in), feet (ft), kilometers (km), or miles (mi).

Example 1. Compute the perimeter of a rectangle with sides 3 meters and 5 meters long.

Solution: If we think fencing, we mentally walk around a rectangle-shaped property to figure out how much fencing to buy. That is the same as simply adding the lengths of all four sides. The lengths of only two sides were given, but this should not be a problem. The opposite sides of a rectangle are equally long. Thus we can compute the perimeter as



$$P = 3\text{m} + 5\text{m} + 3\text{m} + 5\text{m} = 16\text{m}$$

So the perimeter of this rectangle is $\boxed{P = 16\text{m}}$.

Example 1v. Find the perimeter and area of the rectangle with sides 11ft and 9ft long. Solution - Youtube link

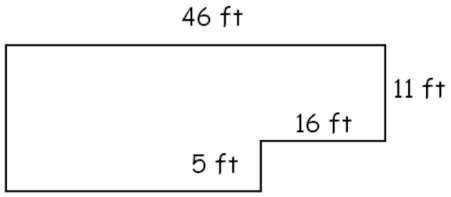
In general, the perimeter of a quadrilateral with sides a, b, c, and d is $P = a + b + c + d$. In case of a rectangle, the opposite sides are equally long, so $c = a$ and $d = b$ and this makes the primeter formula simpler.

Theorem: The perimeter of a rectangle with sides a and b is $P = 2a + 2b$.

This means that in our previous example, the perimeter of a 3m by 5m rectangle is

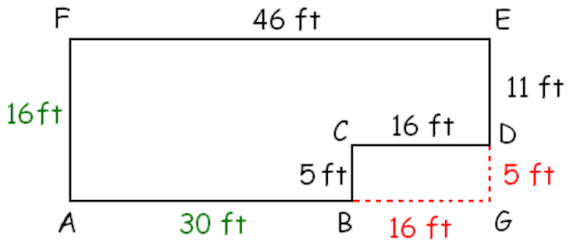
$$P = 2 \cdot 3\text{m} + 2 \cdot 5\text{m} = 6\text{m} + 10\text{m} = 16\text{m}$$

Example 2. Find the perimeter of the object shown. Angles that look like right angles are right angles.



Solution: It is much easier to discuss geometry if we label points and sides. Consider the picture shown. We are clearly missing the lengths of some sides for the perimeter, so we need to figure out those lengths first.

We first draw line AB beyond point B and line ED beyond point D as shown. These lines intersect each other in point G. Since $BGDC$ is a rectangle, its opposite sides are equally long. Therefore, $BG = 16$ft and $DG = 5$ft.



The quadrilateral $AGEF$ is also a rectangle, and so FA is the same length as EG.

$FA = 16\text{ft}$, because $FA = EG = ED + DG = 11\text{ft} + 5\text{ft} = 16\text{ft}$. Also, AG is as long as FE.

$$\begin{aligned} AG &= FE \\ AB + BG &= 46\text{ft} \qquad \text{we know } BG = 16\text{ft} \\ AB + 16\text{ft} &= 46\text{ft} \qquad \text{subtract } 16\text{ft} \\ AB &= 30\text{ft} \end{aligned}$$

We can now compute the perimeter, starting from point A, and moving counterclockwise.

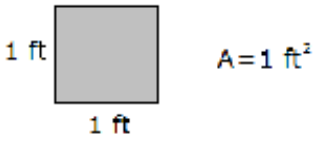
$$P = AB + BC + CD + DE + EF + FA = 30\text{ft} + 5\text{ft} + 16\text{ft} + 11\text{ft} + 46\text{ft} + 16\text{ft} = \boxed{124\text{ft}}$$

Part 3 - Area

The **area** of a geometric object is a measurement of its surface.

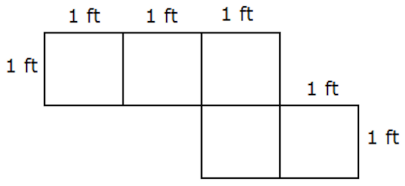
Understanding and remembering area formulas are easier if we know how they were derived. While we could think about perimeter as a fencing problem, area can be thought of as follows. Suppose a geometric object is a room. How many tiles do we need to buy to cover the entire room? Of course, we have to first agree on the size of the tiles we use to measure area.

Definition: The **area** of a 1ft by 1ft square (shown on the picture) is defined to be 1ft^2 (square-foot). Similar definitions can be formulated with mi^2, cm^2, in^2, etc. The area of any object, measured in ft^2, is the number of these 1ft by 1ft square tiles needed to cover the object, cutting and pasting allowed.



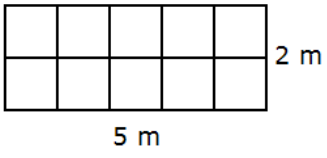
Area is not a length like perimeter. Area is measured in square-meters (m^2), square-centimeters (cm^2), square-inches (in^2), square-feet (ft^2), square-kilometers (km^2), or square-miles (mi^2). etc., and is usually denoted by A.

Example 3. Find the area of the figure shown on the picture.



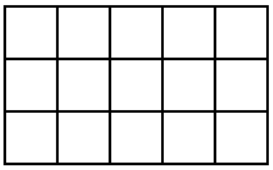
Solution: We simply count the tiles we need to cover this object. Since the figure can be covered using five unit tiles, its area is $\boxed{A = 5\text{ft}^2}$.

Example 4. Find the area of the figure shown on the picture.



Solution: Since the figure can be covered using ten 1m by 1m squares, its area is $\boxed{A = 10\text{m}^2}$.

Theorem: The area of a rectangle with sides x and y is $A = xy$.



Proof: We will not formally prove this theorem. Instead, we will just informally argue for this formula. The main idea should be clear from the previous example. Consider a rectangle with sides 3m and 5m. The area of this rectangle will be as many square-meters as many 1m by 1 m square tiles are needed to cover it.

Once we place a grid on the rectangle, it is easy to see how many such squares are needed. The rectangle is composed of five columns of squares, where each column consists of three squares. Thus we split the rectangle into fifteen unit tiles, and so the area is 15m^2. This shows that as long as the lengths of the sides are integers, we can place a grid on it, and the number of unit square tiles is the product of the length of the two sides.

In reality, this theorem is very difficult to prove, because not all side lengths happen to be integers. Mathematicians proved that this formula is true even if the sides of the rectangle are not integers.

Another interesting fact is that logically, we counted how many meter^2 we have. The computation for the area, however, is slightly different with regards to units. Instead of counting meter^2, we literally multiply meter by meter.

$$A = xy = 3\text{m}\,(5\text{m}) = 15\text{m}^2 \quad \text{and not} \quad 15 \cdot 1\text{m}^2$$

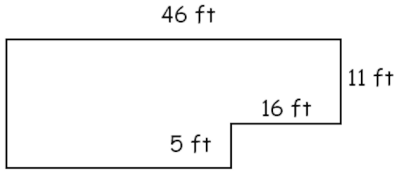
Area computation will always yield for the right unit.

Example 5. Find the area of a rectangle with sides 13 in and 7 in.

Solution: We apply the formula.

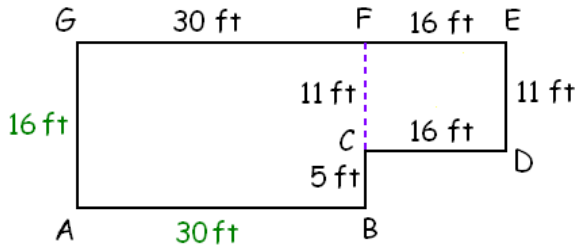
$$A = xy = 13\text{in}\,(7\text{in}) = \boxed{91\text{in}^2}$$

Example 6. Find the area of the object shown. Angles that look like right angles are right angles.



Solution: To compute the area, we have two options.

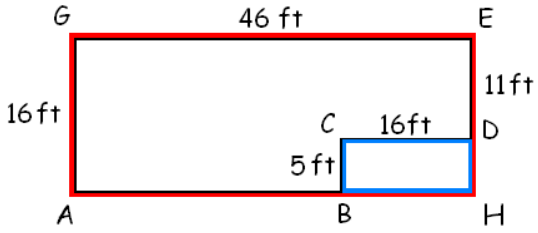
Method 1: We can think of the region as the sum of two rectangles: $CDEF$ is a rectangle with sides 11ft by 16ft, and $ABFG$ is a rectangle with sides 30ft by 16ft. So the area is



$$\begin{aligned} A &= A_{CDEF} + A_{ABFG} \\ &= 11\text{ft} \cdot 16\text{ft} + 16\text{ft} \cdot 30\text{ft} \\ &= 176\text{ft}^2 + 480\text{ft}^2 = \boxed{656\text{ft}^2} \end{aligned}$$

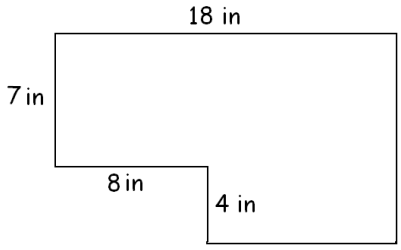
Method 2: We can also think of the region as a big rectangle from which a corner was removed.

The big (red) rectangle, $AHEG$ has sides 16ft and 46ft long. The smaller (blue) rectangle has sides 5ft by 16ft. So the area is the difference.

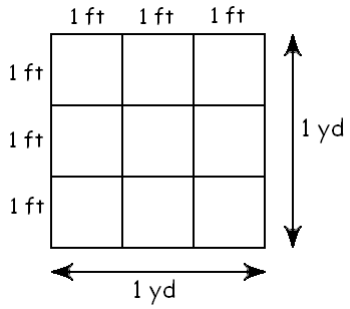


$$\begin{aligned} A &= A_{AHEG} - A_{BHDC} \\ &= 46\text{ft} \cdot 16\text{ft} - 16\text{ft} \cdot 5\text{ft} \\ &= 736\text{ft}^2 - 80\text{ft}^2 = \boxed{656\text{ft}^2} \end{aligned}$$

Example 7. Find the perimeter and area of the figure shown. Solution - Youtube link



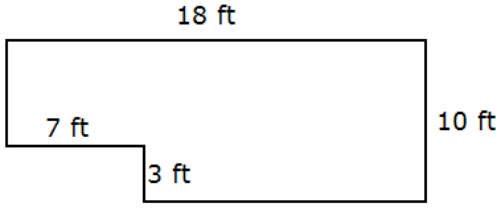
Discussion: One yard equals to three feet. Consider the picture below and discuss: how many square-feet is one square-yard? Can you show the same result algebraically?



Practice Problems

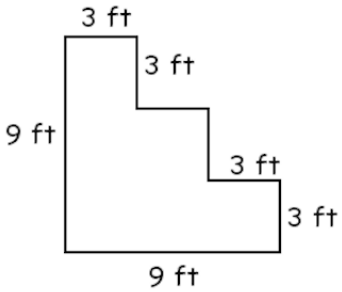
1. Compute the perimeter and area of a rectangle with sides 12cm by 8cm.

2. Consider the figure shown. Angles that look like right angles are right anges.



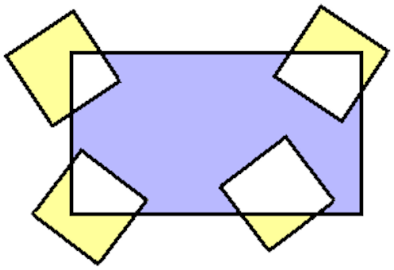
   a) Compute the perimeter of the figure. Include units in your computation and answer.
   a) Compute the perimeter of the figure. Include units in your computation and answer.
   b) Compute the area of the figure. Include units in your computation and answer.
   b) Compute the area of the figure. Include units in your computation and answer.

3. Consider the figure shown. Angles that look like right angles are right angles.



The rectangle shown on the picture has sides 2cm and 3cm long. The four other shapes are all squares with sides 1cm. Which region has a greater area, the blue or the yellow?

Enrichment



Problem Set 1

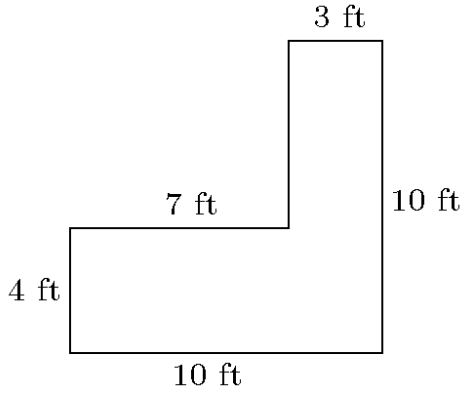
Simplify each of the following expressions by applying the order of operations agreement. **Show all steps. For each step, write a separate line!**

1. $8-2+3$
2. $\dfrac{15-2^3+3}{3^2-2^3}$
3. $120\div\left(4+3\left(5\cdot 2^2-2\left(5+2^2\right)\right)\right)$
4. $\dfrac{3^2-2^2}{(3-2)^2}$
5. $\left(3-\left(10-3^2\right)^2\right)^2$
6. $\dfrac{100\div 5\cdot 2}{25-10+5}$
7. $32-3\left(28-2^2\left(20-5\cdot 3\right)\right)$
8. $\left(\left((10-8)^2-1\right)^2-2\right)^2$
9. $1^2+1^3-1^4+1^5$
10. 2^2+2^3
11. $2^5-3\left(12-3^2\right)^2$
12. $5\cdot 2^3-\left(10-(7-2\cdot 3+1)\div 2+2^2\right)$
13. $\left(2-\left(2-\left(10-3^2\right)^2\right)^2\right)^2$

14. a) One angle in an isosceles triangle is 30°. What can be the measures of the other two angles?

    a) One angle in an isosceles triangle is 100°. What can be the measures of the other two angles?

15. Find the perimeter and area of the object shown.



16. Insert parentheses in the expression on the left-hand side to make the equation true.

$$4\ +\ 2\ \cdot\ 5\ -3\ ^2=11$$

17. Leon has to dress in the dark this morning. His sock drawer contains 20 white socks (10 pairs), 10 black (5 pairs), and 8 green (4 pairs). All the socks are loose in the drawer. If we randomly pull individual socks without seeing their colors, then

    a) How many socks do we need to take out of the drawer so that we have a pair of socks in our hand for certain?

b) How many socks do we need to take out of the drawer so that we have a white sock in our hand for certain?

Chapter 2

2.1 Rounding

If we look it up in the internet, we find that the population of Illinois in 2020 was about 12 000 000 (twelve million). If we think about it, it is hard to have an exact number as people are born and die every minute, and so the number of people living in Illinois is never quite a fixed number we could find. Even if we could, that number, twelve million is too round to be true. Look up any state's population, they all look that round. It would be starnge indeed to find that the population of Illinois is 12 020 034. California's population is about forty million. If we are comparing such large numbers, that 34 at the end of the number 12 020 034 is useless.

When we are talking about numbers, replacing 12 020 034 by 12 000 000 is called **rounding**. In this case, we round because numbers are so large such that after the first few digits the numbers have no meaning in giveing an idea of the side of the number. Sometimes we round numbers to estimate results of computations. Sometimes we round numbers because the numbers are results of measurement that cannot be precise anyway.

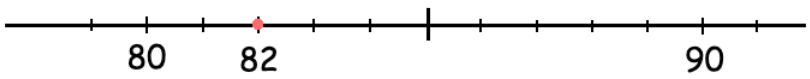
So, how do we round? There are several ways to round, depending on how precise of a number we want to round to. Let us consider the number 13 582.

Example 1. Round 13 582 to the nearest ten.

Solution: By tens we mean the numbers 10, 20, 30, 40, ...

Imagine that we count in this manner until we first reach a number greater than 13 582.

This happens when we reach 13 560, 13 570, $\boxed{13\,580,\ 13\,590}$. Our number, 13 582 is between 13 580 and 13 590. Rounding 13 582 to the nearest ten means selecting 13 580 or 13 590, whichever is closer to our number. Ignoring the beginning digits and focusing on the end of the numbers, the question is this: Which number is 82 closer to, 80 or 90?



As the number line on the picture shows, 82 is closer to 80. Therefore, 13 582, rounded to the nearest ten is $\boxed{13\,580}$.

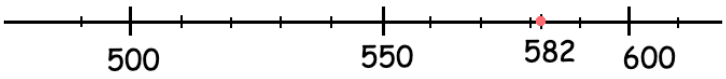
Based on the picture, 81, 82, 83, and 84 would be rounded down to 80, and 86, 87, 88 , and 89 would be rounded up to 90. The only question remains: what about 85? This number is exactly five units away from both 80 and 90. Mathematicians agreed that in such case, we will round up. So, 85 would be rounded up to 90.

Example 2. Round 13 582 to the nearest hundred.

By hundreds, we mean the numbers 100, 200, 300, 400, ...

Imagine that we count in this manner until we first reach a number greater than 13 582.

This happens when we reach 13 300, 13 400, $\boxed{13\,500,\ 13\,600}$. Our number, 13 582 is between 13 500 and 13 600. Rounding 13 582 to the nearest hundred means selecting 13 500 or 13 600, whichever is closer to our number. Ignoring the beginning digits and focusing on the end of the numbers, the question is this: Which number is 582 closer to, 500 or 600?



As the number line on the picture shows, 582 is closer to 600. Therefore, 13 582, rounded to the nearest hundred is $\boxed{13\,600}$.

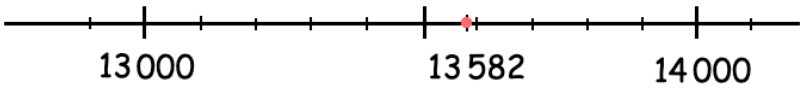
Based on the picture, 501, 502, 503, and 549 would be rounded down to 500, and 551, 552, 553 , and 599 would be rounded up to 600. The only question remains: what about 550? This number is exactly fifty units away from both 500 and 600. Mathematicians agreed that in such case, we will round up. So, 550 would be rounded up to 600.

Example 3. Round 13 582 to the nearest thousand.

Solution: By thousand we mean the numbers 1000, 2000, 3000, 4000, ...

Imagine that we count in this manner until we first reach a number greater than 13 582.

This happens when we reach 11 000, 12 000, $\boxed{13\,000,\ 14\,000}$. Our number, 13 582 is between 13 000 and 14 000. Rounding 13 582 to the nearest thousand means selecting 13 000 or 14 000, whichever is closer to our number.



As the number line on the picture shows, 13 582 is closer to 14 000. Therefore, 13 582, rounded to the nearest thousand is $\boxed{14\,000}$.

Based on the picture, all numbers from 13 000 to 13 499 would be rounded down to 13 000, and all numbers from 13 501 to 13 999, would be rounded up to 14 000. The only question remains: what about 13 500? This number is exactly five hundred units away from both 13 000 and 14 000. Mathematicians agreed that in such case, we will round up. So, 13 500 would be rounded up to 14 000.

Example 4. Round 13 582 to the nearest ten thousand.

Solution: Our number is between 10 000 and 20 000. Rounding 13 582 to the nearest ten thousand means selecting 10 000 or 20 000, whichever is closer to our number. We can now focus on the beginning of the number, and so the question can be rephrased: 13 is closer to 10 or 20? Since 13 is closer to 10 than to 20, the number 13 582 rounded to the nearest ten thousand is $\boxed{10\,000}$.

what about 15 000? This number is exactly five thousand units away from both 10 000 and 20 000. Mathematicians agreed that in such case, we will round up. So, 15 000 would be rounded up to 600.

The agreement in rounding up in these special cases is no more precise than rounding down. However, it comes with a great benefit: we can decide whether to round up or down by looking at a single digit. If that digit is 5 or more, we round up. If that digit is 4 or below, we round down.

In summary,

13 58$\boxed{2}$ rounded to the nearest ten is 13 580 (2 is less than 5)

13 5$\boxed{8}$2 rounded to the nearest hundred is 13 600 (8 is more than 4)

13$\boxed{5}$82 rounded to the nearest thousand is 14 000 (5 is more than 4)

1$\boxed{3}$582 rounded to the nearest ten is 10 000 (3 is less than 5).

Example 5. Round 12 502

a) to the nearest ten b) to the nearest hundred

Solution: If we were to count by tens: 10, 20, 30,... and so on, we would first reach a number greater than 12 502, it would be 12 500 and 12 510. Our number is closer to 12 500, so it is our answer.

If we were to count by hundreds: 100, 200, 300,... and so on, we would first reach a number greater than 12 502, it would be 12 500 and 12 600. Our number is closer to 12 500, so it is our answer.

It is possible to round to the nearest ten and nearest hundred and get the same result. This is because we hit 100, 200, and so on, whenever we count by tens.

Consider the number 572 001. We would get the same result after rounding to the nearest ten, or hundred, or thousand.

Practice Problems

1. Round 6 825 023

   a) to the nearest ten
   b) to the nearest hundred
   c) to the nearest thousand
   d) to the nearest ten thousand
   e) to the nearest hundred thousand
   f) to the nearest million

2. Round 12 479 502

   a) to the nearest ten
   b) to the nearest hundred
   c) to the nearest thousand
   d) to the nearest ten thousand
   e) to the nearest hundred thousand
   f) to the nearest million

2.2 Divisibility Rules

Definition: Suppose that N and m are any two integers. If there exists an integer k such that $N = mk$, then we say that m is a **factor** or **divisor** of N. We also say that N is a **multiple** of m or that N is **divisible** by m. Notation: $m|N$

For example, 3 is a factor of 15 because there exists another integer (namely 5) so that $3 \cdot 5 = 15$. Notation: $3|15$.

Example 1. Label each of the following statements as true or false.

a) 2 is a factor of 10
b) 3 is divisible by 3
c) 14 is a factor of 7
d) 0 is a multiple of 5
e) every integer is divisible by 1
f) every integer n is divisible by n

Solution: a) $10 = 2 \cdot 5$ and so 2 is a factor of 10. This statement is true.

b) $3 = 3 \cdot 1$ and so 3 is divisible by 3. This statement is true.

c) $14 = 7 \cdot 2$ and so 14 is a multiple of 7, not a factor. Can we find an integer k so that $7 = 14 \cdot k$? This is not possible. $k = \frac{1}{2}$ would work, but $\frac{1}{2}$ is not an integer. This statement is false.

d) Since $0 = 5 \cdot 0$, it is indeed true that 0 is a multiple of 5. This statement is true.

e) For any integer n, $n = n \cdot 1$ and so every integer n is divisible by 1. This statement is true.

f) For any integer n, $n = 1 \cdot n$ and so every integer n is divisible by n. This statement is true.

We will later prove all of the following statements. They will cut down on the work as we look for divisors of a number.

Theorem (Dvisibility Tests)

A number n is divisible by 2 if its last digit is 0, 2, 4, 6, or 8.

A number n is divisible by 3 if the sum of its all digits is divisible by 3.

A number n is divisible by 4 if the two-digit number formed by its last two digits is divisible by 4.

A number n is divisible by 5 if its last digit is 0, or 5.

A number n is divisible by 6 if it is divisible by 2 and by 3.

A number n is divisible by 9 if the sum of its all digits is divisible by 9.

A number n is divisible by 10 if its last digit is 0.

A number n is divisible by 12 if it is divisible by 3 and by 4.

Example 2. Consider the numbers 2016, 183422, 606060, 123321, and 38115. Find all numbers from this list that are divisible by a) 3 b) 9 c) 4 d) 5

Solution: a) We need to add all digits. If the sum of the digits is divisible by 3, so is the number.

$2+0+1+6 = 9$. Since 9 is divisible by 3,so is 2016.

$1+8+3+4+2+2 = 20$. Since 20 is not divisible by 3, neither is 183422.

$6+0+6+0+6+0 = 18$. Since 18 is divisible by 3, so is 606060.

$1+2+3+3+2+1 = 12$. Since 12 is divisible by 3, so is 123321.

$3+8+1+1+5 = 18$. Since 18 is divisible by 3, so is 38115.

Therefore, the numbers divisible by 3 are: $\boxed{2016,\ 606060,\ 123321,\ 38115}$.

b) The divisibility test for 9 also requires the sum of all digits. We just worked those out.

The sum of digits of 2016 is 9. Since 9 is divisible by 9, so is 2016.

The sum of digits of 183 422 is 20. Since 20 is not divisible by 9, neither is 183 422.

The sum of digits of 606 060 is 18. Since 18 is divisible by 9, so is 606 060.

The sum of digits of 123 321 is 12. Since 12 is not divisible by 9, neither is 123 321.

The sum of digits of 38 115 is 18. Since 18 is divisible by 9, so is 38 115.

Therefore, the numbers divisible by 9 are: $\boxed{2016,\ 606\,060,\ 38\,115}$.

c) The divisibility test for 4 requires to look at the two-digit number formed from the last two digits. If that two-digit number is divisible by 4, so is the number.

2016 ends in 16. Since 16 is divisible by 4, so is 2016.

183 422 ends in 22. Since 22 is not divisible by 4, neither is 183 422.

606 0606 ends in 60. Since 60 is divisible by 4, so is 606 060.

123 321 ends in 21. Since 21 is not divisible by 4, neither is 123 321.

38 115 ends in 15. Since 15 is not divisible by 4, neither is 2016.

Therefore, the numbers divisible by 4 are: $\boxed{2016,\ 606\,060}$.

d) The divisibility test for 5 requires to look at the last digit. If it is 0 or 5, the number is divisible by 5.

Among the numbers 2016, 183 422, 606 060, 123 321, and 38 115, only 606 060 and 38 115 ends in 0 or 5.Therefore, the numbers divisible by 5 are: $\boxed{606\,060,\ 38\,115}$.

Example 2v. Consider the numbers 159, 289, 292, 444, 591, 50 020. Find all numbers from the list that are divisible by a) 3 b) 4 c) 12 **Solution - Youtube link**

Enrichment

More Divisibility Rules

A number n is divisible by 11 if the difference of the sum of digits at odd places and the sum of its digits at even places, is divisible by 11. For example, $393\,822$ is divisible by 11, because $(3+3+2)=8$ and $9+8+2=19$, and $19-8=11$ is divisible by 11

A number m is divisible by 7 if: First we create a smaller number by cutting off the last digit. From this number, we subtract the double of the last digit. If this difference is divisible by 7, so is our number. For example, 2023 is divisible by 7, because $202-2\cdot 3=196$, and (repeat the rule) $19-2\cdot 6=7$ is divisible by 7.

Note that in case of divisibility by 11 and 7, the difference between the two groups does not need to be zero, only to be divisible by the number.

Practice Problems

1. Consider the following numbers. $128,\ 80, 75, 270,\ 64$

   a) Find all numbers on the list that are divisible by 5.

   b) Find all numbers on the list that are divisible by 3.

   c) Find all numbers on the list that are divisible by 4.

2. Consider the following numbers: $4181,\ 9800,\ 1296,\ 420,\ 55\,050$

   a) Find all numbers from the list that are divisible by 3.

   b) Find all numbers from the list that are divisible by 10.

   c) Find all numbers from the list that are divisible by 6.

3. Consider the following numbers: $28\,072,\ 67\,808,\ 60\,610, 1296$, and 7620.

   a) Find the numbers from the list that are divisible by 4.

   b) Find the numbers from the list that are divisible by 5.

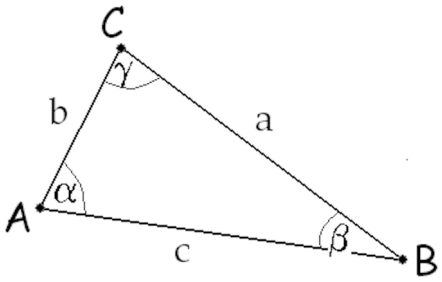
   c) Find the numbers from the list that are divisible by 9.

2.3 Perimeter and Area of a Right Triangle

Part 1 – Standard Labeling

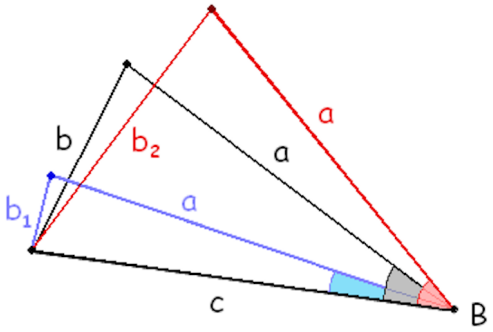
The perimeter of any triangle is simply the sum of the lengths of its three sides. We will have to learn much more for a discussion of the area of right triangles. But before we do that, let us agree first on a method of notation that avoids confusion and lengthy explanations in geometry problems. This agreement is called *standard labeling*, and it establishes a connection between the labels of sides, vertices, and angles in triangles. Every triangle has three of the following three components.

1. **vertices** (singular: vertex)
   Points are usually denoted by uppercase letters. In case of triangles, we often use A, B, and C.
2. **angles**
   Angles are usually denoted by lowercase Greek letters. In case of triangles, we often use α (alpha), β (beta), and γ (gamma).
3. **sides**
   Lines and line segments are usually denoted by lowercase letters. In case of triangles, we often use a, b, and c.



In case of standard labeling, we automatically associate sides, vertices, and angles. A vertex is associated with the angle located at that vertex. These two are associated with the side opposite these. For example, angle α is always assumed to be located at point A, and side a is always assumed to be the side oppsite to point A and angle α. Point B, angle β, and side b are similarly grouped. Unless otherwise indicated, we should always assume standard labeling when presented with data that uses these letters.

Standard labeling is a smart approach to triangles, because there is a natural connection between an angle in a triangle and the side opposite that angle. Consider, for example, the triangle shown above with standard labeling. What if we fixed sides a and c and only modified angle β? Imagine that we have two rods in the lengths of a and c attached to each other at one end and we can freely change the angle between them. If we increase the angle between sides a and c (see the red lines), the side opposite will also increase. If we decrease the angle between sides a and c (see the blue lines), the side opposite will also decrease. So, there seems to be a natural correspondance between side b and angle β.



Theorem: In any triangle ABC, there is a correspondance between the length of a side and the measure of the angle opposite that side:
The longest side is opposite the greatest angle, and vica versa: the greatest angle is opposite the longest side. The shortest side is opposite the smallest angle, and vica versa: the smallest angle is opposite the shortest side.

So, the order between the three sides is the same as the order between the corresponding angles, and vica versa. We recommend that sides in triangles are tracked by their corresponding sides. This is because we can perceive the difference in angles much better than in side lengths.

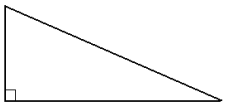
Example 1. Suppose that ABC is a triangle with $\alpha = 82^\circ$ and $\gamma = 39^\circ$. List the length of the sides of the triangle in an increasing order.

Solution: Recall that the three angles in a triangle add up to 180°. This means that if two angles are given, we can compute the third one. $\beta = 180^\circ - (82^\circ + 39^\circ) = 180^\circ - 121^\circ = 59^\circ$. Now we can see the order between the angles. γ is the smallest angle, β is in the middle, and α is the greatest angle. In short: $\gamma < \beta < \alpha$. The order between the lengths of the sides is the same: c is the shortest side, b is in the middle, and a is the longest side. In short: $c < b < a$.

There is an easy but important consequence of this property.

Theorem: In any triangle ABC, if two angles have equal measure, then the sides opposite them have equal length. If two sides are equally long, then the angles opposite those sides have equal mesaures. Such a triangle is called **isosceles**.

Definition: A **right triangle** is a triangle with a right angle. A right angle measures 90°.



We should never assume that a triangle is right, unless it is formally stated either in the problem, or on a picture. Just because an angle appears to be a right angle, it could have a measure of 89.5°, and that is not right. The measure must be exactly 90°.

Theorem: In a right triangle, there can only be one right angle and it is the greatest angle in the triangle.

Discussion:

1. Find an algebraic and geometric argument for the theorem stated above. Why is it true?
2. What does that mean for the sides of a right triangle?

Definition: In any right triangle, there is a single longest side, and it is opposite of the right angle. This side is called the **hypotenuse** of the right triangle. The shorter sides are called **leg**s.

The hypotenuse of a triangle is often denoted by c, but this is just a convenient habit and not a rule! We should never assume that the rigth angle in a triangle is γ. For all we now, it could be any of α, β, or γ.

We took two pages to establish that there is a longest side in a rigth triangle. The perimeter and are formulas will be very easy.

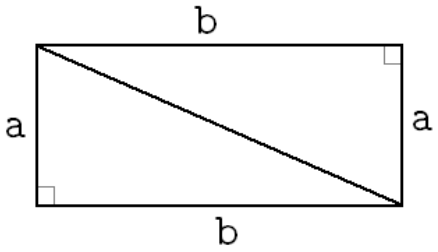
Part 2 - Perimeter and Area Formulas

Recall that the perimeter is just the length of the boundary. This means that the perimeter of any triangle can be computed by adding the lengths of its three sides.

Theorem: The perimeter of a triangle ABC can be computed as $P = a+b+c$.

When we reviewed the area formula for rectangles, we have mentioned that that was a very difficult formula to prove. It is often the case in mathematics that, once we worked very hard for a formula, we use that result over and over. When it comes to area, *all* we know are rectangles. In other words, every area formula was derived from the area formula of the rectangle.

Every right triangle is half of a rectangle. In other words, given any right triangle, we can use two identical copies of it to form a rectangle. We know how to find the area of the rectangle: $A = ab$.
Because the rectangle consists of two identical (also called congruent) right triangles, it naturally follows that each takes up half of the area of the rectangle. Thus the area of the right triangle is $A = \dfrac{ab}{2}$.



Theorem: The area of a right triangle ABC, where c denotes the hypotenuse, is $A = \dfrac{ab}{2}$.

What is unusual about this formula is that we don't need the length of the hypotenuse, only the lengths of the other two sides. This means that given the three sides of a right triangle, we need to know to only use the lengths of the two shorter sides.

Example 2. Compute the perimeter and area of a right triangle with sides 8ft, 15ft, and 17ft.

Solution: The perimeter is just the sum of all three sides.

$$P = a+b+c = 8\text{ft}+15\text{ft}+17\text{ft} = \boxed{40\text{ft}}$$

For the area, we only need the two shorter sides.

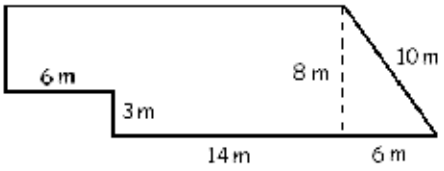
$$A = \frac{ab}{2} = \frac{8\text{ft}\cdot 15\text{ft}}{2} = \frac{120\text{ft}^2}{2} = \boxed{60\text{ft}^2}.$$

Although we will derive formulas for the area of more complicated shapes, we can often avoid those by cutting our objects into rectangles and right triangles.

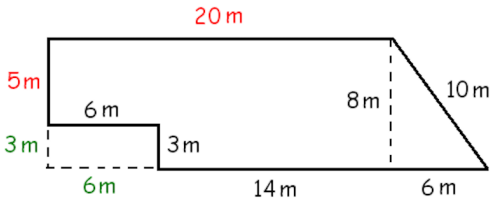
Example 2v. Find the perimeter and area of the right triangle with sides 10in, 24in, and 26in long.

Solution - Youtube link

Example 3. Compute the perimeter and area of the figure shown on the picture. Angles that look like right angles are right angles. Units are in meters.



Solution: For the perimeter, we need to find two missing sides. The vertical side is 5m long since the opposite sides of rectangles are equally long. Similarly, the missing horizontal side is 20m long. We can now compute the perimeter. Notice that the vertical line in the inside is not part of the boundary. We will add all the sides, starting with the long horizontal side.



$$P = 20\text{m} + 10\text{m} + 6\text{m} + 14\text{m} + 3\text{m} + 6\text{m} + 5\text{m} = \boxed{64\text{m}}$$

Now for the area: We will compute the area of the big rectangle and from it we will subtract the area of the smaller rectangle.

$$A_1 = 8\text{m} \cdot 20\text{m} - 3\text{m} \cdot 6\text{m} = 160\text{m}^2 - 18\text{m}^2 = 142\text{m}^2$$

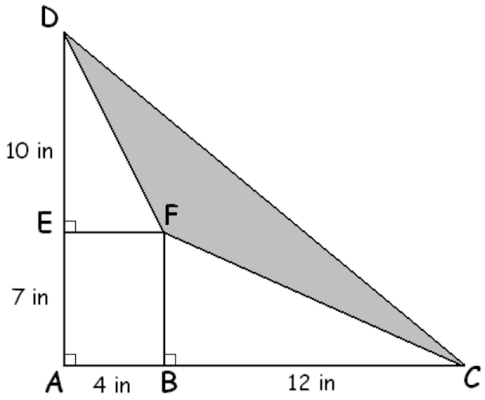
To this we add the area of the right triangle.

$$A_2 = \frac{6\text{m} \cdot 8\text{m}}{2} = \frac{48\text{m}^2}{2} = 24\text{m}^2$$

The area of the entire figure is the sum of the two areas:

$$A = A_1 + A_2 = 142\text{m}^2 + 24\text{m}^2 = \boxed{166\text{m}^2}$$

Example 4. Compute area of the shaded region shown on the picture. Units are in inches.



Solution: First we will compute the area of right triangle ACD. $AC = 16\text{in}$ and $AD = 17\text{in}$.

$$A_{\text{ACD}} = \frac{16\text{in} \cdot 17\text{in}}{2} = \frac{272\text{in}^2}{2} = 136\text{in}^2$$

To get the area of the shaded region, we will subtract the areas of the rectangle $ABFE$ and right triangles DEF and BCF.

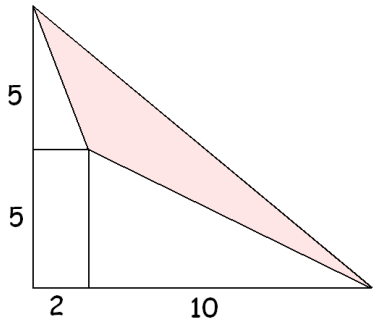
$$\begin{aligned} A_{\text{ABFE}} &= 7\text{in} \cdot 4\text{in} = 28\text{in}^2 \\ A_{\text{DEF}} &= \frac{10\text{in} \cdot 4\text{in}}{2} = 20\text{in}^2 \\ A_{\text{BCF}} &= \frac{7\text{in} \cdot 12\text{in}}{2} = \frac{84\text{in}^2}{2} = 42\text{in}^2 \end{aligned}$$

So the shaded area is when we subtract the white areas from the big right triangle.

$$\begin{aligned} A &= A_{\text{ACD}} - (A_{\text{ABFE}} + A_{\text{DEF}} + A_{\text{BCF}}) \\ &= 136\text{in}^2 - \left(28\text{in}^2 + 20\text{in}^2 + 42\text{in}^2\right) = 136\text{in}^2 - 90\text{in}^2 = \boxed{46\text{in}^2} \end{aligned}$$

Example 4v. Find the area of the shaded region.

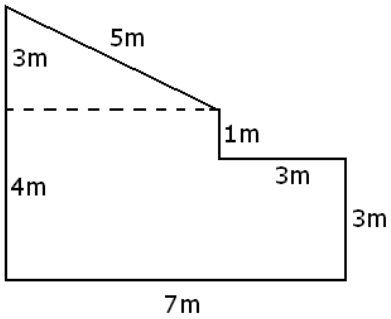
Solution - Youtube link



Practice Problems

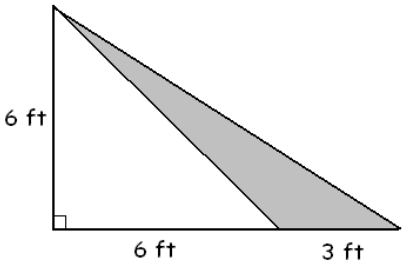
1. Compute the perimeter and area of a right triangle with sides 7ft, 25ft, and 24ft. Include units in your computation and answer.

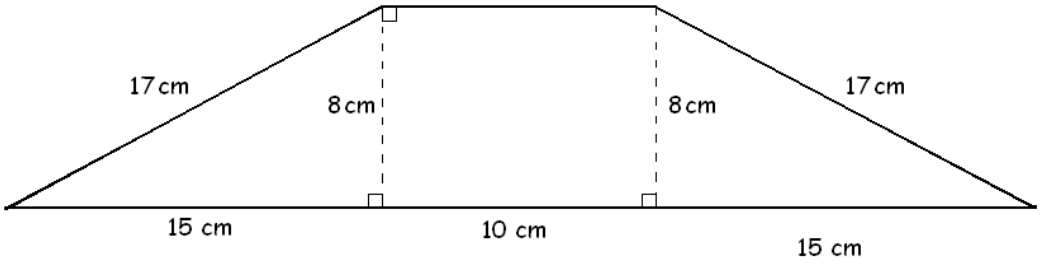
2. Compute the perimeter and area of the figure shown on the picture. Angles that look like right angles are right angles. Units are in meters. Include units in your computation and answer.



3. Compute the area of the shaded region shown on the picture. Include units in your computation and answer.

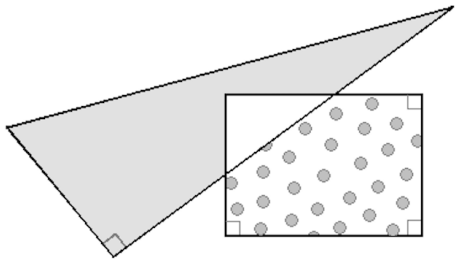
4. Compute the perimeter and area of the figure shown on the picture below. Units are in centimeters. Include units in your computation and answer.





Enrichment

Consider the figure shown on the picture. The shorter sides of the right triangle are 6cm and 10cm long. The sides of the rectangle are 5cm and 6cm long. Which region has the greater area, the shaded or the dotted?

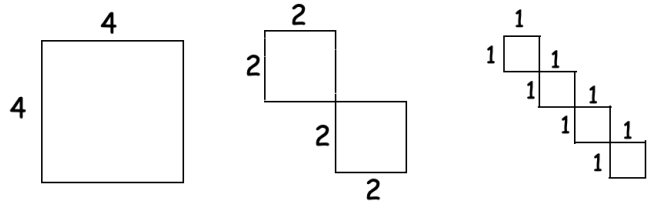


Problem Set 2

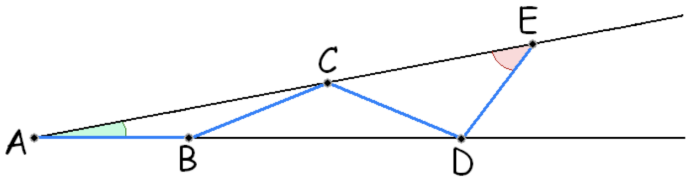
Simplify each of the following expressions by applying the order of operations agreement. **Show all steps. For each step, write a separate line!**

1. $\dfrac{12-3+2}{3^2-2^3}$
2. $15-3\left(12-3^2\right)$
3. $10-2(9-6)$
4. $(10-2)(9-6)$
5. $30-5\left(20-4\left(6^2-48\div 3\cdot 2\right)\right)$
6. $\dfrac{8^2-2^2}{8-2}$
7. $2\cdot\left(3^2-2^2\right)^2-2\cdot\dfrac{10^2-3^2}{10-3}4$
8. $\left(12-\dfrac{100-8^2}{2^3+1}\right)^2$
9. $3\left(12-2^2\right)+5(8-6)$
10. $3\left(\left(12-2^2\right)+5(8-6)\right)$
11. $\left(\left(\left(30-3^3\right)^2-4\right)^2-3\cdot 5\right)^2$
12. $2\left(19-4^2+3\left(5^2-6\cdot 3\right)-7\left(2^4-48\div 6\cdot 2\right)^2\right)$

13. Find the perimeter and area of each of the shapes shown. Angles that look like right angle are right angles. Units are in meters.



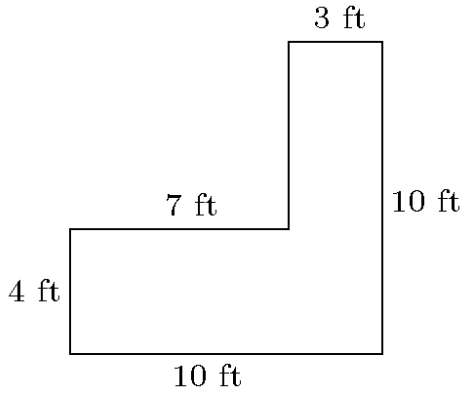
14. The blue line segments are all of equal length. If the angle marked green is 12°, what is the measure of the pink angle?



15. a) One angle in an isosceles triangle is 30°. What can be the measures of the other two angles?

    a) One angle in an isosceles triangle is 100°. What can be the measures of the other two angles?

16. Find the perimeter and area of the object shown.



17. Insert parentheses in the expression on the left-hand side to make the equation true.

$$4 + 2 \cdot 5 - 3^{2} = 11$$

18. Leon has to dress in the dark this morning. His sock drawer contains 20 white socks (10 pairs), 10 black (5 pairs), and 8 green (4 pairs). All the socks are loose in the drawer. If we randomly pull individual socks without seeing their colors, then

    a) How many socks do we need to take out of the drawer so that we have a pair of socks in our hand for certain?

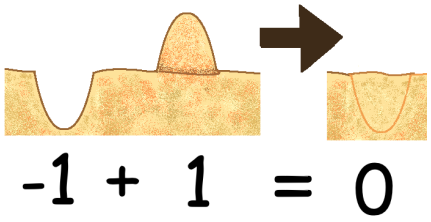
    b) How many socks do we need to take out of the drawer so that we have a white sock in our hand for certain?

Chapter 3

3.1 Introduction To Signed Numbers

With the numbers we have so far, subtraction is the same as "take away". $8-3=5$ means that if we start with 8 tally-marks and take away 3, we will be left with 5 tally marks. And that works, until we have something like $4-7$. We cannot take away 7 from only four. But we needed some sort of a number, even if it is a conceptual one, because this situation came up too often.

Suppose we are playing in the sand. We can think of a pile of sand as a positive thing, 1. So three piles would rperesent the number 3. But we can also have holes of the same size, representing negative numbers. A hole would represent -1.



If we have a pile and a hole, we can use up the pile to fill up the hole and are left with nothing. What would then $-7+4$ be? We have 4 piles and 7 holes. As long as we have both, we can use one pile to fill up one hole. If we use 1 pile to fill up one whole, we are left with one less of each: 3 piles and 6 holes. We can again fill up the hole: then we would have 2 piles and 5 holes left. We can then use the remaining 2 piles to fill up two holes. Now we are left with 3 holes, representing -3 (negative three). And we can think of -3 as the number, to which we need to add 3 so that the sum is zero.

There is a less physical, more conceptual way to think of signed numbers, that unfortunately we all know too well. This would be the concept of debt and money. We can think if positive numbers as money and negative numbers as debt. What would $-4+10$ be? Imagine we have 10 dollars but we also have an outstanding debt of 4 dollars. So we take four dollars from the ten and use it to pay off our debt. We now have no debt and 6 dollars. Therefore, $-4+10=6$.

Definition: The **opposite** of 3 is a negative number, written as -3. For any number, the sum of the number and its opposite is zero.

The opposite of 3 is -3. The opposite of -3 is 3. The opposite of zero is zero itself.

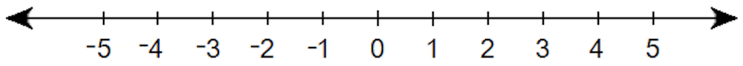
The negative sign already has a meaning, that of subtraction. We now are facing an ambiguity that is often the source of confusion. Does a negative sign denote the opposite of a number, or does it denote subtraction? This is a question that we often need to ask ourselves. While the answer always clearly exists, it very much depends on the context. For example, the negative sign in -3 clearly denotes that we are talking about the opposite of 3 or negative 3. However, if we place a number in front of it, like in $8-3$, the same sign here denotes subtraction.

And what about $8(-3)$? Now the parentheses tells us that the negative sign does not denote subtraction, rather it describes the number after it as negative.

-3	$8-3$	$8(-3)$
the opposite of 3	subtraction	the opposite of 3

Definition: The counting numbers, their opposites, and zero all **integers**.

We often depict integers on a number line.



Definition: (Ordering with Integers) Between two integers, the one on the right is greater.

This is easy to see as well. Being in a debt of 100 dollars makes use more poor than being in a debt of just 2 dollars. Any amount of debt is worse than any amount of money. To determine which number is greater, we just have to ask ourselves:: who is richer? $2 < 10$ was obvious, and also that $-5 < 3$, but now we see that between -100 and -2, -2 is greater. After all, the person who has no money and only 2 dollars of debt is better off than another person who has no money and a 100 dollars of debt. And so $-100 < -2$.

We can swap inequalities, as long as the smaller part of the inequality sign points to the smaller number.

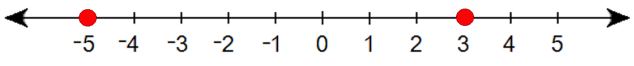
$-100 < -2$	read: -100 is less than -2	$-100 \leq -2$	read: -100 is less than or equal to -2
$-2 > -100$	read: -2 is greater than -100	$-2 \geq -100$	read: -2 is greater than or equal to -100

Wether we plot them on a number line or think money and debt, we will agree that -1000000 (negative one million) is less than 5. But what if we wanted to compare the size of numbers, ignoring their signs? Suppose we want to say that a million dollar debt is a lot of debt. In this case, we use the concept of the absolute value of a number.

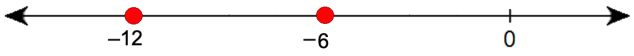
Example 1. Compare the given numbers by inserting $<$ or $>$ between them.

a) $3 \quad -5$ b) $-12 \quad -6$ c) $1 \quad 14$ d) $7 \quad -10$

Solution: a) Who is richer? The person with no debt and 3 dollars, or the person with no money and a debt of 5 dollars? In this case, the person with the money is clearly richer. So we insert the inequality sign in a way so that the wide part is near the greater number. $3 > -5$. Also, if we draw a number line and plotted 3 and -5, we find that between the two numbers, 3 is on the right, therefore it is the greater number.



b) Who is richer? The person with no money and a debt of 12 dollars, or the person with no money and a debt of 6 dollars? In this case, the person with 'lesser debt' is clearly richer, so -6 is greater than -12. So we insert the inequality sign in a way so that the wide part is near the greater number. $-12 < -6$. Also, if we draw a number line and plotted -12 and -6, we find that between the two numbers, -6 is on the right, therefore it is the greater number.



c) This one was already clear: 14 is greater than 1, so $1 < 14$.

d) Who is richer? The person with no debt and 7 dollars, or the person with no money and a debt of 10 dollars? In this case, the person with the money is clearly richer. So we insert the inequality sign in a way so that the wide part is near the greater number. $7 > -10$. Also, if we draw a number line and plotted 7 and -10, we find that between the two numbers, 7 is on the right, therefore it is the greater number.

Definition: The **absolute value of a number** is its distance from zero on the number line. We denote the absolute value of a number x by $|x|$.

Distances can never be negative. -5 is 5 units away from zero on the number line. So is 5, only it is in the other direction. So, the absolute value of both 5 and -5 is 5. In other words, now that we have established that 5 is greater than -5, we would want to say: OK, but they are the same size.

This is typical to mathematics, and in general, progress. We run into a problem, i.e. what is 4 take away 7, and when we solve the problem, we create new ones. Now that we declared the ordering between negative and positive numbers, we will have to do extra work to compare just the size of them. In a sense, the absolute value is the answer to the question: I don't care positive or negative (money or debt), I just want to know how much.

Example 2. Compute each of the following.

a) $|-2|$ b) $|2|$ c) $|0|$ d) $-|-5|$

Solution: a) The number -2 is 2 units away from zero on the number line. Thus $|-2| = 2$.

b) The number 2 is 2 units away from zero on the number line. Thus $|2| = 2$.

c) The distance between zero and zero on the number line is zero. Thus $|0| = 0$.

d) This is an example for two negatives not making a positive. The way we can read this as: *the opposite of the absolute value of negative five*. The absolute value of -5 is 5. The opposite of that is -5. Using notation, $-|-5| = -5$.

Addition of Integers: Again, think money and debt. Positive numbers represent money, negative numbers represent debt. Adding zero to anything will leave the other number unchanged.

Example 3. Compute each of the following sums.

a) $-8+15$ b) $-3+(-8)$ c) $3+(-14)$ d) $-7+2$ e) $-2+0$

Solution: a) We think of $-8+15$ as follows. We start with a person who has no money and is in debt by 8 dollars. To this, we add 15 dollars. So the person pays off all that 8 dollar debt and is still left with 7 dollars. So $-8+15=7$.

b) We think of $-3+(-8)$ as follows. We start with a person who has no money and is in debt by 3 dollars. To this, we add another debt of 8 dollars. So this person is still in debt, now by 11 dollars. So $-3+(-8)=-11$.

c) We think of $3+(-14)$ as follows. We start with a person who has 3 dollars. To this, we add a debt of 14 dollars. So the person pays off all the debt he can - that is 3 dollars and is still in debt by 11 dollars. So $3+(-14)=-11$.

d) We think of $-7+2$ as follows. We start with a person who has no money and is in debt by 7 dollars. Then this person gets 2 dollars. So the person pays off all the debt she can. After she pays off 2 dollars of debt, she has no money and is still left with 5 dollars of debt. So $-7+2=-5$.

e) Adding zero to any number leaves the other number unchanged. Therefore, $-2+0=-2$.

We already know how to add two positive numbers, and we know that the sum is positive. Adding two negative numbers is similar, we are summing debts. So we know that the sum of two negative numbers is also negative. If we add a negative and a positive number, the result may be positive or negative, depending on which number's size (or absolute value) is greater.

Adding zero to any number leaves that number unchanged. In other words, for any integer x, $x+0=x$ and $0+x=x$.

Definition: When added to any integer, zero has no effect. Same is true when subtracting zero.

Discussion:
Can you find a number that behaves with multiplication like zero behaves with addition?

Now that we can add integers, we need to return to absolute values. The absolute value sign is also a grouping symbol that overwrites order of operations. So if there is a sum (or any other expression) within the absolute value sign, we need to perform those until we are left with just a number within the absolute value sign. Then we take the absolute value of that number.

Example 4. Compute each of the following.

a) $|-9+4|$ b) $|-9|+|4|$ c) $|8|+|-7|$ d) $|8+(-7)|$

Solution: a) The absolute value sign is also a pair of parentheses. We perform the addition $-9+4$ inside, and get -5. Then we take the absolute value of -5.
$|-9+4|=|-5|=5$

b) In this example we take the absolute values and then add.
$|-9|+|4|=9+4=13$

c) We take the absolute values and then add.
$|8|+|-7|=8+7=15$

d) We first perform the addition inside and then take the absolute value.

$$|8+(-7)| = |1| = 1$$

Changes in Notation

With the introduction of negative numbers, our notation will have to be modified. It is a widely accepted convention that if there are several signs (operations or negative) between two numbers, a pair of parentheses must separate them. For this reason, until a few decades ago, we used to put a pair of parentheses around *every negative number*.

$-2+-6$	$\rightarrow$	$(-2)+(-6)$	$\rightarrow$	$-2+(-6)$
$+-$ is not allowed		old style		more modern style

The development of mathematical notation is an ongoing process. The most important goal in notation is clarity. As long as clarity is not jeopardized, mathematicians are in the habit of omitting things. A few decades ago we stopped putting the parentheses around the first negative number in the line or inside a parentheses, because there was no risk that we would read the sign incorrectly as subtraction. Also, there is rarely an operation sign in front of the first number. (Subtract from what?) The most common modern style is minimalistic, ommitting as much as possible, as long as confusion is avoided. This can lead to apparent irregularities in notation. For example, our notation will be $2+(-3)$, but when we swap the two, it will be $-3+2$.

Practice Problems

1. Place an inequality sign between the given numbers to make the statement true.

   a) $5 \quad -7$ b) $-12 \quad -4$ c) $0 \quad -8$ d) $-1 \quad -4$ e) $-7 \quad -7$

2. Simplify each of the following.

   a) $|-5|$ b) $|5|$ c) $-|5|$ d) $-|-5|$ e) $|0|$ f) $|-12+9|$ g) $|-12|+|9|$

3. Perform the indicated operations.

   a) $-2+5$ b) $2+(-5)$ c) $3+0$ d) $-3+3$ e) $7+(-4)+(-1)$

4. Perform the indicated operations.

   a) $-7+4$
   b) $|-7|+|4|$
   c) $|-7+4|$
   d) $12+(-2)+8$
   e) $5+(-13)$
   f) $-8+3$
   g) $-3+(-10)$
   h) $9+|-1|$
   i) $|9+(-1)|$

5. * $-100+(-99)+(-98)+....+99+100+101+102$

3.2 Factors of a Number

Definition: Suppose that N and m are any two integers. If there exists an integer k such that $N = mk$, then we say that m is a **factor** or **divisor** of N. We also say that N is a **multiple** of m or that N is **divisible** by m. Notation: $m|N$

For example, 3 is a factor of 15 because there exists another integer (namely 5) so that $3 \cdot 5 = 15$. Notation: $3|15$.

Example 1. Label each of the following statements as true or false.

a) 2 is a factor of 10
b) 3 is divisible by 3
c) 14 is a factor of 7
d) 0 is a multiple of 5
e) every integer is divisible by 1
f) every integer n is divisible by n

Solution: a) $10 = 2 \cdot 5$ and so 2 is a factor of 10. This statement is true.

b) $3 = 3 \cdot 1$ and so 3 is divisible by 3. This statement is true.

c) $14 = 7 \cdot 2$ and so 14 is a multiple of 7, not a factor. Can we find an integer k so that $7 = 14 \cdot k$? This is not possible. $k = \frac{1}{2}$ would work, but $\frac{1}{2}$ is not an integer. This statement is false.

d) Since $0 = 5 \cdot 0$, it is indeed true that 0 is a multiple of 5. This statement is true.

e) For any integer n, $n = n \cdot 1$ and so every integer n is divisible by 1. This statement is true.

f) For any integer n, $n = 1 \cdot n$ and so every integer n is divisible by n. This statement is true.

We will later prove all of the following statements. They will cut down on the work as we look for divisors of a number.

Example 2. List all positive factors of the number 28.

Solution: We start counting, starting at 1.
Is 1 a divisor of 28? Yes, because $28 = 1 \cdot 28$.
We note both factors we found.

28	
1	28

We continue counting. Is 2 a divisor of 28? Yes, because $28 = 2 \cdot 14$.
We note both factors we found.

28	
1	28
2	14

We continue counting. Is 3 a divisor of 28? No. We can divide 28 by 3 and the answer is not an integer.

We continue counting. Is 4 a divisor of 28? Yes, because $28 = 4 \cdot 7$. We note both factors we found.

28	
1	28
2	14
4	7

We continue counting. Is 5 a divisor of 28? No. (We can check with the calculator.) Is 6 a divisor of 28? No. Now we arrive to 7, a number that is already listed as a factor. That's our signal that we have found all of the divisors of 28. We list the divisors in order:

factors of 28: $\boxed{1, 2, 4, 7, 14, 28}$

Example 2v. List all factors of 75. Solution - Youtube link

Discussion: Discussion: What do you think about the argument shown below?

3 is a divisor of 21 because there exists another integer, namely 7, so that $21 = 3 \cdot 7$. As we established that 3 is a divisor of 21, we also found that 7 is also a divisor of 21. In other words, divisors always come in pairs. For example, 28 has six divisors that we found in three pairs: 1 with 28, 2 with 14, and 4 with 7. Consequently, every positive integer has an even number of positive divisors.

Definition: An integer is a **prime number** if it has exactly two divisors: 1 and itself.

For example, 37 is a prime number.

Practice Problems

1. List all the factors of 48.
2. Which of the following is NOT a prime number? $53, 73, 91, 101, 139$

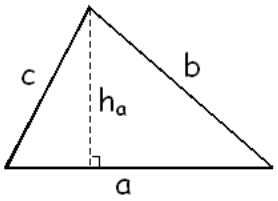
Enrichment

1. Two mathematicians are having a conversation. Mathematician A asks B about his kids. B answers: "I have three children, the product of their ages is 36." A says: "I still don't know the ages of your children." Then B tells A the sum of his three kids' ages. A answers: "I still don't know how old they are. Then B adds: "The youngest one has red hair." Now A knows the ages of all three children. Do you?
2. A king has his birthday. So he decides to let go some of his prisoners. He actually has 100 prisoners at the moment. They are each in a separate cell, numbered from 1 to 100. Well, he is a high tech king. He can close or open any prison door by a single click on the cell's number on his royal laptop. When he clicks at a locked door, it opens. When he clicks at an open door, it locks. At the beginning, every door is locked. First the king clicks on every number from 1 to 100 (therefore opening every door). Then he clicks on every second number from 1 to 100, (i.e. 2, 4, 6, 8, 10,...). Then he clicks on every third number. And so on. Finally, he only clicks on the number 100. Then he orders that the prisoners who find their door open may go free. Who gets to go and who has to stay?

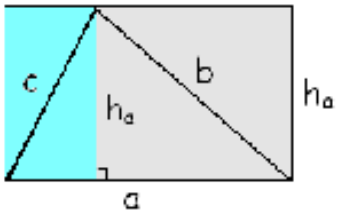
3.3 Area of a Triangle

Let us now consider general triangles.

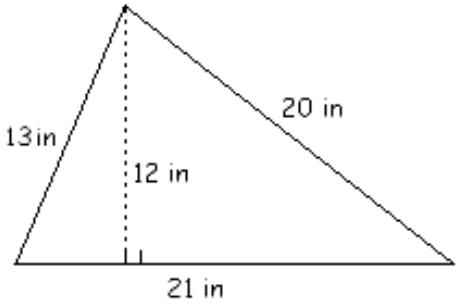
Theorem: The area of a general triangle with sides a, b, c and height h_a as shown on the picture is $A = \dfrac{ah_a}{2}$.



Proof: As before, we will use a previously obtained result. Since the general triangle no longer has a right angle, we create rigth angles by drawing in the altitude or height belonging to side a. Now we split our triangle into two right triangles, and each of them is half of a rectangle. Our triangle makes up for half of the rectangle, with sides a and h_a. Thus $A = \dfrac{ah_a}{2}$.



Example 1. Find the area of the triangle shown on the picture.



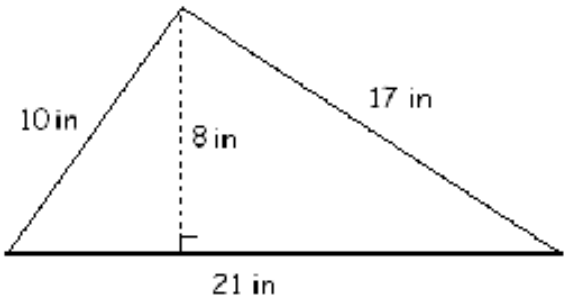
Solution: It is important to notice that we will not need all the information given. We apply the formula for the area.

$$A = \frac{ah}{2} = \frac{21\text{in}\,(12\text{in})}{2} = \frac{252\text{in}^2}{2} = 126\text{in}^2$$

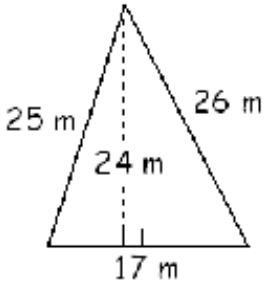
Practice Problems

Find the perimeter and area of each of the triangles shown on the picture. Include units in your computation and answer.

1.

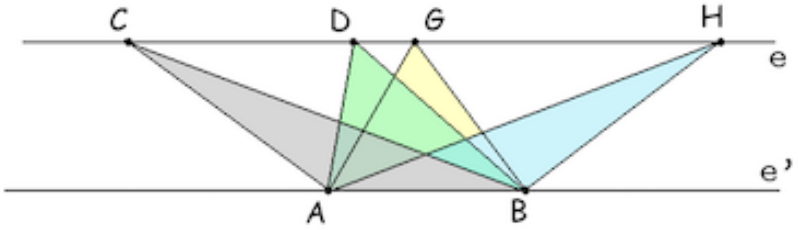


2.



Enrichment

1. Lines e and e' are parallel. Which triangle has the greatest area, ABC or ABD or ABG or ABH?



Chapter 4

4.1 Subtracting Signed Numbers

Recall the signed numbers or integers we have introduced before. We also learned how to add them. Today, we will define subtraction.

> **Definition:** The **opposite** of 3 is written as -3. For any number, the sum of the number and its opposite is zero.

The opposite of 3 is -3. The opposite of -3 is 3. The opposite of zero is zero itself.

> **Definition:** The **absolute value of a number** is its distance from zero on the number line. We denote the absolute value of a number x by $|x|$.

Distances can never be negative. -5 is 5 units away from zero on the number line. So is 5, only it is in the other direction. So, the absolute value of 5 and -5 are both 5.

> **Definition:** The set of all integers is the collection of all counting numbers, their opposites, and zero.
>
> $$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots.$$

> **Definition:** When added to any integer, zero has no effect. Same is true for subtracting zero.

The following statement is always true, and is often extremely useful.

> **To subtract is to add the opposite.**

Of course, we don't always use this fact. In the subtraction $10-3$, we would only complicate things by applying this fact. It would still get us the right result. Instead of subtracting positive 3, we add its opposite, negative 3.

$$10-3=7 \quad \text{and also,} \quad 10+(-3)=7$$

Consider the subtraction $100-(-20)$. We are asked to subtract negative 20. To subtract is to add the opposite. So, instead of subtracting negative 20, we will add its opposite, positive 20.

$$100-(-20)=100+20=120$$

Why is $100-(-20)=100+20$? Even if we understand how to compute this, it would be nice to understand why this is correct. So here is one way to think about this.

Imagine that we have both a bank account a credit card with a bank. Suppose that at the moment, we have 150 dollars in the bank but we also owe 50 dollars to the bank on the credit card. So our net worth is 100 dollars.

Money in bank	Debt on credit card	Total Net worth
150	50	100

Suppose now that we have collected enough bonus points on the credit card to earn rewards. So the bank reduces our credit card debt by 20 dollars. (i.e. subtracts 20 debt, i.e. subtracts negative 20). We still have our 150 in cash, but now our debt is reduced to 30 dollars. So our net worth is now 120 dollars.

	Money in bank	Debt on credit card	Total Net worth
before	150	50	100
after	150	30	120

That is 20 dollars more than before. After all, reducing our debt by 20 dollars is almost the same as if someone gave us 20 dollars so that we can pay off some of our debts.

Example 1. Compute each of the following.

a) $-7-8$ b) $-9-(-5)$ c) $1-7$ d) $6-(-3)$

Solution: a) First, the negative sign in front of the 7 cannot denote subtraction. We are asked to subtract positive 8 from negative 7. To subtract is to add the opposite. Instead of subtracting positive 8, we will add its opposite, negative 8.

$$-7-8=-7+(-8)=-15$$

b) To subtract is to add the opposite. Instead of subtracting negative 5, we will add its opposite, positive 5.

$$-9-(-5)=-9+5=-4$$

c) To subtract is to add the opposite. Instead of subtracting positive 7, we will add its opposite, negative 7.

$$1-7=1+(-7)=-6$$

d) To subtract is to add the opposite. Instead of subtracting negative 3, we will add its opposite, positive 3.

$$6-(-3)=6+3=9$$

Revisiting Changes in Notation

Recall that if there are several signs (any operation or negative) between two numbers, a pair of parentheses must separate them.

$-2+-6$	$-5--3$
$+-$ is not allowed	$--$ is not allowed

For this reason, until just a few decades ago, we used to put a pair of parentheses around *every negative number*.

$(-2)+(-6)$	$(-5)-(-3)$
old style	old style

Then we stopped putting the parentheses around the first negative number in the line or inside a parentheses, because there was no risk that we would read the sign incorrectly as subtraction. Subtract from what? There is rarely an operation sign in front of the first number.

$-2+(-6)$	$-5-(-3)$
more modern	more modern

Practice Problems

1. Perform the indicated operations.

a) $-2+7$
b) $-7-(-4)-2$
c) $12-((-2)-7)$
d) $5-(-3)$
e) $-8+0$
f) $-3-(-10)$
g) $-20+(4-(-1))$
h) $-12-3-(12-3)$
i) $-4+7-6$
j) $-6-|-7|$
k) $-3-(-3)$
l) $|9|+(-1)$
m) $-3-0$
n) $0-4$
o) $-5-1+7$
p) $-5-(1+7)$
q) $9+|-1|$
r) $|9+(-1)|$

2. Perform the indicated operations.

a) $2-7$
b) $|2-7|$
c) $|2|-|7|$
d) $5-(-6)$
e) $|5-(-6)|$
f) $|5|-|-6|$
g) $-2-8$
h) $|-2-8|$
i) $|-2|-|8|$
j) $-4-(-4)$
j) $|-4-(-4)|$
k) $|-4|-|-4|$

4.2 Prime Numbers

Definition: Suppose that N and m are any two integers. If there exists an integer k such that $N = mk$, then we say that m is a **factor** or **divisor** of N. We also say that N is a **multiple** of m or that N is **divisible** by m. Notation: $m|N$

For example, 3 is a factor of 15 because there exists another integer (namely 5) so that $3 \cdot 5 = 15$. Notation: $3|15$.

Example 1. Label each of the following statements as true or false.

a) 2 is a factor of 10 b) 3 is divisible by 3 c) 14 is a factor of 7 d) 0 is a multiple of 5

e) every integer is divisible by 1 f) every integer n is divisible by n

Solution: a) $10 = 2 \cdot 5$ and so 2 is a factor of 10. This statement is true.

b) $3 = 3 \cdot 1$ and so 3 is divisible by 3. This statement is true.

c) $14 = 7 \cdot 2$ and so 14 is a multiple of 7, not a factor. Can we find an integer k so that $7 = 14 \cdot k$? This is not possible. $k = \frac{1}{2}$ would work, but $\frac{1}{2}$ is not an integer. This statement is false.

d) Since $0 = 5 \cdot 0$, it is indeed true that 0 is a multiple of 5. This statement is true.

e) For any integer n, $n = n \cdot 1$ and so every integer n is divisible by 1. This statement is true.

f) For any integer n, $n = 1 \cdot n$ and so every integer n is divisible by n. This statement is true.

Discussion: What do you think about the argument shown below?

3 is a divisor of 21 because there exists another integer, namely 7 so that $21 = 3 \cdot 7$. As we established that 3 is a divisor of 21, we also found that 7 is also a divisor of 21. In other words, divisors always come in pairs. For example, 28 has six divisors that we found in three pairs: 1 with 28, 2 with 14, and 4 with 7. Consequently, every positive integer has an even number of positive divisors.

Definition: An integer is a **prime number** if it has exactly two divisors: 1 and itself.

For example, 37 is a prime number. Prime numbers are a fascinating study within mathematics. Let us first recall the definition. Given a number n, we can find all of its divisors. For example, $n = 20$ has six divisors: $1, 2, 4, 5, 10$, and 20. Prime numbers have a very short list of divisors: only the trivial divisors, 1 and the number itself. For example, 7 is a prime number. 6 is not a prime number since it has divisors other than 1 and 6.

It is important to notice that 1 is not a prime number. The first few prime numbers are: $2, 3, 5, 7, 11, 13, 17, 19, \ldots$ With respect to multiplication, prime numbers are the basic building blocks of numbers.

Suppose we are given a number N and asked to determine if the number is a prime number or not. In theory, this is an easy question to answer. We just take the numbers, 1, 2, 3, . . . N and check each whether they are a divisor of N or not. If we checked all these numbers and find that the only divisors are the trivial divisors 1 and N, the number is a prime number. If we find a non-trivial divisor, the number is not a prime.

One interesting thing we can notice is that the level of difficulty depends on the answer. For example, we can use the divisibility rule for 3 to find that the number 10300800050703 is divisible by 3 (A number is divisible by 3 if and oly if the sum of its digits is divisible by 3). So 3 is a non-trivial divisor and so 10300800050703 is not a prime number. Consider now the number 10300800050719. If this is a prime number, that means that as we check all numbers from 2 to 10300800050711, none of them would be a divisor. So we would have to keep

going and check millions and millions of numbers. So whether 10300800050703 is a prime or not is an easy question, while the same question for 10300800050719 (assuming it is a prime) is a very difficult question.

While in theory we just need to check all numbers between 1 and the given number, this can be very difficult in practice. Even in the age of computers, if a number is great enough, in practice it is nearly impossible to answer this question. *Cryptography* is a branch of mathematics that studies secret codes. To this day, the safest, most difficult to brake codes are based on very large prime numbers.

If we are given a number N and we need to determine whether it is a prime number or not, there are two important ways we can reduce our work.

Theorem: Given an integer $N \geq 2$. If N is not a prime number, then the smallest non-trivial divisor is a prime number.

This means that *if we check the numbers,* $2,3,4,...,N$ *in this order*, then we can skip all composite numbers and check only the prime numbers.

Suppose we are given a number N and we need to see if it is a prime number. Let us also suppose that we are looking for divisors starting at 2, then 3, and so on, in an increasing order.

Suppose that our number N is divisible by 77. Then there exists an integer k, such that $N = 77k$. We can express 77 as $7 \cdot 11$. Then

$$N = 77k = 7 \cdot 11 \cdot k = 7\,(11k)$$

Therefore, if N is divisible by 77, then it is also divisible by 7.

We are not asked to list all divisors, we only want to know if our number is a prime number. Therefore, we can stop searching the moment we find just one non-trivial divisor. If we are checking for divisors in an increasing order, reaching 77 means that all previous numbers, including 7 and 11 failed to be divisors. So, if we didn't find a divisor until 77, we can skip 77 because divisibility by 77 means that N is also divisible by 7 - but then we would have stopped right there.

So, **when checking for divisors of N among $2,3,....,N$, in an increasing order, we can skip all composite numbers and check for prime divisors only**.

Theorem: Given an integer $N \geq 2$. Suppose further than x is an integer such that $x^2 > N$. Then if we did not find a divisor between 2 and x, the number is a prime number.

Recall that divisors come in pairs. Let us consider the number 36. The pair of divisors are $1 \cdot 36$, $2 \cdot 18$, $3 \cdot 12$, $4 \cdot 9$ and $6 \cdot 6$. In every pair of divisors, (except 6 with itself), the smaller one is less than 6, the greater one is greater than 6.

Suppose a and b are integers, both greater than 6. Then ab is greater than 36. So if we have a pair of divisors and one is greater than 6, then the other one must be less than 6.

Suppose we would like to determine if 307 is a prime number or not. If we enter $\sqrt{307}$ into the calculator, we get a decimal $\sqrt{307} = 17.52141.....$ This means that 307 is between 17^2 and 18^2. Indeed, $17^2 = 289$ and $18^2 = 324$. Let us round the number $17.52141...$ up, we get 18. Since $18^2 = 324$, (greater than 301), in any pair of divisors, one of them is must be 18 or less. This is because if a,$b > 18$, then $ab > 324$.

Suppose we would like to decide whether 307 is a prime number. If we don't find a non-trivial divisor until 18, then our number, 307, is a rime number.

Example 2. For each of the numbers given, determine whether it is a prime number or not.

a) 307 b) 57 c) 59 Solution - Youtube link

Solution: Given the two facts established above, 307 is a prime number if none of the prime numbers between 2 and 18 is a divisor of 307. These primes are: $2,3,5,7,11,13$, and 17. We can use the claculator or long division. For example, twhen we check 7, and enter $307 \div 7$, the result is $43.8571\ldots$ This is not an integer, therefore 7 is not a divisor of 307. .When we check all of $2,3,5,7,11,13$, and 17, none of them is a divisor of 307. Therefore, $\boxed{307 \text{ is a prime number}}$.

Example 3. Determine whether 401 is a prime number or not.

Solution: We enter $\sqrt{401}$ into the calculator. The result is $\sqrt{401} = 20.02498\ldots$. This means that 21 is the smallest number such that its square is grreater than 400. Therefore, 401 is a prime number if none of 2, 3, 5, 7, 11, 13, 17, and 19 is a divisor of 401. We check all these numbers and find that none of them is a divisor. Therefore, $\boxed{401 \text{ is a prime number}}$.

Example 4. For each of the numbers given, determine whether it is a prime number or not.

a) 1037 b) 1139 Solution - Youtube link

Solution: We enter $\sqrt{1037}$ into the calculator. The result is $\sqrt{1037} = 32.202484\ldots$. This means that 33 is the smallest number such that its square is grreater than 1037. Therefore, 1037 is a prime number if none of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31 is a divisor of 1037. We check all these numbers and find $1037 \div 17 = 61$. Therefore, $\boxed{1037 \text{ is not a prime number}}$ since it is divisible by 17, so it has more than two divisors.

Example 5. Find all prime numbers from $21,37,77,89,93$. Solution - Youtube link

Practice Problems

1. Consider the numbers.$64,\ 75,\ 80,\ 128,\ 270$ From this list, find all numbers that are divisible by

   a) 5 b) 3 c) 4.

2. List all factors of 48.

3. Which of $53,\ 73,\ 91,\ 101,\ 139$ is NOT a prime number?

4. Given the numbers, determine about each of them whether it is a prime number or not.

   a) 2017 c) 2021 e) 2027 g) 4003

   b) 2019 d) 2023 f) 4001 h) 4009

5. What is the smallest prime number greater than

   a) 100 b) 1000?

Enrichment

1. Magic: think of a three digit number. Enter a six-digit number into your calculator by repeating your three-digit number twice. For example, if you thought of the three-digit number 275, then enter 275275 into your calculator.

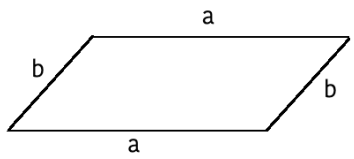
   Done? No matter what number you used to start, the number in your calculator is divisible by 7. Divide by 7. The number in your calculator now is still divisible by 11. Divide it by 11. The number in your calculator is still divisible by 13. Divide it by 13. What do you see? Can you explain it?

2. Two mathematicians are having a conversation. Mathematician A asks B about his kids. B answers: "I have three children, the product of their ages is 36." A says: "I still don't know the ages of your children." Then B tells A the sum of his three kids' ages. A answers: "I still don't know how old they are. Then B adds: "The youngest one has red hair." Now A knows the ages of all three children. Do you?

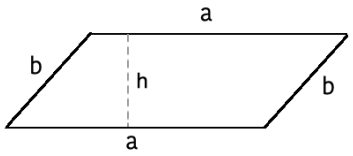
3. A king has his birthday. So he decides to let go some of his prisoners. He actually has 100 prisoners at the moment. They are each in a separate cell, numbered from 1 to 100. Well, he is a high tech king. He can close or open any prison door by a single click on the cell's number on his royal laptop. When he clicks at a locked door, it opens. When he clicks at an open door, it locks. At the beginning, every door is locked. First the king clicks on every number from 1 to 100 (therefore opening every door). Then he clicks on every second number from 1 to 100, (i.e. 2, 4, 6, 8, 10,...). Then he clicks on every third number. And so on. Finally, he only clicks on the number 100. Then he orders that the prisoners who find their door open may go free. Who gets to go and who has to stay?

4.3 Area of a Parallelogram

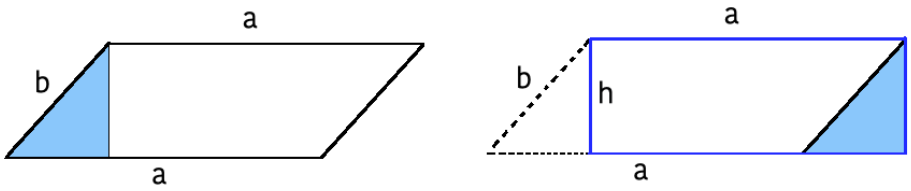
Definition: A parallelogram is a four sided polygon with two pairs of parallel sides.



Theorem: The area of a parallelogram with sides a, b and height h belonging to a is $A = ah$.

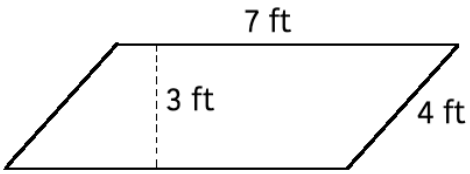


Proof: We will use a previously proven result, the method of computing the area of a rectangle. If we cut off a triangle and paste it back as shown on the picture below, we obtain a rectangle with sides a and h. Notice that b does not have any role in this formula.



Thus the area of the parallelogram equals to the area of a rectangle with sides a and h.

Example 1. Find the area of the parallelogram shown on the picture.

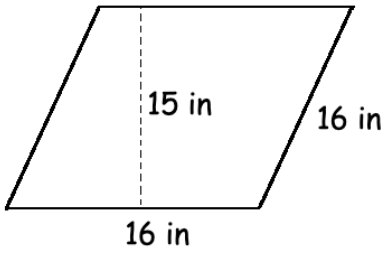


Solution: We apply the formula for the area of a parallelogram.

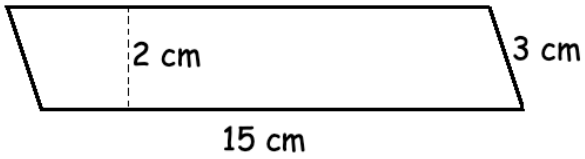
$$A = ah = 7\text{cm}\,(3\text{cm}) = 21\text{cm}^2$$

Practice Problems

1. Find the perimeter and area of the parallelogram shown on the picture. A special case of a parallelologram is when all four sides are equally long. Such a parallelogram is called a **rhombus**.



2. Find the perimeter and area of the parallelogram shown on the picture.



3. Is a rectangle a parallelogran? Explain your answer.

Chapter 5

5.1 Multiplying and Dividing Positive and Negative Numbers

Here is how we multiply signed numbers: first multiply the absolute values. If two integers have the same sign, their product is positive. If two integers have different signs, their product is negative. If any of the factors is zero, the product is zero.

Example 1. Compute each of the following.

a) $-3 \cdot 5$ b) $-4(-5)$ c) $10(-2)$ d) $0(-3)$ e) $-1(8)$

Solution: a) The product of a negative and a positive number is negative.
$-3 \cdot 5 = -15$

b) The product of two negative numbers is positive.
$-4(-5) = 20$

c) The product of a positive and a negative number is negative.
$10(-2) = -20$

d) If any of the factors is zero, the product is zero.
$0(-3) = 0$

e) The product of a negative and a positive number is negative.
$-1(8) = -8$

Notice that if we multiply any integer by -1, the result is the opposite of that integer. This will be very useful later.

Why do these rules work this way? Here is one possible explanation. Multiplication is defined as repeated addition. For example, $4 \cdot 7$ means that we add 7 to itself, 4 times.

$$4 \cdot 7 = 7 + 7 + 7 + 7 = 28$$

Consider now $4 \cdot (-7)$. This means that we add -7 to itself, 4 times

$$4 \cdot (-7) = -7 + (-7) + (-7) + (-7) = -28$$

The logic becomes a bit tortured, but it also works with the first factor being negative. Consider now the product $-5 \cdot 8$. We can interpret the first negative sign as repeated subtraction. So, we are subtracting 8 repeatedly, 5 times. If we feel that we don't have anything to subtract the first 8 from, we can fix that by inserting a zero. We

know that adding zero will not change any value.

$$\begin{aligned} -5\cdot 8 &= -8-8-8-8-8 \\ &= 0-8-8-8-8-8 && \text{to subtract is to add the opposite} \\ &= 0+(-8)+(-8)+(-8)+(-8)+(-8) \\ &= -40 \end{aligned}$$

The most interesting case is probably when we are multiplying two negative numbers. Consider the product $-4\cdot(-10)$. The first negative sign is interpreted as repeated subtraction, the second one is that we are subtracting negative numbers. So we are subtracting negative 10 repeatedly, 4 times. If we need something to subtract the first negative 10 from, we will just insert a zero at the beginning.

$$\begin{aligned} -4\cdot(-10) &= 0-(-10)-(-10)-(-10)-(-10) && \text{to subtract is to add the opposite} \\ &= 0+10+10+10+10=40 \end{aligned}$$

Division of integers: We will deal with zero later. For the quotient of any two non-zero integers, the rules are very simple and similar to those of multiplication. Divide the absolute values. If the the integers have the same sign, the quotient is positive. If they have different signs, the quotient is negative.

Example 2. Compute each of the following.

a) $14\div(-2)$ b) $-24\div(-6)$ c) $-10\div 5$

Solution: a) Divide the absolute values. The quotient of a positive and a negative number is negative.
$14\div(-2)=-7$

b) Divide the absolute values. The quotient of two negative numbers is positive.
$-24\div(-6)=4$

c) Divide the absolute values. The quotient of a negative and a positive number is negative.
$-10\div 5=-2$

Division is often denoted with a horizontal bar. The same computations can also be written as

$$\frac{14}{-2}=-7 \text{ and } \frac{-24}{-6}=4 \text{ and } \frac{-10}{5}=-2$$

Why do these rules work this way? Division is defined in terms of multiplication backward. In other words,

$$\frac{20}{4}=5 \text{ is true because } 4\cdot 5=20$$

Let us apply this idea. What is the result of $14\div(-2)$?

$$\frac{14}{-2}=\boxed{?} \text{ would be true because } -2\cdot\boxed{?}=14$$

Since $-2\cdot 7$ would result in -14, we can only choose -7 to make the multiplication backward work. $-2(-7)=14$, therefore $\dfrac{14}{-2}=-7$.

$$\text{Similarly, } \frac{-24}{-6}=\boxed{?} \text{ would be true because } -6\cdot\boxed{?}=-24$$

We need to multiply -6 by a positive number to get a negative product. Only positive 4 will work, and so $\frac{-24}{-6} = 4$.

Division by Zero: Now that we understand that division is defined in terms of multiplication backward, we can easily deal with zero. The expressions $\frac{0}{3}$ and $\frac{3}{0}$ look very similar, and yet they are very different.

$$\frac{0}{3} = \boxed{?} \text{ would be true because } 3 \cdot \boxed{?} = 0$$

In this case, we can only use zero to make the multiplication backward work. Let us investigate the other case.

$$\frac{3}{0} = \boxed{?} \text{ would be true because } 0 \cdot \boxed{?} = 3$$

Now we are in trouble. If we multiply any number by zero, the product is zero. Therefore, we can not meaningfully complete this division, and so we say that $\frac{3}{0}$ is undefined. In written notation, $\frac{3}{0} =$ undefined.

We have established that we cannot divide a non-zero number by zero. What about $\frac{0}{0}$?

$$\frac{0}{0} = \boxed{?} \text{ would be true because } 0 \cdot \boxed{?} = 0$$

Now the problem is that *every* number would work, because any number times zero is zero. Mathematicians prefer one clean answer as a result of an operation. We do not like an operation that results in several numbers, let alone every number! So, one fundamental rule of mathematics is that division by zero is not allowed. Indeed, division by zero is not just an error: it is one of the worst errors.

The first commandment of mathemtics: *Thou shall not divide by zero. Ever...*

Changes in Notation

With the introduction of negative numbers, our notation will have to be modified. It is a widely accepted convention that if there are several signs (operations or negative) between two numbers, a pair of parentheses must separate them.

$-2+-6$	$-5--3$	$-3\cdot-4$	$-30\div-5$
$+-$ is not allowed	$--$ is not allowed	$\cdot-$ is not allowed	$\div-$ is not allowed

For this reason, until a few decades ago, we used to put a pair of parentheses around *every negative number*.

$(-2)+(-6)$	$(-5)-(-3)$	$(-3)\cdot(-4)$	$(-30)\div(-5)$
old style	old style	old style	old style

The development of mathematical notation is an ongoing process. The most important goal in notation is clarity. As long as clarity is not jeopardized, mathematicians are in the habit of omitting things. A few decades ago we stopped putting the parentheses around the first negative number in the line or inside a parentheses, because there was no risk that we would read the sign incorrectly as subtraction. Also, there is rarely an operation sign in front of the first number.

$-2+(-6)$	$-5-(-3)$	$-3\cdot(-4)$	$-30\div(-5)$
more modern	more modern	more modern	more modern

In the case of multiplication, we can omit one more thing. Recall that multiplication is the default operation; if

we see two numbers with *nothing* between them, that indicates multiplication. For example, there is no operation sign or parentheses in $2x$ or ab and yet it is clear that the operation is multiplication. Now that most negative numbers must be placed in parentheses, we can often omit the dot indicating multiplication.

$-2+(-6)$	$-5-(-3)$	$-3(-4)$	$-30\div(-5)$
cannot omit anything	cannot omit anything	we can omit the dot	cannot omit anything

The most common modern style is minimalistic, ommitting as much as possible, as long as confusion is avoided. This can lead to apparent irregularities in notation. For example, our notation will be $2(-3)$, but when we swap the two factors, it will be $-3\cdot 2$.

Practice Problems

Perform the indicated operations.

1. $-2+7$
2. $-7-(-4)$
3. $12\div(-2)$
4. $5(-3)$
5. $-8\cdot 0$
6. $-3-(-10)$
7. $-20\div 0$
8. $-12\div 3$
9. $-4\cdot 7$
10. $-6-|-7|$
11. $-3\div(-3)$
12. $|9|+(-1)$
13. $-3-0$
14. $0(-4)$
15. $\dfrac{-5}{0}$
16. $0\div(-1)$
17. $9+|-1|$
18. $|9+(-1)|$

5.2 Order of Operations with Positive and Negative Numbers

The good news is that order of operations on the set of all integers ($\mathbb{Z}$) is the same as on the set of all natural numbers ($\mathbb{N}$). This is not by chance: when mathematicians enlarge our number system, they make certain that the new numbers and operations defined are in harmony with properties of the smaller system. We will refer to this consistent effort as **the expansion principle**. The bad news is that notation became more complex with the introduction of zero and the negative numbers.

The Negative Sign

The negative sign has a dual role: it denotes subtraction or it describes the number after it. This ambiguity never leads to confusion, otherwise it wouldn't be allowed. It is extremely important that we correctly interpret these roles before starting a computation. Here is a neat trick: when facing a negative sign, we should ask ourselves if it could denote subtraction. If it can denote subtraction, then it *does* denote subtraction. If it cannot, then it is a sign to be read as '*negative*' or '*the opposite of*'. A negative sign denoting subtraction is one that is loose between two numbers.

-3	$-(-3)$	$8-3$	$8(-3)$	$8-(-3)$	$8(-(-3))$
negative 3	the opposite of negative 3	subtraction	negative 3	subtract negative 3	8 times the opposite of negative 3

Example 1. Simplify each of the given expressions

a) $20-3(-5)$ b) $7-2(-3)$

Solution: a) The first negative sign indicates subtraction, the second one describes 5 as being negative. We start with the multiplication. However, we do not include the subtraction sign in the multiplication. $3(-5)=-15$, and that is what we will subtract.

$$\begin{aligned} 20-3(-5) &= 20-(-15) && \text{to subtract is to add the oposite} \\ &= 20+15 && \text{addition} \\ &= \boxed{35} \end{aligned}$$

b) $7-2(-3)$ Solution - Youtube link

Example 2. Consider the expression $20-3\,(-4-(-2)\,5)$.

a) Pair the opening and closing parentheses. Classify them as grouping symbols or clarifying prentheses.

b) How many operations are indicated?

c) Perform the indicated operations. For each step, write a separate line.

Solution: a) There are two pairs of parentheses, nested inside each other. The outside pair is a grouping symbol, the one enclosing -2 is a clarifying parentheses.

b) We scan the expression left to right and count the operations. At this point we must be clear on the role of each negative sign. The negative signs before 4 and 2 do not denote subtraction.

1. subtraction between 20 and 3 (although it is not going to be 3 what we are subtracting)
2. multiplication between 3 and the expression in parentheses
3. subtraction between -4 and (-2) (although it is not going to be -2 what we are subtracting)
4. multiplication between -2 and 5

So there are four operations.

c) We will write a separate line for each step. There are two advantages of this. First, at every point we just need to find the *one* operation to execute. Second, each line we write is a new and easier problem we need to solve. Within the parentheses, there is a subtraction and a multiplication. We start with the multiplication $-2\cdot 5$.

$$\begin{aligned} 20-3\,(-4-(-2)\,5) &= 20-3\,(-4-(-10)) && \text{to subtract is to add the opposite}\\ &= 20-3\,(-4+10) && \text{addition (also, drop parentheses)}\\ &= 20-3\cdot 6 && \text{multiplication}\\ &= 20-18 && \text{subtraction}\\ &= \boxed{2} \end{aligned}$$

The Trouble with Exponents - It's an Order of Operations Thing

Recall that a negative sign in front of anything can be interpreted as '*the opposite of*', which can also be interpreted as mutliplication by -1. We can interpret -3 as $-1\cdot 3$, and so we can re-interpret the original question from comparing -3^2 and $(-3)^2$ to a question comparing $-1\cdot 3^2$ and $(-1\cdot 3)^2$. The rest is really just an order of operations problem.

Recall that in our order of operations agreement, exponentiation superseeds multiplication. So, when presented by multiplication and exponentiation, we first execute the exponentiation and then the multiplication.

If there is no parentheses, we have				If we have parentheses, then			
-3^2	$=$	$-1\cdot 3^2$	exponentiation first	$(-3)^2$	$=$	$(-1\cdot 3)^2$	multiplication first
	$=$	$-1\cdot 9$	multiplication		$=$	$(-3)^2$	square the number -3
	$=$	-9			$=$	9	

The difference between -3^2 and $(-3)^2$ is truly an order of operations thing: we are talking about taking the opposite and squaring, but in different orders.

-3^2 is the opposite of the square of 3 $(-3)^2$ is the square of the opposite of 3

Example 3. Simplify each of the given expressions.

a) -2^4 b) $(-2)^4$ c) -1^3 d) $(-1)^3$ e) $-(-2)^2$ f) $-\left(-2^2\right)$

Solution: a) $-2^4 = -1\cdot 2^4 = -1\cdot(2\cdot 2\cdot 2\cdot 2) = -1\cdot 16 = -16$
-2^4 can be read as the opposite of 2^4.

b) $(-2)^4 = (-1\cdot 2)^4 = (-1\cdot 2)(-1\cdot 2)(-1\cdot 2)(-1\cdot 2) = (-2)(-2)(-2)(-2) = 16$
$(-2)^4$ can be read as the fourth power of -2.

c) $-1^3 = -1\cdot 1^3 = -1\cdot 1\cdot 1\cdot 1 = -1$

d) $(-1)^3 = (-1)(-1)(-1) = -1$

e) $-(-2)^2 = -1\cdot\left((-1\cdot 2)^2\right) = -1\cdot\left((-2)^2\right) = -1\cdot 4 = -4$

f) Careful! The exponent is inside the parentheses. This is squaring 2 and then taking the opposite of the result twice.
$-\left(-2^2\right) = -1\cdot\left(-1\cdot 2^2\right) = -1\cdot(-1\cdot 4) = -1(-4) = 4$

Example 3v. Simplify each of the given expressions.

a) -3^3 b) $(-3)^2$ c) $-(-3)^2$ Solution - Youtube link

Example 4. Evaluate each of the given expressions.

a) $5(-2)^2$ b) $5\left(-2^2\right)$

Solution: We need to pay attention to the subtle difference between the two expressions: the exponent is inside the parentheses in part b. In that second example, the parentheses is not necessary for the exponentiation, without it we would be looking at subtraction.

a) $5(-2)^2 = 5\cdot 4 = 20$ b) $5\left(-2^2\right) = 5(-4) = -20$

Discussion: Explain why in the expression $-(-5)^2$, the two negatives do not cancel out to a positive.

Example 5v. Simplify each of the given expressions.

a) $-5-7(-4-6)$

b) $-(5-7(-4-6))$

c) $-(5-7(-4))-6$

d) $(-5-7)(-4-6)$

e) $-(5-7(-4)-6)$

f) $-(5-7-(4-6))$

g) $-5-7-(4-6)$

h) $-(5-7)(-4-6)$

i) $-(5-7)-4(-6)$

j) $-5(-7)-4(-6)$

Solution - Youtube link

Example 6. Simplify the given expression. Write a separate line for each step. $-3^2-2\left(3\left(10-(-2)^2\right)-4^2\right)^2$

Solution: We start with the exponentiation in the innermost parentheses.

$$-3^2-2\left(3\left(10-(-2)^2\right)-4^2\right)^2 =$$

$$\begin{aligned}
&= -3^2-2\left(3(10-4)-4^2\right)^2 && \text{subtraction in innermost parentheses}\\
&= -3^2-2\left(3\cdot 6-4^2\right)^2 && \text{exponentiation in parentheses}\\
&= -3^2-2(3\cdot 6-16)^2 && \text{multiplication in parentheses}\\
&= -3^2-2(18-16)^2 && \text{subtraction in parentheses}\\
&= -3^2-2\cdot 2^2 && \text{first exponentiation left to right;}\quad -3^2=-9\\
&= -9-2\cdot 2^2 && \text{exponentiation}\\
&= -9-2\cdot 4 && \text{multiplication}\\
&= -9-8 && \text{subtraction}\\
&= \boxed{-17}
\end{aligned}$$

Example 7v. Perform the indicated operations.

a) $-((-3)+(5-1))-7^2$ Solution - youtube link

b) $\dfrac{5^2-3^2}{14-3\cdot 2^2-2}$ Solution - Youtube link

c) $-2(-1+10(7-5))-(-7)^2$ Solution - Youtube link

The Trouble with the Absolute Value Sign

Absolute value signs are more difficult to pair off because their shapes do not tell us whether they are opening or closing. The following order of operations problems illustrate that very similarly looking problems can actually be quite different. The key to solving these types of is to first pair off the absolute value signs.

Example 8. Evaluate each of the following expressions.

a) $|-5-3|-|7+2|$ c) $-5-||3-7|+2|$

b) $||-5-3|-7|+2$ d) $-5\,|-3-|7+2||$

Solution: a) Consider the expression $|-5-3|-|7+2|$.

We will not be able to get far in solving this problem without pairing the absolute values signs. Naturally, the first vertical bar can only be an opening sign. The second one can not be an another opening sign, becuase $|-|$ does not make any sense. Therefore, we have two pairs, one after the other. We should indicate the pairing using different colors or different sizes for the two pairs of absolute values signs. Keep in mind, they also serve as grouping symbols. After we paired up the signs, the problem becomes quite easy.

a) $|-5-3|-|7+2|=$

$$\begin{aligned}
&= \left|-5-3\right|-|7+2| && \text{subtraction in first parentheses}\\
&= \left|-8\right|-|7+2| && \text{find the absolute value of } -8\\
&= 8-|7+2| && \text{addition in parentheses}\\
&= 8-|9| && \text{find the absolute value of 9}\\
&= 8-9 && \text{subtraction}\\
&= \boxed{-1}
\end{aligned}$$

b) Consider the expression $||-5-3|-7|+2$.

It is easy to see that the first and second absolute value signs can not be a pair, because $||$ does not make sense. Therefore, since they are at the beginning, they are both opening. This means that we have one pair of absolute value signs nested inside the other. The one that opens last has to close first. After we have established that, the problem becomes much easier.

$$\begin{aligned}
||-5-3|-7|+2 &= \left||-5-3|-7\right|+2 && \text{subtraction in innermost parentheses}\\
&= \left||-8|-7\right|+2 && \text{evaluate the absolute value of } -8\\
&= |8-7|+2 && \text{subtraction in parentheses}\\
&= |1|+2 && \text{evaluate the absolute value of 1}\\
&= 1+2=\boxed{3} && \text{addition}
\end{aligned}$$

c) Consider the expression $-5-||3-7|+2|$.

We again see that $||$ in front of 3 can not be a pair. Thus they are both opening, and so we have one pair inside the other.

$$\begin{aligned}
-5-||3-7|+2| &= -5-\Big||3-7|+2\Big| && \text{subtraction in innermost parentheses}\\
&= -5-\Big||-4|+2\Big| && \text{evaluate the absolute value of } -4\\
&= -5-|4+2| && \text{addition in parentheses}\\
&= -5-|6| && \text{evaluate the absolute value of 6}\\
&= -5-6=\boxed{-11} && \text{subtraction}
\end{aligned}$$

d) Consider the expression $-5|-3-|7+2||$. We again see that $||$ after 2 can not be a pair. Thus they are both closing, and so we have one pair inside the other.

$$\begin{aligned}
-5|-3-|7+2|| &= -5\Big|-3-|7+2|\Big| && \text{addition in innermost parentheses}\\
&= -5\Big|-3-|9|\Big| && \text{evaluate the absolute value of 9}\\
&= -5|-3-9| && \text{subtraction in parentheses}\\
&= -5|-12| && \text{evaluate the absolute value of } -12\\
&= -5\cdot 12=\boxed{-60} && \text{multiplication}
\end{aligned}$$

Example 8v. Simplify each of the given expressions.

a) $|-3-8|-|11+9|$

b) $|-3-8-|11+9||$

c) $|-3-8|-11+9||$

d) $|-3-|8-11+9||$

e) $|-3|-8-11+9||$

Solution - Youtube link

Practice Problems

Simplify each of the given expressions by applying the order of operations agreement. Write a separate line for each step.

1. $-2-3\left(1-4^2\right)$
2. $-2-3(1-4)^2$
3. $(-2-3(1-4))^2$
4. $(-2-3)(1-4)^2$
5. $-2(-3(1-4))^2$
6. $\dfrac{-7-(-2)^3-(-1)^4}{-2(5-(-5))}$
7. $\dfrac{-24\div 2(-3)}{5-(-1)^2}$
8. $(2-5)\left(12-(-4)^2\right)$
9. $2-5\left(12-(-4)^2\right)$
10. $13-5(7-10)$
11. $13-5|7-10|$
12. $\dfrac{12-3(8-5)}{2^4-(-4)^2}$
13. $-3^2-\left((-2)^2-3(-1)^3\right)$
14. $(-3)^2-2\left(3-4\left(5-2^3\right)\right)$

1. $\left(\left(\left(5-(-2)^2\right)^2-3\right)^2+1\right)^2-1$
2. $(-2)^2-(-2)^3-(-2)^4$
3. $-(-5)^2-2\left(3-5^2\right)$
4. $-(-5)^2-2(3-5)^2$
5. $-\left((-5)^2-2(3-5)^2\right)$
6. $\dfrac{5(-2)-3\left|-5+(-1)^2\right|}{-2(3-(-2))-(-1)^2}$
7. $\dfrac{10-2\left(-3-(-1)^3\right)}{(-2)^3+(-1)^4}$
8. $\dfrac{12-2(3(4(5-7)+5)-3)}{-3+2\left(7-(-3)^2\right)-(-1)^3}$
9. $4-\left(6-\left(8-\left(10-\left(12-4^2\right)\right)\right)\right)$

Absolute value signs are more difficult to pair off because their shapes do not tell us whether they are opening or closing. The following order of operations problems can be solved by first pairing off the absolute value signs.

1. $5-2||-4+7|-10|$
2. $5|-2|-4+7||-10$
3. $5-2|-4+|7-10||$
4. $5|-2-4+|7-10||$
5. $5-|2-|4+7|-10|$
6. $|5-|2-4||+7-10$
7. $|5-2|-|4+7-10|$
8. $||5-2|-4|+7-10$
9. $||5-2|-4+7|-10$

Enrichment

1. We would hope that our order of operations agreement is void of ambiguities. This is not true. In case of the absolute value sign, we cannot tell whether it is opening or closing. Because of this, it is possible to find expressions that can be interpreted correctly in several ways, and the results are different! Evaluate each of the given expressions correctly in two different ways. What are your results?

   a) $|5-2|-4+7|-10|$ b) $2|-1-5|-3|-4|$

2. Enter pairs of parentheses or absolute value signs into the expression on the left-hand side to make the follwing statements true.

   a) $5-2-3\ ^2-4-7\ ^2+2=9$ b) $5-2-3\ ^2-4-7\ ^2+2=-9$

5.3 Prime Factorization

Recall the definition of a prime number.

> **Definition:** An integer is a **prime number** if it has exactly two divisors: 1 and itself.

For example, 37 is a prime number. Prime numbers are a fascinating study within mathematics. Let us first recall the definition. Given a number n, we can find all of its divisors. For example, $n = 20$ has six divisors: $1, 2, 4, 5, 10$, and 20. Prime numbers have a very short list of divisors: only the trivial divisors, 1 and the number itself. For example, 7 is a prime number. 6 is not a prime number since it has divisors other than 1 and 6.

It is important to notice that 1 is not a prime number. The first few prime numbers are: $2, 3, 5, 7, 11, 13, 17, 19, \ldots$ With respect to multiplication, prime numbers are the basic building blocks of numbers.

> **Theorem:** (*Fundamental Theorem of Arithmetic*) Every integer greater than 1 can be written as a product of prime numbers, and this decomposition is unique up to order of factors.

For example, $20 = 2 \cdot 2 \cdot 5$. According to the fundamental theorem of arithmetic, there is no other way to write 20 as a product of prime numbers. We usually use exponential notation: $20 = 2^2 \cdot 5$.

Example 1. Find the prime factorization of 180.

Solution: We start with the smallest prime number, 2 and ask: is 180 divisible by 2? If the answer is yes, we write down and divide 180 by 2.

180	2
90	

Now we ask: is 90 divisile by 2? The answer is yes. So we write 2 next to 90 and divide by 2.

180	2
90	2
45	

Now we ask: is 45 divisile by 2? The answer is no. So, we are done with the prime factor 2 and move on to the next prime number, 3. Since 45 is divisible by 3, we write it next to 45 and divide by 3.

180	2
90	2
45	3
15	

Now we ask: is 15 divisile by 3? The answer is yes. So we write 3 next to 15 and divide by 3.

180	2
90	2
45	3
15	3
5	

The only prime divisor of 5 is 5 itself. We write 5 next to 5 and then divide.

180	2
90	2
45	3
15	3
5	5
1	

Once we wrote 1, we are done. The prime factorization of 180 is therefore

$$180 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \text{ or } 180 = 2^2 \cdot 3^2 \cdot 5$$

Example 2. a) Find the prime factorization of 300.

b) Find the prime factorization of 2250. Solution - Youtube link

Solution: We start with the first prime number, 2. Is our number, 300 divisible by 2? If yes, we divide 300 by 2. Since $300 = 2 \cdot 150$, we now have one prime factor, 2 and we must find the prime factorization of 150.

300	2
150	2
75	3
25	5
5	5
1	

We ask next: is the number 150 divisible by 2? If yes, we divide 150 by 2. So now $300 = 2 \cdot 2 \cdot 75$ and we are looking for the prime factorization of 75.

We ask next: is the number 75 divisible by 2? This time, the answer is no. We have exhausted the prime factor 2. So we roll up to 3 and ask: is 75 divisible by 3? If yes, we divide 75 by 3. Since $75 \div 3 = 25$, so now $300 = 2 \cdot 2 \cdot 3 \cdot 25$ and we are looking for the prime factorization of 25. Although we know the final answer now, we continue the process. Every time we find a factor, we write it down and divide. The quotient is written down under the pair in the first column. Once that column reaches 1, the second column is the prime factorization of our number. Thus $300 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 = \boxed{2^2 \cdot 3 \cdot 5^2}$

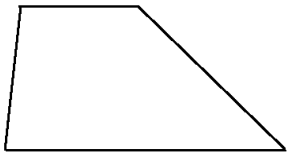
Practice Problems

Find the prime factorization for each of the given numbers.

1. 600 2. 5500 3. 2016 4. 2015

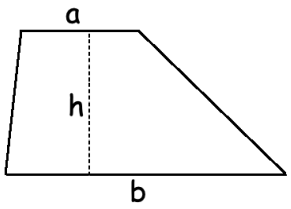
5.4 Area of a Trapezoid

Definition: A **trapezoid** is a four sided polygon with one pair of parallel sides.

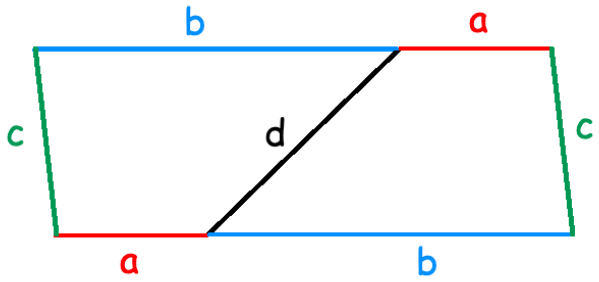


Note that having one pair of parallel sides means one or more pairs. So, even a square is a trapezoid because it does have a pair of parallel sides. We will find the area of the trapezoid using area formulas we already have. In this case, that of a parallelogram.

Theorem: The area of a trapezoid, with sides and height labeled as on the picture below, is $A = \frac{a+b}{2}h$.

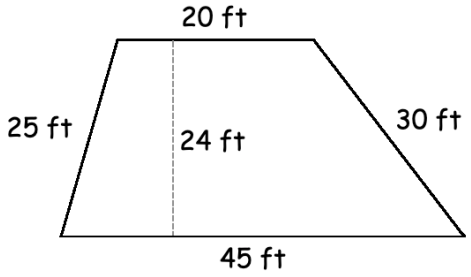


Proof: If we use two identical trapezoids, we can make a parallelogram as shown on the picture below.



We already know that the area of this parallelogram is $A = (a+b)h$. Since our trapezoid is exactly half of the parallelogram, its area is $A = \frac{(a+b)h}{2}$.

Example 1. Find the perimeter and area of the trapezoid shown on the picture.



Solution: For the perimeter, we simply add the four sides.

$$P = 20\text{ft} + 30\text{ft} + 45\text{ft} + 25\text{ft} = 120\text{ft}$$

For the area, we apply our new formula. It is important to notice that we will not need all data given. For the area, we only need the lengths of the parallel sides and the height connecting them.

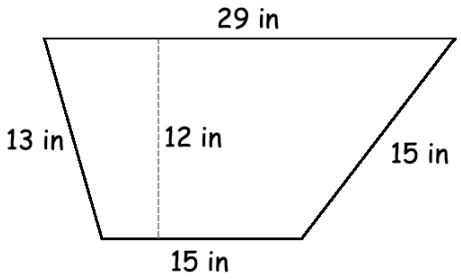
With that data, we use the formula $A = \dfrac{a+b}{2}h$.

$$A = \frac{a+b}{2}h = \frac{20\text{ft} + 45\text{ft}}{2} \cdot 24\text{ft} = \frac{65\text{ft}}{2} \cdot 24\text{ft} = 780\text{ft}^2$$

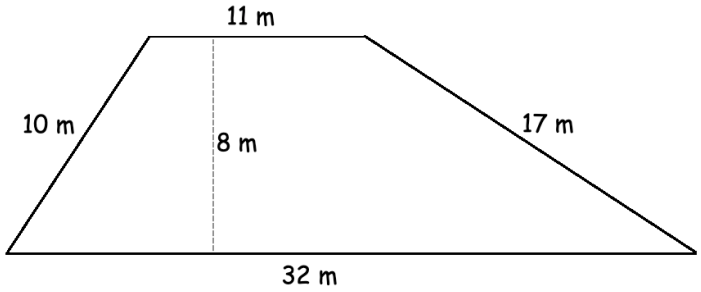
Practice Problems

Compute the perimeter and area of the given trapezoids.

1.



2.

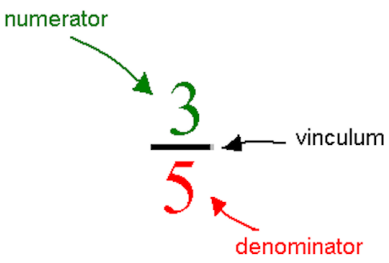


Chapter 6

6.1 Introduction to Fractions and Percents

A fraction has three components as shown on the picture. The important parts are the **numerator** above and the **denominator** below the little line that is called the vinculum.

At first let us not even consider a fraction alone. We will just define a **fraction of something**.



Definition: $\frac{3}{5}$ of a quantity can be obtained as follows.

Step 1. We first divide the quantity into 5 equal shares.

Step 2. Let us take 3 such shares. That is $\frac{3}{5}$ of our quantity.

So the numerator tells us how many shares we have. The denominator tells us how big each share is.

Example 1. Find $\frac{3}{5}$ of 100 dollars.

Solution: Step 1. Divide 100 dollars into five equal shares. This means that we exchange a 100 dollar bill into five twenty-dollar bills. In other words, $\frac{1}{5}$ of 100 dollars is 20 dollars.

Step 2. To obtain $\frac{3}{5}$, we take three such shares. In this case this means taking three twenty-dollar bills. That is 60 dollars. In other words, $\frac{3}{5}$ of 100 dollars is $\boxed{60 \text{ dollars}}$.

Example 2. Compute $\frac{4}{7}$ of 42.

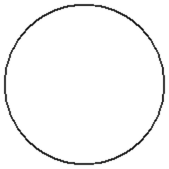
Solution: Step 1. We divide 42 into seven equal shares. $\frac{1}{7}$ of 42 is 6.

Step 2. We take four such shares. Thus $\frac{4}{7}$ of 42 is $4 \cdot 6 = 24$. The answer is $\boxed{24}$.

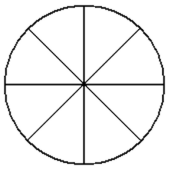
Example 2v. Find each of the following.

a) $\frac{1}{6}$ of 18 b) $\frac{5}{6}$ of 18 Solution - Youtube link

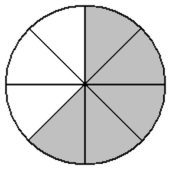
Example 3. Shade the region on the picture that corresponds to $\frac{5}{8}$ of the circle.



Solution: Step 1. Divide the circle into 8 equal parts.

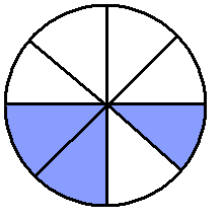


Step 2. We shade five such parts.

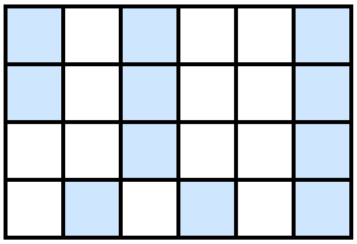


Example 3v. Express the shaded area as a fraction.

a) Solution - Youtube link



b) Solution - Youtube link



Example 4. A cake was sliced into equal slices. Amy ate 2 slices and Betsy ate 3 slices. If 2 slices were remaining, what fraction of the cake was eaten?

Solution: We need to first figure out how many slices made up the cake. If 2 were eaten by Amy and 3 by Betsy and 2 more were left, then there were all together $2+3+2=7$ slices. 5 slices were eaten which were $\frac{5}{7}$ of the cake. So the answer is that $\boxed{\frac{5}{7} \text{ of the cake was eaten}}$.

Example 4v. There are 800 students enrolled in a school. If $\frac{3}{4}$ of the students are boys, how many students are girls? Solution - Youtube link

Example 5. Compute $\frac{8}{100}$ of 2000 dollars.

Solution: We divide 2000 dollars into 100 equal shares. Then each share is 20 dollars. (We divide 2000 by 100). Then we take 8 such shares, that is $8 \cdot 20$ dollars $= 160$ dollars. Thus $\frac{8}{100}$ of 2000 dollars is $\boxed{160 \text{ dollars}}$.

Definition: We often use fractions with 100 in the denominator. Such a fraction is also called a **percent** and is denoted by %.

For example, computing 9% of a quantity is exactly the same as computing $\frac{9}{100}$ of it.

Example 5v. Find 9% of 200. Solution - Youtube link

Example 6. In 2012, there were approximately 235 000 000 people eligible to vote in the USA. If 59% of them voted in 2012, how many people voted?

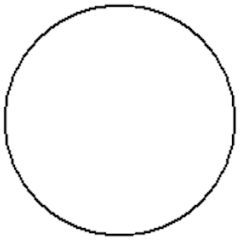
Solution: We need to compute 59% of 235 000 000. That is the same as computing $\frac{59}{100}$ of 235 000 000. We first divide 235 000 000 by 100. Division by 100 is very easy in this case: just cut off the last two zeroes. So, 1% or $\frac{1}{100}$ of the number is 2 350 000. Now we take 59 of that, we multiply 2 350 000 by 59. The result is 138 650 000. Thus, approximately 138 650 000 people voted in 2012.

Example 6v. A $500 TV went on a 6% off sale. What is the sale price? Solution - Youtube link

Practice Problems

1. Compute $\frac{4}{9}$ of 63.
2. Compute $\frac{5}{6}$ of 24.

1. Shade $\frac{5}{9}$ of the circle given.



1. The price of a TV is $400. We want to raise the price by 5%. What is the new price?
2. The price of a couch is $700. Next week, it will go on a 15% off sale. What is the new price? (A 15% sale means that the price of the item is lowered by 15%.)
3. Find $\frac{3}{4}$ of 56.
4. We placed $2000 into a bank account with 6% yearly interest rate. How much money do we have in the bank after one year?
5. This problem is about a method of comparing fractions.

   a) Compute $\frac{3}{7}$ of 420.

   b) Compute $\frac{4}{10}$ of 420.

   c) Based on the results of parts a) and b), which fraction is larger, $\frac{3}{7}$ or $\frac{4}{10}$?
6. Bert has made $54 000 last year. If he has to pay 32% of his income in taxes, how much taxes does he owe and how much of his income will he keep?
7. Sally used to make $2400 per month, but now she got a 3% raise. How much is her monthly salary now?
8. Mr. X won $600 000 in the lottery two yers ago. By now he has spent some of the money. When he was asked what happened, he said the following. "*I didn't spend it all. I spent* $\frac{1}{3}$ *of the money by taking a luxury yacht trip around the world. Then I put the rest in the bank. Later I decided to buy a house I really liked. So I took* $\frac{3}{4}$ *of the money out of the bank and baught the house. For half of what was left, I purchased stocks that completely lost their value. Finally, I gave* $\frac{2}{5}$ *of what's left to my niece for her college education.*" How much money is left from the winnings?

6.2 The Greatest Common Factor and Least Common Multiple

Suppose that two natural numbers are given. The greatest common factor and least common multiple of two numbers are important quantities that can be obtained using the prime factorization them. In what follows, we will see how these can be computed.

Definition: Suppose that n and m are any two positive integers. The **greatest common factor** (GCF) of the two numbers is the greatest integer that is a factor of both numbers.

Common factors always exist, because 1 is a common factor of all integers. The greatest common factor (or greatest common divisor) of n and m is denoted by $\gcd(n,m)$.

Example 1. Find the greatest common factor of 60 and 96.

Solution: We can list the factors of 60 and 96.

$$\begin{aligned} \text{factors of } 60 &: \quad 1,2,3,4,5,6,10,12,15,20,60 \\ \text{factors of } 96 &: \quad 1,2,3,4,6,8,12,16,24,32,48,96 \end{aligned}$$

We can list the common factors:

$$\text{common factors:} \quad 1,2,3,4,6,12$$

Thus the greatest common factor of 60 and 96 is $\boxed{12}$.

While the method presented above works, it will be increasingly difficult to apply it for larger numbers. There is a more efficient method that uses the prime factorization.

Solution 2: We will use the prime factorization of these numbers.

$$60 = 2^2 \cdot 3 \cdot 5 \quad \text{and} \quad 96 = 2^5 \cdot 3$$

If d is a divisor of 60, then d can not have a prime divisor other than 2 or 3 or 5. If d is a divisor of 96, then d can not have a prime divisor other than 2 and 3. So, common factors can only have 2 and 3 in their prime factorization.

Now we just need to find the exponents of those prime factors. Consider first the prime factor 2. 60 has 2^2 in its prime factorization. This means that 60 is divisible by 4 but not by 8. So, any factor of 60 can not have more than two $2-$factors. 96 has 2^5 in its prime factorization. This means that 96 is divisible by 32 but not by 64. So, any factor of 96 can not have more than five $2-$factors. In order to be a factor of both, the exponent can be at most 2. We look at the exponents of 3 and in both prime factorizations, there is exactly one $3-$factor. Thus the greatest common factor is $2^2 \cdot 3 = \boxed{12}$.

Example 2. Find the greatest common factor of 240 and 135.

Solution: We will use the prime factorization of these numbers.

$$240 = 2^4 \cdot 3 \cdot 5 \quad \text{and} \quad 135 = 3^3 \cdot 5$$

If d is a divisor of 240, then d can not have a prime divisor other than 2 or 3 or 5. If d is a divisor of 135, then d can not have a prime divisor other than 3 and 5. So, common factors can only have 3 and 5 in their prime factorization. We now just need to find the exponents of those prime factors. Consider first the prime factor 3. 240 has one $3-$factor in its prime factorization. This means that 240 is divisible by 3 but not by 9. So, any factor of 240 can have at most one $3-$factor. 135 has 3^3 in its prime factorization. This means that 135 is divisible by 27 but not by 81. So, any factor of 135 can have at most three $3-$factors. In order to be a factor of both, the exponent can be at most 1. We look at the exponents of 5 and in both prime factorizations, there is exactly one $5-$factor. Thus the greatest common factor is $3 \cdot 5 = \boxed{15}$.

Example 3. Find the greatest common factor of 40 and 63.

Solution: We will use the prime factorization of these numbers.

$$40 = 2^3 \cdot 5 \quad \text{and} \quad 63 = 3^2 \cdot 7$$

If d is a divisor of 40, then d can not have a prime divisor other than 2 or 5. If d is a divisor of 63, then d can not have a prime divisor other than 3 and 7. So, the common factors can not have any of these in their prime factorization. This means that the two numbers share no divisors besides 1. The greatest common factor of these numbers is $\boxed{1}$. When the greatest common factor of two numbers is 1, we say that the numbers are **relatively prime.**

To compute the greatest common factor of two or more numbers, we list all prime factors that are common to the prime factorization of all numbers. With each such prime number, we use the lowest occuring exponent from the prime factorizations.

Definition: Suppose that n and m are any two positive integers. The **least common multiple** (LCM) of the two numbers is the least positive integer that is a multiple of both numbers.

Common multiples always exist, because the product mn is a common multiple of both integers. The least common multiple of n and m is denoted by $\text{lcm}(n,m)$.

Example 4. Find the greatest common factor of 60 and 96.

Solution: We will use the prime factorization of these numbers.

$$60 = 2^2 \cdot 3 \cdot 5 \quad \text{and} \quad 96 = 2^5 \cdot 3$$

If T is a multiple of 60, then T must have the prime divisors 2, 3, and 5 in its prime factorization. If T is a multiple of 96, then T must have the prime divisors 2 and 3 in its prime factorization. So, common multiples all contain 2, 3, and 5 in their prime factorization. We now just need to find the exponents of those prime factors. Consider first the prime factor 2. 60 has 2^2 in its prime factorization. This means that 60 is divisible by 4 but not by 8. So, any multiple of 60 must have at least two 2−factors. 96 has 2^5 in its prime factorization. This means that 96 is divisible by 32. So, any multiple of 96 must have at least five 2−factors. In order to be a multiple of both, the exponent can be at least 5. We look at the exponents of 3 and in both prime factorizations, there is exactly one 3−factor. Because 60 is divisible by 5, we need a factor of 5 as well. Thus the least common multiple is $2^5 \cdot 3 \cdot 5 = \boxed{480}$.

Example 5. Find the least common multiple of 240 and 135.

Solution: We will use the prime factorization of these numbers.

$$240 = 2^4 \cdot 3 \cdot 5 \quad \text{and} \quad 135 = 3^3 \cdot 5$$

If T is a multiple of 240, then T must have 2, 3, and 5 in its prime factorization. If T is a multiple of 135, then T must have 3 and 5 in its prime factorization. So, common multiples must have 2, 3, and 5 in their prime factorization. We now just need to find the exponents of the least common multiple. Furthermore, in order to be a multiple of 240, T must have at least four 2−factors in its prime factorization. Similarly, in order to be a multiple of 135, T must have at least three 3−factors in its prime factorization. Thus the least common multiple is $2^4 \cdot 3^3 \cdot 5 = \boxed{2160}$.

Example 6. Find the greatest common factor of 40 and 63.

Solution: We will use the prime factorization of these numbers.

$$40 = 2^3 \cdot 5 \quad \text{and} \quad 63 = 3^2 \cdot 7$$

If T is a multiple of 40, then T must have 2 and 5 in its prime factorization. If T is a multiple of 63, then T must have 3 and 7 in its prime factorization. So, the common multiples must have 2, 3, 5, and 7 in its prime factorization. For the exponents: T must have at least three 2−factors if it is a multiple of 40 and T must have at least two 3−factors if it is a multiple of 63. Thus the least common multiple is $2^3 \cdot 3^2 \cdot 5 \cdot 7 = \boxed{2520}$.

Example 7v. Find the greatest common factor and least common multiple of each of the given pairs of numbers.

a) 36 and 45. Solution - Youtube link

b) 189 and 126 Solution - Youtube link

Discussion:

Consider our results for examples 1-6. In each case, find the product of the two numbers and the product of their least common multiple and greatest common factor. For example, in case of 60 and 96, $\text{lcm}(60,96) = 480$ and $\gcd(60,96) = 12$. So, $60 \cdot 96 = 5760$ and $480 \cdot 12 = 5760$. Is it a coincidence that the products are the same?

Practice Problems

1. Find the greatest common factor and least common multiple of each of the given pairs of numbers.

   a) 72 and 300 b) 240 and 3600 c) 100 and 420

2. Is it possible that the greatest common factor and least common multiple of two numbers are equal to each other?

3. Find all possible positive integer values of n if we know that $\gcd(6,120,n) = 6$ and $\text{lcm}(6,120,n) = 120$.

6.3 Algebraic Expressions

We will now start studying the expressions and statements frequently made in algebra.

Definition: A **numerical expression** is an expression that combines numbers and operations. To **evaluate** a numerical expression is to compute its value.

For example, $3 \cdot 5^2$ is a numerical expression. So are $-\dfrac{12}{3+1}$ and $5^2 - 2^2$ and $-|-5|$. We can evaluate numerical expressions by performing the operations indicated in the expression. Naturally, we must correctly apply the order of operations agreement. For that, we must clearly understand notation. It is a fundamental principle that notation can be objectively understood in only *one* possible way.

Example 1. Evaluate each of the given numerical expressions.

a) $3 \cdot 5^2$ b) $-\dfrac{12}{3+1}$ c) $3^2 + 2^2$ d) $(3+2)^2$

Solution: a) Between exponentiation and multiplication, we first perform the exponentiation. $3 \cdot 5^2 = 3 \cdot 25 = \boxed{75}$

b) The addition in the denominator must be performed before we divide. (Why?)

$$-\frac{12}{3+1} = -\frac{12}{4} = \boxed{-6}$$

c) $3^2 + 2^2 = 9 + 4 = \boxed{13}$

d) $(3+2)^2 = 5^2 = \boxed{25}$

Note: The error of confusing $3^2 + 2^2$ with $(3+2)^2$ is called the "Freshman's Dream Error".

Definition: An **algebraic expression** is an expression that combines numbers, operations, and variables.

Variables are letters that represent numbers. They are subjects to the same rules as numbers. We use variables for different reasons. Often because the variable represents a number we do not know but would like to know. However, there are other reasons. Sometimes we use variables because we would like to discuss general statements. Consider the following statement as an example. "*For every numbers x and y*, $x + y = y + x$". This statement is not about the numbers x and y; it is rather about the operation addition. No matter what two numbers we add, the order of the numbers in the addition does not matter. We express this property of addition by saying that addition is commutative. In this case, we use variables because we would like to talk about all the numbers at the same time.

In our modern language, we prefer to use x as a variable, especially if it is the type of unknown we want to find. A fundamental principle is that within the context of a problem, a variable has one fixed meaning. Suppose that we are solving a word problem that involves the number of books and the number of pencils. In this case, we can not label both of them x, unless we are certain that the number of books is the same as the number of pencils. **To denote quantities that may be different, we must use different letters.** For example, we can denote the number of books by x and the number of pencils by y. We might find out later in the problem that the values of x and y might be equal. That is perfectly fine. We must use different variables for quantities that *could* be different.

For example, $3x^2 - 1$ is an algebraic expression. So are $x + 3$ and $2a - b$ and $5y + 3$. If the values of all variables

in an expression are given, we can evaluate it. To evaluate an algebraic expression, we substitute the given values of the variables into the expression, and evaluate the resulting numerical expression.

When substituting a number into an algebraic expression, it is our responsibility to preserve the indicated operations and their order correctly. Consider the expression $2x$. If $x = 5$, we can not write 25 instead of $2x$, because our notation would indicate a two-digit number with no operation.

To evaluate an algebraic expression, we can perform the following steps.

Step 1. Copy the entire expression with one modification: replace each variable by an empty pair of parentheses.

Step 2. Insert the values into the parentheses. Now the problem became an order of operations problem.

Step 3. Drop the unnecessary parentheses and work out the order of operations problem. (It may appear awkward first to create all these parentheses but they are extremely helpful.)

Example 2. Evaluate the algebraic expression $3x^2 - x + 5$ if $x = 4$.

Solution: We first copy the entire expression, replacing the letter x by little pairs of parentheses.

$$3x^2 - x + 5 = 3(\ )^2 - (\ ) + 5$$

Then we insert the number 4 into each pair of parentheses.

$$3x^2 - x + 5 = 3(4)^2 - (4) + 5$$

Now the problem became an order of operations problem. We start with the exponent.

$$\begin{aligned} 3 \cdot 4^2 - 4 + 5 &= 3 \cdot 16 - 4 + 5 && \text{perform multiplication} \\ &= 48 - 4 + 5 && \text{subtraction: } 12 - (-2) = 12 + 2 = 14 \\ &= 44 + 5 && \text{addition} \\ &= \boxed{49} \end{aligned}$$

Example 2v. Evaluate each of the following.

a) $3a^2 - 2b + 4$ if $a = 2$ and $b = 5$

b) $3a^2 - 2b + 4$ if $a = 5$ and $b = 2$

c) $3x^2 - 5x + 8$ if $x = 3$

Solution - Youtube link

Example 3. Evaluate the algebraic expression $3(p+q)^2-(5q-2p)^4$ if $p=7$ and $q=3$

Solution:

$$(p+q)^2-(5q-2p)^4=[(7)+(3)]^2-[5(3)-2(7)]^4$$

$$\begin{aligned}
&= (7+3)^2-(5\cdot 3-2\cdot 7)^4 && \text{addition in parentheses}\\
&= 10^2-(5\cdot 3-2\cdot 7)^4 && \text{multiplications in parentheses,}\\
&= 10^2-(15-2\cdot 7)^4 && \text{left to right}\\
&= 10^2-(15-14)^4 && \text{subtraction in parentheses}\\
&= 10^2-1^4 && \text{exponents, left to right}\\
&= 100-1^4 && \text{careful! } 1^4\neq 4\\
&= 100-1\\
&= 99
\end{aligned}$$

Practice Problems

Evaluate each of the algebraic expressions when $x=6$ and $y=8$.

1. $19-y+x$
2. $19-(y+x)$
3. $2x^2-5y+3$
4. x^2+y^2
5. $(x+y)^2$
6. $3y-x$
7. $3(y-x)$
8. $\dfrac{5x-y}{2}$
9. $5x-\dfrac{y}{2}$
10. $\dfrac{x^2-5x+4}{y-3}$

10. Consider the expression $\dfrac{6a-3b-ab+2a^2}{2a-b}-3$. Evaluate this expression if

a) if $a=5$ and $b=1$

b) if $a=5$ and $b=2$

c) if $a=5$ and $b=3$

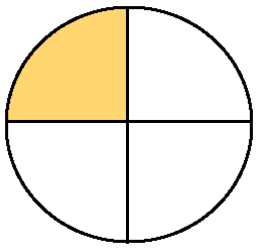
d) if $a=4$ and $b=1$

Chapter 7

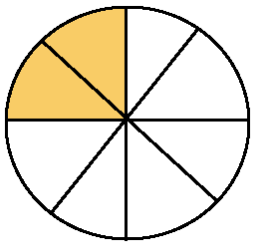
7.1 Equivalent Fractions

So far, we didn't look at a standalone fraction on its own. We only defined a fraction as expressing part of something. We will continue to do this. A warning to the reader: do not read this section while hungry. We will be talking a lot about cakes and pizzas.

Suppose that Ann, Bethany, Cecile and Desire are about to share a pizza. They cut the pizza into four equal slices. Ann happily looks at her slice (the shaded region). What fraction can express this slice? The answer is $\frac{1}{4}$. We are talking about one slice, so big so that four equal slices make up the whole.



Just before they start to eat, four other friends arrive. They decide to share, so everyone cuts their own slice in two equal pieces. Before she gives her friend half of her food, Ann looks at her plate. Her share now looks like this. What fraction could express this? Since we cut each slice in two, the four slices became eight slices. So, we are now looking at $\frac{2}{8}$.



The fractions $\frac{1}{4}$ and $\frac{2}{8}$ express the same part of a given quantity. Such fractions are called **equivalent** to each other. Given any fraction, we can create infitely many equaivalent fractions by multiplying both numerator and denominator by the same non-zero number. For example, if we start with $\frac{2}{3}$, the following fractions are all equivalent:

$\frac{2}{3}=\frac{4}{6}$	$\frac{2}{3}=\frac{2\cdot 2}{3\cdot 2}=\frac{6}{9}$	$\frac{2}{3}=\frac{2\cdot 3}{3\cdot 3}=\frac{8}{12}$	$\frac{2}{3}=\frac{2\cdot 5}{3\cdot 5}=\frac{10}{15}$
multiply both numerator and denominator by 2	multiply both numerator and denominator by 3	multiply both numerator and denominator by 4	multiply both numerator and denominator by 5

Theorem: (also called the Fundamental Property of Fractions) Given a fraction $\frac{a}{b}$ where a and b are integers, $b \neq 0$, and any non-zero number c, $\frac{a}{b}$ and $\frac{ac}{bc}$ are equivalent fractions, i.e.

$$\frac{a}{b} = \frac{ac}{bc}$$

Another way to express this is as follows: One thing we can always do to a fraction without changing its value is to multiply both numerator and denominator by the same number. Equivalent fractions are essentially different representations of the same number.

Example 1. Re-write $\frac{4}{5}$ with a denominator of 20.

Solution: The only thing we can do is to multiply both numerator and denominator by the same number. If we want a new denominator of 20, that means that we have to multiply the old denominator, 5, by 4. Then we have no other option than multiplying the numerator by the same number. $4 \cdot 4 = 16$ and $5 \cdot 4 = 20$. Thus the equivalent fraction we are looking for is:

$$\frac{4}{5} = \frac{4 \cdot 4}{5 \cdot 4} = \boxed{\frac{16}{20}}$$

Example 2. Re-write $\frac{4}{5}$ as a percent.

Solution: The only thing we can do is to multiply both numerator and denominator by the same number. Since we must write $\frac{4}{5}$ as a percent, that measn that we want a new denominator of 100. What number should we multiply by 5 to get to 100? This is a division problem, $100 \div 5 = 20$. This means that we will multiply both numerator and denominator by 20.

$$\frac{4}{5} = \frac{4 \cdot 20}{5 \cdot 20} = \frac{80}{100}$$

and $\frac{80}{100}$ is the same as $\boxed{80\%}$.

Now that we know that each fraction can be represented in infinitely many forms, we should find an "official" one, hopefully the simplest.

We now know that a fraction's value does not change if we multiply both numerator and denominator by the same non-zero number.

$$\frac{a}{b} = \frac{ac}{bc} \qquad b, c \neq 0$$

Reading the same equality from right to left as $\frac{ac}{bc} = \frac{a}{b}$ tells us that we are also allowed to *divide* both numerator and denominator by the same non-zero number, and we did not change the value of the fraction. This gives us the concept of the simplest form of a fraction.

Definition: Given a fraction $\frac{a}{b}$ where a and b are integers, $b \neq 0$, the fraction is **reduced** or **in lowest terms** if a and b have no common divisor greater than 1. If this is the case, we also say that a and b are **relatively prime**.

It feels true that every fraction has a unique reduced form. Since every fraction can be reduced, and the reduced form is clearly its simplest form, **we always present fractions in lowest terms as final answers**.

Example 3. Reduce $\frac{12}{28}$ to lowest terms.

Solution: We need to look for a common factor between 12 and 28. The greatest common divisor is 4. Indeed, when we divide both numerator and denominator by 4, we get that

$$\frac{12}{28} = \frac{4 \cdot 3}{4 \cdot 7} = \frac{\cancel{4} \cdot 3}{\cancel{4} \cdot 7} = \frac{3}{7}$$

When we divide both numerator and denominator by the same number, we say that we **cancel** or **cancel out** that factor.

Note that we do not have to get it right for the first time. Suppose we only notice the common factor 2. Then we have $\frac{12}{28} = \frac{6}{14}$. This is not reduced yet, because both 6 and 12 are even. So we go again: we divide both both numerator and denominator by 2 again: $\frac{12}{28} = \frac{6}{14} = \frac{3}{7}$. Now the fraction is in lowest terms, so the answer is $\boxed{\frac{3}{7}}$.

Example 4. Re-write 35% as a fraction in lowest terms.

Solution: First, $35\% = \frac{35}{100}$. Clearly both numerator and denominator are divisible by 5, so let's get rid of that 5 by dividing both numerator and denominator by 5.

$$\frac{35}{100} = \frac{5 \cdot 7}{5 \cdot 20} = \frac{\cancel{5} \cdot 7}{\cancel{5} \cdot 20} = \frac{7}{20}$$

This is now in lowest form since 7 and 20 do not share any divisor besides the obvious one, 1. So 35% as a reduced fraction is $\boxed{\frac{7}{20}}$.

Until now, we only looked at fractions that were a part of a whole. If we cut a pizza into six equal slices and take all six slices, how can we express this as a fraction? Based on what we learned so far, $\frac{6}{6}$ is the fraction expressing this. Dividing both numerator and denominator by 6, we get $\frac{6}{6} = \frac{1}{1}$ or simply 1. 1 whole pizza.

So, the reduced form of fractions such as $\frac{3}{3}$ or $\frac{7}{7}$ or $\frac{12}{12}$ is simply 1.

What if we buy two pizzas, we slice them both into 6 equal slices and take 10 such slices? That could be expressed as $\frac{10}{6}$. And if we take all 12 slices? That would be $\frac{12}{6} = \frac{2}{1} = 2$, or two 2 whole pizzas.

This means that every positive integer can be expressed as a fraction, with infinitely many equaivalent forms. For example, $5 = \frac{5}{1} = \frac{30}{6}$. Indeed, if we buy 5 whole pizzas and cut each into 6 equal slices, we will have 30 slices, each so big so that 6 of them makes an entire pizza.

Definition: If the numerator of a fraction is less than its denominator, we call such a fraction **proper**. A proper fraction always expresses less than a unit of a given quantity. Examples of proper fractions are $\frac{2}{3}$, $\frac{3}{10}$, $\frac{6}{15}$, or 55%.

If the numerator of a fraction is greater than or equal to its denominator, we call the fraction **improper**. An improper fraction always expresses a whole unit or more of a quantity. Examples of improper fractions are $\frac{5}{5}$, $\frac{12}{10}$, $\frac{40}{2}$, or 120%.

In spite of the suggested value judgement, we don't see anything improper about an improper fraction. Improper fractions can be presented as final values, as long as they are in lowest terms.

Example 5. Reduce $\frac{30}{12}$ to lowest terms.

Solution: Both 30 and 12 are divisible by 6. So we divide both numerator and denominator by 6 and we obtain that

$$\frac{30}{12} = \frac{5 \cdot \cancel{6}}{2 \cdot \cancel{6}} = \frac{5}{2}$$

Since 5 and 2 are obviously relatively primes (i.e. share no divisor greater than 1), this is the final answer. So the reduced form of $\frac{30}{12}$ is $\boxed{\frac{5}{2}}$.

Example 6. Re-write 160% as a fraction in lowest terms.

Solution: $160\% = \frac{160}{100}$. We can easily divide both numerator and denominator by 10, that is just chopping off the last zero. $160\% = \frac{160}{100} = \frac{16}{10}$. This is still not reduced because both 16 and 10 are even. So we divide both numerator and denominator by 2 and get $160\% = \frac{160}{100} = \frac{16}{10} = \frac{8}{5}$. Since 5 and 8 are obviously relatively primes (i.e. share no divisor greater than 1), this is the final answer. The reduced fraction form of 160% is $\boxed{\frac{8}{5}}$.

Consider now the expression $\frac{15}{5}$. Are we looking at two integers with an operation, namely division between them, or are we looking at a single fraction, expressing 15 slices, where the slices are so big that five of them makes up a whole? There is a tremendeous duality here: we can read the same expression in two fundamentally different way. This is only allowed in mathematics if the duality never causes conflict or confusion. That is, every question has the same answer, no matter which of the two interpretations are we using. Instead of confusion, this duality is really a source of power. We can always look at a fraction as a division between integers, and we can always look at a division between integers as if it was a fraction.

Before the time of calculators, people used a lot of mental techniques to help in computation. The following is

just an example of such a technique using the duality described above.

Example 7. Perform the division $\frac{140}{5}$ without the help of a calculator or paper.

Solution: Even though this is a division between two integers, we will look at it for a while as if it was a fraction. If so, we are allowed to multiply numerator and denominator by the same number, and our chosen number is 2. This is because division by 10 is very easy: just chop off a zero at the end.

$$\frac{140}{5} = \frac{140 \cdot 2}{5 \cdot 2} = \frac{280}{10} = 28$$

Practice Problems

1. a) Re-write $\frac{2}{9}$ with a denominator of 45, i.e. find x so that $\frac{2}{9} = \frac{x}{45}$.

   b) Verify your result by taking $\frac{2}{9}$ of 90 and $\frac{x}{45}$ of 90. If you get the same result, your answer is probably correct.

2. a) Re-write $\frac{5}{6}$ with a numerator of 30.

   b) Re-write $\frac{5}{6}$ with a denominator of 30.

3. Re-write each of the given percents as a reduced fraction.

   a) 60% b) 25% c) 100% d) 50% e) 150% f) 220%

4. Re-write each of the given fractions as a percent.

   a) $\frac{3}{10}$ b) $\frac{17}{20}$ c) $\frac{60}{25}$ d) $\frac{3}{50}$ e) 5

5. Use the duality between division of integers and standalone fractions to perform the given divisions without the aid of a calculator. a) $\frac{120}{5}$ b) $\frac{2050}{50}$ c) $\frac{1200}{25}$

6. a) Bring the fractions $\frac{7}{10}$ and $\frac{2}{3}$ to a common denominator. Compare the numerators.

   Which fraction is greater, $\frac{7}{10}$ or $\frac{2}{3}$?

   b) Compute both $\frac{7}{10}$ and $\frac{2}{3}$ of 60. Is your answer in conflict with part a)?

7.2 Equations and Inequalities

The most frequently made statements in algebra are equations and inequalities.

Definition: An **equation** is a pair of numerical or algebraic expressions connected with an equal sign.

For example, $2+3=9$ and $1+2a=5b$ are equations. So are $x+y=y+x$, and $x^3+4=x^2+4x$. Some equations have variables in them, some don't.

Definition: A **solution of an equation** is a value (or ordered pair of values) of the unknown(s) that, when substutited into both sides of the equation, makes the statement of equality true.

Example 1. Consider the equation $x^3+4=x^2+4x$. Evaluate both sides of the equation with the given value of x to determine whether it is a solution of the equation or not.

a) $x=3$ b) $x=2$

Solution: a) We substitute $x=3$ into both sides of the equation $x^3+4=x^2+4x$ and compare the values. We will denote the left-hand side by LHS and the right-hand side by RHS.

If $x=3$, then

$\text{LHS} = 3^3+4$	$\text{RHS} = 3^2+4\cdot 3$
$= 27+4=31$	$= 9+12=21$

The left-hand side is 31, and the right-hand side is 21. Since these are not equal, $x=3$ is not a solution of the given equation. In short, LHS$=31\neq 21=$RHS, so 3 is not a solution.

b) We substitute $x=2$ into both sides of the equation $x^3+4=x^2+4x$.

If $x=2$, then

$\text{LHS} = 2^3+4$	$\text{RHS} = 2^2+4\cdot 2$
$= 8+4=12$	$= 4+8=12$

Since the two sides are equal, 21 is a solution of the equation.

We have seen that both 2 and -2 are solutions of the equation $x^3+4=x^2+4x$. We will leave it for the reader to verify that $x=1$ is also a solution. So, it is not true that equations can have only one solution! To solve an equation means to find *all* solutions of it.

Definition: An **inequality** is a pair of numerical or algebraic expressions connected with an inequality sign.

For example, $3x-1\geq 5x-7$, and $2x<4y-5$, and $2x^2\leq 7x-4$ are all inequalities.

Definition: A **solution of an inequality** is a value of the unknown that, when substutited into both sides of the inequality, makes the statement of inequality true.

Example 2. Consider the equation $3x-1\geq 5x-7$. Evaluate both sides of the equation with the given value of x to determine whether it is a solution of the inequality or not.

a) $x = 5$ b) $x = 3$

Solution: a) We evaluate both sides of the inequality $3x - 1 \geq 5x - 7$ with $x = 5$ to see whether the inequality statement is true.

If $x = 5$, then

$\text{LHS} = 3 \cdot 5 - 1$	$\text{RHS} = 5 \cdot 5 - 7$
$= 15 - 1 = 14$	$= 25 - 7 = 18$

This means that if $x = 5$, then $3x - 1 \geq 5x - 7$ becomes

$$14 \geq 18$$

This statement is false, therefore 5 is not a solution of the inequality.

b) We evaluate both sides of the inequality with $x = 3$.

If $x = 3$, then

$\text{LHS} = 3 \cdot 3 - 1$	$\text{RHS} = 5 \cdot 3 - 7$
$= 9 - 1 = 8$	$= 15 - 7 = 8$

This means that if $x = 3$, then $3x - 1 \geq 5x - 7$ becomes $8 \geq 8$. This statement is true, therefore 3 is a solution of the inequality.

Sample Problems

1. Consider the equation $2x^2+x+34=21x-8$.

   a) Is the number 1 a solution of the equation?
   b) Is 3 a solution of the equation?
   c) Is $x=4$ a solution of the equation?
   d) Is 7 a solution of the equation?

2. Consider the equation $3a-2b-1=(a-b)^2+4$.

   a) Is the pair of numbers $a=8$ and $b=5$ a solution of the equation?
   b) Is the pair of numbers $a=10$ and $b=7$ a solution of the equation?

3. Consider the inequality $x^2+2x-1\leq 8x+6$..

   a) Is $x=2$ a solution of the inequality?
   b) Is $x=7$ a solution of the inequality?
   c) Is $x=10$ a solution of the inequality?

Practice Problems

1. Consider the equation $\dfrac{2x^2-11x-21}{2x+3}=3x-(2x+7)$

   a) Is the number 8 a solution of the equation?
   b) Is 13 a solution of the equation?
   c) Is $x=10$ a solution of the equation?

2. Consider the equation $y=\dfrac{5x-3}{2}$.

   a) Is the pair of numbers $x=1$ and $y=1$ a solution of the equation?
   b) Is the pair of numbers $x=9$ and $y=4$ a solution of the equation?
   c) Is the pair of numbers $x=3$ and $y=6$ a solution of the equation?
   d) Is the pair of numbers $x=17$ and $y=41$ a solution of the equation?

3. Consider the equation $(p-q)^2+\dfrac{3p-1}{6-q}=4(p+1)$.

   a) Is the pair of numbers $p=8$ and $q=5$ a solution of the equation?
   b) Is the pair of numbers $p=7$ and $q=1$ a solution of the equation?

4. Consider the inequality $x^2+x+10\geq 10x+2$

   a) Is $x=0$ a solution?
   b) Is $x=1$ a solution?
   c) Is $x=4$ a solution?
   d) Is $x=10$ a solution?

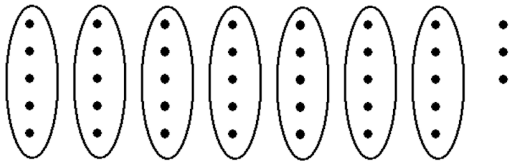
7.3 Division with Remainder

Theorem: (Division with Remainder) For every integer N and positive integer m, there exist unique integers q and r such that

$$N = mq + r \qquad \text{where } 0 \le r \text{ and } r < m$$

For example, if $N = 38$ and $m = 5$, then the quotient is $q = 7$ and the remainder is $r = 3$. The picture illustrates the division $38 \div 5 = 7$ R 3, or, in other form: $38 = 5 \cdot 7 + 3$. Note that there are two different ways to represent this division:

$38 \div 5 = 7$ R 3 and $\dfrac{38}{5} = 7\dfrac{3}{5}$.



Example 1. Perform the given division with remainder. $6071 \div 17$

Solution: One way would be is to perform long division. Another is to utilize our calculator. First we enter the division into the calculator and see the result as a decimal: $357.117\,647\,058\,8$. Note that this is NOT division with remainder. However, we can quickly adjust this. Since the result is between 357 and 358, we already know that the quotient is 357 and now we need to only find the remainder.

$$6071 \div 17 = 357 \text{ R } ?$$

If the quotient is 357, then we have already accounted for $17 \cdot 357 = 6069$. How much is missing from 6071? Clearly 2. Thus the answer is: $6071 \div 17 = 357$ R 2.

How can we check? We should look for two things: did we account for all 6071? Indeed, $17 \cdot 357 + 2 = 6071$. The other question is: Is the remainder less than the divisor? Indeed, $2 < 17$ and so our solution is correct.

Example 2. Perform the given division with remainder. $8271 \div 45$

Solution: First we enter the division into the calculator and see the result as a decimal: 183.8. Note that this is NOT division with remainder. However, we can quickly adjust this. Since the result is between 183 and 184, we already know that the quotient is 183 and now we need to only find the remainder. If the quotient is 183, then we have already accounted for $183 \cdot 45 = 8235$. How much is missing from 6071? $8271 - 8235 = 36$ Thus the answer is: $8271 \div 45 = 184$ R 36.

It is always a good idea to check: $183 \cdot 45 + 36 = 8271$, and $36 < 45$ and so the remainder is less than the divisor. So our solution is correct.

Example 2b. Perform the given divisions with remainder.

a) $137 \div 4$ Solution - Youtube link

b) $325 \div 7$ Solution - Youtube link

An Application: Orbits

Example 3. What is the last digit of 2^{99}? Solution - Youtube link

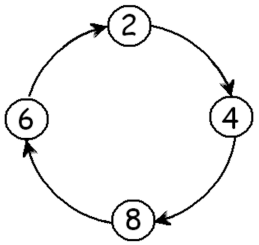
Solution: Let us first investigate what the last digits of smaller 2−powers are.

$$\begin{array}{lll} 2^1 = 2 & 2^5 = 3\boxed{2} & 2^9 = 51\boxed{2} \\ 2^2 = 4 & 2^6 = 6\boxed{4} & 2^{10} = 102\boxed{4} \\ 2^3 = 8 & 2^7 = 12\boxed{8} & 2^{11} = 204\boxed{8} \\ 2^4 = 1\boxed{6} & 2^8 = 25\boxed{6} & 2^{12} = 409\boxed{6} \end{array}$$

Before the 2-powers become uncomfortably large, we can observe a repeating pattern: the last digits being

$$2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6, ...$$

There are only 4 numbers appearing as last digits, following each other in a repeating pattern, we also call an **orbit**. The question only remains: where does 99 land in this pattern?



One way to determine that is to apply division with remainder. Consider the last row:

$$2^4 = 1\boxed{6}, \quad 2^8 = 25\boxed{6}, \quad 2^{12} = 409\boxed{6},$$

We can see that if the exponent is divisible by 4, then the last digit is 6. Consider now the first row:

$$2^1 = \boxed{2}, \quad 2^5 = 3\boxed{2}, \quad 2^9 = 51\boxed{2},$$

These exponents, $1, 5, 9, 13, ...$ are right after a number divisible by 4. These are numbers that result in a remainder 1 when divided by 4. In the next row,

$$2^2 = \boxed{4}, \quad 2^6 = 6\boxed{4}, \quad 2^{10} = 102\boxed{4},$$

the exponents are even but not divisible by 4. These can also be expressed as numbers that result in a remainder 2 when divided by 4. In the last row,

$$2^3 = \boxed{8}, \quad 2^7 = 12\boxed{8}, \quad 2^{11} = 204\boxed{8},$$

the exponents have a remainder 3 when divided by 4. We amend our results with these labels.

$2^1 = 2$	$2^5 = 3\boxed{2}$	$2^9 = 51\boxed{2}$	remainder: 1	last digit: 2
$2^2 = 4$	$2^6 = 6\boxed{4}$	$2^{10} = 102\boxed{4}$	remainder: 2	last digit: 4
$2^3 = 8$	$2^7 = 12\boxed{8}$	$2^{11} = 204\boxed{8}$	remainder: 3	last digit: 8
$2^4 = 1\boxed{6}$	$2^8 = 25\boxed{6}$	$2^{12} = 409\boxed{6}$	remainder: 0	last digit: 6

We can find where 99 lands in this pattern if we divide it by 4 and focus on the remainder.

$$99 \div 4 = 24 \text{ R } 3$$

The remainder is 3. Therefore, 99 lands in the second to last row with exponents $3, 7, 11, 15, ...$, and so the last digit of 2^{99} is 8.

Practice Problems

1. Perform the indicated divisions with remainders.

   a) $132 \div 7$ b) $1145 \div 12$ c) $918 \div 8$ d) $201 \div 12$

2. Find the last digit of each of the following numbers.

   a) 7^{2017} b) 9^{120} c) 2^{1001} d) 3^{286}

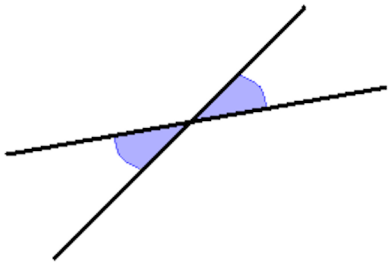
3. If it is 10 AM right now, what will be the time exactly 100 hours from now?

7.4 Angles in a Triangle

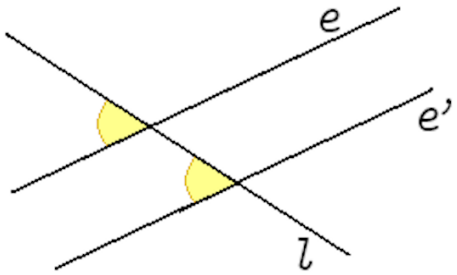
Theorem: The sum of the interior angles in any triangle is 180°.

Proof: We will use the following facts.

When two lines intersect each other, the opposite angles (as shown) formed between them have equal measure. We say that these angles are **vertical angle**s.



When a third line intersects a pair of parallel lines, the **corresponding angles** (as shown) have equal measure. This third line is called the transversal for the pair of parallel lines.



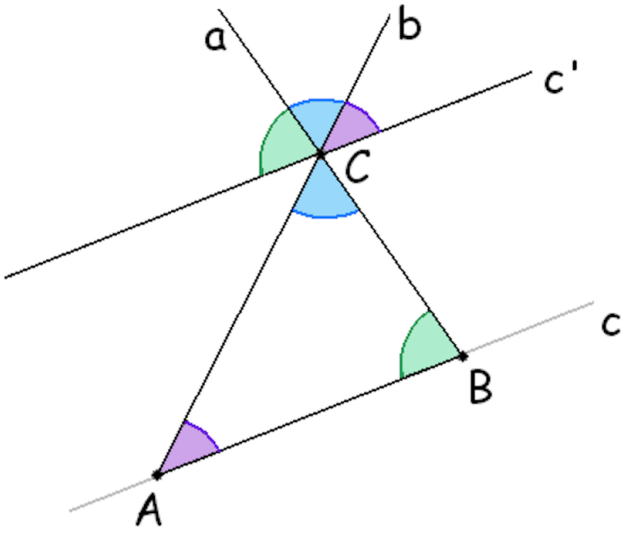
We are now ready to pove our claim. Let ABC be any triangle. We extend lines a and b beyond point C We also draw a line through point C that is parallel to side AB

The purple angles at point A and point C are equal because line b is a transversal for the parallel lines c and c'.

Similarly, the green angles at points B and C are eqiual because line a is also a transversal for the parallel lines c and c'. Finally, the two blue angles at C are equal because they are vertical angles.

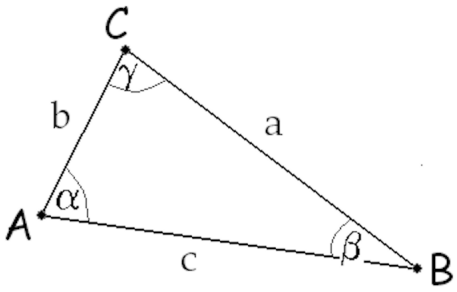
To complete our proof, we need to observe two things.

First, the sum of the three angles in the triangle is the same as the sum of the other three triangles outside of the triangle, above point C. Second, those three angles add up to 180° because next to each other, they form a straight angle, 180°.



This completes our proof.

Standard labeling is a method of notation that automatically associates line segments, points and angles in a triangle. Standard labeling helps us avoiding confusion or lengthy explanations in geometry problems. This agreement is called *standard labeling*, and it establishes a connection between the labels of sides, vertices, and angles in triangles. Every triangle has three of the following three components.



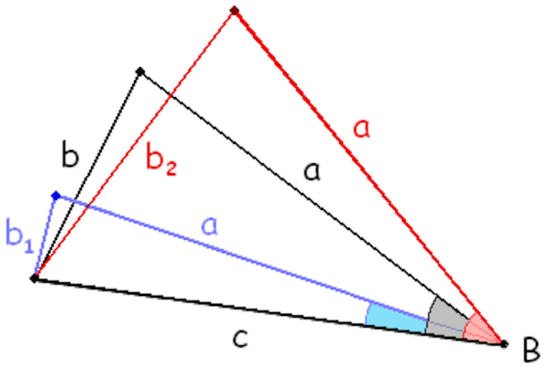
vertices :(singular: vertex) Points are usually denoted by uppercase letters. In case of triangles, we often use A, B, and C.

angles: Angles are usually denoted by lowercase Greek letters. In case of triangles, we often use α (alpha), β (beta), and γ (gamma).

sides: Lines and line segments are usually denoted by lowercase letters. In case of triangles, we often use a, b, and c.

In case of standard labeling, we automatically associate sides, vertices, and angles. A vertex is associated with the angle located at that vertex. These two are associated with the side opposite these. For example, angle α is always assumed to be located at point A, and side a is always assumed to be the side oppsite to point A and angle α. Point B, angle β, and side b are similarly grouped. Unless otherwise indicated, we should always assume standard labeling when presented with data that uses these letters.

Standard labeling is a smart approach to triangles, because there is a natural connection between an angle in a triangle and the side opposite that angle. Consider, for example, the triangle shown above with standard labeling. What if we fixed sides a and c and only modified angle β? Imagine that we have two rods in the lengths of a and c attached to each other at one end and we can freely change the angle between them. If we increase the angle between sides a and c (see the red lines), the side opposite will also increase. If we decrease the angle between sides a and c (see the blue lines), the side opposite will also decrease. So, there seems to be a natural correspondance between side b and angle β.



Theorem: In any triangle ABC, there is a correspondance between the length of a side and the measure of the angle opposite that side: The longest side is opposite the greatest angle, and vica versa: the greatest angle is opposite the longest side. The shortest side is opposite the smallest angle, and vica versa: the smallest angle is opposite the shortest side.

So, the order between the three sides is the same as the order between the corresponding angles, and vica versa. We recommend that sides in triangles are tracked by their corresponding sides. This is because we can perceive the difference in angles much better than in side lengths.

Example 1. Suppose that ABC is a triangle with $\alpha = 82^\circ$ and $\gamma = 39^\circ$. List the length of the sides of the triangle in an increasing order.

Solution: Recall that the three angles in a triangle add up to 180°. This means that if two angles are given, we can compute the third one. $\beta = 180^\circ - (82^\circ + 39^\circ) = 180^\circ - 121^\circ = 59^\circ$. Now we can see the order between the angles. γ is the smallest angle, β is in the middle, and α is the greatest angle. In short: $\gamma < \beta < \alpha$. The order between the lengths of the sides is the same: c is the shortest side, b is in the middle, and a is the longest side. In short: $c < b < a$.

There is an easy but important consequence of the connections between an angle in a triangle and the side opposite.

Theorem: In any triangle ABC, if two angles have equal measures, then the sides opposite them have equal length. Conversely, if two sides are equally long, then the angles opposite those sides have equal mesaures. Such a triangle is called **isosceles**.

We will see that isosceles triangles are very commonly occuring objects in geometry problems. In many cases, we prove that two angles are equal to each other and the conclude that the sides opposite these angles have equal length. Or vica versa, if we prove that two sides are equally long, we can then conclude that the opposite angles are also equal.

Theorem: In any triangle ABC, if all three angles have equal measures, then all three sides have equal length. Conversely, if three sides in a triangle are equally long, then all angles have equal mesaures. Such a triangle is called **regualar** or **equilateral or equiangular.**.

The angles in a regular triangle measure 60°. If the three angles add up to 180° and are equal, $3x = 180 \implies x = 60^\circ$.

Example 2. Suppose that one angle in an isosceles triangle is 40°. Find the measure of the other angles in the triangle.

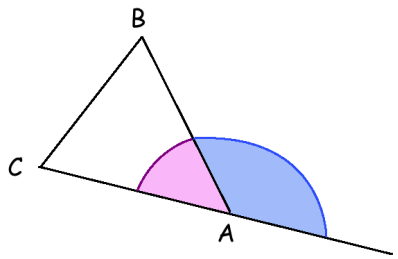
Solution: As it is phrased, this problem has several solutions. If the triangle is isosceles because it has two 40° angles, then $40 + 40 = 80$ and so the missing third angle must be 100°.

If the other two angles are different from 40°, they must be equal to each other. Two equal angles must add up to 140°, because the three angles add up to 180°. Therefore, each angle must be 70°.

So the angles in the triangle are either 40°, 70°, 70°, or 40°, 40°, and 100°.

We check: $40 + 40 + 100 = 180$ and the triangle is isosceles. Similarly, $70 + 70 + 40 = 180$ and the triangle is isosceles. So our answer is: the other angles are either 40° and 100°, or both 70°.

Definition: In a triangle ABC, if we extend one side (either one), we see two angles. The angle marked purple is called the **interior angle**. The angle marked blue is called the **exterior angle**.



An interior angle and exterior angle together form a straight angle, 180°. So at any vertex, the sum of interior and exterior angles is 180°.

Example 3. Suppose that ABC is a triangle. What is the exterior angle at point A if the angle at point B is 65° and at C is 85°.

Solution: Given are the interior angles at B and C. They add up to 150°. Therefore, the interior angle at A is 30°. Recall that the three angles add up to 180°. If the interior angle is 30°, then the exterior angle is 150°.

Notice the following. Given an interior angle at A in a triangle, the interior angles at B and C, when added to our angle, the sum is 180°. The exterior angle does the same: the sum is 180° when added to our angle. Therefore, in any triangle, an exterior angle is always the same as the sum of the other two interior angles.

Example 4. What is the sum of the exterior angles in a triangle?

Solution: Let us first add all six angles! At each vertex of the triangle, the interior and exterior angles add up to 180°. So the sum of all six angles is $3 \cdot 180° = 540°$. To get the exterior angles, let us subtract the interior angles. all three of them add up to 180°. Then we get $540° - 180° = 360°$. So the sum of exterior angles is 360°.

Practice Problems

1. What is the missing angle in a triangle with angles 50° and 100°.
2. Find the angles in an isosceles triangle if we know that one exterior angle is
   a) 48° b) 160°

Chapter 8

8.1 Improper Fractions and Mixed Numbers

Recall the definition of an improper fraction.

Definition: If the numerator of a fraction is less than its denominator, we call it a **proper fraction**. A proper fraction always expresses less than a unit of a given quantity. Examples of proper fractions are $\frac{2}{3}$, $\frac{3}{10}$, $\frac{6}{15}$, or 55%.

If the numerator of a fraction is greater than or equal to its denominator, we call it an **improper fraction**. An improper fraction always expresses a whole unit or more of a quantity. Examples of improper fractions are $\frac{5}{5}$, $\frac{12}{10}$, $\frac{40}{2}$, or 120%.

Also recall that if we multiply both numerator and denominator of a fraction by the same non-zero number, the resulting fraction is equivalent to the original fraction. Equivalent fractions express the same amount. We can also write any integer as a fraction. For example, 1 can be written as $\frac{1}{1}$, $\frac{3}{3}$, $\frac{5}{5}$, $\frac{100}{100}$ or 100%. The integer 5 can be written as $\frac{5}{1}$, $\frac{10}{2}$, $\frac{15}{3}$, or 500%. We can also divide both numerator and denominator by the same number. When a fraction's numerator and denominator share no divisor greater than 1, the fraction is **in lowest terms**.

With the introduction of improper fractions, our notation includes a new (and huge) duality. We can look at the expression $\frac{20}{4}$ as a division between two integers, i.e. two objects and an operation. We can also interpret the expression $\frac{20}{4}$ as an improper fraction, i.e. a single object. This ambiguity is only allowed because every question we can ever ask has the same answer, no matter which interpretations we used. As final result, fractions must be presented in their simplest form. For example, the fraction $\frac{15}{18}$ must be reduced to lowest terms and presented as $\frac{5}{6}$. The fraction $\frac{20}{4}$ must be presented as the integer 5 because it is a much simpler presentation than $\frac{5}{1}$. But how do we simplify (if we even can) the improper fraction $\frac{7}{3}$?

The name improper fraction already suggests that there is something wrong with such a fraction. We strongly disagree with this notion. However, this might have not been the general opinion when the concept of mixed numbers was developed.

A dime is a common nickname for the silver colored, small ten-cent coin. 10 dimes are worth a dollar, so one dime can be represented as $\frac{1}{10}$ of a dollar. Suppose we have 42 dimes. How can we express this fact? As an improper fraction, we can of course write $\frac{42}{10}$.

Suppose we go to the bank and change all dimes we can for dollar bills. In this case, we could exchange 40 dimes for four dollar bills. Using this idea, we can write $\frac{42}{10}$ as a mixed number as $4\frac{2}{10}$. The integer part expresses the bills, the fraction part expresses the coins. Of course we can also bring the fraction part to lowest terms and get $4\frac{1}{5}$. For some, this is the only way to present the fraction $\frac{42}{10}$ in its simplest form.

Definition: A **mixed number** is an alternative representation for improper fractions that can not be simplified as integers. A mixed number has an integer part and a fraction part. For example, $2\frac{1}{3}$ is a mixed number where the integer part is 2 and the fraction part is $\frac{1}{3}$.

Not every country uses mixed number notation. In countries that don't use mixed numbers, $2\frac{1}{3}$ would appear as $2+\frac{1}{3}$. In the USA, mixed numbers are fairly common, but their usefullness is debated. Let us also note that this notation is an example where two objects are written next to each other with no operation between them, and it does *not* represent multiplication.

Example 1. Re-write the improper fraction $\frac{87}{10}$ as a mixed number.

Solution: We can think of this as a person with 87 dimes in their pocket. How much money can be exchanged to dollar bills? Since 10 dimes are worth a dollar, we can exchange 80 dimes for eight dollars in paper money, and are left with seven dimes. This can be expressed as $\boxed{8\frac{7}{10}}$.

Example 2. Re-write the improper fraction $\frac{513}{100}$ as a mixed number.

Solution: We can think of this as a person with 403 pennies in their pocket. How much money can be exchanged to dollar bills? Since 100 pennies are worth a dollar, we can exchange 500 pennies for five dollars in paper money, and are left with thirteen pennies. This can be expressed as $\boxed{5\frac{13}{100}}$.

Notice that in converting an improper fraction to a mixed number, we apply division with remainder. The division $87 \div 10 = 8$ R 7 is behind the conversion $\frac{87}{10} = 8\frac{7}{10}$ and the divsion $513 \div 100 = 5$ R 13 is behind the conversion $\frac{513}{100} = 5\frac{13}{100}$. The idea behind mixed numbers is simplifying, which means that the integer part needs to be reduced to lowest terms.

Example 3. Re-write the improper fraction $\frac{28}{10}$ as a mixed number.

Solution: We can think of this as a person with 28 dimes in their pocket. We perform the division with remainder:
$28 \div 10 = 2$ R 8. This is the same as saying that $\frac{28}{10} = 2\frac{8}{10}$, just as in the previous examples. However, this mixed number needs to be simplified where we bring the fraction part to lowest terms. Clearly $\frac{8}{10} = \frac{4}{5}$, so $\frac{28}{10} = \boxed{2\frac{4}{5}}$.

Let us see an example beyond coins. Suppose that we work in a pizza place where each pizza is cut into 6 slices. So, one slice of pizza can be represented as $\frac{1}{6}$. A whole pizza can be represented as 1 (imagine we are not cutting it into slices) or $\frac{6}{6}$ (if we do cut it into slices). How can we express $\frac{25}{6}$ as a mixed number? We perform the division with remainder: $25 \div 6 = 4$ R 1 and we convert $\frac{25}{6}$ to the mixed number $4\frac{1}{6}$. We can interpret this as 4 whole pizzas, and one additional slice. This is correct, 4 whole pizzas can be represented as $4 = \frac{4}{1} = \frac{24}{6}$ that is, 24 slices.

Example 4. Re-write the improper fraction $\frac{17}{6}$ as a mixed number.

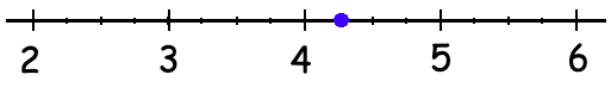
Solution: We perform the division with remainder: $17 \div 6 = 2$ R 5. Thus $\frac{17}{6} = \boxed{2\frac{5}{6}}$.

Example 5. Draw a picture representing the improper fraction $\frac{7}{3}$. Use your picture to convert $\frac{7}{3}$ to a mixed number.

Solution: To represent a fraction with denominator three means that we will draw circles and slice them into three equal part. Because this is an improper fraction, we will need more than one circle. We keep starting new units until we have seven slices. Our picture shows that the mixed number corresponding to $\frac{7}{3}$ is $\boxed{2\frac{1}{3}}$.

Example 6. Re-write the improper fraction $\frac{17}{4}$ as a mixed number. Use your result to plot $\frac{17}{4}$ on the number line.

Solution: We perform the divsion with remainder: $17 \div 4 = 4$ R 1. Therefore, $\frac{17}{4} = \boxed{4\frac{1}{4}}$. When we plot $4\frac{1}{4}$, we first find 4 and 5. We split the line segment between 4 and 5 into four equal part, and count one from 4.



Example 7. Re-write the mixed number $3\frac{2}{5}$ as an improper fraction.

Solution: We can imagine a pizza place where each pizza is cut into five slices. Then the question is: if someone orders three full pizzas and two more slices, how many slices were ordered? Three full pizzas will account for 15 slices, so all together we have 17 slices. Algebraically, $3 = \frac{3}{1} = \frac{15}{3}$ and so $3\frac{2}{5} = \boxed{\frac{17}{3}}$.

Example 8. Re-write the mixed number $2\frac{5}{8}$ as an improper fraction.

Solution: We can imagine a pizza place where each pizza is cut into eight slices. Then the question is: if someone orders two full pizzas and five more slices, how many slices were ordered? Two full pizzas will account for 16 slices, so all together we have $16+5=21$ slices. Algebraically, $2=\frac{2}{1}=\frac{16}{8}$ and so $2\frac{5}{8}=\boxed{\frac{21}{8}}$.

In this course, we will always have to present fractions in their simplest possible form as final answers. **However, we will not view mixed numbers as more simplified than improper fractions.** This means that improper fractions can always be presented in their improper form as a final answer, as long as it is in lowest terms. For example, $\frac{42}{10}$ is not acceptable as final answer, but $\frac{21}{5}$ is. The mixed number $4\frac{2}{10}$ is not simplified, $4\frac{1}{5}$ is. However, $4\frac{1}{5}$ is not considered more simplified than $\frac{21}{5}$. They are equally acceptable, and, from an algebraic point of view, $\frac{21}{5}$ is actually preferred. We will see that with just a very few exceptions, improper fractions have nicer properties than mixed numbers.

Practice Problems

1. Re-write each of the following improper fractions as mixed numbers. Bring the fraction part to lowest terms.
   a) $\frac{10}{3}$ b) 150% c) $\frac{35}{10}$ d) $\frac{120}{7}$ e) $\frac{42}{8}$ f) $\frac{715}{100}$ g) 320%
2. Re-write each of the mixed numbers as improper fractions in lowest terms.
   a) $3\frac{4}{5}$ b) $1\frac{3}{8}$ c) $5\frac{6}{7}$ d) $3\frac{4}{10}$ e) $10\frac{5}{7}$ f) $5\frac{1}{4}$
3. Draw a picture to represent each of the given fractions.
   a) $\frac{13}{4}$ b) $2\frac{3}{5}$ c) $\frac{7}{2}$ d) $\frac{10}{7}$
4. Plot each of the given fractions on the number line.
   a) $\frac{8}{3}$ b) $3\frac{2}{5}$ c) $\frac{9}{2}$ d) $\frac{7}{4}$

8.2 One- and Two-Step Equations

We now start to learn more about equations. Let us first recall a few definitions.

> **Definition**: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign. A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality true.

For example, 4 is a solution of the equation $3x^2 - x = 4x + 28$..

> **Definition**: To **solve an equation** is to find *all* solutions of it.

Caution! Finding one solution for an equation is not the same as solving it. For example, the number 2 is a solution of the equation $x^3 + 30 = 4x^2 + 11x$. We can fnd this solution by trial and error. However, 2 isnot the only solution of this equation. So is 5. Trial and error is rarely a legitimate method because there is no way for us to gurantee that there are no other solutions are there. For example, the equation above has a third solution. It is impossible for us to try all numbers because there are infinitely many of them, and we have finite lives.

So we will need to develop systematic methods to solve equations. We will start with the easiest group of equations, linear equations. There are several types of linear equations, and we will start with the easiest type that is called one-step equations.

To solve a linear equation, we isolate the unknown by applying the same operation(s) to both sides. Consider, for example, Ann and Dewitt who has the same monthly salary. This month they both get a 40 dollar raise. Who is making more money now? It is clear that if we start with two equal quantities and we add the same amount to them, they will still remain equal. This is the underlying principle of solving equations. We always apply the same operations to both sides in an effort to bring the equation in a simple form such as $x = 2$. .

Example 2. Solve each of the given equations. Make sure to check your solutions.

a) $x - 8 = 10$ b) $3y = 120$ c) $\frac{x}{3} = 8$ d) $m + 9 = 20$

Equations like these are called **one-step equations** because they can be solved in only one step. We need to isolate the unknown on one side. In order to do that, we perform the inverse operation. (The inverse operation of additon is subtraction and vica versa. The inverse operation of multiplication is division and vica versa.)

Solution: a) Consider this. Given a number, we get 10 if we subtract 8 from it. What is this number? If we add the 8 back, we get 18.. Indeed, if we subtract 8 from 18, we do get 10. We reverse the operation. In the case of $x - 8 = 10$, we add 8 to both sides.

$$\begin{aligned} x - 8 &= 10 \qquad \text{add } 8 \\ x &= 18 \end{aligned}$$

So the only solution of this equation is 18. We should check; if $x = 18$, the left-hand side is

$$\text{LHS} = x - 8 = 18 - 8 = 10 = \text{RHS} \checkmark$$

So our solution, $\boxed{x = 18}$ is correct.

b) What number must we multiply by 3 to get 120? To find this out, we divide. If the unknown was multiplied by 3, we reverse it by dividing both sides by 3.

$$\begin{aligned} 3y &= 120 \qquad \text{dividde by 3} \\ y &= 40 \end{aligned}$$

So the only solution of this equation is 40. We check; if $y = 40$, then

$$\text{LHS} = 3y = 3 \cdot 40 = 120 = \text{RHS} \checkmark$$

So our solution, $\boxed{y = 40}$ is correct.

c) The unknown was divooded by 3. To reverse this, we will multiply both sides by 3.

$$\begin{aligned} \frac{x}{3} &= 8 \qquad \text{multiply by 3} \\ x &= 24 \end{aligned}$$

So the only solution of this equation is 24. We check; if $x = 24$, then

$$\text{LHS} = \frac{24}{3} = 8 = \text{RHS} \checkmark$$

So our solution, $\boxed{x = 24}$ is correct.

d) In order to isolate the unknown, we we reverse the addition by subtracting 9 from both sides.

$$\begin{aligned} m + 9 &= 20 \qquad \text{subtract 9} \\ m &= 11 \end{aligned}$$

So the only solution of this equation is 11. We check; if $m = 11$, then

$$\text{LHS} = m + 9 = 11 + 9 = 20 = \text{RHS}$$

So our solution, $\boxed{m = 11}$ is correct.

Discussion: Solve each of the following equations. How are these unusual?

a) $5x = 5$ b) $5x = 0$ c) $x - 4 = -4$ d) $\frac{x}{3} = 0$

Example 3. One side of a rectangle is 12 feet long. Find the length of the other side if the area of the rectangle is 60 square-feet.

Solution: Let us denote the missing side by x. We will write and solve an equation expressing the area of the rectangle.

$$\begin{aligned} 12x &= 60 \qquad \text{divide by 12} \\ x &= 5 \end{aligned}$$

Thus the other side is $\boxed{5 \text{ feet}}$ long. Note that if we carry the units in the computation, they will work out perfectly.

$$(12\text{ft})\, x = 60\text{ft}^2 \qquad \text{divide by 12ft} \qquad \text{margin work:} \quad \frac{60\text{ft}^2}{12\text{ft}} = \frac{5\text{ft} \cdot \cancel{\text{ft}}}{\cancel{\text{ft}}} = 5\text{ft}$$

$x = 5\text{ft}$

Part 2 – Two-Step Equations

Suppose we decide to hide a small object, say a coin. We put the coin on the table, then place an envelope over it, and then, just to be sure, we place a hat on top of the envelope. Let us find the coin! To do that, what do we need to remove, and in what order? We would first remove the hat and then the envelope, right?

This is the basis of solving two-step equations. To isolate the unknown, we will perform the inverse operations, in the reverse order. What happened last can be undone first.

Example 4. Solve each of the given equations. Make sure to check your solutions.

a) $10 = 3x - 11$ b) $3x - 8 = 7$ c) $\dfrac{t-7}{2} = 8$ d) $\dfrac{x}{3} + 4 = 7$

Solution: a) The equation $10 = 3x - 11$ looks unusual in the sense that two-step equations often contain the unknown on the left-hand side. We are always allowed to swap two sides of an equation. If $A = B$, then clearly, also $B = A$. We will do this first. This is an optional step that is always available.

$$\begin{aligned} 10 &= 3x - 11 \quad \text{we swap the two sides} \\ 3x - 11 &= 10 \end{aligned}$$

We now look at the side that contains x and ask: *What happened to the unknown*? The answer is: *Multiplication by* 3 *and then subtraction of* 11. We need to apply the inverse operations, in reverse order. In this case, this means that we will add 11 to both sides and then divide both sides by 3.

$$\begin{aligned} 3x - 11 &= 10 \quad \text{add 11} \\ 3x &= 21 \quad \text{divide by 3} \\ x &= 7 \end{aligned}$$

So the only solution of this equation is 7. We check: if $x = 7$, then

$$\text{LHS} = 3x - 11 = 3 \cdot 7 - 11 = 21 - 11 = 10 = \text{RHS} \checkmark$$

So our solution, $\boxed{x = 7}$ is correct.

b) As we look at the equation $3x - 8 = 7$, we ask: *What happened to the unknown*? The answer is: *Multiplication by* 3 *and then addition of* 8. We need to apply the inverse operations, in a reverse order. In this case, this means that we will subtract 8 from both sides and then divide both sides by 3.

$$\begin{aligned} 3x - 8 &= 7 \quad \text{add 8} \\ 3x &= 15 \quad \text{divide by 3} \\ x &= 5 \end{aligned}$$

So the only solution of this equation is 5. We check: if $x = 5$, then

$$\text{LHS} = 3 \cdot 5 - 8 = 15 - 8 = 7 = \text{RHS} \checkmark$$

So our solution, $\boxed{x = 5}$ is correct.

c) What happened to the unknown? On the left-hand side, there was a subtraction of 7 and then a division by 2. To reverse that, we will multiply both sides by 2 and then add 7 to both sides.

$$\begin{aligned} \frac{t-7}{2} &= 8 && \text{multiply by 2} \\ t-7 &= 16 && \text{add 7} \\ t &= 23 \end{aligned}$$

So the only solution of this equation is 23. We check: if $t = 23$, then

$$\text{LHS} = \frac{23-7}{2} = \frac{16}{2} = 8 = \text{RHS} \checkmark$$

So our solution, $\boxed{t = 23}$ is correct.

d) What happened to the unknown? On the left-hand side, there was a division by -3 and then an addition of 4. To reverse that, we will subtract 4 from both sides by and then multiply both sides by -3.

$$\begin{aligned} \frac{x}{3}+4 &= 7 && \text{subtract 4} \\ \frac{x}{3} &= 3 && \text{multiply by 3} \\ x &= 9 \end{aligned}$$

So the only solution of this equation is 9. We check: if $x = 9$, then

$$\text{LHS} = \frac{9}{3}+4 = 3+4 = 7 = \text{RHS} \checkmark$$

So our solution, $\boxed{x = 9}$ is correct.

Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x-5 = 17$
2. $\dfrac{a-10}{5} = 3$
3. $\dfrac{t}{4} - 10 = 2$
4. $\dfrac{t-5}{12} = 4$
5. $5x-7 = 3$
6. $\dfrac{x+6}{3} = 2$
7. $\dfrac{x}{3}+8 = 20$
8. $3(x+7) = 36$
9. $\dfrac{x}{3}+2 = 2$

Practice Problems

Solve each of the following equations. Make sure to check your solutions.

1. $7x-3=11$
2. $\frac{x-3}{2}=5$
3. $5x-3=2$
4. $\frac{x+8}{7}=4$
5. $\frac{x}{2}-3=1$
6. $4x+3=23$
7. $\frac{a+7}{4}=9$
8. $5x+2=32$
9. $\frac{x}{7}-8=2$
10. $x+5=5$
11. $\frac{2x-1}{7}=7$
12. $5(x-2)=20$
13. $\frac{x-8}{7}=2$
14. $3b+13=25$
15. $\frac{x}{3}-4=4$

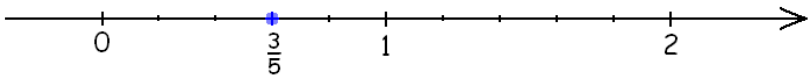
Chapter 9

9.1 Adding and Subtracting Fractions

Part 1 - Fractions as Part of Our Number System

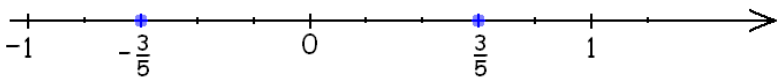
Until now, we have only looked at fractions as expressing a part of a quantity. Before we define the four basic operations on fractions, it is best to start looking at them as part of our number system.

We have defined what it means $\frac{3}{5}$ of somthing. What about $\frac{3}{5}$ alone, as a number? We understand $\frac{3}{5}$ alone as $\frac{3}{5}$ of the number 1. So, we can actually place fractions on the number line. In case of $\frac{3}{5}$, we divide the unit between zero and one into five equal parts, and place $\frac{3}{5}$ after three such line segments.



We can also interpret all integers as fractions. For example, 4 is the same as $\frac{8}{2}$ or $\frac{4}{1}$. The expression $\frac{4}{1}$ could mean division between the integers 4 and 1, and also, the fraction with numerator 4 and denominator 1. Thus duality is fundamental, and only allowed because all answers are always the same when worked with the two different meaning.

How about negative numbers? Negative fractions? We can clearly think if $-\frac{3}{5}$ as the opposite of $\frac{3}{5}$.



A *simplified* or *reduced* fraction cannot have more than one negative sign.

Let us consider the fraction $\frac{-2}{-7}$. Recall the fundamental property of fractions: that we can multiply both numerator and denominator by the same non-zero number without changing the value of the fraction. Let us proceed with -1.

$$\frac{-2}{-7} = \frac{-2(-1)}{-7(-1)} = \frac{2}{7}$$

So, a fraction with a negative number in the numerator and denominator can be simplified, and the result is simply a positive fraction. Notice that the rule is the same as with division between two integers.

The negative sign of a fraction can be located in three places. All three of the fractions shown below are equaivalent.

$$-\frac{2}{3} \quad \text{or} \quad \frac{-2}{3} \quad \text{or} \quad \frac{2}{-3}$$

The first form, $-\frac{2}{3}$ $\left(\text{ the opposite of } \frac{2}{3}\right)$ is considered simplified, and it is an elegant way to present a fraction as final answer. However, it is not the recommended form for computing with signed fractions.

The second form, $\frac{-2}{3}$ (the numerator is -2, the denominator is 3) is both simplified and highly recommended for computations. Indeed, when performing operations with signed fractions, we recommend that the negative sign is interpreted as belonging to the numerator.

The third form, $\frac{2}{-3}$ (the numerator is 2, the denominator is -3) is not considered simplified, and not recommended for computations either. Thus we recommend that when faced with such a fraction, we immediately multiply both numerator and denominator by -1, thereby switching to the second form, $\frac{-2}{3}$. **Fractions with a negative sign in their denominators are not acceptable as final answers.**

Example 1. Simplify each of the given fractions.

a) $\frac{10}{-28}$ b) $\frac{-12}{1}$ c) $\frac{-3}{-18}$ d) $\frac{4}{-7}$

Solution: a) The fraction itself can be simplified by dividing both numerator and denominator by 2. But then $\frac{5}{-14}$ is not simplified because the negative sign is in the wrong place. Multiplying both numerator and denominator by -1 will give us the answer in the correct form, $\frac{-5}{14}$. We can also present this as $-\frac{5}{14}$.

$$\frac{10}{-28} = \frac{5}{-14} = \boxed{\frac{-5}{14} \text{ or } -\frac{5}{14}}$$

Eventually we will be able to speed up the process by dividing numerator and denominator of $\frac{10}{-28}$ by -2.

b) The fraction $\frac{-12}{1}$ is not simplified, because it can be simplified to -12. Integers should never be presented as fractions. Clearly -12 is simpler than $\frac{-12}{1}$. So, the correct answer is $\boxed{-12}$.

c) We can simplify $\frac{-3}{-18}$ by dividing numerator and denominator by -3.

$$\frac{-3}{-18} = \boxed{\frac{1}{6}}$$

d) The fraction $\frac{4}{-7}$ is noot simplified only because the negative sign is in the denominator. We can fix that by multplying numerator and denominator by -1.

$$\frac{4}{-7} = \frac{4\cdot(-1)}{-7\cdot(-1)} = \boxed{\frac{-4}{7} \text{ or } -\frac{4}{7}}$$

Example 2. Re-write the mixed number $-4\frac{5}{8}$ as an improper fraction.

Solution: We simply ignore the negative sign, convert the mixed number to an improper fraction, and then put the negative sign back.

$$-4\frac{5}{8} = -\left(\frac{4\cdot 8+5}{8}\right) = -\frac{37}{8}$$

So the correct answer is $\boxed{-\frac{37}{8} \text{ or } \frac{-37}{8}}$.

Part 2 - Adding and Subtracting Fractions with the Same Denominator

Recall that the numerator and denominator play very different roles. The numerator counts how many slices we have. The denominator only describes how large each slice is. For example, in the fraction $\frac{3}{8}$, we are dealing with three slices. Each slice is so large so that eight of them makes up a whole. So, $\frac{3}{8}$ and $\frac{3}{5}$ both express three slices, only the slices in $\frac{3}{5}$ are bigger that those in $\frac{3}{8}$.

To add or subtract fractions with the same denominator, we write a single denominator and express the addition or subtraction of the signed numbers in the numerator.

For example, the addition $\frac{2}{7}+\frac{3}{7}$ is simple: we are talking about the same size of slices, and we add three slices to two slices. Thus $\frac{2}{7}+\frac{3}{7} = \frac{2+3}{7} = \frac{5}{7}$.

Consider now the addition $\frac{4}{5}+\frac{2}{5}$. Again, the same denominator indicates that the slices are of the same size, so we simpliy add two slices to the four. In short, $\frac{4}{5}+\frac{2}{5} = \frac{4+2}{5} = \frac{6}{5}$.

The answer is an improper fraction. However, we do not see anything improper about it, as long as it is in lowest terms. We could present the final answer as $1\frac{1}{5}$ but $\frac{6}{5}$ is correct, simplified, and is actually preferred.

In case of signed numbers, the negative sign should always be kept in the numerator. Then the addition or subtraction of signed fractions will be just a matter of addition or subtraction of (signed) integers in the numerator.

Example 3. Perform the addition or subtraction of the fractions as indicated.

a) $\frac{3}{8}-\frac{7}{8}$ b) $\frac{2}{5}-\frac{4}{5}-\left(-\frac{3}{5}\right)$ c) $\frac{3}{7}-\frac{9}{7}$ d) $\frac{3}{10}-\frac{7}{10}+\left(-\frac{1}{10}\right)$ e) $2\frac{1}{3}-5\frac{2}{3}$

Solution: a) We re-write the subtraction as one in the numerator. Then, if we can, we simplify the answer.

$$\frac{3}{8}-\frac{7}{8}=\frac{3-7}{8}=\frac{-4}{8}=\boxed{\frac{-1}{2} \text{ or } -\frac{1}{2}}$$

b) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{2}{5}-\frac{4}{5}-\left(-\frac{3}{5}\right)=\frac{2-4-(-3)}{5}=\frac{-2-(-3)}{5}=\frac{-2+3}{5}=\boxed{\frac{1}{5}}$$

c) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{3}{7}-\frac{9}{7}=\frac{3-9}{7}=\boxed{\frac{-6}{7} \text{ or } -\frac{6}{7}}$$

d) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{3}{10}-\frac{7}{10}+\left(-\frac{1}{10}\right)=\frac{3-7+(-1)}{10}=\frac{-4+(-1)}{10}=\frac{-5}{10}=\boxed{\frac{-1}{2} \text{ or } -\frac{1}{2}}$$

e) In case of adding or subtracting mixed numbers, we recommend the use of improper fractions.

$$2\frac{1}{3}-5\frac{2}{3}=\frac{7}{3}-\frac{17}{3}=\frac{7-17}{3}=\boxed{\frac{-10}{3} \text{ or } -\frac{10}{3}}$$

There are other ways of handling this, but we recommend the method shown above. However, there is a way to do this subtraction by separating integer and fraction parts. We need to be careful though to correctly interpret the negative signs and subtraction signs to both integer and fraction parts.

$$2\frac{1}{3}-5\frac{2}{3}=\underset{\text{integer part}}{2-5}+\underset{\text{fraction part}}{\frac{1}{3}-\frac{2}{3}}=-3+\left(-\frac{1}{3}\right)=-3\frac{1}{3}$$

We got lucky with the last problem because both integer and fraction part ended up with the same sign in the result. What about a subtraction such as $4\frac{2}{5}-1\frac{4}{5}$?

$$4\frac{2}{5}-1\frac{4}{5}=\underset{\text{integer part}}{4-1}+\underset{\text{fraction part}}{\frac{2}{5}-\frac{4}{5}}=3+\left(-\frac{2}{5}\right)$$

We can not present a mixed number with mixed signs. The sign of the entire mixed number is determined by the integer part, because it has more value than the fraction part. This answer should be positive. How do we express 3 and $-\frac{2}{5}$ as a mixed number? We borrow 1 from the integer part, express it as $\frac{5}{5}$ and throw it in with the integer part.

$$3+\left(-\frac{2}{5}\right)=2+1+\left(-\frac{2}{5}\right)=2+\frac{5}{5}+\left(-\frac{2}{5}\right)=2+\frac{5+(-2)}{5}=2+\frac{3}{5}=2\frac{3}{5}$$

This can get quite complicated, which is why we recommend to use improper fractions instead of mixed numbers.

$$4\frac{2}{5}-1\frac{4}{5}=\frac{22}{5}-\frac{9}{5}=\frac{22-9}{5}=\frac{13}{5} \qquad \left(\text{same as } 2\frac{3}{5}\right)$$

Part 3 - Adding and Subtracting Fractions with Different Denominators

How do we add the fractions $\frac{5}{6}$ and $\frac{3}{4}$? It is true that we are adding five slices and three slices, but they do not have the same size. So, how do we add fractions with different denominators? Well, we just don't. Instead, we use the fundamental property of fractions to bring the fractions to a common denominator.

The fundamental property of fractions is that we can multiply both numerator and denominator by the same non-zero number without changing the value of the fraction. Therefore, we can find fractions equivalent to $\frac{5}{6}$ with all denominators that are multiples of 6. For example, $\frac{5}{6}=\frac{10}{12}=\frac{15}{18}=\frac{20}{24}=\frac{30}{36}=\frac{50}{60}$ and so on. Similarly, we can find fractions equivalent to $\frac{3}{4}$ with all denominators that are multiples of 4.
For example, $\frac{3}{4}=\frac{6}{8}=\frac{9}{12}=\frac{12}{20}=\frac{15}{20}=\frac{18}{24}$, and so on.

If we multiply 6 and 4, we get a common multiple, 24. However, we should always attempt to work with **the least common multiple**, and that is 12 in this case. To bring $\frac{5}{6}$ to a denominator 12, we multiply numerator and denominator by 2. To bring $\frac{3}{4}$ to a denominator 12, we multiply numerator and denominator by 3.

$$\frac{5}{6}=\frac{5\cdot 2}{6\cdot 2}=\frac{10}{12} \quad \text{and} \quad \frac{3}{4}=\frac{3\cdot 3}{4\cdot 3}=\frac{9}{12} \quad \text{Thus we have that} \quad \frac{5}{6}+\frac{3}{4}=\frac{10}{12}+\frac{9}{12}=\frac{10+9}{12}=\frac{19}{12}$$

This method also works with adding fractions with integers, because we can always re-write an integer as a fraction with denominator 1.

Example 4. Perform the addition or subtraction of the fractions as indicated.

a) $\frac{3}{10}-\frac{4}{5}$ b) $4-\frac{2}{3}-\left(-\frac{1}{2}\right)$ c) $\frac{2}{5}-3$ d) $\frac{2}{3}-\frac{5}{6}-\left(-\frac{1}{2}\right)$ e) $4\frac{1}{6}-3\frac{2}{3}$

Solution: a) The least common denominator between 10 and 5 is 10. We re-write $\frac{4}{5}$ as a fraction with denominator 10. by multiplying both numerator and denominator by 2. $\frac{4}{5}=\frac{4\cdot 2}{5\cdot 2}=\frac{8}{10}$

$$\frac{3}{10}-\frac{4}{5}=\frac{3}{10}-\frac{8}{10}=\frac{3-8}{10}=\frac{-5}{10}=\boxed{\frac{-1}{2} \text{ or } -\frac{1}{2}}$$

b) The integer 4 is not a fraction, but that can be easily fixed, by re-writing it as $\frac{4}{1}$. We are now looking for the least common multiple of 1, 3, and 2. Clearly, 6 will be good as the least common multiple. Then $\frac{4}{1}$ can be re-written as $\frac{4}{1}=\frac{4\cdot 6}{1\cdot 6}=\frac{24}{6}$. Similarly, $\frac{2}{3}=\frac{2\cdot 2}{3\cdot 2}=\frac{4}{6}$ and $\frac{-1}{2}=\frac{-1\cdot 3}{2\cdot 3}=\frac{-3}{6}$. Once the fractions have the same denominator, we perform the additions and subtractions in the numerator. If there is a negative sign, we keep it in the numerator.

$$4-\frac{2}{3}-\left(-\frac{1}{2}\right)=\frac{4}{1}-\frac{2}{3}-\left(\frac{-1}{2}\right)=\frac{24}{6}-\frac{4}{6}-\left(\frac{-3}{6}\right)=\frac{24-4-(-3)}{6}=\frac{20-(-3)}{6}=\boxed{\frac{23}{6}}$$

c) We re-write 3 as $\frac{3}{1}$. The least common multiple of 1 and 5 is 5. To re-write $\frac{3}{1}$ with a denominator 5, we multiply both numerator and denominator by 5. $\frac{3}{1}=\frac{3\cdot 5}{1\cdot 5}=\frac{15}{5}$

$$\frac{2}{5}-3=\frac{2}{5}-\frac{3}{1}=\frac{2}{5}-\frac{15}{5}=\frac{2-15}{5}=\boxed{\frac{-13}{5} \text{ or } -\frac{13}{5}}$$

We can express the answer as a mixed number, but there is no need, unless we are specifically asked for it.

d) The least common multiple of 3, 6, and 2 is 6. So we will re-write each fraction with a denominator of 6. If there is a negative sign, we will keep it in the numerator.

$$\text{So, } \frac{2}{3}=\frac{2\cdot 2}{3\cdot 2}=\frac{4}{6} \quad \text{and} \quad \frac{5}{6}=\frac{5}{6} \quad \text{and} \quad -\frac{1}{2}=\frac{-1}{2}=\frac{-1\cdot 3}{2\cdot 3}=\frac{-3}{6}$$

We can now perform the subtractions, left to right.

$$\frac{2}{3}-\frac{5}{6}-\left(-\frac{1}{2}\right)=\frac{4}{6}-\frac{5}{6}-\left(\frac{-3}{6}\right)=\frac{4-5-(-3)}{6}=\frac{-1-(-3)}{6}=\frac{-1+3}{6}=\frac{2}{6}=\boxed{\frac{1}{3}}$$

e) In case of mixed numbers, we again recommend to switch to improper fractions. We re-write $4\frac{1}{6}$ as $\frac{25}{6}$. Notice that the addition $4+\frac{1}{6}$ gives us exactly the same result. $3\frac{2}{3}$ can be re-written as $\frac{11}{3}$. The least common denominator between 6 and 3 is 6.

$$4\frac{1}{6}-3\frac{2}{3}=\frac{25}{6}-\frac{11}{3}=\frac{25}{6}-\frac{22}{6}=\frac{25-22}{6}=\frac{3}{6}=\boxed{\frac{1}{2}}$$

Practice Problems

Perform the indicated operations. Present your answer as an integer or a reduced fraction. You do not need to convert improper numbers to mixed numbers.

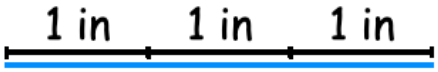
1. $\frac{2}{3}+\frac{5}{6}$
2. $\frac{2}{3}-\frac{5}{6}$
3. $-2-\frac{2}{3}$
4. $-\frac{2}{5}-\left(-\frac{5}{2}\right)$
5. $\frac{3}{4}-\left(-\frac{5}{8}\right)$
6. $\frac{5}{4}-\frac{1}{6}$
7. $2-\frac{2}{3}$
8. $-2+\frac{5}{8}$
9. $2\frac{4}{5}-3\frac{1}{2}$
10. $-3\frac{1}{4}-2\frac{5}{6}$
11. $\frac{1}{3}-\frac{3}{4}+\frac{5}{12}$
12. $-\frac{3}{4}-\frac{5}{6}$
13. $\frac{5}{12}-\frac{3}{4}$
14. $\frac{19}{3}-2\frac{5}{6}$
15. $\frac{5}{2}-\frac{7}{10}+\frac{6}{5}$

Solve each of the given equations.

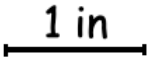
16. $x-\frac{2}{3}=\frac{1}{5}$
17. $x+\frac{3}{4}=\frac{1}{2}$
18. $x-3\frac{1}{5}=\frac{3}{10}$

9.2 Volumes

Perimeter is a measure of length. The length of an object, measured in inches, is the number of 1-inch line segments we need to cover our object.



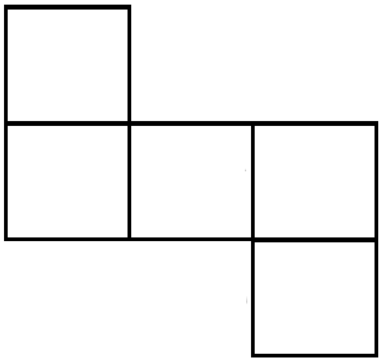
This blue line is 3 inches long.



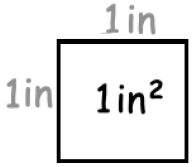
Unit inch

Perimeter, and all lengths are measured in meters (m), inches (in), feet (ft), miles (mi), etc.

Area is a measure of surface. Area of an object, measured in square-inches, is the number of 1-inch by 1-inch squares we need to cover our object.



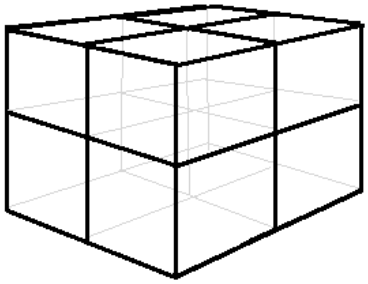
This area is 5 square-inches.



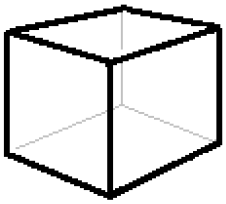
1 square-inch

Area is measured in square-meters (m^2), square-inches (in^2), square-feet (ft^2), square-miles (mi^2), etc.

Volume is a measure of space. The volume of an object, measured in cubic inches, is the number of 1 inch by 1 inch by 1 inch cubes we need to build a replica of our object. Another way to imagine this is to make an ice-sculpture replica of our object, melt it and then freeze it to unit cubes and count.



The volume of this object is 8 cubic-inches.



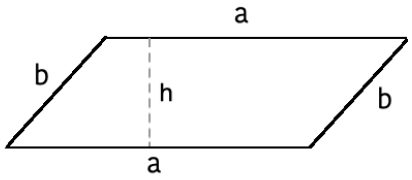
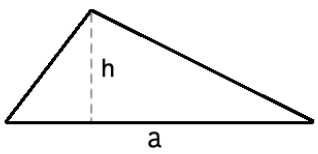
1 cubic inch

Volumes are measured in cubic meters (m^3), cubic inches (in^3), cubic feet (ft^3), cubic miles (mi^3), etc.

Another way of thinking of all this is that length is one-dimensional, part of a line while area, part of a plane is two-dimensional, and volume, part of space is three-dimensional. The space we live in can be described as three-dimensional.

This idea might help us to develop an easy method of remembering volume formulas. Let us first develop the concept of "solid objects" and "pointy object". A solid would start in a shape and end in the same shape. A pointy object would start in a shape and end in a point.

For example, in the plane (2-dimensions), a solid could be a parallelogram while a pointy object would be a triangle.

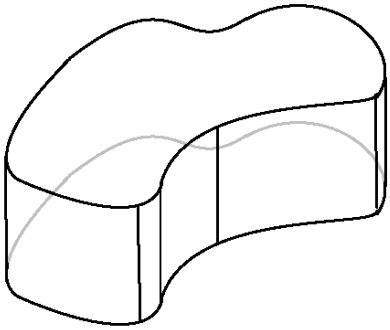
2-dimensional solid	2-dimensional pointy
Area is base times height	Area is base times height over 2
$A = ah$	$A = \frac{ah}{2}$
	

3-dimensional solid

Volume is base area times height

$$V = Bh$$

where B is the area of the base

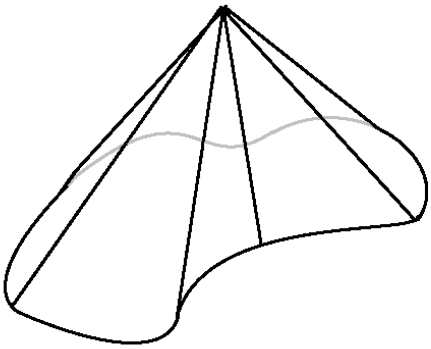


3-dimensional pointy

Volume is base area times height over 3

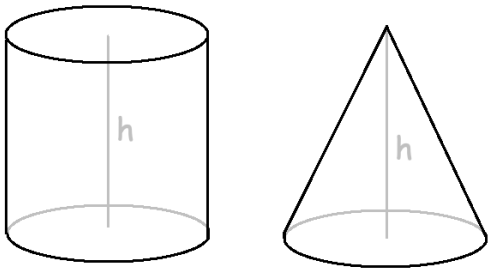
$$V = \frac{Bh}{3}$$

where B is the area of the base



Maybe we cannot visualize objects in 4 dimension, but their 'volume' formula is probably $\frac{Vh}{4}$ base volume times the height over four. Either way, using this general rule is easier than memorizing many volume formulas.

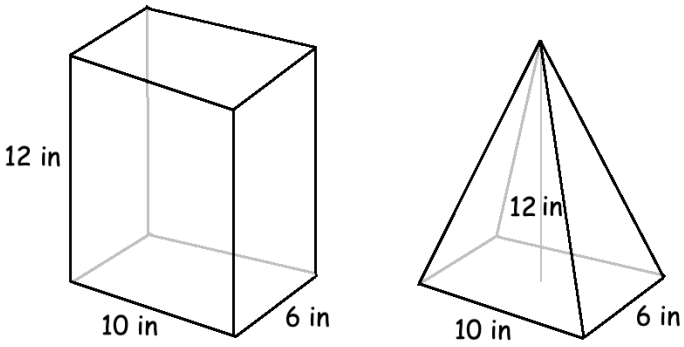
Example 1. The solid shown on the picture is called a cylinder. The pointy object is called a cone. Find the volume of each if the base is a circle with area 30in^2 and the height is 12in.



Solution: We do not yet know the formula for the area of the circle, but this area is given. We apply our new volume formulas.
For the cylinder, $V = Bh = 30\text{in}^2 \cdot 12\text{in} = 360\text{in}^3$

For the cone, $V = \dfrac{Bh}{3} = \dfrac{30\text{in}^2 \cdot 12\text{in}}{3} = 120\text{in}^3$.

Example 2. The solid shown on the picture is a rectangular prism. The pointy object is called a rectangular pyramid. Find the volume of each objects.



Solution: We apply our new volume formulas.

In both cases, the base is a rectangle with sides 6in and 10in.

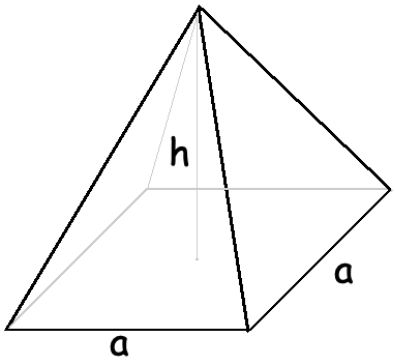
The area of the base is $B = 6\text{in} \cdot 10\text{in} = 60\text{in}^2$. The height is, in both cases, 12in.

For the rectangular prism, $V = Bh = 60\text{in}^2 \cdot 12\text{in} = 720\text{in}^3$

For the pyramid, $V = \dfrac{Bh}{3} = \dfrac{60\text{in}^2 \cdot 12\text{in}}{3} = 240\text{in}^3$.

Practice Problems

1. What is the volume of a square with edges 5ft long?
2. Find the volume of the square based pyramid shown on the picture, where $a = 4\text{cm}$ and $h = 6\text{cm}$.
3. Suppose that the volume of a rectangular prism is 24m^3. How many such prisms are possible if all three sides are whole numbers?



Chapter 10

10.1 Multiplying Fractions

Theorem: To multiply two fractions, we multiply one numerator by the other and one denominator by the other.

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

We will refer to multiplying numerator by numerator and denominator by denominator as mutiplying straight across.

Example 1. Perform each of the multiplications as indicated.

a) $\frac{3}{4}\cdot\frac{5}{12}$ b) $-2\cdot\frac{5}{4}$ c) $-3\frac{1}{3}\cdot 1\frac{1}{2}$ d) $\left(\frac{2}{5}\right)^2$

Solution: a) We multiply straight across: numerator by numerator and denominator by denominator. Then we reduce the fraction to lowest terms before presenting our final answer.

$$\frac{3}{4}\cdot\frac{5}{12}=\frac{3\cdot 5}{4\cdot 12}=\frac{15}{48}$$

The result is not in lowest terms. We reduce it before presentig it as our answer by dividing both numerator and denominator by 3.

$$\frac{15}{48}=\boxed{\frac{5}{16}}$$

We often save time and effort by simplifying the fraction before performing the multiplications. Consider the problem we just solved example. We were able to divide both numerator and denominator by 3, because the numerator in $\frac{3}{4}$ and the denominator in $\frac{5}{12}$ are both divisible by 3. So, we can divide out by that 3 before we perform the multiplication. We just re-write 12 as $3\cdot 4$ and cross out a factor of 3 from both numerator and denominator.

$$\frac{3}{4}\cdot\frac{5}{12}=\frac{3\cdot 5}{4\cdot 12}=\frac{-3\cdot 5}{4\cdot -3\cdot 4}=\frac{5}{16}$$

Crossing out the same factor from the products in both numerator and denominator is the same as dividing both by that factor. This is often called cancellation.

b) We can interpret integers as fractions with denominator 1. If there is a negative sign, we keep it with the numerator for all computations. Therefore, we will re-write -2 as $\frac{-2}{1}$.

$$-2\cdot\frac{5}{4}=\frac{-2}{1}\cdot\frac{5}{4}=\frac{-2\cdot 5}{1\cdot 4}=\frac{-10}{4}=\boxed{-\frac{5}{2}}$$

c) While it is possible to multiply mixed numbers, it takes a lot of work and it will not be shown here. We can simply re-write mixed numbers as improper fractions. If there is a negative sign, we keep it in the numerator during computations.

$$-3\frac{1}{3}\cdot 1\frac{1}{2}=\frac{-10}{-3}\cdot\frac{-3}{2}=\frac{-10}{2}=\boxed{-5}$$

d) If we square a fraction, we must use a pair of parentheses. Other wise, $\frac{2}{5}^2$ can be interpreted as division between integers $\frac{2^2}{5}$. The parentheses guarantees that we all understand that the entire fraction $\frac{2}{5}$ is being suared, and not just its denominator.

$$\left(\frac{2}{5}\right)^2=\frac{2}{5}\cdot\frac{2}{5}=\boxed{\frac{4}{25}}$$

As we are defining operations on fractions, let us note that the notation and order of operations agreement remain the same with fractions as they were with integers. Some issues, such as the difference between -3^2 and $(-3)^2$ will also remain.

Example 2. Perform the indicated operations. $-\frac{2}{3}-\frac{4}{5}\left(-2\frac{6}{7}\right)$

Solution: We see a multiplication and a subtraction. We start with the multiplication, but first re-write the mixed number as an improper fraction. We keep the negative sign in the numerator. $2\frac{6}{7}=\frac{20}{7}$ and so $-2\frac{6}{7}=\frac{-20}{7}$.

$$-\frac{2}{3}-\frac{4}{5}\left(-2\frac{6}{7}\right)=-\frac{2}{3}-\frac{4}{5}\left(\frac{-20}{7}\right)=-\frac{2}{3}-\frac{4(-20)}{5\cdot 7}=-\frac{2}{3}-\frac{4(-4)\cdot -5}{-5\cdot 7}=-\frac{2}{3}-\frac{-16}{7}$$

Next we bring the fractions to the least common denominator and perform the subtraction of signed numbers in the numerator. The least common denominator is 21.

$$-\frac{2}{3}-\frac{-16}{7}=-\frac{14}{21}-\frac{-48}{21}=\frac{-14-(-48)}{21}=\frac{-14+48}{21}=\boxed{\frac{34}{21}}$$

Part 2 - Multiplying by One

It is a frequently occuring task in algebra to re-write an expression without changing its value. There are just about two operations that guarantee this in case of any quantity: we can always *add zero* to it or *multiply* it *by one*. (Actually, we can also subtract zero and divide by one.) In abstract algebra, an element that does not make any change under the operation to any quantity, is called an identity element. Zero is called the **additive identity**, as it does nothing in addition. The number 1 is called the **multiplicative identity** because it does nothing in multiplication.

For every real number x, $x+0=x$. This is the **additive identity** property of zero.
For every real number x, $x\cdot 1=x$. This is the **multiplicative identity** property of one.

Many algebraic techniques are based zero or multiplication (or division) by 1. With fractions, multiplication by 1 is extremely useful. We can even explain the fundamental property of fractions in terms of multiplication by 1. Recall the fundamental property of fractions: we can multiply both numerator and denominator by the same non-zero number and the value of the fraction would remain the same. In the addition $\frac{2}{3}+\frac{1}{2}$, we would need to re-write $\frac{2}{3}$ with a denominator of 6.

$$\frac{2}{3}=\frac{2}{3}\cdot 1=\frac{2}{3}\cdot\frac{2}{2}=\frac{2\cdot 2}{3\cdot 2}=\frac{4}{6}$$

Theorem: The fractions $-\frac{2}{3}$, $\frac{-2}{3}$, and $\frac{2}{-3}$ all have the same value.

Proof. We already knew that $\frac{-2}{3}$ and $\frac{2}{-3}$ are equivalent fractions. We can get from one to the other one by multiplying both numerator and denominator by -1. As a general habit, we should rarely tolerate a negative denominator.

$$\frac{2}{-3}=\frac{2\cdot(-1)}{-3\cdot(-1)}=\frac{-2}{3}$$

But only now can we connect $\frac{-2}{3}$ to $-\frac{2}{3}$.

$-\frac{2}{3}$ is the opposite of $\frac{2}{3}$. That means multiplication by -1, and we will express -1 as $\frac{-1}{1}$.

$$-\frac{2}{3}=-1\cdot\frac{2}{3}=\frac{-1}{1}\cdot\frac{2}{3}=\frac{-1\cdot 2}{1\cdot 3}=\frac{-2}{3}$$ ■ (End of Proof)

The negative sign is not the only thing that can freely move in a fraction. Other factors can only move between being a factor in the numerator to being a multiplyer of a fraction.

Theorem: The algebraic expressions $\frac{3}{5}x$ and $\frac{3x}{5}$ are equivalent.

Proof. The key here is that we can re-write x as $\frac{x}{1}$. Division by one never changes the value of a number.

$$\frac{3}{5}x=\frac{3}{5}\cdot x=\frac{3}{5}\cdot\frac{x}{1}=\frac{3x}{5}.$$ ■ (End of Proof)

Units behave the same way as x does. Kilogramm is a measure of mass, 1 kilogram is about 2.2 pounds. If we measure mass in kilograms, (denoted by kg), we can have the unit either in the numerator, or after the number. For example, $\frac{2}{3}$kg is the same as $\frac{2\text{kg}}{3}$. Although these fractions express slightly different things, they are equaivalent, and we can freely move between the two forms. Inside computations we often prefer $\frac{2\text{kg}}{3}$ and we would present

the final answer as $\frac{2}{3}$kg.

Part 3 - The Reciprocal

Example 3. Perform the multiplication $\frac{3}{8} \cdot \left(2\frac{2}{3}\right)$.

Solution: Before multiplying stratight across, we re-write the mixed number as an improper fraction.

$$\frac{3}{8} \cdot \left(2\frac{2}{3}\right) = \frac{3}{8} \cdot \frac{8}{3} = \frac{24}{24} = \boxed{1}$$

> **Definition:** If the product of two numbers is 1, we call such a pair of numbers **reciprocal**s of each other. For example, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, and the reciprocal of $\frac{3}{2}$ is $\frac{2}{3}$. The reciprocal is also called the **multiplicative inverse**.

There is an obvious symmetry between $\frac{2}{3}$ and $\frac{3}{2}$. It is easy to see why those two fractions would multiply to 1. This is because the product of them is

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1.$$

So, we simply flip a fraction upside down to get to its reciprocal. However, we should always remember that the definition of the reciprocal of a number x is another number so that their product is 1.

Example 4. Find the reciprocal for each of the following.

a) $\frac{2}{5}$ b) $-\frac{3}{7}$ c) 2 d) x e) $1\frac{3}{5}$

Solution: a) To find the reciprocal of $\frac{2}{5}$, we need to flip it upside down. So the reciprocal of $\frac{2}{5}$ is $\boxed{\frac{5}{2}}$. Indeed, the product of these two fractions is $\frac{2}{5} \cdot \frac{5}{2} = \frac{10}{10} = 1$.

b) How does the flipping work with the negative sign? The short story is that the reciprocal of $-\frac{3}{7}$ is $\boxed{-\frac{7}{3}}$.

The long story is that we can re-write $-\frac{3}{7}$ as $\frac{-3}{7}$. Then we flip for the reciprocal: we get from $\frac{-3}{7}$ to $\frac{7}{-3}$. Negative signs are not acceptable in the denominator, so we immediately lift it up to the numerator. (Just mutiply numerator and denominator by -1.) So the reciprocal of $\frac{-3}{7}$ is $\frac{7}{-3}$, which is $\frac{-7}{3}$. That's the same as $-\frac{7}{3}$.

The reciprocal of $-\frac{3}{7}$ is the flip of $\frac{-3}{7}$. That is $\frac{7}{-3} = \frac{-7}{3} = \boxed{-\frac{7}{3}}$

This algebraic gymnastics with the negative sign works, but we can also think of reciprocals in terms of the definition. A number and its reciprocal multiply to 1. That's a positive product. Therefore, the reciprocal of a negative number is negative so that the product can be positive. For the reciprocal of $-\frac{3}{7}$, we do two things: flip the fraction $\frac{3}{7}$ and decide that the reciprocal of $-\frac{3}{7}$ must be negative. The reciprocal is then $-\frac{7}{3}$.

c) How do we take the reciprocal of something that is not necessarily a fraction? We can re-write 2 as $\frac{2}{1}$ and then we can flip. The reciprocal of 2 is the same as the reciprocal of $\frac{2}{1}$, which is $\boxed{\frac{1}{2}}$.

d) Let x be any non-zero number. We re-write it as $\frac{x}{1}$ and flip. So the reciprocal of x is $\boxed{\frac{1}{x}}$. This expression is meaningful as long as x is not zero. Therefore, every non-zero number has a unique reciprocal and is denoted by $\frac{1}{x}$. To check if x and $\frac{1}{x}$ are really reciprocals, we use the definition. The product of a number and its reciprocalis 1. Indeed,

$$x \cdot \frac{1}{x} = \frac{x}{1} \cdot \frac{1}{x} = \frac{x}{x} = 1 \qquad \text{This is true as long as } x \text{ is not zero.}$$

e) We can only find the reciprocal of a mixed number by converting it to an improper fraction and then flip it. $1\frac{3}{5} = \frac{8}{5}$, so

$$\text{the reciprocal of } 1\frac{3}{5} \text{ is the reciprocal of } \frac{8}{5}, \text{ which is } \boxed{\frac{5}{8}}.$$

Notice that separately flipping the integer- and fraction parts in a mixed number does not pruduce the right results: $1\frac{3}{5}$ and $1\frac{5}{3}$ are not reciprocals. Their product is

$$1\frac{3}{5} \cdot 1\frac{5}{3} = \frac{8}{5} \cdot \frac{8}{3} = \frac{64}{15} \text{ and not } 1.$$

To get the reciprocal, we *have to* convert mixed numbers to improper fractions.

Practice Problems

1. Perform the indicated operations. Present your answer as an integer or a reduced fraction. You do not need to convert improper numbers to mixed numbers.

a) $-\frac{8}{15} \cdot \frac{3}{4}$

b) $\left(-\frac{1}{2}\right)^2$

c) $\frac{4}{15} \cdot \left(2\frac{1}{7}\right)$

d) $-\left(-\frac{2}{3}\right)^2$

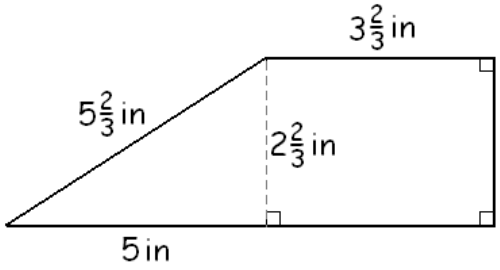
e) $\frac{1}{6} - \frac{8}{25}\left(-1\frac{2}{3}\right)$

f) $\frac{1}{6} - 2\left(-\frac{1}{2}\right)^2$

g) $-\frac{5}{6} - \left(3 - \frac{2}{5}\right)\left(-\frac{1}{2}\right)$

h) $\left(4 - \frac{2}{3}\right) \cdot \frac{2}{5}$

2. Find the perimeter and area of the object shown on the picture.



3. Find the reciprocal of each of the following.

   a) $\frac{2}{5}$ b) $3\frac{1}{3}$ c) $-\frac{5}{7}$ d) 4

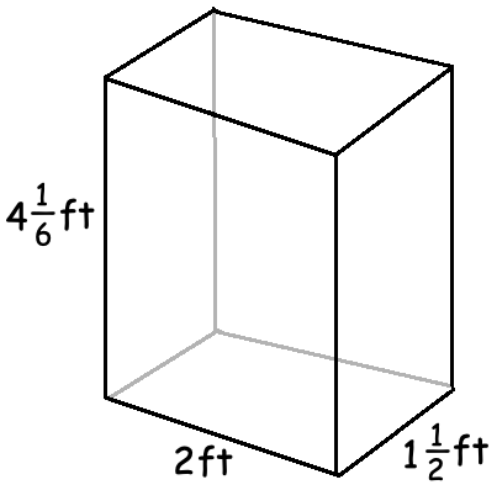
4. Evaluate the expression $x^2 - 6x - 1$ if

   a) $x = -\frac{1}{2}$ b) $x = \frac{5}{2}$ c) $x = -\frac{2}{3}$

5. Evaluate the expression $-2x^2 + x + 3$ if

   a) $x = \frac{1}{2}$ b) $x = \frac{3}{2}$ c) $x = -\frac{2}{3}$ d) $x = -\frac{5}{6}$

6. Compute the volume of the rectangular prism shown on the picture.



Enrichment

1. (Enrichment) Contributed by Prof. Abdallah Shuaibi.

   Two travelers meet a third one, who is very hungry. He offers the two travelers 8 dollars for a meal. One traveler has three pieces of bread, the other one has five. So the hungry man gives them the 8 dollars, they all sit down and eat all 8 pieces of bread together. Afterwards, the two get into an argument about how to divide up the money. The one who contributed 5 pieces of bread wants to split it to 5 and 3. The other wants to divide the money evenly, 4 and 4. They go to a wise man for advice. They tell him their story and ask him to divide the money between them. The wise man gives the man who had 3 pieces of bread 1 dollar and 7 to the man with 5 pieces of bread. Is this a just or even reasonable decision?

10.2 Conversion Factors

In physics, in some applications of mathematics, and even in every day life, we dealing with numbers with units. For example, length can not be determined or communicated by a number only. 1 mile is much much longer than 1 inch. Also, 5 minutes is much shorter than 5 years.

The cancellation of factors in numerator and denominator of fractions can be used to convert a quantity from one unit to another one. Recall that we never change the value of any number if we multiply it by 1. For this reason, 1 is sometimes called the **multiplicative identity**.

Consider now the statement 1 hour$=$ 60 minutes. In physics, hour is denoted by h and minutes by min. If we divide any non-zero quantity by itself, the result is 1. Thus, using the equality $1\text{h} = 60\text{min}$, we can write two fractions, both of value 1.

$$\frac{1\text{h}}{60\text{min}} \qquad \text{and} \qquad \frac{60\text{min}}{1\text{h}}$$

These two fractions are called **conversion factors** or unit multiplyers, and we use them to converting time measurements from minutes to hours or backward, from hours to minutes.

Example 1. Convert $\frac{7}{12}$ hours to minutes.

Solution: We will re-write $\frac{7}{12}$ hours as $\frac{7\text{h}}{12}$ and multiply it by one of the conversion factors shown above. Since we would like to get rid of *hours*, we will select the conversion factor that has *hour* in its denominator.

$$\frac{7}{12}\text{h} = \frac{7\text{h}}{12} \cdot 1 = \frac{7\text{h}}{12} \cdot \frac{60\text{min}}{1\text{h}}$$

Now the *hour* unit is a factor in both numerator and denominator, so we can cancel it out, and we will be left with just minutes in the numerator.

$$\frac{7\cancel{\text{h}}}{12} \cdot \frac{60\text{min}}{1\cancel{\text{h}}} = \frac{7 \cdot 60\text{min}}{12} = \frac{7 \cdot (5 \cdot 12)\,\text{min}}{12} = \frac{35\text{min}}{1} = \boxed{35\text{min}}$$

Example 2. Convert 9000 minutes to hours.

Solution: We first re-write 9000 minutes as a fraction, and keep the unit with the denominator. If needed, we can always write a denominator 1. Then we multiply it by 1.

$$9000\text{ min} = \frac{9000\text{ min}}{1} = \frac{9000\text{ min}}{1} \cdot 1$$

The fraction expressing 1 is going to be the conversion factor that has minutes in its denominator. The minutes unit is cancelled out, and we are left with hours instead.

$$9000\text{ min} = \frac{9000\text{ min}}{1} \cdot \frac{1\text{h}}{60\text{min}} = \frac{9000\ \cancel{\text{min}}}{1} \cdot \frac{1\text{h}}{60\cancel{\text{min}}} = \frac{9000\text{h}}{60} = \frac{150\text{h}}{1} = \boxed{150\text{ h}}$$

Conversion factors can also be used in groups.

Example 3. Convert 10 years to seconds.

Solution: We will go from years to days, then from days to hours, then to minutes and finally to seconds. The entire computation can be done in a single line. Let us assume that a year is 365 days long.

$$10\text{y} = \frac{10\text{y}}{1} = \frac{10\cancel{\text{y}}}{1} \cdot \frac{365\cancel{\text{d}}}{1\cancel{\text{y}}} \cdot \frac{24\cancel{\text{h}}}{1\cancel{\text{d}}} \cdot \frac{60\text{min}}{1\cancel{\text{h}}} \cdot \frac{60\text{s}}{1\text{min}} = \frac{10 \cdot 365 \cdot 24 \cdot 60 \cdot 60\text{s}}{1} = \boxed{315\,360\,000\text{s}}$$

Example 4. Convert 5 square-yards to square-feet.

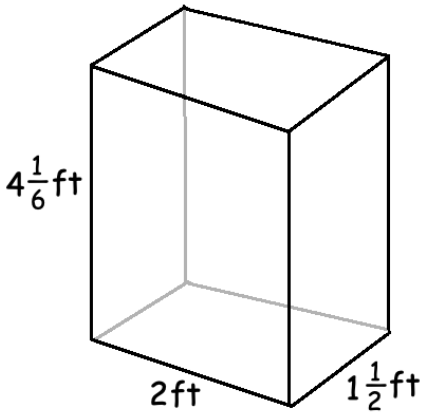
Solution: The unit of the area, square-yards is the product of yard with itself. Therefore, we will use the conversion factor from yard to feet twice.

$$5\text{y}^2 = \frac{5\text{y}}{1} \cdot \frac{1\text{y}}{1} = \frac{5\cancel{\text{y}}}{1} \cdot \frac{3\text{ft}}{1\cancel{\text{y}}} \cdot \frac{1\cancel{\text{y}}}{1} \cdot \frac{3\text{ft}}{1\cancel{\text{y}}} = \frac{45\text{ft}^2}{1}$$

Practice Problems

Perform each of the following conversions.

1. 210 hours to weeks
2. $\frac{3}{28}$ weeks to minutes
3. 6000 inches to feet, given that $12\text{in} = 1\text{ft}$
4. 900 000 square-inches to square-feet. (Hint: $1\text{ft}^2 = 1\text{ft} \cdot 1\text{ft}$)
5. $81\frac{\text{mi}}{\text{h}}$ (miles per hour) to meter per second $\left(\frac{\text{m}}{\text{s}}\right)$. Hint: $1\text{mi} \approx 1600\text{m}$
6. Compute the volume of the given rectangular prism in cubic inches.



Chapter 11

11.1 Dividing Fractions

As it turns out, we never divide fractions. Every non-zero number has a reciprocal. We interpret division as multiplication by the reciprocal.

Theorem: To divide is to multiply by the reciprocal.

Proof. Let x be any non-zero number. Let us multiply y by $\frac{1}{x}$.

$$y \cdot \frac{1}{x} = \frac{y}{1} \cdot \frac{1}{x} = \frac{y \cdot 1}{1 \cdot x} = \frac{y}{x}$$ is the same as division by x. ■

Example 1. Perform the divisions of the fractions as indicated.

a) $\frac{5}{6} \div \frac{2}{3}$ b) $\frac{2}{5} \div (-4)$ c) $4\frac{1}{5} \div 1\frac{1}{6}$ d) $-2 \div \left(-\frac{3}{8}\right)$

Solution: a) To divide is to multiply by the reciprocal. Instead of dividing by $\frac{2}{3}$, we will multiply by its reciprocal, $\frac{3}{2}$.

$$\frac{5}{6} \div \frac{2}{3} = \frac{5}{6} \cdot \frac{3}{2} = \frac{5 \cdot 3}{6 \cdot 2} = \frac{5 \cdot -3}{2 \cdot -3 \cdot 2} = \boxed{\frac{5}{4}}$$

b) To divide is to multiply by the reciprocal. Instead of dividing by -4, we multiply by its reciprocal. $-4 = \frac{-4}{1}$, therefore, its reciprocal is $\frac{1}{-4}$. However, we immediately lift the negative sign and as always, we keep it in the numerator. Thus, the reciprocal of -4 is $-\frac{1}{4}$ and we will write it as $\frac{-1}{4}$.

$$\frac{2}{5} \div (-4) = \frac{2}{5} \cdot \frac{-1}{4} = \frac{-2}{20} = \boxed{-\frac{1}{10}}$$

c) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{3}{7} - \frac{9}{7} = \frac{3-9}{7} = \boxed{\frac{-6}{7} \text{ or } -\frac{6}{7}}$$

d) We can not take the reciprocal of a mixed number. Therefore, we immediately convert each mixed number to an improper fraction, and instead of division, we multiply by the reciprocal.

$$4\frac{1}{5} \div 1\frac{1}{6} = \frac{21}{5} \div \frac{7}{6} = \frac{21}{5} \cdot \frac{6}{7} = \frac{(-7 \cdot 3) \cdot 6}{5 \cdot -7} = \boxed{\frac{18}{5}}$$

Example 2. a) Perform the division $10 \div \frac{1}{4}$ b) How many quarters are there in a $10-$dollar roll of quarters?

Solution: a) To divide is to multiply by the reciprocal. The reciprocal of $\frac{1}{4}$ is $\frac{4}{1}$ or 4.

$$10 \div \frac{1}{4} = 10 \cdot 4 = \boxed{40}$$

b) Each dollar can be exchanged for four quarters, so ten dollars would be the same as ten times four, or forty quarters.

From this last example we see that division still has its same old meaning: how many times can we fit $\frac{1}{4}$ into 10?

Example 3. Simplify the given expression: $\dfrac{\frac{2}{3}+\frac{3}{4}}{\frac{5}{6}-\frac{1}{3}}$

Solution: Recall that if a fraction bar (or division bar) is stretching under entire expressions, then it also serves as (invisible) parentheses. Thus

$$\frac{\frac{2}{3}+\frac{3}{4}}{\frac{5}{6}-\frac{1}{3}} \text{ is the same as } \left(\frac{2}{3}+\frac{3}{4}\right) \div \left(\frac{5}{6}-\frac{1}{3}\right)$$

Therefore, we will first perform the addition in the numerator and the subtraction in the denominator. We will finally divide. For additions and subtractions, we need to use a common denominator.

$$\frac{\frac{2}{3}+\frac{3}{4}}{\frac{5}{6}-\frac{1}{3}} = \frac{\frac{8}{12}+\frac{9}{12}}{\frac{5}{6}-\frac{2}{6}} = \frac{\frac{8+9}{12}}{\frac{5-2}{6}} = \frac{\frac{17}{12}}{\frac{3}{6}} = \frac{17}{12} \cdot \frac{6}{3} = \frac{17 \cdot 6}{(6 \cdot 2) \cdot 3} = \frac{17}{2 \cdot 3} = \boxed{\frac{17}{6}}$$

Example 4. Perform each of the given divisions. a) $\dfrac{\frac{2}{3}}{5}$ b) $\dfrac{2}{\frac{3}{5}}$

Solution: a) If needed, we can always re-write an integer with a denominator of 1.

$$\frac{\frac{2}{3}}{5} = \frac{\frac{2}{3}}{\frac{5}{1}} = \frac{2}{3} \cdot \frac{1}{5} = \boxed{\frac{17}{6}}$$

b) This time we will need to re-write 2 as a fraction. $\dfrac{2}{\frac{3}{5}} = \dfrac{\frac{2}{1}}{\frac{3}{5}} = \dfrac{2}{1}\cdot\dfrac{5}{3} = \boxed{\dfrac{10}{3}}$.

As the previous example shows, $\dfrac{\frac{a}{b}}{c}$ and $\dfrac{a}{\frac{b}{c}}$ are different expressions, with different values. Because of this, we have to be careful with expressions with more than one fraction bar (or division bar). Order of operations must be clearly indicated by the different sizes of the fraction bars. A fraction or division problem with more than one fraction bar is called a **complex fraction.** Complex fractions can always be simplified to a form with just one fraction bar.

Example 5. Simplify each of the given expressions. a) $\dfrac{\frac{a}{b}}{c}$ b) $\dfrac{a}{\frac{b}{c}}$

Solution: a) In this case, we need to re-write c as $\frac{c}{1}$. Then we multiply by the reciprocal.

$$\frac{\frac{a}{b}}{c} = \frac{\frac{a}{b}}{\frac{c}{1}} = \frac{a}{b}\cdot\frac{1}{c} = \boxed{\frac{a}{bc}}$$

b) In this case, we need to re-write a as $\frac{a}{1}$. Then we multiply by the reciprocal.

$$\frac{a}{\frac{b}{c}} = \frac{\frac{a}{1}}{\frac{b}{c}} = \frac{a}{1}\cdot\frac{c}{b} = \boxed{\frac{ac}{b}}$$

Practice Problems

Perform the indicated operations. Present your answer as an integer or a reduced fraction. You do not need to convert improper numbers to mixed numbers.

1. $-\dfrac{8}{15}\div\left(\dfrac{6}{25}\right)$

2. $\left(-\dfrac{1}{2}\right)^2 - \dfrac{2}{5}\div\dfrac{3}{10}$

3. $\dfrac{\frac{3}{8}-\frac{2}{3}}{\frac{1}{2}-\frac{5}{6}}$

4. $\dfrac{1}{6}-\dfrac{8}{15}\cdot\dfrac{1}{4}\div\left(-\dfrac{2}{5}\right)$

5. $\dfrac{-3\div\frac{2}{5}+3}{\left(1\frac{1}{2}\right)^2}$

6. $\left(-2\dfrac{1}{3}\right)\div\left(-1\dfrac{2}{3}\right)^2$

7. $\dfrac{2}{7}-\dfrac{2}{3}\div\left(1\dfrac{2}{5}\right)$

8. $\dfrac{2}{7}\div\dfrac{2}{3}-\left(1\dfrac{2}{5}\right)$

9. $\left(3\dfrac{1}{3}\right)\div\left(2\dfrac{1}{2}\right)$

10. Evaluate the expression $\dfrac{-2x^2+x+3}{2x-3}$

a) $x=\dfrac{1}{2}$ b) $x=\dfrac{3}{2}$ c) $x=-\dfrac{2}{3}$ d) $x=-\dfrac{5}{6}$

11.2 Introduction to Ratios

Before the digital age, pictures were developed into photographs. A standard photograph would be a 3 by 5, meaning that the horizontal longer side is 5 inches long, and the vertical side is 3 inches long.

Suppose we want to enlarge a standard photograph. If we just increase one side but not the other, or just randomly increase the two sides, the image becomes distorted. The only way to enlarge the picture and preserve the image on it is by multiplying both sides by the same number. For example, we can double both sides: the horizontal side would become 10 inches and the vertical side would become 6 inches long. We can 'triple' the image, for sides 15 inches and 9 inches long. In either case, the ratio of the horizontal side to the vertical side remains the same.

Definition: A **ratio** is a comparison of two quantities by division.

In this case, the ratio of the vertical side to the horizontal side should remain the same. 3 to 5, or 6 to 10, or 9 to 15. A ratio can be written in several different ways:

$$3 \text{ to } 5 \quad \text{or} \quad 3:5 \quad \text{or} \quad \frac{3}{5} \quad \text{or } 3 \div 5.$$

Example 1. The ratio 2 to 7 can be written as $\frac{2}{7}$ or $2:7$ or $2 \div 7$.

Example 2. The ratio 20 to 15 can be written as $\frac{20}{15}$. We can reduce this fraction to lowest terms as $\frac{4}{3}$. Therefore, the ratio 20 to 15 is the same as 4 to 3.

Example 3. The computer lab at our school had 42 PCs and 35 Macintosh. Express and simplify each of the given ratios. a) the ratio of PCs to Macs b) the ratio of Macs to PCs

Solution: a) The ratio of PCs to Macs is 42 to 35. As a fraction, that is $\frac{42}{35}$. We can reduce this fraction and get $\frac{6}{5}$ or 6 to 5.

b) The ratio of Macs to PCs is 35 to 42. As a fraction, that is $\frac{35}{42}$. We can reduce this fraction and get $\frac{5}{6}$ or 5 to 6.

Example 4. Simplify the ratio $\frac{3}{7}$ to $\frac{4}{7}$.

Solution: We can write the ratio as a fraction and simplify.

$$\frac{\frac{3}{7}}{\frac{4}{7}}$$ to divide is to multiply by the reciprocal.

$$\frac{\frac{3}{7}}{\frac{4}{7}} = \frac{3}{7}\cdot\frac{7}{4} = \frac{3}{4}$$ or 3 to 4.

Example 5. Simplify the ratio $3\frac{1}{3}$ to $2\frac{1}{2}$.

Solution: We can write the ratio as a fraction and simplify. Before we do that, we need to re-write the mixed numbers as improper fractions.

$$\frac{3\frac{1}{3}}{2\frac{1}{2}}$$ to divide is to multiply by the reciprocal.

$$\frac{\frac{10}{3}}{\frac{5}{2}} = \frac{10}{3}\cdot\frac{2}{5} = \frac{4}{3}$$ or 4 to 3.

Example 6. Express and simplify the ratio $4\frac{1}{6}$ hours to 100 minutes.

Solution: We will use the fact that 1 hour is 60 minutes and convert from hours to minutes. We will use a conversion factor.

$$4\frac{1}{6}\text{ hour} = \frac{25\,\cancel{\text{hr}}}{6}\cdot\frac{60\text{ min}}{1\,\cancel{\text{hr}}} = 250\text{ minutes}$$

Then the ratio as a fraction is $\frac{250\text{ min}}{100\text{ min}} = \frac{25}{10} = \frac{5}{2}$. Thus the ratio is 5 to 2.

Example 7. Express and simplify the ratio $1\frac{1}{4}$ feet to 10 inches.

Solution: We will use the fact that 1 feet is 12 inches and convert from feet to inches. We will use a conversion factor.

$$1\frac{1}{4}\text{ft} = \frac{5\text{ft}}{4} = \frac{5\,\cancel{\text{ft}}}{4}\cdot\frac{12\text{in}}{1\,\cancel{\text{ft}}} = 15\text{in}$$

Then the ratio as a fraction is $\frac{15\text{in}}{10\text{in}} = \frac{3}{2}$ Thus, the reduced ratio is $\frac{3}{2}$.

In many, many, many aspects of our life, quantities related to each other have naturally constant ratios. For example, the quantity of a product we purchase and the amount of money we pay for them. If 2 pounds of rice costs 3 dollars, then the ratio of the money we pay and the weight of the rice is constant.

quantity	money we pay
2 pounds	3 dollars
4 pounds	6 dollars
20 pounds	30 dollars

We call that constant ratio the unit price: $\dfrac{3 \text{ dollar}}{2 \text{ pounds}} = 1\dfrac{1}{2}$ dollars per pound

Example 8. According to a survey, four out of five people when asked, claims to have a hobby.

a) If we asked 1000 people, how many claimed to have a hobby?

b) If 1000 people claimed to have a hobby, how many people were asked as part of the survey?

Solution: a) The ratio of people with a hobby and the people asked is constant, 4 to 5. Let us denote the number of people with a hobby by x. Then the ratio can be expressed in two different ways, giving us an equation.

$$\frac{4}{5} = \frac{x}{1000}$$

There are several ways to solve this equation. We will treat it as a one-step equation where x was divided by 1000. So we isolate x by multiplying both sides by 1000.

$$x = \frac{4}{5} \cdot 1000 = \frac{4000}{5} = 800$$

Therefore, 800 people claimed to have a hobby out of the 1000 asked. Indeed, the ratio 800 to 1000 is the same ratio as 4 to 5.

b) The ratio of the number of people with a hobby and of the people asked is constant, 4 to 5. Let us denote the number of people asked by y. Then the ratio can be expressed in two different ways, giving us an equation.

$$\frac{4}{5} = \frac{1000}{y}$$

If $A = B$, then $\dfrac{1}{A} = \dfrac{1}{B}$. One nice thing about ratios is that if we don't like the fact that the unknown is in the denominator, we can take the reciprocal of both sides. That would be the same as looking at the ratio of the number of people asked to the number of people with hobbies.

$$\frac{5}{4} = \frac{y}{1000}$$

So we isolate y by multiplying both sides by 1000.

$$x = \frac{5}{4} \cdot 1000 = \frac{5000}{4} = 1250$$

Therefore, 1250 people were asked. Indeed, the ratio 1000 to 1250 is the same ratio as 4 to 5.

Example 9. We have a recipe for cookies. The quantities described in the recipe are for 6 servings, but we want to bake for only four people. The recipe calls for $1\frac{2}{3}$ cups of flour and $\frac{3}{4}$ cups of sugar. How much flour and sugar should we use?

Solution: The constant ratio in this situation is the amount used to the number of people. These ratios can be expressed as

$$\text{for flour: } \frac{1\frac{2}{3}\text{ cups}}{6\text{ people}} \text{ or } \frac{1\frac{2}{3}}{6} \quad \text{and} \quad \frac{\frac{3}{4}\text{ cups}}{6\text{ people}} \text{ or } \frac{\frac{3}{4}}{6} \text{ for sugar.}$$

Let us denote the amount of flour we need for 4 people by x. The constant ratio then can be expressed in two different ways, giving us an equation in x.

$$\frac{1\frac{2}{3}\text{ cups}}{6} = \frac{x}{4}$$

We can solve this equation by multiplying both sides by 4.

$$\frac{1\frac{2}{3}\text{ cups}}{6} \cdot 4 = x$$

So we get that

$$x = \frac{1\frac{2}{3}\text{ cups}}{6} \cdot 4 = \frac{4 \cdot 1\frac{2}{3}\text{cups}}{6} = \frac{4 \cdot \frac{5}{3}}{6}\text{ cups} = \frac{\frac{20}{3}}{\frac{6}{1}}\text{ cups} = \frac{20}{3} \cdot \frac{1}{6}\text{ cups} = \frac{10}{9}\text{ cup} = 1\frac{1}{9}\text{ cups}$$

Similarly, if we denote the amount of flour we need for 4 people by y. The constant ratio then can be expressed in two different ways, giving us an equation in y.

$$\frac{\frac{3}{4}\text{ cups}}{6} = \frac{y}{4}$$

We can solve this equation by multiplying both sides by 4.

$$\frac{\frac{3}{4}\text{ cups}}{6} \cdot 4 = y$$

So we get that $y = \frac{\frac{3}{4}\text{ cups}}{6} \cdot 4 = \frac{4 \cdot \frac{3}{4}\text{cups}}{6} = \frac{3}{6}\text{ cups} = \frac{1}{2}\text{ cup}$

Practice Problems

1. Re-write each of the given ratios as a reduced fraction.

   a) 8 to 18

   b) 18 to 8

   c) 5 to 15

   d) $\frac{2}{3}$ to $\frac{5}{6}$

   e) 60% to $\frac{3}{4}$

   f) 14 inches to one feet

   g) 300 minutes to three hours

   h) 30 nickels to 5 dollars

2. The party was a great success. There were 120 women and 100 men in attendance. Find each of the given ratios.

   a) the number of men to the number of women

   b) the number of men to the number of guests

   c) the number of women to the number of guests

3. There is no student at our school who takes both French and Spanish. For every four students studying French at our school, there are 14 students who take Spanish

   a) If 378 students are taking French, how many are taking Spanish?

   b) If 378 students are taking Spanish, how many are taking French?

   c) If all together 378 students are taking French or Spanish, how many are taking French? And Spanish? (Hint: if 4 students take French and 14 take Spanish, then 4 out 18 students are taking French.)

4. On our map, a distance of 3 inches represents 500 miles.

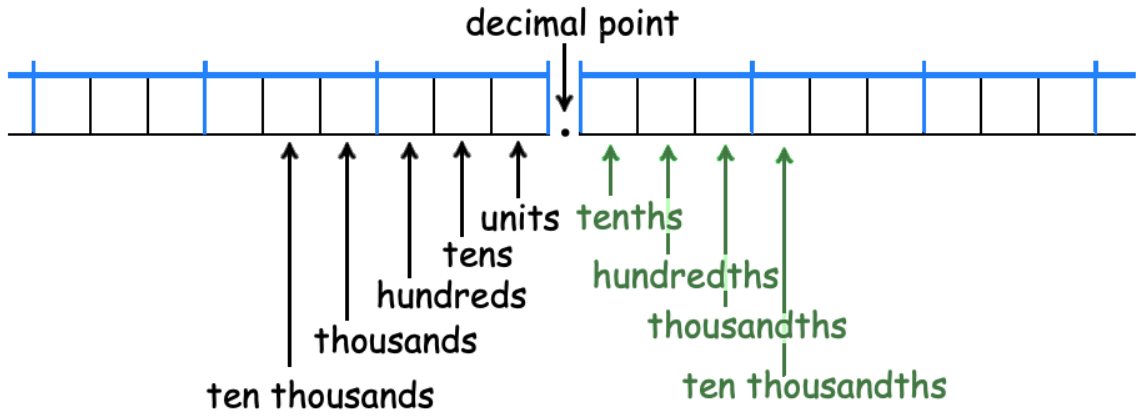
   a) What distance is denoted by $4\frac{1}{5}$ inches?

   b) If two cities are 125 miles from each other, what is their distance on the map?

Chapter 12

12.1 Introduction to Decimals

Decimals are standardized fractions with denominators 10, 100, 1000, etc. We express decimals by extending our system using place values and digits. The numerators are the digits, and the denominators are indicated by the place values.



1 is pronounced one

10 is pronounced hundred

1000 is pronounced one thousand

10 000 is pronounced ten thosuand

100 000 is pronounced hundred thousand

1 000 000 is pronounced one million

0.1 is pronounced one-tenth

0.01 is pronounced one-hundredth

0.001 is pronounced one-thousandth

0.0001 is pronounced one ten-thousandth

0.000 01 is pronounced one hundred-thousandth

0.000 001 is pronounced one-millionth

Example 1. Consider the decimal 35.702.

a) Express the number using words. b) Re-write the number as a fraction of two integers.

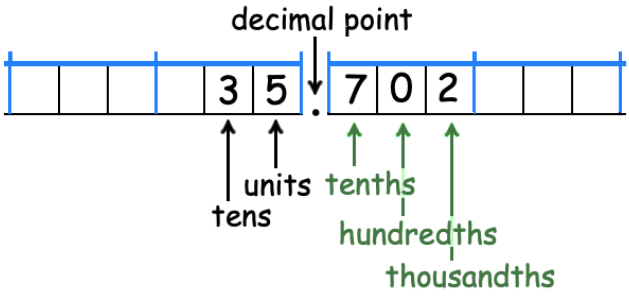
Solution: a) We can always read out a number following the steps shown below.

Step 1. Read out the number until the decimal point.

Step 2. For the decimal point, we say "and"

Step 3. After the decimal point, we read out the number as if it was a regular decimal with regular place values.

Step 4. We name the place value of the last digit.



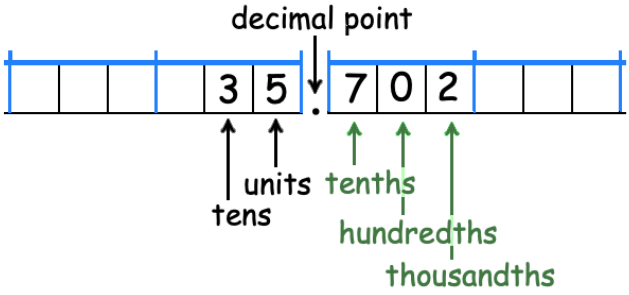
Step 1	Step 2	Step 3	Step 4
thirty-five	and	seven hundred two	one-thousandths

So the decimal is pronounced thirty-five and seven hundred two one-thousandths.

b) If we can read out the number, we have automatically converted it into a mixed number.

thirty-five and seven hundred two one-thousandths can be written as $35\frac{702}{1000}$ or $35+\frac{702}{1000}$

Note that we get the same correct result if we interpret the decimal digit by digit and then add the fractions.



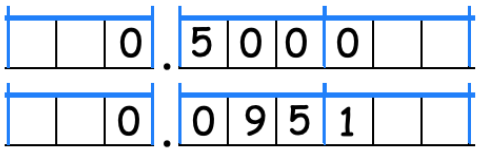
$$35+\frac{7}{10}+\frac{0}{100}+\frac{2}{1000}=35+\frac{700+0+2}{1000}=35+\frac{702}{100}$$

Example 2. Compare the decimals 0.5 and 0.0951

Solution: The number 951 is greater than 5 but that would be the incorrect comparison, because we need to consider the place values. We can re-write both numbers as fractions and compare them.

0.5 can be read as zero and five tenths, or $\frac{5}{10}$. The other decimal, 0.0951 can be read as zero and nine hundred fifty-one ten-thousandths, or as $\frac{951}{10\,000}$. To compare the decimals, we need to bring them to the common denominator: Compare $\frac{5000}{10\,000}$ and $\frac{951}{10\,000}$. 5000 is greater than 951 and so 0.5 is greater than 0.0951.

We can compare decimals in their decimal form as well, but we need to consider the place values. We can do that by making both numbers 'equally long':



Again, we are comparing 5000 and 951.

Rounding Decimals

The basic principle and technique of rounding decimals are the same as it is with whole numbers. After identifying the place value to be rounded to, we round down if the next digit is 4 or less, and we round up if the next digit is five or more.

Example 3. Consider the number 628.4073096. Round this number to the nearest

a) hundreds b) tens c) units d) one-tenths e) one-hundredths f) one-thousandths g) ten-thousandths

Solution: a) Our number, 628.4073096 is between 600 and 700. To correctly round, we need to figure out which one is nearest to our number. After the hundreds digit we see the digit 2. This digit is four or less, there fore, we round down and the answer is 600.

b) Our number, 628.4073096 is between 620 and 630. To correctly round, we need to figure out which one is nearest to our number. After the tens digit we see the digit 8. This digit is five or more, there fore, we round up and the answer is 630. Indeed, 628.4073096 is closer to 630 than to 620.

c) Our number, 628.4073096 is between 628 and 629. To correctly round, we need to figure out which one is nearest to our number. After the units digit we see the digit 4. This digit is four or less, there fore, we round down and the answer is 628. Indeed, 628.4073096 is closer to 628 than to 629.

d) Our number, 628.4073096 is between 628.4 and 628,5. To correctly round, we need to figure out which one is nearest to our number. After the one-tenths digit we see the digit 0. This digit is four or less, there fore, we round down and the answer is 628.4 Indeed, 628.4073096 is closer to 628.4 than to 628.5.

e) Our number, 628.4073096 is between 628.40 and 628,41. To correctly round, we need to figure out which one is nearest to our number. After the one-hundredths digit we see the digit 7. This digit is five or more, there fore, we round up and the answer is 628.41 Indeed, 628.4073096 is closer to 628.41 than to 628.40.

f) Our number, 628.4073096 is between 628.407 and 628,408. To correctly round, we need to figure out which one is nearest to our number. After the one-thousandths digit we see the digit 3. This digit is four or less, there fore, we round down and the answer is 628.407 Indeed, 628.4073096 is closer to 628.407 than to 628.408.

g) Our number, 628.4073096 is between 628.4073 and 628,4074. To correctly round, we need to figure out which one is nearest to our number. After the ten–thousandths digit we see the digit 0. This digit is four or less, there fore, we round down and the answer is 628.4073 Indeed, 628.4073096 is closer to 628.4073 than to 628.4074.

Practice Problems

1. Write each of the given numbers using digits.
   a) twenty-eight and three thousand two hundred thirteen millionths
   b) eighyty thousand four and four hundred-thousandths
   c) forty-two and four thousand twenty-five ten-thousandths
2. Write each of the given decimals using letters.
   a) 0.25 b) 21.8005 c) 2.0021 d) 100.000301 e) 195.61023
3. Write each of the given decimals as a reduced fraction.
   a) 0.5 b) 0.32 c) 0.025
4. Round 3802.75428 to the

   a) nearest thousands
   b) nearest hundreds
   c) nearest tens
   d) nearest units
   e) nearest one-tenths
   f) nearest one-hundredths
   g) nearest one-thousandths
   h) nearest ten-thousandths

12.2 Basic Percent Problems

Consider the following problem: Find $\frac{3}{5}$ of 100.

Solution: $\frac{1}{5}$ of 100 can be obtained by splitting 100 into 5 equal shares. If we think of money, this is an easy task: 5 twenty-dollar bill make up a hundred dollar bill. In short, $\frac{1}{5}$ of 100 is 20.

We need $\frac{3}{5}$ of 100, which means that we need to take 3 shares out of the 5. This is $3(20) = 60$. Thus, $\frac{3}{5}$ of 100 is 60.

Notice that we obtain the same result if we simply multiply $\frac{3}{5}$ by 100.

$$\frac{3}{5} \cdot 100 = \frac{3}{5} \cdot \frac{100}{1} = \frac{300}{5} = 60$$

We will establish a language we will use to solve word problems involving fractions. Consider the statement

$$\frac{3}{5} \text{ of } 100 \text{ is } 60.$$

These three quantities are always present in word problems involving fractions. The following definitions are not very elegant but seem to be useful. The fraction $\frac{3}{5}$ will be called F for fraction. The quantity we are splitting up (in this case, 100) will be called the (of) number, because the word *of* is always near it. The result will be called the (is) number, since the word *is* is always near it. Then all word problems involving fractions can be solved using one formula,

$$\boxed{(\text{is}) = (\text{Fraction}) \cdot (\text{of})}$$

In the following problems we are given two quantities, and are to find the third one. Consequently, there are three types of word problems involving fractions. We will call a problem type 1 if we have to find the (is) number, type 2 if we have to find F (the fraction), and type 3 if we have to find the (of) number.

Example 1. Find $\frac{2}{9}$ of 450.

Solution: The formula we use is $\boxed{(\text{is}) = (\text{Fraction}) \cdot (\text{of})}$. First we identify which two quantities are given. We will denote the third one by x.

$(\text{is}) = x$
$F = \frac{2}{9}$
$(\text{of}) = 450$

We will substitute these into the formula and solve for x. The fraction and the (of) are given, and so

$$(\text{is}) = F \cdot (\text{of}) \quad \text{becomes} \quad x = \frac{2}{9} \cdot 450 = \frac{900}{9} = 100$$

So the answer is $\boxed{100}$.

Because the fraction and the (of) numbers were given and the (is) number was to be found, this problem is type 1.

Example 2. 75 is what fraction of 400?

Solution: The formula we use is $\boxed{(\text{is}) = (\text{Fraction}) \cdot (\text{of})}$. We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x. In this case,

$(\text{is}) = 75$
$F = x$
$(\text{of}) = 400$

We will substitute these into the formula and solve for x.

$$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 75 &= x \cdot 400 && \text{Solve for } x \text{ by dividing both sides by 400} \\ \frac{75}{400} &= x && \text{simplify the result} \\ x &= \frac{3}{16} \end{aligned}$$

The answer is $\boxed{\frac{3}{16}}$. We check by computing $\frac{3}{16}$ of 400. It is indeed 75, and so our solution is correct.

Because the (is) and the (of) numbers were given and the fraction was to be found, this problem is type 2.

Example 3. $\frac{4}{11}$ of a number is 36. Find this number.

Solution: The formula we use is $\boxed{(\text{is}) = (\text{Fraction}) \cdot (\text{of})}$ We first identify the two numbers given and call the missing number x.

$(\text{is}) = 36$
$F = \frac{4}{11}$
$(\text{of}) = x$

The fraction and the (is) are given, and so we label the (of) number as x.

$$\begin{aligned} (\text{is}) &= (\text{Fraction}) \cdot (\text{of}) \\ 36 &= \frac{4}{11} \cdot x && \text{Divide both sides by } \frac{4}{11} \\ \frac{36}{\frac{4}{11}} &= x && \text{To divide is to multiply by the reciprocal:} \\ 99 &= x && \frac{36}{\frac{4}{11}} = 36 \cdot \frac{11}{4} = 99 \end{aligned}$$

The answer is $\boxed{99}$. We check: is it true that $\frac{4}{11}$ of 99 is 36? Since $\frac{4}{11}(99) = 36$, our solution is correct.

Because the (is) number and the fraction were given and the (of) number was to be found, this problem is type 3.

We can now solve percent problems as well, because percents are fractions with denominator 100.

Example 4. Find 15% of 400.

Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x. In this case,

$(\text{is}) = x$
$F = 15\% = \dfrac{15}{100}$
$(\text{of}) = 400$

We will substitute these into the formula and solve for x.

$$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ x &= \frac{15}{100} \cdot 400 \\ x &= 60 \end{aligned}$$

Thus 15% of 400 is $\boxed{60}$. This problem is type 1.

Example 5. 21 is what percent of 350?

Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x. In this case,

$(\text{is}) = 21$
$F = x$
$(\text{of}) = 350$

We will substitute these into the formula and solve for x.

$$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 21 &= x \cdot 350 \qquad \text{divide by } 350 \\ \frac{21}{350} &= x \end{aligned}$$

We obtained the value of x, but not as a percent. We will convert x into a percent.

$$x = \frac{21}{350} = \frac{7 \cdot 3}{7 \cdot 50} = \frac{3}{50} = \frac{3 \cdot 2}{50 \cdot 2} = \frac{6}{100} = 6\%$$

Thus 21 is $\boxed{6\%}$ of 350. We can check by computing 6% of 350: $\dfrac{6}{100} \cdot 350 = 21$. Thus our solution, 6% is correct. This problem is type 2.

Example 6. 24% of what number is 72?

Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x. In this case,

$(\text{is}) = 72$
$F = 24\% = \dfrac{24}{100}$
$(\text{of}) = x$

$$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 72 &= \frac{24}{100} \cdot x \qquad \text{divide by } \frac{24}{100} \\ \frac{72}{\frac{24}{100}} &= x \\ 300 &= x \end{aligned}$$

The computation is $72 \div \dfrac{24}{100} = \dfrac{72}{1} \cdot \dfrac{100}{24} = \dfrac{3 \cdot 24}{1} \cdot \dfrac{100}{24} = \dfrac{300}{1} = 300$. Thus 72 is 24% of $\boxed{300}$.

We can check by computing 24% of 300: $\dfrac{24}{100} \cdot 300 = \dfrac{24}{100} \cdot \dfrac{300}{1} = 24 \cdot 3 = 72 = 72$. Thus our solution is correct.

We have focused on the steps of solving such equations. Computations can be simplified by reducing fractions or working with decimals instead of fractions.

Sample Problems

1. Find 16% of 3600.
2. 27 is what percent of 18?
3. 120% of what number is 150?
4. What do we get if we increase 600 by 150%?
5. We placed $ 8000 into a bank account with an annual 7% of interest rate. How much money do we have in the account a year later?
6. The population of a town has decreased from 80000 to 68000. What percent of a decrease does this represent?
7. Tom got a 4% raise in his job. Now he makes 2496 per month. How much was he making before the raise?
8. A TV set went on a 35% off sale. The sale price is $312. Find the original price.

Practice Problems

1. What fraction of $2\frac{1}{8}$ is $\frac{1}{4}$?
2. Compute 87% of 300.
3. 130% percent of what number is 78?
4. Fifteen percent of what number is 105?
5. What percent of 450 is 288?
6. What number do we get if we increase 80 by 240%?
7. What percent of 460 is 1472?
8. 347% of what number is 2429?
9. Fifteen percent of the town's population are students. If there are 1800 students living in the town, how many people live there?
10. Paul earned $128 this week in his part time job. If this was a sixty percent increase from last week, how much money did he make last week?
11. A TV went on a 14% off sale. The sale price is $516. Find the original price of the TV.
12. Overnight, the number of bacteria increased by one hundred sixty percent. There are now 650000 bacteria. How many was there yesterday?
13. We want to increase quantity Q by 2%. Which of the following is the correct expression for that?

    A) $1.2Q$ B) $1.12Q$ C) $1.02Q$ D) $3Q$ E) $2.2Q$
14. We want to increase quantity Q by 20%. Which of the following is a correct expression for that?

    A) $1.2Q$ B) $1.12Q$ C) $1.02Q$ D) $3Q$ E) $2.2Q$
15. We want to increase quantity Q by 12%. Which of the following is a correct expression for that?

    A) $1.2Q$ B) $1.12Q$ C) $1.02Q$ D) $3Q$ E) $2.2Q$

16. We want to increase quantity Q by 120%. Which of the following is a correct expression for that?

    A) $1.2Q$ B) $1.12Q$ C) $1.02Q$ D) $3Q$ E) $2.2Q$

17. We want to increase quantity Q by 200%. Which of the following is a correct expression for that?

    A) $1.2Q$ B) $1.12Q$ C) $1.02Q$ D) $3Q$ E) $2.2Q$

18. Last month, Randy was making $1800 a month in his sales job. Based on his performance, he could get a raise at any time.

    a) Two weeks ago, he got a raise of 10%. How much was he making after this increase?

    b) A week ago, he got a second raise of 10%. How much is he making now?

    c) Look at Randy's current pay and compare it to the original $1800. What percent increase does his current pay represent?

19. The population of a town has increased 20% in the 80s. During the next ten years, the town's population further grew 15%. What percent of an increase occurred during these twenty years?

20. A stock loses 60% of its value. What must the percent of increase be to recover all of its lost value? (Hint: if no value for the stock is given, make up a few different numbers.)

Chapter 13

13.1 Operations on Decimals

Part 1. Addition and Subtraction.

Addition and subtraction of decimals happen the same way as with whole numbers. In order to add the values of the corresponding placeholders, align the numbers by the decimal point. This is the same as adding fractions with common denominators.

Example 1. Perform the given additions.

a) $8.32 + 1.02$ b) $8.32 + 10.2$

Solution: We align the numbers by the decimal point. The sum will have the decimal point at the same place as the numbers to be added.

a)
$$\begin{array}{r} 8.32 \\ +\ 1.02 \\ \hline 9.34 \end{array}$$

b)
$$\begin{array}{r} 8.32 \\ +\ 10.2 \\ \hline 18.52 \end{array}$$

Note that aligning place values is the same as adding fractions with common denominators. For part a), we can re-write the numbers as fractions

$$8.32 = 8 + \frac{3}{10} + \frac{2}{100} \quad \text{and } 1.02 = 1 + \frac{0}{10} + \frac{2}{100}$$

Adding the first column is $8 + 1 = 9$, adding the second column is $\frac{3}{10} + \frac{0}{10} = \frac{3}{10}$, and the third column is $\frac{2}{100} + \frac{2}{100} = \frac{4}{100}$.

The process is the same if digits add up to more than ten, and we carry over.

Example 2. Perform the addition $37.94 + 85.27$

Solution: We align the numbers by the decimal point. The sum will have the decimal point at the same place as the numbers to be added.

$$\begin{array}{r} 37.94 \\ +\ 85.27 \\ \hline 123.21 \end{array}$$

In case of subtraction too, the procedure remains the same as with whole numbers. We align the place values by the decimal point.

Example 3. Perform the subtraction $74.26 - 28.57$

Solution: We align the numbers by the decimal point. The difference will have the decimal point at the same place as the numbers to be subtracted.

$$\begin{array}{r} 74.26 \\ -\ 28.57 \\ \hline 45.69 \end{array}$$

Part 2. Multiplying and dividing decimals by 10-powers.

Let us recall that the decimal 0.037 is the same as the fraction $\frac{37}{1000}$. Let us see what happens when we mutiply this decimal by ten-powers.

$$\frac{37}{1000} \cdot 10 = \frac{370}{1000} = \frac{37}{100} \qquad \text{Therefore, } 0.037 \cdot 10 = 0.37$$

$$\frac{37}{1000} \cdot 100 = \frac{3700}{1000} = \frac{37}{10} = 3\frac{7}{10} \qquad \text{Therefore, } 0.037 \cdot 100 = 3.7$$

$$\frac{37}{1000} \cdot 1000 = \frac{37000}{1000} = 37 \qquad \text{Therefore, } 0.037 \cdot 1000 = 37$$

$$\frac{37}{1000} \cdot 10000 = \frac{370000}{1000} = 370 \qquad \text{Therefore, } 0.037 \cdot 10000 = 370$$

To multiply a decimal by a $10-$power, we simply move the decimal point to the right. The number of decimal places to be moved is the same as the number of zeroes after the 1 in the $10-$power. So, to multiply by 1000, we move the decimal point three places to the right. If the decimal point reaches the end of the number, we add zeroes to the end.

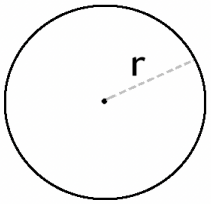
Division by $10-$powers will work similarly, only we will move the decimal to the left. C

o

Chapter 15

15.1 Circumference and Area of Circles

Definition: A circle is the set of all points in a plane that are equidistant to a fixed point. That equal distance is called the radius of the circle, that fixed point is called the center of the circle.

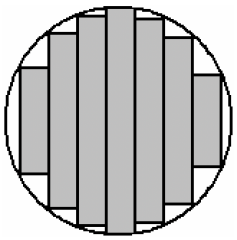


Theorem: The circumference of a circle with radius r is $C = 2\pi r$.
The area of a circle with radius r is $A = \pi r^2$.

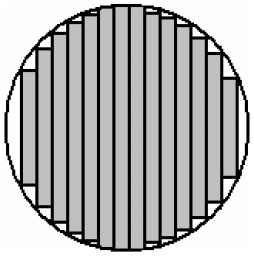
The letter π (read as 'pie') is the lower case letter 'p' in the Greek alphabet. It represents a number that is associated with the area and circumference of circles. It is a tricky number with strange and interesting properties, and an unpleasant decimal form. $\pi \approx 3.14159365....$ As it turns out, these decimals go on forever. Worse yet, the decimals do not repeat like they do in the decimal presentation of $\frac{1}{3}$. So, if we want to be precise without writing tediously long decimals, it is practical to just use the symbol π. We will refer to this symbolic presentation as the exact value, as opposed to approximate values. Writing down the character π doesn't take up a lot of space or time, and the reader can use their calculator to find the answer in a decimal form, up to the precision they need.

Most calculators have a π button. Whenever it is possible, it should be used instead of entering 3.14. The button carries a much more precise approximation.

At this level of mathematics, we do not have the tools necessary to prove these formulas. The proof actually uses (again) the formula for the area of a rectangle. The basic idea is to first find an estimation for the area using rectangles.



We can compute the grey area since it is composed of rectangles. But of course this is a crude underestimation of the actual area. The trick is that the more rectangles we use, the more accurate the approximation becomes. Using more and more rectangles to estimate the area of the semi-circle, the approximations become better and better.



Calculus offers tools to find a unique number these approximations approach. This number is defined to be the area of the circle.

Example 1. Find the circumference and area of a circle of radius 7 ft.

Solution: We apply the formulas: $C = 2\pi r$ and $A = \pi r^2$. We get better precision if we use the π button on the calculator instead of entering 3.14.

$$C = 2\pi r^2 = 2\pi\,(7\text{ ft}) = 14\pi\text{ ft} \approx 43.9823$$

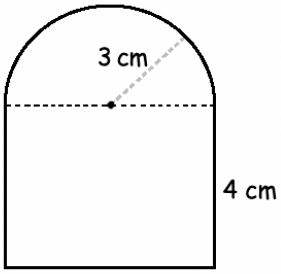
and

$$A = \pi r^2 = \pi\,(7\text{ft})^2 = 49\pi\text{ ft}^2 \approx 153.93804\text{ ft}^2$$

The exact values are 14π ft and 49π ft^2. It is nice to present these to the reader in case they will need a better precision than that in our approximate values, 43.9823 ft and 153.93804 ft^2.

Example 2. Find the area of the figure shown on the picture.

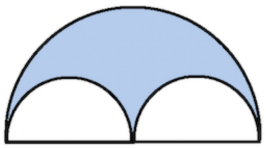
Solution: The area is the sum of the areas of a rectangle and a semi-circle. What is interesting in this problem is that we need our geometric intuition to find the horizontal side of the rectangle. Because they align, we can conclude that the horizontal side is the same as the diameter of the circle. (The diameter d is the longest line segment that can be drawn inside, in case of a circle it is simply twice the radius).



We can now apply the appropriate formulas and then add the results. Because the top is a semicircle, we need to take half of both the area and circumference of the circle. The horizontal side of the rectangle is 6 cm since we can fit exactly two radii on it.

$$\begin{aligned} A_{\text{rectangle}} &= ab = 4\text{ cm}\,(6\text{ cm}) = 24\text{ cm}^2 \\ A_{\text{semicircle}} &= \frac{\pi r^2}{2} = \frac{\pi\,(3\text{ cm})^2}{2} \approx 14.137\text{ cm}^2 \\ A &= A_{\text{rectangle}} + A_{\text{semicircle}} \approx 24\text{ cm}^2 + 14.137\text{ cm}^2 = 38.137\text{ cm}^2 \end{aligned}$$

Example 3. The picture shows three semi-circles. The smaller semi-circles both have a radius of 6 cm. Find the area of the shaded region.



Solution: The shaded area can be found by computing the area of the big semi-circle and subtracting the areas of the smaller semi-circles.

The tricky part is that the radius of the larger semi-circle is not given. We have to figure that one out from the geometric relationships shown on the picture.

The diameter of the larger semi-circle is as long as the two diameters of the smaller semi-circles. So the large semi-circle's diameter is 24 cm and thus and so its radius is 12 cm. We are now ready to compute the area.

The area of the big semi-circle is:

$$A_{\text{big}} = \frac{\pi\ (12\text{ cm})^2}{2} = \frac{144\pi\text{ cm}^2}{2} = 72\pi\text{ cm}^2 \approx 226.1947\text{ cm}^2$$

The area of each of the smaller semi-circle is:

$$A_{\text{small}} = \frac{\pi\ (6\text{ cm})^2}{2} = \frac{36\pi\text{ cm}^2}{2} = 18\pi\text{ cm}^2 \approx 56.5487\text{ cm}^2$$

The area of each of the two smaller semi-circles arre:

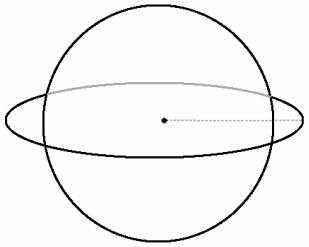
$$A_{\text{two semi-circles}} = 2\cdot 18\pi\text{ cm}^2 \approx 113.0973\text{ cm}^2$$

So the shaded area is

$$A \approx 226.1947\text{ cm}^2 - 113.0973\text{ cm}^2 = 113.0973\text{ cm}^2$$

Example 4. A satellite is always above the Earth's equator. It revolves around Earh at a height of 600 miles. The radius of the Earth is 3960 miles long. How many miles long is the satellite's circular orbit? Round your answer to the nearest mile.

Solution: The orbit of the satellite is a circle with a radius longer than that of the Earth. Its radius is in fact the sum of the radius of Earth and the height. So $r = 3969\text{ mi} + 600\text{ mi} = 4560\text{ mi}$. The rest of the problem is easy, we just need to find the circumference of the circle with this radius.

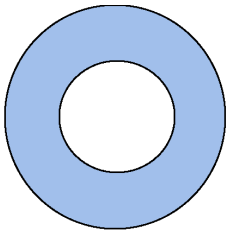


$$C = 2\pi r = 2\pi\,(4560\text{ mi}) = 9120\pi\text{ mi} \approx 28\,561.325\text{ mi}$$

When rounding to the nearest mile, we get 28 651 miles.

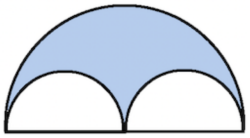
Practice Problems

1. Find the area of the ring shown on the picture. The circles have the same center and the radii are 3 m and 4 m.



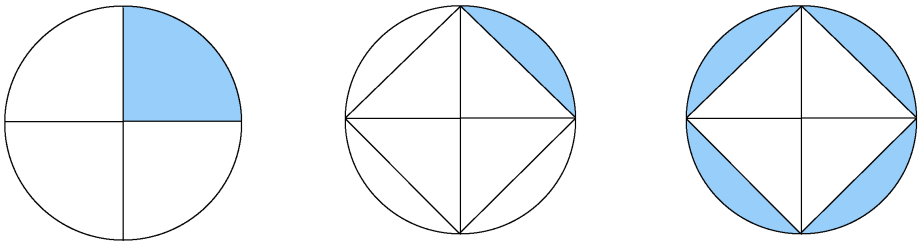
2. Suppose that every pizza we make today will have the same thickness. What contains then more calories, a 12-inch-diameter pizza or two 6-inch-diameter pizzas?

3. The picture shows three semicircles. The radius of both of the smaller semicircles is 2 m. Find the area of the shaded region.



4. All three circles shown below have a radius of 10 units. In each case, find the approximate value of the area of the shaded region.

a) b) c)



5. A square, a quarter circle, and a semi-circle. Find the area of the shaded region if the square has sides 20 cm long.

Appendix A – Answers

1.1 – Place Value

Practice Problems

1. a) hundreds b) millions c) ten thousands d) tens
2. a) 37 023 b) 211 602 780 c) 500 005 d) 20 035 090 165
3. a) thirty-two million, three hundred sixty thousand, eight hundred forty-one
   b) three hundred thousand, five hundred nine thousand, ninety-six.
   c) nineteen million, thirty-five thousand, eight hundred fifty-one

1.3 – The Order of Operations Agreement

Sample Problems 1. 5 2. 8 3. 13 4. 55 5. 25 6. 81 7. 7

Practice Problems 1. 43 2. 51 3. 9 4. 12 5. 4 6. 4 7. 1 8. 40 9. 15 10. 5 11. 3
12. 20 13. 85 14. 2 15. 1

1.4 – Perimeter and Area of a Rectangle

Discussion

1. $1\,\text{yd}^2 = 9\text{ft}^2$

   Algebraically: if $1\,\text{yd} = 3\text{ft}$, then $1\,\text{yd}^2 = 1\,\text{yd} \cdot 1\,\text{yd} = 3\text{ft} \cdot 3\text{ft} = 9\text{ft}^2$

Practice Problems 1. $P = 40\text{cm},\ A = 96\text{cm}^2$ 2. $P = 56\text{ft},\ A = 159\text{ft}^2$ 3. $P = 36\text{ft},\ A = 54\text{ft}^2$

Problem Set 1

1. 9 2. 10 3. 12 4. 5 5. 4 6. 2 7. 8 8. 49 9. 2 10. 12 11. 5 12. 27 13. 1

14. a) $30°, 120°$ or $75°, 75°$ b) $40°, 40°$ 15. $P = 40\text{ft}$ $A = 58\text{ft}^2$

16. a) 4 b) 19

    a) Imagine the absolute worst case scenario. To have a pair, we need to pull two socks of the same color. There are three colors, so worst case scenario, three different colors were pulled. But then the fourth sock must have a color matching one of the three.

    b) In the worst case scenario, we pull out all the other socks before a white. There are 18 socks of other colors. So, if we have 18 non-white socks in our hands, the 19th must be white.

17. $4+2\left((5-3)^2\right)=12$

2.1 – Rounding

Practice Problems

1. a) 6825020 b) 6825000 c) 6825000 d) 6830000 e) 6800000 f) 7000000
2. a) 12479500 b) 12479500 c) 12480000 d) 12480000 e) 12500000 f) 12000000

2.2– Divisibility Rules

Practice Problems

1. a) 80, 75, 270 b) 75, 270 c) 128, 80, 64
2. a) 1296, 420, 55050 b) 9800, 420, 55050 c) 1296, 420, 55050
3. a) 28072, 67808, 1296, 7620 b) 60610, 7620 c) 1296 d) 28072, 60610 , 7620

2.3 – Perimeter and Area of a Right Triangle

Practice Problems

1. $P = 56\text{ft}, A = 84\text{ft}^2$ 2. $P = 26\text{m}$, $A = 31\text{m}^2$ 3. $A = 9\text{ft}^2$ 4. $P = 84\text{cm}, A = 200\text{cm}^2$

Enrichment The two areas are equal. Both the triangle and the rectangle have area 30cm^2, and then they both lose the same region. So they remain equal.

Problem Set 2

14. $36°$

3.1 – Introduction to Signed Numbers

1. a) $5 > -7$ b) $-12 < -4$ c) $0 > -8$ d) $-1 > -4$ e) $-7 \geq -7$ or $-7 \leq -7$
2. a) 5 b) 5 c) -5 d) -5 e) 0 f) 3 g) 21
3. a) 3 b) $2+(-5)$ c) $3+0$ d) $-3+3$ e) $7+(-4)+(-1)$
4. a) -3 b) 11 c) 3 d) 18 e) $5+(-13)$ f) -5 g) -13 h) 10 i) 8
5. 203 (hint: pair the numbers cleverly)

3.2 – Factors of a Number

Practice Problems 1. $1,2,3,4,6,8,12,16,24,48$ 2. $91 = 7 \cdot 13$

4.1 – Subtracting Signed Numbers

1. a) 5 b) -5 c) 21 d) 8 e) -8 f) 7 g) -15 h) -24 i) -3 j) -13 k) 0 l) 8 m) -3 n) -4 o) 1 p) -13 q) 10 r) 8
2. a) -5 b) 5 c) -5 d) 11 e) 11 f) $|5| - |-6|$ g) 8 h) 10 i) -6 j) 0 j) 0 k) 0

4.2 – Prime Numbers

Practice Problems

1. $1,2,3,4,6,8,12,16,24,48$ 2. 91 3. a) 80, 75, 270 b) 75, 270 c) $128,80,64$
4. a) yes b) no, $2019 = 3 \cdot 673$ c) no, $2021 = 43 \cdot 47$ d) no, $2023 = 7 \cdot 289$ e) yes f) yes g) yes h) no, $4009 = 19 \cdot 211$
5. a) 101 b) 1009, because $1001 = 7 \cdot 143$, $1003 = 17 \cdot 59$ and $1007 = 19 \cdot 53$ (1002, 1004, 1006, and 1008 are divisible by 2 and 1005 by 5)

4.3 – Area of a Parallelogram

Practice Problems

1. $P = 64\text{in}$ and $A = 240\text{in}^2$
2. $P = 36\text{cm}$ and $A = 30\text{cm}^2$
3. Yes, a rectangle does have two pair of parallel sides. A rectangle is a special case of a parallelogram, where the angles are right angles.

5.1 – Multiplying and Dividing Positive and Negative Numbers

Practice Problems 1. 5 2. -3 3. -6 4. -15 5. 0 6. 7 7. undefined 8. -4 9. -28 10. -13 11. 1 12. 8

13. -3 14. 0 15. undefined 16. 0 17. 10 18. 8

5.2 – Order of Operations with Positive and Negative Numbers

Practice Problems 1. 43 2. -29 3. 49 4. -45 5. -162 6. 2 7. 0 8. 9 9. 12 10. 22 11. 28 12. -2 13. undefined 14. -16 15. -21 16. 24 17. -4 18. 19 19. -33 20. -17 21. -2 22. -6 23. -8 24. -9 25. 20 26. 3 27. 15 28. -14 29. 0 30. 2 31. -2 32. -4

Enrichment 1. a) 69 and 11 b) 0 and 40

The pairs are colored differently. If your printout is black and white, look at the document at your computer.

a) $|5-2|-4+7|-10| = 69$ and $|5-2|-4+7|-10| = 11$

b) $2|-1-5|-3|-4| = 0$ and $2|-1-5|-3|-4| = 40$

2. a) $5-2\left((-3)^2-\left((4-7)^2+2\right)\right) = 9$ b) $5-\left((2-3)^2-(4-7)\right)^2+2 = -9$

5.3 – Prime Factorization

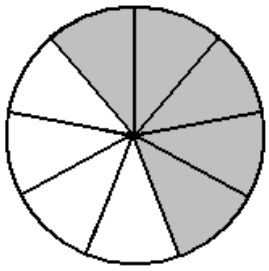
Practice Problems 1. $600 = 2^3 \cdot 3 \cdot 5^2$ 2. $5500 = 2^2 \cdot 5^3 \cdot 11$ 3. $2016 = 2^5 \cdot 3^2 \cdot 7$ 4. $2015 = 5 \cdot 13 \cdot 31$

5.4 – Perimeter and Area of a Trapezoid

Practice Problems 1. $P = 72\text{in}$ and $A = 264\text{in}^2$ 2. $P = 70\text{m}$ and $A = 172\text{m}^2$

6.1 – Introduction to Fractions and Percents

Practice Problems 1. 16 2. 20

3. 

4. \$420 5. \$595 6. 42 7. \$2120

8. a) 180 b) 168 c) $\frac{3}{7}$ is larger

9. \$17 280 in taxes and will keep \$36 720.

10. \$2472 11. \$30 000

6.2 – The Greatest Common Factor and Least Common Multiple

Practice Problems 1. a) 12 and 1800 b) 240 and 3600 c) 20 and 2100

2. Yes, it is possible. For example, $\gcd(4,4) = 4$ and $\text{lcm}(4,4) = 4$.
But if $m \neq n$, then the two will never be equal. 3. $6, 12, 24, 30, 60, 120$

6.3 – Algebraic Expressions

Practice Problems 1. 17 2. 5 3. 35 4. 100 5. 196 6. 18 7. 6 8. 11 9. 26 10. 2

11. a) 5 b) 5 c) 5 d) 4

7.1 – Equivalent Fractions

Practice Problems

1. a) $\frac{2}{9} = \frac{10}{45}$ b) $\frac{2}{9}$ of 90 is 20, and so is $\frac{10}{45}$ of 90.
2. a) $\frac{30}{36}$ b) $\frac{25}{30}$
3. a) $\frac{3}{5}$ b) $\frac{1}{4}$ c) 1 d) $\frac{1}{2}$ e) $\frac{3}{2}$ f) $\frac{11}{5}$
4. a) 30% b) 85% c) 240% d) 6% e) 500%
5. a) 24 b) 41 c) 48
6. a) $\frac{7}{10} = \frac{21}{30}$ and $\frac{2}{3} = \frac{20}{30}$. Clearly, $\frac{21}{30} > \frac{20}{30}$ and so $\frac{7}{10} > \frac{2}{3}$.
   b) $\frac{7}{10}$ of 60 is 42, and $\frac{2}{3}$ of 60 is 40. Since $42 > 40$, we get again that $\frac{7}{10}$ is greater.

7.2 – Equations and Inequualities

Sample Problems

1. a) no, since $37 \neq 13$ b) yes, since $55 = 55$ c) no, since $70 \neq 76$ d) yes, since $139 = 139$
2. a) yes, since $13 = 13$ b) no, since $15 \neq 13$

Practice Problems

1. a) yes, since $1 = 1$ b) yes, since $6 = 6$ c) yes, since $3 = 3$
2. a) yes, since $1 = 1$ b) no, since $21 \neq 4$ c) yes, since $6 = 6$ d) yes, since $41 = 41$
3. a) no, since $32 \neq 36$ b) yes, since $40 = 40$
4. a) yes , $10 \geq 2$ b) yes, $12 = 12$ c) no, $30 \ngeq 42$ d) yes, $120 \geq 102$

7.3 – Division with Remainder

Practice Problems

1. a) 18 R 6 b) 95 R 5 c) 114 R 6 d) 16 R 9
2. a) 7 b) 1 c) 2 d) 9 3. 2 PM

7.4 – Angles in a Triangle

Practice Problems

1. 30° 2. a) $24^\circ, 24^\circ$, and 132° b) $20^\circ, 20^\circ, 140^\circ$ and $80^\circ, 80^\circ, 20^\circ$ are both possible

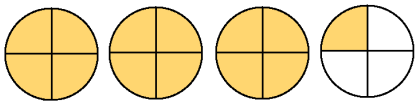
8.1 – Improper Fractions and Mixed Numbers

Practice Problems

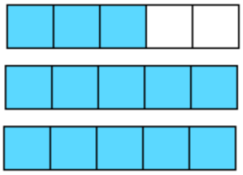
1. a) $3\frac{1}{3}$ b) $1\frac{1}{2}$ c) $3\frac{1}{2}$ d) $17\frac{1}{7}$ e) $5\frac{1}{4}$ f) $7\frac{3}{20}$ g) $3\frac{1}{5}$

2. a) $\frac{19}{5}$ b) $\frac{11}{8}$ c) $\frac{41}{7}$ d) $\frac{17}{5}$ e) $\frac{75}{7}$ f) $\frac{21}{4}$

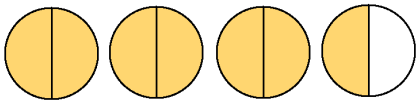
3. a) $\frac{13}{4} = 3\frac{1}{4}$



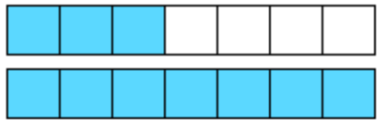
b) $2\frac{3}{5} = \frac{13}{5}$



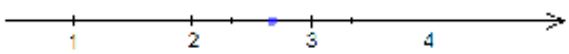
c) $\frac{7}{2} = 3\frac{1}{2}$



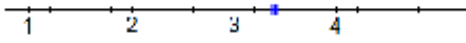
d) $\frac{10}{7} = 1\frac{3}{7}$



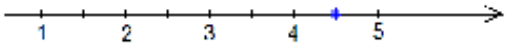
4. a) $\frac{8}{3} = 2\frac{2}{3}$



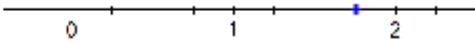
b) $3\frac{2}{5}$



c) $\frac{9}{2} = 4\frac{1}{2}$



d) $\frac{7}{4} = 1\frac{3}{4}$



8.2 – One- and Two-Step Equations

Discussion a) 1 b) 0 c) 0 d) 0

One thing that is unusual in this problem is the idea of cancellation. Cancellation results in 0 or 1, depending on the operation.

Sample Problems 1. 11 2. 25 3. 48 4. 53 5. 2 6. 0 7. 36 8. 5 9. 0

Practice Problems 1. 2 2. 13 3. 1 4. 20 5. 8 6. 5 7. 29 8. 6 9. 70 10. 0

11. 25 12. 6 13. 22 14. 4 15. 24

9.1 – Adding and Subtracting Fractions

Practice Problems 1. $\frac{3}{2}$ 2. $-\frac{1}{6}$ 3. $-\frac{8}{3}$ 4. $\frac{21}{10}$ 5. $\frac{11}{8}$ 6. $\frac{13}{12}$ 7. $\frac{4}{3}$ 8. $-\frac{11}{8}$ 9. $-\frac{7}{10}$ 10. $-\frac{73}{12}$

11. 0 12. $-\frac{19}{12}$ 13. $-\frac{1}{3}$ 14. $\frac{7}{2}$ 15. 3 16. $\frac{13}{15}$ 17. $-\frac{1}{4}$ 18. $\frac{7}{2}$

9.2 – Volumes

Practice Problems

1. $V = 125\text{ft}^3$ 2. 32cm^3
2. 6 possibilities: $(1,1,24),(1,2,12),(1,3,8),(1,4,6),(2,2,6),(2,3,4)$

10.1 – Multiplying Fractions

Practice Problems

1. a) $-\frac{2}{5}$ b) $\frac{1}{4}$ c) $\frac{4}{7}$ d) $-\frac{4}{9}$ e) $\frac{7}{10}$ f) $-\frac{1}{3}$ g) $\frac{7}{15}$ h) $\frac{4}{3}$
2. $P = \frac{62}{3}\text{in}$ $A = \frac{148}{9}\text{in}^2$ 3. a) $\frac{5}{2} = 2\frac{1}{2}$ b) $\frac{3}{10}$ c) $-\frac{7}{5} = -1\frac{2}{7}$ d) $\frac{1}{4}$
4. a) $\frac{9}{4}$ b) $-\frac{39}{4}$ c) $\frac{31}{9}$ 5. a) 3 b) 0 c) $\frac{13}{9}$ d) $\frac{7}{9}$ 6. $\frac{25}{2}\text{in}^3$

10.2 – Unit Conversions

Practice Problems

1. $\frac{5}{4} = 1\frac{1}{4}$ weeks 2. 1080 minutes 3. 500ft 4. 6250ft^2 5. $36\frac{\text{m}}{\text{s}}$ 6. $21\,600\text{in}^3$

11.1 – Dividing Fractions

Practice Problems

1. $-\frac{20}{9}$ 2. $-\frac{13}{12}$ 3. $\frac{7}{8}$ 4. $\frac{1}{2}$ 5. -2 6. $-\frac{21}{25}$ 7. $-\frac{4}{21}$ 8. $-\frac{34}{35}$ 9. $\frac{4}{3}$
10. a) $-\frac{2}{3}$ b) undefined c) $\frac{1}{3}$ d) $-\frac{1}{6}$

11.2 – Introduction to Ratios

Practice Problems

1. a) $\frac{4}{9}$ b) $\frac{9}{4}$ c) $\frac{1}{3}$ d) $\frac{4}{5}$ e) $\frac{4}{5}$ f) $\frac{7}{6}$ g) $\frac{5}{3}$ h) $\frac{3}{10}$ 2. a) $\frac{5}{6}$ b) $\frac{5}{11}$ c) $\frac{6}{11}$
3. 3. a) 1323 b) 108 c) 84 take French and 294 take Spanish 4. a) 700 miles b) $\frac{3}{4}$ inch

12.1 – Introduction to Decimals

Practice Problems

1. a) 28.003213 b) 80 004.00004 c) 42.4025
1. a) zero and twenty-five hundredths b) twenty-one and eight thousand five ten-thousandths c) two and twenty-one ten-thousandths d) one hundred and three hundred one one-millionths e) one hundred ninety-five and sixty-one thousand, twenty-three hundred-thousandths
2. a) $\frac{1}{2}$ b) $\frac{8}{25}$ c) $\frac{1}{40}$
3. a) 4000 b) 3800 c) 3800 d) 3803 e) 3802.8 f) 3802.75 g) 3802.754 h) 3802.7543

12.2 – Basic Percent Problems

Sample Problems

1. 576 2. 150% 3. 125 4. 1500 5. $8560 6. 15% decrease 7. $2400 per month 8. $480

Practice Problems

1. $\frac{2}{17}$ 2. 261 3. 60 4. 700 5. 64% 6. 272 7. 320% 8. 700 9. 12000 10. $80
11. $600 12. 260000 13. C 14. A 15. B 16. E 17. D
18. a) $1980 b) $2178 c) 21% increase 19. 38% 20. 150%

15.1 – Circumference and Area of Circles

Practice Problems

1. 21.991149 m^2

2. (pizzas) Comparing the areas, two 3-inch radius pizzas will have a combined area of

$$A = 2\pi r^2 = 2\pi(3\text{ in})^2 = 2\cdot 9\pi \text{in}^2 = 18\pi \text{in}^2 \approx 56.54867 \text{in}^2$$

and one 6-inch radius pizza will have an area of

$$A = \pi R^2 = \pi(6\text{ in})^2 = 36\pi \text{in}^2 \approx 113.097 \text{in}^2$$

and so the 12-diameter pizza will contain more calories.

3. 4π $m^2 \approx 12.5664$ m^2

4. a) 78.5398 units b) 28.5398 units c) 114.159 units

5. 157.0796 cm^2

Appendix B – Solutions

Solutions for 1.3 – The Order of Operations Agreement

1. $2 \cdot 3^2 - \left(6^2 - 2 \cdot 5\right) \div 2$

   Solution: We start with the parentheses. We will work within the parentheses until the entire expression within it becomes one number. In the parentheses, there is an exponentiation, a subtraction, and a multiplication. Since it is stronger, we start with the exponent.

   $$\begin{aligned} 2 \cdot 3^2 - \left(6^2 - 2 \cdot 5\right) \div 2 &= && \text{exponent within parentheses} \\ 2 \cdot 3^2 - (36 - 2 \cdot 5) \div 2 &= && \text{multiplication within parentheses} \\ 2 \cdot 3^2 - (36 - 10) \div 2 &= && \text{subtraction within parentheses} \\ 2 \cdot 3^2 - (26) \div 2 &= && \text{we may drop parentheses now} \\ 2 \cdot 3^2 - 26 \div 2 &= \end{aligned}$$

   Now that there is no parentheses, we perform all exponents, left to right. There is only one, so we have

   $$\begin{aligned} 2 \cdot 3^2 - 26 \div 2 &= && \text{exponent} \\ 2 \cdot 9 - 26 \div 2 &= \end{aligned}$$

   Now we execute all multiplications, divisions, left to right

   $$\begin{aligned} 2 \cdot 9 - 26 \div 2 &= && \text{multiplication} \\ 18 - 26 \div 2 &= && \text{division} \\ 18 - 13 &= && \text{subtraction} \\ &= & \boxed{5} \end{aligned}$$

2. $18 - 7 - 3$

   Solution: It is a common mistake to subtract 4 from 18. This is not what order of operations tell us to do. The two subtractions have to be performed left to right.

   $$\begin{aligned} 18 - 7 - 3 &= && \text{first subtraction from left} \\ 11 - 3 &= && \text{subtraction} \\ &= & \boxed{8} \end{aligned}$$

3. $5^2 - 2\left(10 - 2^2\right)$
Solution: We start with the parentheses

$$\begin{aligned} 5^2 - 2\left(10 - 2^2\right) &= \text{exponent in parentheses} \\ 5^2 - 2(10 - 4) &= \text{subtraction in parentheses} \\ 5^2 - 2(6) &= \text{drop parentheses} \\ 5^2 - 2 \cdot 6 &= \text{exponents} \\ 25 - 2 \cdot 6 &= \text{multiplication} \\ 25 - 12 &= \text{subtraction} \\ &= \boxed{13} \end{aligned}$$

4. $8^2 - 3^2$
Solution: There are three operations, two exponents and a subtraction. We start with the exponents, left to right.

$$\begin{aligned} 8^2 - 3^2 &= \text{first exponent from left} \\ 64 - 3^2 &= \text{exponent} \\ 64 - 9 &= \text{subtraction} \\ &= \boxed{55} \end{aligned}$$

5. $(8-3)^2$
Solution: We start with the parentheses

$$\begin{aligned} (8-3)^2 &= \text{subtraction in parentheses} \\ (5)^2 &= \text{drop parentheses} \\ 5^2 &= \text{exponents} \\ &= \boxed{25} \end{aligned}$$

This problem and the previous one tells us a very important thing: $a^2 - b^2$ and $(a-b)^2$ are different expressions! In $a^2 - b^2$ we first square a and b and the subtract. In $(a-b)^2$ we first subtract b from a and then square the difference.for

6. $\left(3^3 - 4 \cdot 5 + 2\right)^2$
Solution: We will work within the parentheses until it becomes a number. Within the parentheses, we start with the exponents.

$$\begin{aligned} \left(3^3 - 4 \cdot 5 + 2\right)^2 &= \text{exponents within parentheses} \\ (27 - 4 \cdot 5 + 2)^2 &= \text{multiplication within parentheses} \\ (27 - 20 + 2)^2 &= \end{aligned}$$

There is an addition and a subtraction in the parentheses. **It is not true that addition comes before subtraction!** Addition and subtraction are equally strong; we execute them left to right. We start with the subtraction within parentheses.

$$\begin{aligned} (27 - 20 + 2)^2 &= (7 + 2)^2 && \text{addition within parentheses} \\ (9)^2 &= 9^2 = \boxed{81} && \text{drop parentheses, exponents} \end{aligned}$$

7. $\dfrac{3+2\left(20-3^2-5\right)}{3^2-2^2}+4^1$

Solution: The division bar stretching over entire expressions is a case of the **invisible parentheses**. It instruct us to work out the top until we obtain a number, the bottom until we obtain a number, and finally divide. The invisible parentheses here means

$$\frac{3+2\left(20-3^2-5\right)}{3^2-2^2}=\left[3+2\left(20-3^2-5\right)\right]\div\left[3^2-2^2\right]$$

And now we see that the invisible parentheses was developed to simplify notation. We will start with the top. Naturally, we stay within the parentheses until they disappear.

$$\begin{aligned}
\frac{3+2\left(20-3^2-5\right)}{3^2-2^2}+4^1 &= && \text{exponent in parentheses}\\
\frac{3+2\left(20-9-5\right)}{3^2-2^2}+4^1 &= && \text{first subtraction from left in parentheses}\\
\frac{3+2\left(11-5\right)}{3^2-2^2}+4^1 &= && \text{subtraction in parentheses}\\
\frac{3+2\left(6\right)}{3^2-2^2}+4^1 &= && \text{drop parentheses}\\
\frac{3+2\cdot 6}{3^2-2^2}+4^1 &= && \text{multiplication on top}\\
\frac{3+12}{3^2-2^2}+4^1 &= && \text{addition on top}\\
\frac{15}{3^2-2^2}+4^1 &= &&
\end{aligned}$$

Now we work out the bottom, applying order of operations

$$\begin{aligned}
\frac{15}{3^2-2^2}+4^1 &= && \text{first exponent from left to right}\\
\frac{15}{9-2^2}+4^1 &= && \text{exponent}\\
\frac{15}{9-4}+4^1 &= && \text{subtraction}\\
\frac{15}{5}+4^1 &= && \text{same as}\\
15\div 5+4^1 &= &&
\end{aligned}$$

We now have a division, an addition, and an exponent. We start with the exponent.

$$\begin{aligned}
15\div 5+4^1 &= && \text{exponent, } 4^1=4\\
15\div 5+4 &= && \text{division}\\
3+4 &= && \text{addition}\\
&= && \boxed{7}
\end{aligned}$$

Solutions for 7.2 – Equations and Inequalities

1. Consider the equation $2x^2+x+34=21x-8$.

   a) Is the number 1 a solution of the equation?

   Solution: We need to substitute 1 for x into both the left-hand side and right-hand side of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

   The left-hand side:

$$\begin{aligned} \text{LHS} &= 2x^2+x+34=2(1)^2+(1)+34 \\ &= 2\cdot 1+1+34=2+1+34=3+34=37 \end{aligned}$$

   The right-hand side:

$$\text{RHS}=21x-8=21(1)-8=21-8=13$$

   Since the values of the two expressions are not equal when $x=1$, 1 is NOT a solution of the equation.

   b) Is 3 a solution of the equation?

   Solution: We need to substitute 3 for x into both the left-hand side and right-hand side of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

   The left-hand side:

$$\begin{aligned} \text{LHS} &= 2x^2+x+34=2(3)^2+(3)+34 \\ &= 2\cdot 9+3+34=18+3+34=21+34=55 \end{aligned}$$

   The right-hand side:

$$\text{RHS}=21x-8=21(3)-8=63-8=55$$

   Since the values of the two expressions are equal when $x=3$, 3 I is a solution of the equation.

   c) Is $x=4$ a solution of the equation?

   Solution: We need to substitute 4 for x into both the left-hand side and right-hand side of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

   The left-hand side:

$$\begin{aligned} \text{LHS} &= 2x^2+x+34=2(4)^2+(4)+34 \\ &= 2\cdot 16+4+34=32+4+34=36+34=70 \end{aligned}$$

   The right-hand side:

$$\text{RHS}=21x-8=21(4)-8=84-8=76$$

   Since the two expressions are not equal when $x=4$, 4 is NOT a solution of the equation.

   d) Is 7 a solution of the equation?

   Solution: We need to substitute 7 for x into both the left-hand side and right-hand side of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

   The left-hand side:

$$\text{LHS}=2x^2+x+34=2(7)^2+(7)+34=2\cdot 49+7+34=98+7+34=105+34=139$$

   The right-hand side:

$$\text{RHS}=21x-8=21(7)-8=147-8=139$$

   Since the values of the two expressions are equal when $x=7$, 7 IS a solution of the equation.

2. Consider the equation $3a-2b-1=(a-b)^2+4$.

a) Is the pair of numbers $a=8$ and $b=5$ a solution of the equation?

Solution: We need to substitute $a=8$ and $b=5$ into both the left-hand side and right-hand side of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

The left-hand side:

$$\begin{aligned} \text{LHS} &= 3a-2b-1=3(8)-2(5)-1=3\cdot 8-2\cdot 5-1 \\ &= 24-2\cdot 5-1=24-10-1=14-1=13 \end{aligned}$$

The right-hand side:

$$\text{RHS}=(a-b)^2+4=((8)-(5))^2+4=(8-5)^2+4=3^2+4=9+4=13$$

Since the values of the two expressions are equal when $a=8$ and $b=5$, this pair IS a solution of the equation.

b) Is the pair of numbers $a=10$ and $b=7$ a solution of the equation?

Solution: We need to substitute $a=10$ and $b=7$ into both the left-hand side and right-hand side of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

The left-hand side:

$$\begin{aligned} \text{LHS} &= 3a-2b-1=3(10)-2(7)-1=3\cdot 10-2\cdot 7-1 \\ &= 30-2\cdot 7-1=30-14-1=16-1=15 \end{aligned}$$

The right-hand side:

$$\text{RHS}=(a-b)^2+4=((10)-(7))^2+4=(10-7)^2+4=3^2+4=9+4=13$$

Since the values of the two expressions are not equal when $a=8$ and $b=5$, this pair is NOT a solution of the equation.

3. Consider the inequality $x^2+2x-1\leq 8x+6$..

a) Is $x=2$ a solution of the inequality?

Solution: We need to substitute 2 for x into both the left-hand side and right-hand side of the inequality and evaluate those algebraic expressions to see whether the left-hand side is lessthan or equal to the right-hand side.

The left-hand side:

$$\text{LHS}=2^2+2\cdot 2-1=4+2\cdot 2-1=4+4-1=7$$

The right-hand side:

$$\text{RHS}=8\cdot 2+6=16+6=22$$

Since the value of the left-hand side is less than the rigth-hand side when $x=2$, 2 is a solution of the equation.

b) Is $x=7$ a solution of the inequality?

Solution: We need to substitute 7 for x into both the left-hand side and right-hand side of the inequality and evaluate those algebraic expressions to see whether the left-hand side is lessthan or equal to the right-hand side.

The left-hand side:

$$\text{LHS}=7^2+2\cdot 7-1=49+2\cdot 7-1=49+14-1=63-1=62$$

The right-hand side:

$$\text{RHS}=8\cdot 7+6=56+6=62$$

Since the value of the left-hand side is equal to the ritght-hand side when $x=7$, 7 is a solution of the equation.

c) Is $x=10$ a solution of the inequality?

Solution: We need to substitute 10 for x into both the left-hand side and right-hand side of the inequality and evaluate those algebraic expressions to see whether the left-hand side is less than or equal to the right-hand side.

The left-hand side:

$$\text{LHS}=10^2+2\cdot 10-1=100+2\cdot 10-1=100+20-1=120-1=119$$

The right-hand side:

$$\text{RHS}=8\cdot 10+6=80+6=86$$

Since the value of the left-hand side is greater than the right-hand side when $x=10$, 10 is not a solution of the equation.

Solutions for 8.2 – One- and Two-Step Equations

Solve each of the following equations. Make sure to check your solutions.

1. $2x-5=17$

   Solution:

$$\begin{aligned} 2x-5 &= 17 && \text{add 5 to both sides} \\ 2x &= 22 && \text{divide by 2} \\ x &= 11 \end{aligned}$$

   We check: if $x=11$, then

$$\text{RHS} = 2(11)-5 = 22-5 = 17 = \text{LHS}$$

   Thus our solution, $\boxed{x=11}$ is correct.

2. $\dfrac{a-10}{5}=3$

   Solution:

$$\begin{aligned} \frac{a-10}{5} &= 3 && \text{multiply both sides by 5} \\ a-10 &= 15 && \text{add 10 to both sides} \\ a &= 25 \end{aligned}$$

   We check: if $a=25$, then

$$\text{LHS} = \frac{25-10}{5} = \frac{15}{5} = 3 = \text{RHS}$$

   Thus our solution, $\boxed{a=25}$ is correct.

3. $\dfrac{t}{4}-10=2$

   Solution:

$$\begin{aligned} \frac{t}{4}-10 &= 2 && \text{add 10 to both sides} \\ \frac{t}{4} &= 12 && \text{multiply both sides by 4} \\ t &= 48 \end{aligned}$$

   We check: if $t=48$, then

$$\text{RHS} = \frac{t}{4}-10 = \frac{48}{4}-10 = 12-10 = 2 = \text{LHS}$$

   Thus our solution, $\boxed{t=48}$ is correct.

4. $\dfrac{t-5}{12}=4$

   Solution:

$$\begin{aligned} \frac{t-5}{12} &= 4 && \text{multiply both sides by 12} \\ t-5 &= 48 && \text{add 5 to both sides} \\ t &= 53 \end{aligned}$$

   We check: if $t=53$, then $\text{LHS} = \dfrac{53-5}{12} = \dfrac{48}{12} = 4 = \text{RHS}.$

   Thus our solution, $\boxed{t=53}$ is correct.

5. $5x-7=3$

   Solution: We apply all operations to both sides.

$$\begin{aligned} 5x-7 &= 3 && \text{add } 7 \\ 5x &= 10 && \text{divide by } 5 \\ x &= 2 \end{aligned}$$

   We check: if $x=2$, then

$$\text{LHS} = 5\cdot 2-7 = 10-7 = 3 = \text{RHS}$$

   Thus our solution, $\boxed{x=2}$ is correct.

6. $\dfrac{x+6}{3}=2$

   Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x+6}{3} &= 2 && \text{multiply by } 3 \\ x+6 &= 6 && \text{subtract } 6 \\ x &= 0 \end{aligned}$$

   We check: $\text{LHS}=\dfrac{0+6}{3}=\dfrac{6}{3}=2=\text{RHS}$

   Thus our solution, $\boxed{x=0}$ is correct.

7. $\dfrac{x}{3}+8=20$

   Solution: We apply all operations to both sides.

$$\begin{aligned} \frac{x}{3}+8 &= 20 && \text{subtract } 8 \\ \frac{x}{3} &= 12 && \text{multiply by } 3 \\ x &= 36 \end{aligned}$$

   We check:

$$\text{LHS} = \frac{36}{3}+8 = 12+8 = 20 = \text{RHS}$$

   Thus our solution, $\boxed{x=36}$ is correct.

8. $3(x+7)=36$

   Solution: We apply all operation to both sides,

$$\begin{aligned} 3(x+7) &= 36 && \text{divide by } 3 \\ x+7 &= 12 && \text{subtract } 7 \\ x &= 5 \end{aligned}$$

   We check: if $x=5$, then

$$\text{LHS} = 3(5+7) = 3\cdot 12 = 36 = \text{RHS}$$

   Thus our solution, $\boxed{x=5}$ is correct.

9. $\frac{x}{3}+2=2$

Solution:

$$\begin{aligned} \frac{x}{3}+2 &= 2 && \text{subtract 2} \\ \frac{x}{3} &= 0 && \text{multiply by 3} \\ x &= 0 \end{aligned}$$

We check: if $x=0$, then

$$\text{LHS} = \frac{0}{3}+2 = 0+2 = 2 = \text{RHS}$$

Thus our solution, $\boxed{x=0}$ is correct.

Solutions for 12.2 - Basic Percent Problems

1. Find 16% of 3600.

   Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x. In this case,

   $(\text{is}) = \quad x$
   $F = \quad 0.16$
   $(\text{of}) = \quad 3600$

   We will substitute these into the formula and solve for x. $0.16 \cdot 360 = 57.6$

   $$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ x &= 0.16 \cdot 3600 \\ x &= 576 \end{aligned}$$

   Thus 16% of 3600 is 576.

2. 27 is what percent of 18?

   Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x. Because the is-number is smaller than the of-number, we should expect a percentage larger than 100%.

   $(\text{is}) = \quad 27$
   $F = \quad x$
   $(\text{of}) = \quad 18$

   We will substitute these into the formula and solve for x.

   $$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 27 &= x \cdot 18 \qquad \text{divide by } 18 \\ \frac{27}{18} &= x \\ x &= 1.5 = 150\% \end{aligned}$$

   Thus 27 is 150% of 18. We can check by computing 150% of 18:

   $$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ (\text{is}) &= 1.5 \cdot 18 = 27 \end{aligned}$$

   Thus our solution, 150% is correct.

3. 120% of what number is 150?

   Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x. In this case,

   $(\text{is}) = \quad 150$
   $F = \quad 1.2$
   $(\text{of}) = \quad x$

   We will substitute these into the formula and solve for x.

   $$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 150 &= 1.2 \cdot x \qquad \text{divide by } 1.2 \\ \frac{150}{1.2} &= x \\ 125 &= x \end{aligned}$$

   We can check by computing 120% of 125:

   $$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ (\text{is}) &= 1.2 \cdot 125 = 150 \end{aligned}$$

Thus our solution, 125 is correct.
The following examples all boil down to one of the basic problems shown above. One advice: before starting computations, re-write the problem to a basic question. Once we have this question, the problem is easy to solve.

4. What do we get if we increase 600 by 150%?

   Solution 1: We compute 150% of 600 and then we add it to 600. To compute 150% of 600 is a type 1 problem:

   $$\begin{aligned}(\text{is}) &= x\\ F &= \frac{150}{100} = 1.5\\ (\text{of}) &= 600\end{aligned}$$

   Now we use our formula to find x.

   $$\begin{aligned}(\text{is}) &= \text{F}\cdot(\text{of})\\ x &= 1.5\cdot 600 = 900\end{aligned}$$

   So after the increase, we have $600+900=1500$.

   Solution 2: This is a neat shortcut that will become very important in other problems. If a quantity is increased by 150%, then it "grew up" from 100% of itself to $100\%+150\% = 250\%$ of itself. We can find the answer quickly if we simply compute 250% of 600. The basic question is: What is 250% of 600?(Type 1)

   $$\begin{aligned}(\text{is}) &= x\\ F &= 250\% = 2.5\\ (\text{of}) &= 600\end{aligned}$$

   We substitute the data into the formula:

   $$\begin{aligned}(\text{is}) &= \text{F}\cdot(\text{of})\\ x &= 2.5\cdot 600\\ x &= 1500\end{aligned}$$

   Thus the answer is 1500.

5. We placed \$8000 into a bank account with an annual 7% of interest rate. How much money do we have in the account a year later?

   Solution 1: We compute 7% of 8000 and add the result to 8000. The basic question is: What is 7% of 8000? (Type 1)

   $$\begin{aligned}(\text{is}) &= x\\ F &= 0.07\\ (\text{of}) &= 8000\end{aligned}$$

   We substitute the data into the formula:

   $$\begin{aligned}(\text{is}) &= \text{F}\cdot(\text{of})\\ x &= 0.07\cdot 8000\\ x &= 560\end{aligned}$$

   Thus we earned \$ 560 in interest, and now we have $\$\ 8000+\$\ 560 = \$\ 8560$.

   Solution 2: This is a neat shortcut that will become very important in other problems. If a quantity is increased by 7%, then it "grew up" from 100% of itself to 107% of itself. We can find the amount in the bank, if we simply compute 107% of 8000. The basic question is: What is 107% of 8000?

   $$\begin{aligned}(\text{is}) &= x\\ F &= 1.07\\ (\text{of}) &= 8000\end{aligned}$$

   We substitute the data into the formula:

   $$\begin{aligned}(\text{is}) &= \text{F}\cdot(\text{of})\\ x &= 1.07\cdot 8000\\ x &= 8560\end{aligned}$$

   Thus we now have \$8560.

6. The population of a town has decreased from 80 000 to 68 000. What percent of a decrease does this represent?

   Solution 1: We subtract 68 000 from 80 000 to determine the change. $80000 - 68000 = 12000$. Now the question is: 12 000 is what percent of 80 000? (Type 2)

   (is) = 12 000
   $F =$ x
   (of) = 80 000

   We substitute the data into the formula:

$$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 12000 &= x \cdot 80000 \\ \frac{12000}{80000} &= x \\ 0.15 &= x \end{aligned}$$

   Thus

$$x = 0.15 = \frac{0.15}{1} = \frac{0.15\,(100)}{1\,(100)} = \frac{15}{100} = 15\%$$

   This is a 15% decrease.

   Solution 2: Instead of comparing the chage to the original condition, we can compare the new condition to the original condition and interpret the result. The question may be re-phrased as: 68 000 is what percent of 80 000? Then

   (is) = 68 000
   $F =$ x
   (of) = 80 000

   We substitute the data into the formula:

$$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 68000 &= x \cdot 80000 \\ \frac{68000}{80000} &= x \\ 0.85 &= x \end{aligned}$$

   Thus

$$x = 0.85 = \frac{0.85}{1} = \frac{0.85\,(100)}{1\,(100)} = \frac{85}{100} = 85\%$$

   Since the population has decreased from 100% of itself to 85% of itself, our result reflects a 15% decrease.

7. Tom got a 4% raise in his job. Now he makes 2496 per month. How much was he making before the raise?

   Solution: It would be a mistake to simply subtract 4% of 2496 from 2496, because the raise was 4% of a smaller, unknown number! The trick is to realize that Tom is now making exactly 104% of his original pay. The question is thus (Type 3) 104% of what number is 2496?

   (is) = 2496
   $F =$ 1.04
   (of) = x

   We substitute the data into the formula:

$$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 2496 &= 1.04 \cdot x \quad \text{divide by } 1.04 \\ \frac{2496}{1.04} &= x \\ 2400 &= x \end{aligned}$$

   Thus his original pay was $ 2400 per month. We can check by taking 104% of 2400 and find that it is indeed 2496.

8. A TV set went on a 35% off sale. The sale price is $ 312. Find the original price.
Solution: This problem is similar to the previous one in the sense that it is type 3, and the method used previously is essential. It would be a mistake to simply add 35% of 312 from 312, because the deduction was 35% of a larger, unknown number! The trick is to realize that the current price is 65% of the original price. The difference is that we subtract 35% this time. The question is thus (Type 3) 65% of what number is 312?

$(\text{is}) = \quad 312$
$\text{F} = \quad 0.65$
$(\text{of}) = \quad x$

We substitute the data into the formula:

$$\begin{aligned} (\text{is}) &= \text{F} \cdot (\text{of}) \\ 312 &= 0.65 \cdot x \qquad \text{divide by } 0.65 \\ \frac{312}{0.65} &= x \\ 480 &= x \end{aligned}$$

Thus the original price was $ 480. We can check by taking 65% of 480 and find that it is indeed 312.