

Beginning and Intermediate Algebra with Geometry - Part 1

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Chapter 1

1.1 Introduction to Intermediate Algebra

Part 1- What is Mathematics?

As a graduate student, I had the annoying habit of asking my teachers and peer students what they think mathematics is. To my surprise, I received many different answers, and, to this day, I agree with many of them. In my eyes, mathematics is many things. In mathematics, we will be talking a lot about things being true or being not true. Although this probably happens in every course in every discipline, mathematical truth can be objectively established and agreed upon. To achieve such an objective approach, we have to develop a language that is objectively understood. In this sense, mathematics is also a language.

Definition: **Mathematics** is a collection of *true statements* that are developed, expressed, and interpreted using an objective *language* and rules of *logic*.

To understand mathematics, we need to first agree on an objective language. Reading and writing mathematical notation correctly will be important.



'Then you should say what you mean', the March Hare went on. 'I do,' Alice hastily replied; 'at least - at least I mean what I say - that's the same thing, you know.' 'Not the same thing a bit!' said the Hatter. 'Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see!'

Lewis Carroll

Alice's Adventures in Wonderland



Part 2 - The Language of Mathematics

Mathematical statements are much like English sentences. As English sentences are built from different kinds of words, mathematical statements usually contain three types of components: **objects, operations, and relations**.

Definition: The concept of an **object** (very much like nouns in English sentences) is usually clearly understood and needs no explanation.

Examples of objects from algebra include the number 2, the number 3, and numbers in general. Objects from geometry include lines, points, triangles, circles, line segments, etc.

Definition: An **operation** is an action (very much like verbs in English sentences) that can be applied to objects and usually result in new objects.

Examples of operations from algebra include addition, subtraction, multiplication, and division. Operations from geometry include reflection to a line, rotation, or translation that can be performed on points, triangles, circles, line segments, etc.

Definition: A **relation** is something we use to compare two objects.

Unlike operations, relations do not produce new things; we use relations to compare already existing objects. For example, the operation addition produces 7 if applied to 2 and 5. Relations in algebra are equal, or less, or greater. Relations from geometry are how geometric objects can be compared to each other: similar, congruent, parallel, perpendicular.

Many statements use at least one of each of these three components. In the statement $2 + 5 = 7$ the numbers 2, 5, and 7 are the objects, addition (denoted by $+$) is the operation, and being equal (denoted by $=$) is the relation.

It is a common misconception to think of mathematics as the study of *only* numbers. Numbers are only certain types of objects. As we progress in the study of mathematics, we will find that there are many other types of interesting objects. For example, sets are objects we will soon study. Furthermore, the study of operations and relations is also interesting and fruitful.

So, what kind of true statements can be established in mathematics? There are three types of true statements in mathematics: **definitions, axioms, and theorems**.

Definition: A **definition** is a labeling statement in which we agree to use an expression to refer to an object, operation, or relation in mathematics.

Definitions are all true statements, because they simply reflect an agreement in the terms of the language. To be precise, these decisions were made without consulting any of us, often decades (if not centuries) before any of us were ever born. An example of a definition would be if we pointed to a clear sky and said: "From now on, let's call this color blue".

Definition: A **theorem** is a statement that we insist on proving before believing that it is true. To **prove** a theorem means to derive it from previously established true statements, using logically correct steps.

If you think about that last definition a little, you will see that no theorem can exist, unless we agree on accepting a few statements to be true, without proving them. These are our "starting true statements".

Definition: An **axiom** is a statement we agree to accept to be true without proof.

Axioms are usually simple, basic statements that are in agreement with our intuition. For example, the statement "*It is possible to draw a straight line from any point to any other point.*" is an axiom. It has been a constant effort to keep the number of axioms to a minimum. We prove a theorem by deriving its statement from statements already established to be true.

To be precise, when we prove our first theorem, we derive its statement from the axioms. When we prove our second theorem, we derive its statement from the axioms and the first theorem. When proving the third theorem, we can use all the axioms, and the first and second theorems. And so on. For our tenth theorem, we have all the axioms and the first nine theorems at our disposal. At this point, we are building a logically sound theory, a unified discipline within mathematics. It is one thing to suspect, to feel, or to have a hunch that something is true. It is entirely different from proving it, with unescapable force of logic.

The ancient Greek mathematician Euclid discussed mathematics in this manner, i.e stating axioms and building a theory by deriving a sequence of theorems from the axioms. (He called axioms postulates.) Mathematicians immediately accepted and embraced this logical approach to the study of mathematics - and it is how it is done still today. Although Euclid has contributed to several parts of mathematics (including geometry and number theory), he completely axiomatized of what we now call classical geometry or Euclidean geometry. He stated five postulates, accepted them to be true and derived most basic theorems of classical geometry.

Part 3 - The Real Number System

We will be studying the properties of the real numbers. Most statements about the real numbers are theorems, and so they can be derived from just a few axioms. These axioms are behind many properties, and we will refer to them often as we review beginning algebra. The axioms of the real numbers are as follows.

A1. Addition is commutative.

For all real numbers x and y ,

$$x + y = y + x$$

A2. Addition is associative.

For all real numbers x, y , and z ,

$$(x + y) + z = x + (y + z).$$

A3. Additive identity.

There exists a real number d such
 that for all real numbers x ,

$$x + d = x.$$

 (This number is 0).

A4. Additive inverse.

For all real numbers x , there exists
 a real number x^* , such that

$$x + x^* = 0.$$

 (We denote this number by $-x$).

M1. Multiplication is commutative.

For all real numbers x and y ,

$$xy = yx.$$

M2. Multiplication is associative.

For all real numbers x, y , and z ,

$$(xy)z = x(yz).$$

M3. Multiplicative identity.

There exists a real number d such
 that for all real numbers x ,

$$xd = x.$$

 (This number is 1.)

M4. Multiplicative inverse.

For all non-zero real number x , there exists
 a real number x^* such that

$$xx^* = 1.$$

 (We denote this number by $\frac{1}{x}$).

D1. The Distributive Law.

For all real numbers x, y , and z ,

$$z(x + y) = zx + zy.$$

Sets that have these properties are called fields and are studied in depth in abstract algebra. We will often re-phrase axiom A4 as '**To subtract is to add the opposite.**' and M4 as '**To divide is to multiply by the reciprocal.**'. We will often re-phrase A3 and M3 as '**Two operations that we can perform on any number without changing its value are: to add zero and to multiply by one.**'

There are also some additional axioms of the real number system about ordering. They are listed here for interested students, but we will mostly focus on the nine axioms listed before.

O1. For all real numbers a and b , either $a \leq b$ or $b \leq a$ or both.

O2. For all real numbers a and b , If $a \leq b$ and $b \leq a$, then $a = b$.

O3. For all real numbers a, b , and c , if $a \leq b$ and $b \leq c$, then $a \leq c$.

O4. For all real numbers a, b , and c , if $a \leq b$ then $a + c \leq b + c$.

O5. For all real numbers a, b , and non-negative c , if $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

A set with all these properties is called an ordered field. However, so far these properties hold for both $\mathbb{Q}$ (the set of all rational numbers) and $\mathbb{R}$ (the set of all real numbers.) What distinguishes these two is the completeness property, that is true for $\mathbb{R}$ but not for $\mathbb{Q}$. This property is studied in higher level mathematics classes such as calculus.

C. Completeness property: Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Identity and Inverse elements

An **identity element** does 'nothing'. It is a unique element of the set that works for every element. The **inverse of an element** is another element such that when the operation is applied to the number and its inverse, the result is the identity. The inverse is different for different numbers.

		Real Numbers with Addition.	Real Numbers with Multiplication
Identity	Systematic Name	additive identity	multiplicative identity
	Defining Property	does 'nothing' in addition	does 'nothing' in multiplication
	Value	0	1
Inverse	Systematic Name	additive inverse of a	multiplicative inverse of a
	Non-Systematic Name	opposite of a	reciprocal of a
	Defining Property	$a + (-a) = 0$ i.e. 'takes' a to the identity	$a \cdot \frac{1}{a} = 1$ i.e. 'takes' a to the identity



Enrichment

1. Look up Euclid's Elements on the internet. (Start at Wikipedia). What is Elements? What is Euclid's contribution to **all** subjects within today's mathematics? What are postulates? List Euclid's five postulates. Explain the significance of these five statements.
2. What is the parallel postulate? Look up the history of Euclid's parallel postulate on the internet. (Start with Wikipedia.) How would we go about proving that an axiom is really a theorem? List statements from geometry that are logically equivalent to the parallel postulate. What exactly could it mean for two axioms to be equivalent to each other?



~ 325 B.C. – 270 B.C

1.2 The Words And and Or

[Youtube link- Lecture](#)

The words *and* and *or* are used differently in mathematics from every day use. Their meaning in mathematics is more restricted than in the English language. First of all, statements or questions in mathematics that use these words are all yes or no questions.

Imagine a cold Monday morning when Sylvia arrives late to her first class, a mathematics class. To make things worse, it is exam day. The teacher stops her at the door and asks: "Did you bring a pen or a pencil?" What does an answer of yes mean? What does an answer of no mean? An answer of yes means that she either has a pen only, or a pencil only, or both. An answer of no means that she has neither.

Her next class is a drawing class where both pen and pencils are needed. There the teacher might ask her: "Did you bring a pen and pencil?" What does an answer of yes mean? What does an answer of no mean? An answer of yes means that she has both a pen and a pencil. An answer of no means that she has only pen, or only pencil, or neither. This is the only allowed use of the words and and or in mathematics.

Suppose that A and B are statements. The statement **A or B** is true when A is true, or when B is true, or when both A and B are true. A or B is false if both A and B are false.

The statement **A and B** is true when both A and B is true. A and B is false when either A is false, or B is false, or both A and B are false. We can express this using truth tables.

Truth table for A or B

A	B	A or B
true	true	true
true	false	true
false	true	true
false	false	false

Truth table for A and B

A	B	A and B
true	true	true
true	false	false
false	true	false
false	false	false

When a single statement is formed by connecting two or more statements with *and* or *or*, we call such a statement a **compound statement**.

Example 1. Determine whether the given statements are true or false.

a) *The sky is blue or the Earth is flat.*

b) *The sky is blue and the Earth is flat.*

Solution: These compound statements are made by connecting two statements. These statements are:

The sky is blue. - this is true. and *The Earth is flat.* - this is false.

a) *The sky is blue **or** the Earth is flat.*

This is true because one true statement is enough for an 'or' statement to be true.

b) *The sky is blue **and** the Earth is flat..*

This is false because for an 'and' statement to be true, both statements must be true.

Example 2. Determine whether the given statements are true or false.

- a) *The number 8 is greater than 8 and is equal to 8.*
- b) *The number 8 is greater than 8 or is equal to 8.*
- c) $3 \leq 3$ d) $5 \leq 3$ e) $2 \leq 10$

Solution: *The number 8 is greater than 8* - this is false. *The number 8 is equal to 8* - this is true.

- a) When we connect a true and a false statement with 'and', the compound statement is false. So, *The number 8 is greater than 8 and is equal to 8.*— is false.
- b) When we connect a true and a false statement with 'or', the compound statement is true. *The number 8 is greater than 8 or is equal to 8.*—is true, and we write it as $8 \geq 8$.
- c) $3 \leq 3$ d) $5 \leq 3$ e) $2 \leq 10$ [Solution - Youtube link](#)

Example 3. Determine whether the given statement is true or false.

- a) *44 is divisible by eight and three is greater than five.*
- b) *44 is divisible by eight or three is greater than five.*
- c) *34 is not divisible by 7, or 39 is divisible by 9.*
- d) *34 is not divisible by 7, and 39 is divisible by 9.*

Solution: a) 44 is divisible by eight. - this statement is false. Three is greater than five. - this statement is also false. If both statements are false, then the compound statement made of those two statements is false, with both *and* and *or* connecting the two.

b) see above with part a).

c), d) [Solution - YouTube Link](#)

Example 4. Consider the numbers 1, 2, 3, 4, 5, 6, and 7.

- a) Find all numbers from the list for which the following statement is true: The number is even and greater than 3.
- b) Find all numbers from the list for which the following statement is true: The number is even or greater than 3.

Solution: a) We simply take all the numbers on the list, one by one, and check whether the statement is true or false.

The number 1 is even and greater than 3. This compound statement is obviously false since both statements making up the compound statement are false. We move on to 2.

The number 2 is even and greater than 3. This compound statement is false because the second statement making up the compound statement is false. In case of *and*, both statements need to be true for a true statement. We continue with 3.

The number 3 is even and greater than 3. This compound statement is false because both statements making up the compound statement are false. We move on to 4.

The number 4 is even and greater than 3. This compound statement is true because both statements making up the compound statement are true. 4 is going to be on our list. We move on to 5.

The number 5 is even and greater than 3. This compound statement is false because the first statement making up the compound statement is false. In case of *and*, both statements need to be true for a true statement. We continue with 6.

The number 6 is even and greater than 3. This compound statement is true because both statements making up the compound statement are true. The number 6 is also going to be on our list. We move on to 7.

The number 7 is even and greater than 3. This compound statement is false because the first statement making up the compound statement is false. In case of *and*, both statements need to be true for a true statement.

We found that 4 and 6 are the numbers on the list for which the given compound statement is true.

- b) This problem is very similar to the previous one. The only difference is the word connecting the two statements is *or* and not *and*. For an *or*-statement to be true, just one of the statements (or both) needs to be true. In this light, we test the numbers from the list again.

The number 1 is even or greater than 3. This compound statement is false because both statements making up the compound statement are false.

2 is even or greater than 3. This compound statement is true because the first statement making up the compound statement is true. In case of *or*, one true statement is enough. 2 is going to be on our list.

3 is even or greater than 3. This compound statement is false because both statements making up the compound statement are false.

4 is even or greater than 3. This compound statement is true because both statements making up the compound statement are true. 4 is going to be on our list.

5 is even or greater than 3. This compound statement is true because the second statement making up the compound statement is true. 5 is going to be on our list.

6 is even or greater than 3. This compound statement is true because both statements making up the compound statement are true. 6 is also going to be on our list.

The number 7 is even or greater than 3. This compound statement is true because the second statement making up the compound statement is true. 7 is going to be on our list.

We found that 2, 4, 5, 6, and 7 are the numbers on the list for which the given compound statement is true.

Example 6. Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Find all numbers x from this list such that

- a) $x \geq 6$ and x is odd b) $x \geq 6$ or x is odd [Solution - Youtube link](#)

Example 7. Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Find all numbers x from this list such that

- a) $x < 7$ or $x > 3$ b) $x < 7$ and $x > 3$ [Solution - Youtube link](#)

Example 8. Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Find all numbers x from this list such that

- a) $x < 7$ and $x < 4$ b) $x < 7$ or $x < 4$ [Solution - Youtube link](#)



Practice Problems

Label each of the given statements as true or false.

1. A week consists of seven days.
2. Every month consist of 31 days.
3. Water is a liquid at room temperature.
4. Water is frozen solid at a temperature of $5F^{\circ}$.
5. $2 + 2 = 5$
6. 5 is an odd number.
7. 3 is an even number.
8. 2 is less than 10.
9. 2 is greater than 10.
10. 2 is equal to 10.
11. A week consists of seven days, or every month consist of 31 days.
12. A week consists of seven days, and every month consist of 31 days.
13. $2 + 2 = 5$, or water is a liquid at room temperature.
14. $2 + 2 = 5$, and water is a liquid at room temperature.
15. 2 is less than 10, or 2 is greater than 10.
16. 3 is an even number or $2 + 2 = 5$.
17. 2 is equal to 10 or 5 is an odd number.
18. 2 is less than 10 and water is frozen solid at a temperature of $5F^{\circ}$.
19. 7 is less than 7 or 7 is equal to 7. (We write this as $7 \leq 7$)
20. 7 is less than 7 and 7 is equal to 7.
21. Consider the numbers 1, 2, 3, 4, 5, 6, 7, and 8.
 - a) Find all numbers from the list for which the following is true: The number is odd and less than 6.
 - b) Find all numbers from the list for which the following is true: The number is odd or less than 6.



Enrichment

1. Interpret A or B or C as $(A \text{ or } B) \text{ or } C$ and create a truth table for this compound statement. How many different cases are there?
2. Interpret A and B and C as $(A \text{ and } B) \text{ and } C$ and create a truth table for this compound statement. How many different cases are there?
3. Create a truth table for the compound statement $(A \text{ and } B) \text{ or } C$. How many different cases are there?
4. Create a truth table for the compound statement $A \text{ and } (B \text{ or } C)$. How many different cases are there?

Problem Set 1

1. Label each of the following statements as true or false.

- | | | |
|--------------------------|-------------------------|------------------------|
| a) 20 is an even number. | d) 8 is greater than 8. | g) 7 is an odd number. |
| b) 32 is an odd number. | e) 10 is less than 20. | h) 7 is less than 9. |
| c) 8 is greater than 7 | f) 7 is an even number. | |

2. Label each of the following statements as true or false.

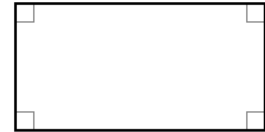
- | | |
|--|--|
| a) 7 is an odd number or 8 is greater than 7. | d) 12 is greater than 5 and 5 is greater than 8. |
| b) 7 is an odd number and 8 is greater than 7. | e) 12 is greater than 5 or 5 is greater than 8. |
| c) 12 is less than or equal to 12. | f) 7 is an odd number or 7 is an even number. |

For the next exercise, let us recall the definitions of a few geometric objects.

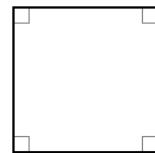
Definition: A **parallelogram** is a four-sided polygon (or quadrilateral) with two pairs of parallel sides.



Definition: A **rectangle** is a four-sided polygon with four right angles.



Definition: A **square** is a rectangle with four equal sides.



1. Label each of the following statements as true or false.

- | | |
|-------------------------------------|---|
| a) Every square is a rectangle. | d) Every rectangle is a parallelogram. |
| b) Every rectangle is a square. | e) Every parallelogram is a square. |
| c) Every square is a parallelogram. | f) If two adjacent sides of a rectangle have equal sides, then the rectangle is a square. |

2. Consider the numbers 1,2,3,4,5,6,7 and 8. Find all numbers from the list with the given property.

- | | |
|---|--|
| a) The number is odd and less than 5. | d) The number is even or greater than 3. |
| b) The number is odd or less than 5. | e) The number is even or the number is odd. |
| c) The number is even and greater than 3. | f) The number is even and the number is odd. |

Chapter 2

2.1 The Set of All Natural Numbers

The **set of all natural numbers**, (sometimes also called the set of all counting numbers), denoted by $\mathbb{N}$, is the set

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

It is easy to imagine that the set of all natural numbers was the first set of numbers at which human beings looked. The four basic operations can be defined and performed on natural numbers as follows.

Addition, denoted by $+$, is defined as we usually think of addition: the addition of two natural numbers is obtaining the total amount of those quantities combined.

In the statement $3 + 7 = 10$, we say that 3 and 7 are **addends** and 10 is called the **sum** of 3 and 7.

$$3 + 7 = 10$$

addend addend sum

If we add two natural numbers, the sum is always a natural number. This is called **closure**. The set of all natural numbers is closed under addition: If n and m are natural numbers, then $n + m$ is also a natural number.

Subtraction, denoted by $-$, is a mathematical operation that represents the operation of removing objects from a collection.

In the statement $18 - 5 = 13$, we say that 18 is the **minuend** and 5 is the **subtrahend**, and 13 is called the **difference** of 18 and 5.

$$18 - 5 = 13$$

minuend subtrahend difference

If we subtract a natural number from another natural number, the difference may or may not exist within $\mathbb{N}$. For example, the subtraction $8 - 5$ results in a natural number but the subtraction $3 - 11$ does not. In other words, the set of all natural numbers is NOT closed under subtraction.



Addition and subtraction have interesting properties we did not discuss here. Highly Recommended! Check out what the great Professor Herbert Gross has to say about these operations. Please note that he refers to the set of whole numbers. Whole numbers are all the natural numbers and zero.



Addition and Subtraction

Multiplication, denoted by $\cdot$, or by $\times$, or by nothing at all between two objects, is defined as we usually think of

multiplication

$$3 \cdot 7 = 21 \quad \text{or} \quad 3 \times 7 = 21 \quad \text{or} \quad (3)7 = 21 \quad \text{or} \quad 3(7) = 21$$

Please note that in the last two equations, the **parentheses do not indicate multiplication**. The parentheses helps us interpret 3 and 7 as two separate numbers and not the number 37. Once we see two numbers with no operation sign between them, that **NOTHING** indicates multiplication.

In the statement $3 \cdot 7 = 21$, we say that 21 is the **product** of 3 and 7. We also say that 3 and 7 are **divisors** or factors of 21.

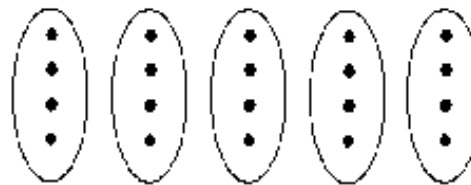
$$3 \cdot 7 = 21$$

↗ factor ↑ factor ↖ product

If we multiply two natural numbers, the product is always a natural number. In other words, the set of all natural numbers is closed under multiplication: if n and m are natural numbers, then nm is also a natural number.

Division, denoted by $\div$ or by $/$ is defined as we usually think of division. One example is: if we have 20 dots and we circle together every four dots, how many packages of four do we obtain? The answer is clearly five, because five packages of four will account for 20 dots.

$$20 \div 4 = 5 \quad \text{or} \quad \frac{20}{4} = 5$$



In the statement $20 \div 4 = 5$, we say that 20 is the **dividend**, 4 is the **divisor**, and 5 is the **quotient** of 20 and 4.

$$20 \div 4 = 5$$

↗ dividend ↑ divisor ↖ quotient

If we divide a natural number by another natural number, the quotient may or may not exist within $\mathbb{N}$. For example, the division $12 \div 3$ results in a natural number, but the division $20 \div 7$ does not. In other words, the set of all natural numbers is **NOT** closed under division.



Multiplication and division have interesting properties we did not discussed here. Highly Recommended! Check out what the great Professor Herbert Gross has to say about these operations.



Multiplication and Division

The video links are part of



Professor Herbert Gross's online self-study arithmetics course. Check it out!

**Discussion:**

1. Consider the multiplication $4 \cdot 9 = 36$ and the division $30 \div 6 = 5$. While in the division all of 30, 6, and 5 have different names, both 4 and 9 are simply called factors in the multiplication. Can you explain why this will not cause any problems?
2. Under which of the four basic operations (addition, subtraction, multiplication, division) is $\mathbb{N}$ closed?

**Enrichment**

If you haven't already done so, read our lecture notes on axioms, because you need to understand the concepts of axioms and theorems for this exercise.

1. The set of all natural numbers, $\mathbb{N}$ was not defined rigorously. Instead, we gave an intuitive description of $\mathbb{N}$. However, mathematicians insisted on axiomatizing $\mathbb{N}$. This means that they established a set of axioms from which many theorems can be derived. The resulting collection of true statements is the same as what we get starting with the intuitive definition.

Research the Peano axioms. Feel free to start at Wikipedia. What are the Peano axioms?

Note: when you look at the Peano axioms, you may notice that according to Wikipedia, 0 is a natural number. In our class, 0 was not defined as a natural number. Do not let yourself be annoyed or confused by the difference. Both conventions are very common.

2.2 The Order of Operations Agreement

The order of operations rule is an agreement among mathematicians, it simplifies notation. P stands for parentheses, E for exponents, M and D for multiplication and division, A and S for addition and subtraction. Notice that M and D are written next to each other. This is to suggest that multiplication and division are equally strong. Similarly, A and S are positioned to suggest that addition and subtraction are equally strong.

P
E
M D
A S

This is the hierarchy, and there are two basic rules.

1. **Between two operations that are on different levels of the hierarchy, we start with the operation that is higher.** For example, between a division and a subtraction, we start with the division since it is higher in the hierarchy than subtraction.
2. **Between two operations that are on the same level of the hierarchy, we start with the operation that comes first from reading left to right.**

Example 1. Perform the indicated operations. $20 - 3 \cdot 4$

Solution: We observe two operations, a subtraction and a multiplication. Multiplication is higher in the hierarchy than subtraction, so we start there.

$$\begin{aligned} 20 - 3 \cdot 4 &= && \text{multiplication} \\ 20 - 12 &= && \text{subtraction} \\ &= && \boxed{8} \end{aligned}$$

Example 2. Perform the indicated operations. $36 \div 3 \cdot 2$

Solution: It is a common mistake to start with the multiplication. The letters M and D are in the same line because they are equally strong; among them we proceed left to right. From left to right, the division comes first.

$$\begin{aligned} 36 \div 3 \cdot 2 &= && \text{division} \\ 12 \cdot 2 &= && \text{multiplication} \\ &= && \boxed{24} \end{aligned}$$

Example 2v. Perform the indicated operations.

$$\text{a) } 40 \div (5 \cdot 2) \quad \text{b) } (40 \div 5) \cdot 2 \quad \text{c) } 40 \div 5 \cdot 2$$

[Solution - Youtube link](#)

Example 3. Perform the indicated operations. $24 \div 4 \div 2$

Solution: It is essential to perform these two divisions left to right. If we proceeded differently, we would get a different result.

$$\begin{aligned} 24 \div 4 \div 2 &= && \text{first division from left} \\ 6 \div 2 &= && \text{division} \\ &= && \boxed{3} \end{aligned}$$

Example 4. Perform the indicated operations. $(8 - 5)^2 - 5 + 2$

Solution: We start with the subtraction within the parentheses. Then we can drop the parentheses.

$$\begin{aligned}
 (8 - 5)^2 - 5 + 2 &= 3^2 - 5 + 2 && \text{exponentiation} \\
 &= 9 - 5 + 2 && \text{subtraction} \\
 &= 4 + 2 && \text{addition} \\
 &= \boxed{6}
 \end{aligned}$$

Example 4v. Simplify each of the given expressions.

a) $14 - 4(3^2 - 7)$ b) $(14 - 4)(3^2 - 7)$

[Solution - Youtube link](#)

Notations for Multiplication

There are different ways we denote multiplication. Suppose we wanted to express the multiplication 2 times 3, our options are $\cdot$ (dot), or $\times$ (cross), or *nothing*.

$2 \cdot 3$	or	2×3	or	$(2)(3)$ or $2(3)$ or $(2)3$
This is the preferred notation		We will almost never use this. It's a dinosaur, it is going away....		We will almost never use this with positive numbers

It is a common misconception to believe that parentheses mean multiplication. This is not true. **Parentheses never denote multiplication.** In the expression $2(3)$, the parentheses tell us that we are not looking at the two-digit number 23, but rather at the separate numbers 2 and 3, with *nothing* between them. In written notation, multiplication is the default operation. In other words, if we see two numbers with nothing between them, the *nothing* indicates multiplication. This becomes more clear if we consider expressions such as $2x$ or ab . There is no operation sign (or parentheses) in these expressions and yet we know that the operation is multiplication. It is the *nothing* between 2 and x and between a and b means multiplication.

Understanding Parentheses

Parentheses have no meaning on their own. They merely help us to understand the context in the written language, like punctuation does in written English. Some parentheses are **grouping symbols** overwriting order of operations. Others clarify boundaries of numbers. For example, it is possible to write $2(3)$ instead of $2 \cdot 3$. The parentheses in $2(3)$ is not a grouping symbol. Rather, it helps us understand that we are not looking at the two-digit number 23. We will refer to this as **clarifying parentheses**. One easy way to tell the difference between the two types of parentheses is that if there is no operation inside a pair of parentheses, then it cannot be a grouping symbol.

We will often see expressions with several pairs of parentheses. Luckily, there are only two possibilities.

$4(3 - 1) - (7 \cdot 2 - 9)$	$20 - (12 - 2(5 \cdot 8 - 6^2) + 1)$
one pair of parentheses follows the other	one pair of parentheses is nested inside the other

When faced with an order of operations problem, *we must pair the parentheses before starting the computation*. Students are encouraged to use colored pencils to make the pairing more visual such as in $20 - (12 - 2(5 \cdot 8 - 6^2) + 1)$. Sometimes authors aim to make notation easier to read by using pairs of parentheses with different shapes, such as $[]$ or $\{ \}$. These different looking parentheses serve as regular grouping symbols, they only made to look differently. For example, $20 - (12 - 2[5 \cdot 8 - 6^2] + 1)$ might be easier to read than $20 - (12 - 2(5 \cdot 8 - 6^2) + 1)$.

When we have several pairs of parentheses following each other, we perform them left to right. When we are dealing with parentheses nested inside others, we start with the innermost one.

Sometimes we have a case of an **invisible parentheses**. When an operation sign encloses entire expressions, it serves as a grouping symbol. For example, there is no parentheses in sight in the expression $\frac{32-4}{9-2}$, and division comes before subtraction. However, the division bar stretching over the entire expression indicates that we must perform the subtraction on the top and on the bottom, and only then divide. This becomes obvious if we switch the notation from the division bar to the symbol $\div$.

$$\frac{32-4}{9-2} = (32-4) \div (9-2)$$

In this example, the division bar is easier to read, so it will be the notation of choice. Note however, that students often make a mistake when entering things like this in the calculator. The correct answer is 7 but the calculator will give us the wrong answer if we enter it incorrectly as $32 - 4 \div 9 - 2$. In these cases, the parentheses must become visible.

Example 5. Consider the expression $4(3-1) - (7 \cdot 2 - 9)$

- How many operations are there in the expression?
- Simplify the expression by applying the order of operations agreement. For each step, write a separate line.

Solution: a) We "scan" the expression left to right, and count the operations.

- | | |
|---|-----------------------------------|
| 1. multiplication between 4 and $(3-1)$ | 4. multiplication between 7 and 2 |
| 2. subtraction $3-1$ | 5. subtracting 9 |
| 3. subtraction between the two expressions within the parentheses | |

So there are five operations.

- We have two pairs of parentheses. We will work them out, left to right. We start with the subtraction $3-1$. Once this subtraction is performed, we no longer need the parentheses. Instead, we will denote the multiplication using the dot notation.

$4(3-1) - (7 \cdot 2 - 9)$	$=$	$4 \cdot 2 - (7 \cdot 2 - 9)$	Now we work on the other parentheses.
			Multiplication before subtraction
	$=$	$4 \cdot 2 - (14 - 9)$	Subtraction within parentheses
			Drop parentheses
	$=$	$4 \cdot 2 - 5$	Multiplication
	$=$	$8 - 5$	Subtraction
	$=$	$\boxed{3}$	

Example 6v. Simplify each of the given expressions.

a) $67 - 4(5 + 3(4 \cdot 7 - 5^2))$

[Solution - Youtube link](#)

b) $\frac{24 \div 4 \cdot 3}{9 - 4 + 1}$

[Solution - Youtube link](#)

c) $2(7 + 3(2 \cdot 8 - 5))$

[Solution - Youtube link](#)

Example 7. Evaluate the expression $\left(7 - \left(5 - \left(11 - (7 - 4)^2\right)^2\right)^2\right)^2$

Solution: This problem is a good example for why it is a good thing to make only one step in each line. Instead of sorting out all the things to do, we just need to find the one operation to do. Once we write down our next line, we have the same goal, but this time we are looking at a simpler problem. The first operation to perform is the subtraction $7 - 4$ in the innermost parentheses. In this example, we will immediately drop parentheses that are no longer necessary. This is the recommended practice.

$$\begin{aligned}
 &\left(7 - \left(5 - \left(11 - (7 - 4)^2\right)^2\right)^2\right)^2 = \\
 &= \left(7 - \left(5 - (11 - 3^2)^2\right)^2\right)^2 && \text{exponentiation in the innermost parentheses} \\
 &= \left(7 - \left(5 - (11 - 9)^2\right)^2\right)^2 && \text{subtraction in the innermost parentheses} \\
 &= \left(7 - (5 - 2^2)^2\right)^2 && \text{exponentiation in the innermost parentheses} \\
 &= \left(7 - (5 - 4)^2\right)^2 && \text{subtraction in the innermost parentheses} \\
 &= (7 - 1^2)^2 && \text{exponentiation in parentheses} \\
 &= (7 - 1)^2 && \text{subtraction in parentheses} \\
 &= 6^2 && \text{exponentiation} \\
 &= \boxed{36}
 \end{aligned}$$

Example 7v. Simplify $\left((3^2 - 5)^2 - 14\right)^2 - 1^3$

[Solution - Youtube link](#)



Sample Problems

Simplify each of the following expressions by applying the order of operations agreement.

1. $2 \cdot 3^2 - (6^2 - 2 \cdot 5) \div 2$

4. $8^2 - 3^2$

7. $\frac{3 + 2(20 - 3^2 - 5)}{3^2 - 2^2} + 4^1$

2. $18 - 7 - 3$

5. $(8 - 3)^2$

3. $5^2 - 2(10 - 2^2)$

6. $(3^3 - 4 \cdot 5 + 2)^2$



Practice Problems

Simplify each of the following expressions by applying the order of operations agreement.

1. $2 \cdot 5^2 - (6 \cdot 5 - 3^2) \div 3$

7. $\frac{(5 - 3)^2}{2^2}$

12. $30 - (2(15 - 2^3) - 2^2)$

2. $10^2 - 7^2$

8. $120 \div 6 \cdot 2$

13. $4(3(2(2^2 - 1) + 1) - 1) + 5$

3. $(10 - 7)^2$

9. $((7 - 4)^2 - 5)^2 - 1$

14. $\frac{2(3^3 - 4 \cdot 5) - 2^2}{4^2 - (3^2 + 2)}$

4. $20 - 7 - 1$

10. $\frac{22 - 3^2 + 2(20 - 3^2 - 5)}{3^2 - 2^2}$

15. $\left(\left(\left(19 - (8 - 2^2)^2\right)^2 - 7\right)^2 - 3\right)^2$

5. $2^3 - (11 - 3^2)^2$

11. $\frac{5 + (5^2 - 3^2)}{3^2 - 2 \cdot 1^8}$

6. $\frac{5^2 - 3^2}{2^2}$



Enrichment

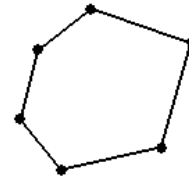
- Place one or more pairs of parentheses into the expression on the left-hand side to make the equation true.

$$12 - 2 \cdot 3 - 1^2 + 2 - 3 + 4 = 20$$

2.3 Perimeter and Area of a Rectangle

Part 1 - Definition

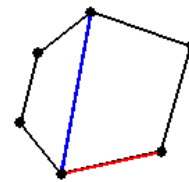
Definition: An n -sided **polygon** is n many points on a plane, connected with straight lines, as shown.



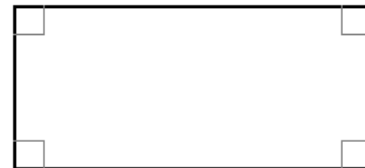
In every polygon, the number of points (also called vertices) and the number of sides are equal. Polygons with four sides are also called **quadrilaterals**.

Definition: A line segment connecting two 'neighboring' points is a **side** of the polygon. A line segment connecting two points, if it is not a side, then we call it a **diagonal**.

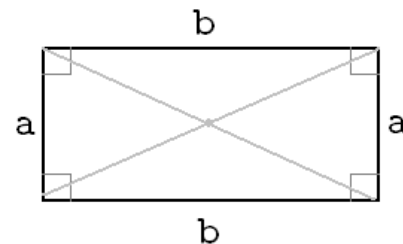
On the picture, the red line segment is a side, and the blue line segment is a diagonal of the polygon.



Definition: A **rectangle** is a four-sided polygon with four right angles.

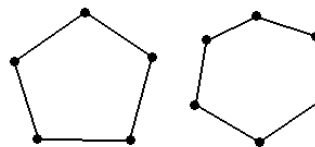


We also refer to four-sided polygons as **quadrilaterals**. Although they are not part of the definition of a rectangle, there are a few important properties of rectangles that we should keep in mind. **In every rectangle, the opposite sides have equal lengths. The two diagonals of any rectangle are equally long and they bisect each other.** (To bisect something means to cut it into two equal parts.)



Enrichment

Draw and count the diagonals in each of the polygons shown.



Part 2 - Perimeter

Definition: The **perimeter** of any geometric object is the length of its boundary.

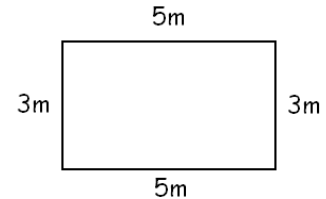
In general, we can always think of the perimeter as a fencing problem. If we have a property, how long of a fence do we need to buy to completely fence around the property? We will denote perimeter by P . Perimeter is a *length*, we measure it in meters (m), centimeters (cm), inches (in), feet (ft), kilometers (km), or miles (mi).

Example 1. Compute the perimeter of a rectangle with sides 3 meters and 5 meters long.

Solution: If we think fencing, we mentally walk around a rectangle-shaped property to figure out how much fencing to buy. That is the same as simply adding the lengths of all four sides. The lengths of only two sides were given, but this should not be a problem. The opposite sides of a rectangle are equally long. Thus we can compute the perimeter as

$$P = 3\text{m} + 5\text{m} + 3\text{m} + 5\text{m} = 16\text{m}$$

So the perimeter of this rectangle is $P = 16\text{m}$.



Example 1v. Find the perimeter and area of the rectangle with sides 11ft and 9ft long. [Solution - Youtube link](#)

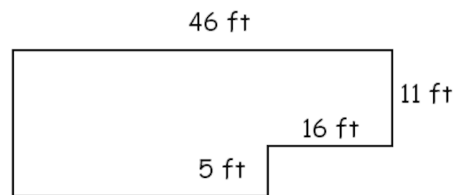
In general, the perimeter of a quadrilateral with sides a , b , c , and d is $P = a + b + c + d$. In case of a rectangle, the opposite sides are equally long, so $c = a$ and $d = b$ and this makes the perimeter formula simpler.

Theorem: The perimeter of a rectangle with sides a and b is $P = 2a + 2b$.

This means that in our previous example, the perimeter of a 3m by 5m rectangle is

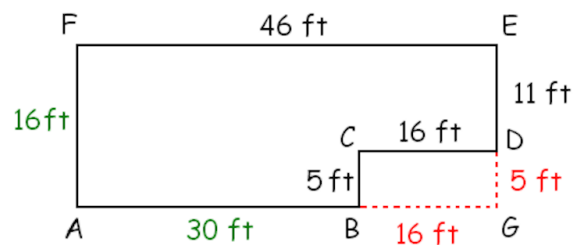
$$P = 2 \cdot 3\text{m} + 2 \cdot 5\text{m} = 6\text{m} + 10\text{m} = 16\text{m}$$

Example 2. Find the perimeter of the object shown. Angles that look like right angles are right angles.



Solution: It is much easier to discuss geometry if we label points and sides. Consider the picture shown. We are clearly missing the lengths of some sides for the perimeter, so we need to figure out those lengths first.

We first draw line AB beyond point B and line ED beyond point D as shown. These lines intersect each other in point G . Since $BGDC$ is a rectangle, its opposite sides are equally long. Therefore, $BG = 16\text{ft}$ and $DG = 5\text{ft}$.



The quadrilateral $AGEF$ is also a rectangle, and so FA is the same length as EG .

$FA = 16\text{ft}$, because $FA = EG = ED + DG = 11\text{ft} + 5\text{ft} = 16\text{ft}$. Also, AG is as long as FE .

$$\begin{aligned}AG &= FE \\AB + BG &= 46\text{ft} && \text{we know } BG = 16\text{ft} \\AB + 16\text{ft} &= 46\text{ft} && \text{subtract 16ft} \\AB &= 30\text{ft}\end{aligned}$$

We can now compute the perimeter, starting from point A , and moving counterclockwise.

$$P = AB + BC + CD + DE + EF + FA = 30\text{ft} + 5\text{ft} + 16\text{ft} + 11\text{ft} + 46\text{ft} + 16\text{ft} = \boxed{124\text{ft}}$$

Part 3 - Area

The **area** of a geometric object is a measurement of its surface.

Understanding and remembering area formulas are easier if we know how they were derived. While we could think about perimeter as a fencing problem, area can be thought of as follows. Suppose a geometric object is a room. How many tiles do we need to buy to cover the entire room? Of course, we have to first agree on the size of the tiles we use to measure area.

Definition: The **area** of a 1ft by 1ft square (shown on the picture) is defined to be 1ft^2 (square-foot). Similar definitions can be formulated with mi^2 , cm^2 , in^2 , etc. The area of any object, measured in ft^2 , is the number of these 1ft by 1ft square tiles needed to cover the object, cutting and pasting allowed.

1 ft

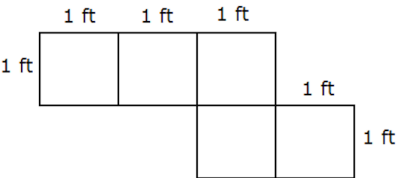
1 ft

$A = 1\text{ ft}^2$

Area is not a length like perimeter. Area is measured in square-meters (m^2), square-centimeters (cm^2), square-inches (in^2), square-feet (ft^2), square-kilometers (km^2), or square-miles (mi^2). etc., and is usually denoted by A .

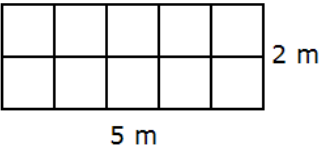
Example 3. Find the area of the figure shown on the picture.

Solution: We simply count the tiles we need to cover this object. Since the figure can be covered using five unit tiles, its area is $A = 5\text{ft}^2$.



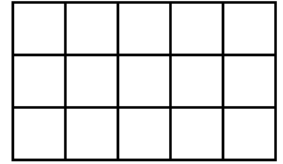
Example 4. Find the area of the figure shown on the picture.

Solution: Since the figure can be covered using ten 1m by 1m squares, its area is $A = 10\text{m}^2$.



Theorem: The area of a rectangle with sides x and y is $A = xy$.

Proof: We will not formally prove this theorem. Instead, we will just informally argue for this formula. The main idea should be clear from the previous example. Consider a rectangle with sides 3m and 5m. The area of this rectangle will be as many square-meters as many 1m by 1 m square tiles are needed to cover it.



Once we place a grid on the rectangle, it is easy to see how many such squares are needed. The rectangle is composed of five columns of squares, where each column consists of three squares. Thus we split the rectangle into fifteen unit tiles, and so the area is 15m^2 . This shows that as long as the lengths of the sides are integers, we can place a grid on it, and the number of unit square tiles is the product of the length of the two sides.

In reality, this theorem is very difficult to prove, because not all side lengths happen to be integers. Mathematicians proved that this formula is true even if the sides of the rectangle are not integers.

Another interesting fact is that logically, we counted how many meter² we have. The computation for the area, however, is slightly different with regards to units. Instead of counting meter², we literally multiply meter by meter.

$$A = xy = 3\text{m}(5\text{m}) = 15\text{m}^2 \quad \text{and not} \quad 15 \cdot 1\text{m}^2$$

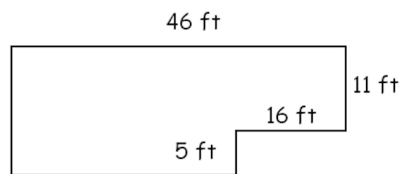
Area computation will always yield for the right unit.

Example 5. Find the area of a rectangle with sides 13 in and 7 in.

Solution: We apply the formula.

$$A = xy = 13\text{in}(7\text{in}) = \boxed{91\text{in}^2}$$

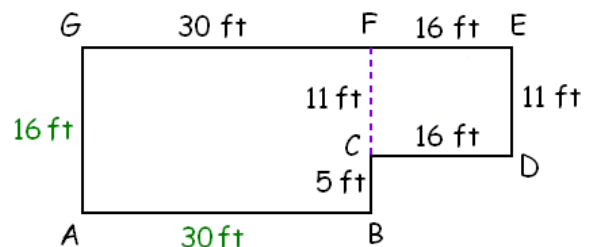
Example 6. Find the area of the object shown. Angles that look like right angles are right angles.



Solution: To compute the area, we have two options.

Method 1: We can think of the region as the sum of two rectangles: $CDEF$ is a rectangle with sides 11ft by 16ft, and $ABFG$ is a rectangle with sides 30ft by 16ft. So the area is

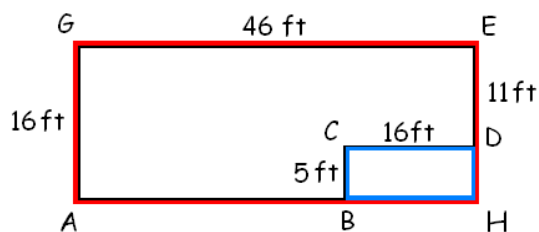
$$\begin{aligned} A &= A_{CDEF} + A_{ABFG} \\ &= 11\text{ft} \cdot 16\text{ft} + 16\text{ft} \cdot 30\text{ft} \\ &= 176\text{ft}^2 + 480\text{ft}^2 = \boxed{656\text{ft}^2} \end{aligned}$$



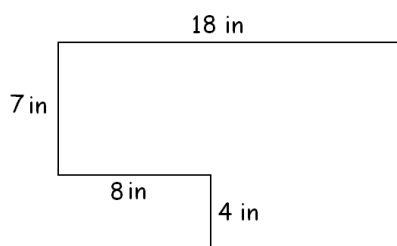
Method 2: We can also think of the region as a big rectangle from which a corner was removed.

The big (red) rectangle, $AHEG$ has sides 16ft and 46ft long. The smaller (blue) rectangle has sides 5ft by 16ft. So the area is the difference.

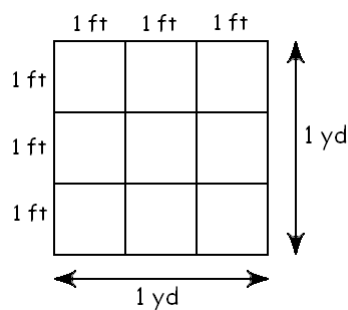
$$\begin{aligned} A &= A_{AHEG} - A_{BHDC} \\ &= 46\text{ft} \cdot 16\text{ft} - 16\text{ft} \cdot 5\text{ft} \\ &= 736\text{ft}^2 - 80\text{ft}^2 = \boxed{656\text{ft}^2} \end{aligned}$$



Example 7. Find the perimeter and area of the figure shown. [Solution - Youtube link](#)



Discussion: One yard equals to three feet. Consider the picture below and discuss: how many square-feet is one square-yard? Can you show the same result algebraically?

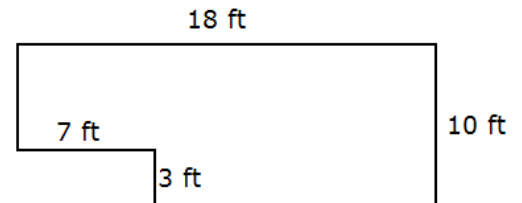




Practice Problems

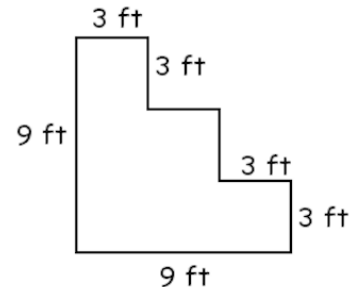
1. Compute the perimeter and area of a rectangle with sides 12cm by 8cm.
2. Consider the figure shown. Angles that look like right angles are right angles.

- a) Compute the perimeter of the figure. Include units in your computation and answer.
- b) Compute the area of the figure. Include units in your computation and answer.



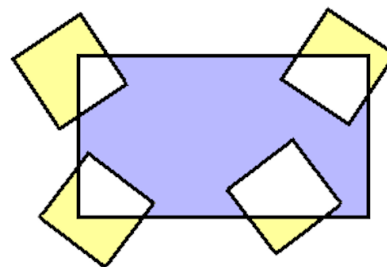
3. Consider the figure shown. Angles that look like right angles are right angles.

- a) Compute the perimeter of the figure. Include units in your computation and answer.
- b) Compute the area of the figure. Include units in your computation and answer.



Enrichment

The rectangle shown on the picture has sides 2cm and 3cm long. The four other shapes are all squares with sides 1cm. Which region has a greater area, the blue or the yellow?



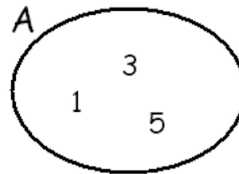
2.4 Introduction to Set Theory

Definition: A **set** is a collection of objects. Two sets are equal if they contain the same objects.

Sets are usually denoted by uppercase letters. There are several ways a set could be given. We can describe a set using English language. In case of small sets, we can also simply list its elements separated by commas and enclosed in braces $\{ \}$.

Example 1. Let M be the set of all one-digit natural numbers. Re-write this set by listing its elements.

We use the braces and list all one-digit natural numbers. $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.



We can also describe a set using a diagram.

We depicted $A = \{1, 3, 5\}$.

Some famous sets have their own set theory label. For example, we already know the infinite set $\{1, 2, 3, 4, \dots\}$. This is the set of all natural numbers or counting numbers, and it is denoted by $\mathbb{N}$.

Sometimes we need to be more descriptive when specifying sets.

Example 2. $S = \{x^2 : x \text{ is a natural number and } x \leq 5\}$.

We read this as S is a set containing x^2 , where x is a natural number and x is less than or equal to 5. Of course, such a small set can be expressed much simpler, by listing its elements, as $S = \{1, 4, 9, 16, 25\}$.

This notation is often very useful when describing infinite sets.

Definition: A set is a collection of objects. The objects that make up the set are called the **elements** or **members** of the set, or that it **belongs** to the set.

Notation: If x is an element of a set S , we write by $x \in S$. If y is not an element of S , we write $y \notin S$.

Suppose that $A = \{1, 3, 5\}$. The following statements are all true.

$3 \in A$	read as: 3 belongs to A	$4 \notin A$	read as: 4 does not belong to A
$5 \in A$	read as: 5 is an element of A	$-6 \notin A$	read as: -6 is not an element of A

Example 3. Suppose that $A = \{1, 3, 5\}$ and that $\mathbb{N}$ is the set of all natural numbers, in short, $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. Determine whether the given statements are true or false.

- a) $1 \in A$ b) $2 \in A$ c) $-1 \in \mathbb{N}$ d) $5 \notin \mathbb{N}$ e) $4 \notin A$

Solution: a) The statement $1 \in A$ reads: *1 belongs to set A*. This is true as 1 is an element of set A.
 b) The statement $2 \in A$ reads: *2 belongs to set A*. This is not true as 2 is not an element of set A.
 c) The statement $-1 \in \mathbb{N}$ reads: *-1 belongs to the set of all natural numbers*.
 (In short, -1 is a natural number). This statement is false.
 d) The statement $5 \notin \mathbb{N}$ reads: *5 does not belong to the set of all natural numbers*.
 (In short, 5 is not a natural number). This statement is false.
 e) The statement $4 \notin A$ reads: *4 does not belong to set A*. This statement is true.

Definition: Two sets are **equal** if they contain the same objects. We use the symbol $=$ to denote equal sets.

When writing a set, the order of listing and repetition of its elements does not change a set.

Example 4. Let A be the set of odd natural numbers between 0 and 6. Suppose further that $B = \{5, 1, 3\}$, and that $C = \{1, 5, 1, 5, 1, 1, 1, 3, 3\}$. Then all three sets are equal to each other, i.e $A = B = C$.

Since we are free to list the elements of a set any way we want to, it is often strategic to keep things organized by listing elements in increasing order. In this case, $A = B = C = \{1, 3, 5\}$. Sometimes we have reasons to part from this convention.

Definition: The **set of all integers**, denoted by $\mathbb{Z}$, is the set containing all natural numbers, their opposites, and zero.

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \dots\}$$

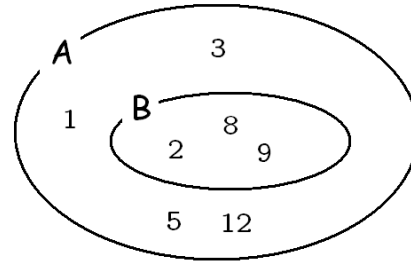
Some people prefer to present the set of all integers sort of organized, as $\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$. The disadvantage here is that both the beginning and the end of this infinite list goes on forever. More precisely, there is no beginning and no end. Both presentations are commonly used.

Definition: There is a unique set that contains no elements. It is called the **empty set** and is denoted by $\emptyset$ or by $\{\}$.

Definition: Set B is a **subset** of set A if all elements of B also belong to A . Notation: $B \subseteq A$

There is another way to express this relationship. B is a subset of A if for all things x in the world, if x is an element of B , then x is also an element of A . This approach might be helpful later.

Example 5. Suppose that $A = \{1, 2, 3, 5, 8, 9, 12\}$ and $B = \{2, 8, 9\}$. Then B is a subset of A .



Example 5v. Which statement is true?

$$\{1, 3, 6\} \subseteq \{1, 2, 3, 4\}$$

$$\{1, 3, 6\} \not\subseteq \{1, 2, 3, 4\}$$

[Solution - Youtube link](#)

Example 5W. Which statement is true?

$$\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$\{2, 4, 6\} \not\subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

[Solution - Youtube link](#)

Example 6. Suppose that $X = \{a, b, d, f\}$ and $Y = \{a, b, c, d, e, f, g\}$. Then $X \subseteq Y$.

Example 6v. Which statement(s) are true?

$$\{b, c, d, e\} \subseteq \{a, b, c, d, e, f, g, h\}$$

$$\{b, c, d, e\} \not\subseteq \{a, b, c, d, e, f, g, h\}$$

[Solution - Youtube link](#)

Example 7. Suppose that $S = \{1, 4, 9, 16\}$. Then $S \subseteq S$.

While this might look strange at first, the definition of subset applies. Every element of S is an element of S . Perhaps this statement is similar to $5 \leq 5$. For every number x , the statement $x \leq x$ is true.

Even more interestingly, the empty set is also a subset of every set. This is because the definition applies, even if strangely so. For every object x in the world, if x is in the empty set (it's not), then it is also in set A . We say that this statement is **vacuously true**.

Theorem: For all sets S , the following are both true: $\emptyset \subseteq S$ and $S \subseteq S$.

Example 8. If $\mathbb{N}$ is the set of all natural numbers and $\mathbb{Z}$ is the set of all integers, then $\mathbb{N} \subseteq \mathbb{Z}$.

Example 9. Suppose that E is the set of all even natural numbers, $E = \{2, 4, 6, 8, 10, \dots\}$, and recall that $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. Then $E \subseteq \mathbb{N}$.

Example 10. Suppose that L is the set of all letters in the English alphabet, and V is the set of vowels in the English alphabet. Then $V \subseteq L$.

Example 11. Let M be the set of all mammals and D the set of all dogs. Then D is a subset of M , $D \subseteq M$.

Example 11v. True or false?

$$\{x \mid x \text{ is a cat}\} \subseteq \{x \mid x \text{ is a black cat}\}$$

$$\{x \mid x \text{ is a cat}\} \not\subseteq \{x \mid x \text{ is a black cat}\}$$

[Solution - Youtube link](#)

Example 12. List all subsets of the given set if

a) $A = \{1\}$ b) $B = \{1, 2\}$ c) $C = \{1, 2, 3\}$

Solution: a) The set $\{1\}$ has two subsets: $\emptyset$ and $\{1\}$.

b) The set $\{1, 2\}$ has four subsets: $\emptyset$, $\{1\}$, $\{2\}$, and $\{1, 2\}$.

c) As the given sets become larger, it becomes important to be systematic when finding the subsets. Let us list the subsets of C by organizing by the number of its elements.

0-element subsets: $\emptyset$

1-element subsets: $\{1\}, \{2\}, \{3\}$

2-element subsets: $\{1, 2\}, \{1, 3\}, \{2, 3\}$

3-element subsets: $\{1, 2, 3\}$

So there are 8 subsets.



Practice Problems

- Suppose that S is a set defined as $S = \{-2, 4, 5, 16\}$ and recall that $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. Determine whether the given statements are true or false.
 - $-2 \in S$
 - $-2 \in \mathbb{N}$
 - $-3 \notin \mathbb{N}$
 - $5 \notin S$
 - $1 \in \mathbb{N}$
- Let $A = \{1, 2, 5, 8, 9\}$ and $B = \{2, 4, 6, 8\}$. Draw a Venn diagram depicting these sets.
- Suppose that $P = \{1, 7, 8\}$ and $Q = \{1, 2, 5, 7, 8\}$. Label each of the following statements as true or false.
 - $P \subseteq \mathbb{Z}$
 - $Q \subseteq P$
 - $P \subseteq Q$
 - $\mathbb{N} \subseteq Q$
 - $\emptyset \subseteq P$
- Find each of the following sets and if possible, present them by listing their elements.
 - $S = \{x : x \text{ is a natural number such that } x < 4 \text{ and } x < 7\}$
 - $A = \{a : a \text{ is a natural number such that } a < 4 \text{ or } a < 7\}$
 - $P = \{y : y \text{ is an integer such that } y > 3 \text{ and } y < 8\}$
 - $M = \{y : y \text{ is an integer such that } y > 3 \text{ or } y < 8\}$
- Suppose that $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 3, 4\}$. Explain why $Y \in X$ is a false statement.
- Find two sets A and B so that both $A \subseteq B$ and $B \subseteq A$ are true.
- List all subsets of $A = \{1, 2, 3, 4\}$.



Enrichment

- Suppose that A and B are sets such that $A \cap B = \{1, 2, 5\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$. How many different sets are possible for A ?
- Our junior class had 60 students. If 42 students took history, 35 students took French, and 19 took both history and French, how many students in the junior class took neither French nor history? (Hint: this is the kind of problem in which a Venn diagram can be very helpful.)

Problem Set 2

1. Suppose that $A = \{2, 5, 8, 9\}$, $B = \{1, 2, 4, 8, 10\}$, and $C = \{2, 8, 9\}$. Label each of the following statements as true or false.

- a) $5 \in A$ c) $8 \subseteq A$ e) $C \subseteq B$ g) $\emptyset \subseteq C$
 b) $5 \notin B$ d) $C \subseteq A$ f) $\emptyset \in A$ h) $B \subseteq B$

2. Label each of the following statements as true or false.

- a) 3 is odd and the set of all natural numbers is closed under addition.
 b) 12 is even and -7 is a natural number.
 c) 5 is greater than 7 or 5 is equal to 7. (We write this as $5 \geq 7$ read: 5 is greater than or equal to 7)
 d) 8 is less than 8 or 8 is equal to 8. (We write this as $8 \leq 8$ read: 8 is less than or equal to 8)

3. List all natural numbers x with the given property.

- a) $x < 6$ b) $x < 7$ and $x > 3$ c) $x < 7$ or $x > 3$ d) $x \leq 10$ and x is even

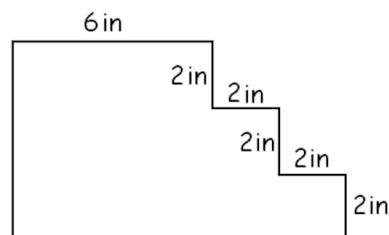
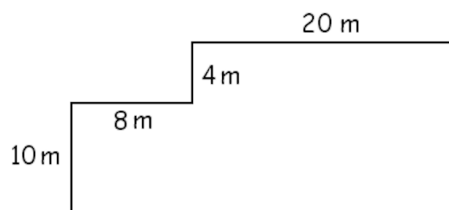
4. Simplify each of the following expressions by applying the order of operations agreement. **Show all steps. For each step, write a separate line!**

- a) $8 - 2 + 3$ g) $32 - 3(28 - 2^2(20 - 5 \cdot 3))$
 b) $\frac{15 - 2^3 + 3}{3^2 - 2^3}$ h) $\left(\left((10 - 8)^2 - 1\right)^2 - 2\right)^2$
 c) $120 \div (4 + 3(5 \cdot 2^2 - 2(5 + 2^2)))$ i) $1^2 + 1^3 - 1^4 + 1^5$
 d) $\frac{3^2 - 2^2}{(3 - 2)^2}$ j) $2^2 + 2^3$
 e) $\left(3 - (10 - 3^2)^2\right)^2$ k) $2^5 - 3(12 - 3^2)^2$
 f) $\frac{100 \div 5 \cdot 2}{25 - 10 + 5}$ l) $5 \cdot 2^3 - (10 - (7 - 2 \cdot 3 + 1) \div 2 + 2^2)$
 m) $\left(2 - \left(2 - (10 - 3^2)^2\right)^2\right)^2$

5. Insert parentheses in the expression on the left-hand side to make the equation true.

$$36 - 2 \cdot 5 - 2^2 + 4 = 10$$

6. Compute the perimeter and area of each of the objects shown. Angles that look like right angles are right angles. Include units in your computation and answer.



Chapter 3

3.1 The Set of All Integers

Recall that the set of all natural numbers (also called counting numbers) is $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. This set was historically the first set that people used. In this course, it will be a recurring theme that a mathematical system or set would be enlarged. This was the case with the natural numbers. Why would mathematicians of past centuries feel the need to step beyond the natural numbers? One reason is closure.

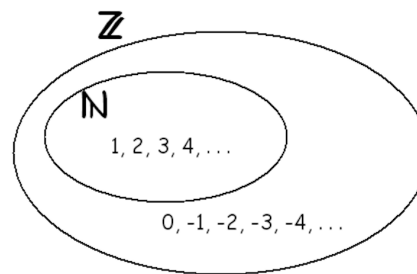
Recall the meaning of closure. The set $\mathbb{N}$ is closed under addition. In other words, the sum of *any* two natural numbers is also a natural number. $\mathbb{N}$ is also closed under multiplication. However, we do not have closure under subtraction and division. We *can* find a subtraction, say $3 - 12$, or a division, $10 \div 7$ that do not result in natural number. If we stay within the set of natural numbers, this means that the results for $3 - 12$ or $10 \div 7$ do not exist. It is a common theme in mathematics to work towards closure.

The set of all natural numbers, $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ is closed under addition and multiplication, but not under subtraction and division.

Definition: The set of all integers, denoted by $\mathbb{Z}$, is the set

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

The set of integers completely contains the set of natural numbers. In other words, the set of all natural numbers is a subset of the set of all integers, $\mathbb{N} \subseteq \mathbb{Z}$. We can also imagine that we started with the natural numbers and added zero and the opposite of each natural number to form the set of all integers.



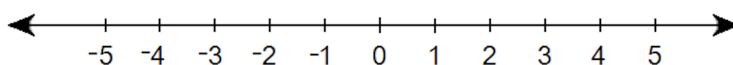
Definition: The **opposite** of 3 is written as -3 . For any number, the sum of the number and its opposite is zero. Another expression for the opposite is the **additive inverse**.

The opposite of 3 is -3 . The opposite of -3 is 3. The opposite of zero is zero itself.

The negative sign already has a meaning, that of subtraction. We now are facing an ambiguity that is often the source of confusion. Does a negative sign denote the opposite of a number, or does it denote subtraction? This is a question that we often need to ask ourselves. While the answer always clearly exists, it very much depends on the context. For example, the negative sign in -3 clearly denotes that we are talking about the opposite of 3 or negative 3. However, if we place a number in front of it, like in $8 - 3$, the same sign here denotes subtraction. And what about $8(-3)$? Now the parentheses tells us that the negative sign does not denote subtraction, rather it describes the number after it as negative.

-3	$8 - 3$	$8(-3)$
the opposite of 3	subtraction	the opposite of 3

We often depict integers on a number line.



Definition: (Ordering on $\mathbb{Z}$) Between two integers, the one on the right is greater.

Another way of envisioning this is to think of a positive number as money and a negative number as debt, and ask: who is richer? $2 < 10$ was obvious, and also that $-5 < 3$, but now we see that between -100 and -2 , -2 is greater. After all, the person who has no money and only 2 dollars of debt is better off than another person who has no money and a 100 dollars of debt. And so $-100 < -2$.

We can swap inequalities, as long as the smaller part of the inequality sign points to the smaller number.

$-100 < -2$	read: -100 is less than -2	$-100 \leq -2$	read: -100 is less than or equal to -2
$-2 > -100$	read: -2 is greater than -100	$-2 \geq -100$	read: -2 is greater than or equal to -100

Whether we plot them on a number line or think money and debt, we will agree that -1000000 (negative one million) is less than 5. But what if we wanted to compare the size of numbers, ignoring their signs? Suppose we want to say that a million dollar debt is a lot of debt. In this case, we use the concept of the absolute value of a number.

Definition: The **absolute value of a number** is its distance from zero on the number line. We denote the absolute value of a number x by $|x|$.

Distances can never be negative. -5 is 5 units away from zero on the number line. So is 5, only it is in the other direction. So, the absolute value of 5 and -5 are both 5.

Example 1. Compute each of the following.

- a) $|-2|$ b) $|2|$ c) $|0|$ d) $-|-5|$

Solution: a) The number -2 is 2 units away from zero on the number line. Thus $|-2| = 2$.

b) The number 2 is 2 units away from zero on the number line. Thus $|2| = 2$.

c) The distance between zero and zero on the number line is zero. Thus $|0| = 0$.

d) This is an example for two negatives not making a positive. The way we can read this as: *the opposite of the absolute value of negative five*. The absolute value of -5 is 5. The opposite of that is -5 .

Using notation, $-|-5| = -5$.

Addition of Integers: Again, think money and debt. Positive numbers represent money, negative numbers represent debt. Adding zero to anything will leave the other number unchanged.

Example 2. Compute each of the following sums.

- a) $-4 + 7$ b) $-3 + (-8)$ c) $3 + (-14)$ d) $-7 + 2$ e) $-2 + 0$

Solution: a) We think of $-4 + 7$ as follows. We start with a person who has no money and is in debt by 4 dollars. To this, we add 7 dollars. So the person pays off all that 4 dollar debt and is still left with 3 dollars. So $-4 + 7 = 3$.

b) We think of $-3 + (-8)$ as follows. We start with a person who has no money and is in debt by 3 dollars. To this, we add another debt of 8 dollars. So this person is still now in debt by 11 dollars. So $-3 + (-8) = -11$.

c) We think of $3 + (-14)$ as follows. We start with a person who has 3 dollars. To this, we add a debt of 14 dollars. So the person pays off all the debt he can - that is 3 dollars and is still in debt by 11 dollars. So $3 + (-14) = -11$.

d) We think of $-7 + 2$ as follows. We start with a person who has no money and is in debt by 7 dollars. Then this person gets 2 dollars. So the person pays off all the debt she can. After she pays off 2 dollars of debt, she has no money and is still left with 5 dollars of debt. So $-7 + 2 = -5$.

e) Adding zero to any number leaves the other number unchanged. Therefore, $-2 + 0 = -2$.

We already know how to add two positive numbers, and we know that the sum is positive. Adding two negative numbers is similar, we are summing debts. So we know that the sum of two negative numbers is also negative. If we add a negative and a positive number, the result may be positive or negative, depending on which number's size (or absolute value) is greater.

Adding zero to any number leaves that number unchanged. In other words, for any integer x , $x + 0 = x$ and $0 + x = x$. In the language of algebra, we refer to a number that has no effect in an operation as an identity or identity element.

Definition: When added to any integer, zero has no effect. Because of this behavior, we call zero an **additive identity**.



Discussion:

Based on its behavior, can you find a multiplicative identity within the set of all integers?

Now that we can add integers, we need to return to absolute values. The absolute value sign is also a grouping

symbol that overwrites order of operations. So if there is a sum (or any other expression) within the absolute value sign, we need to perform those until we are left with just a number within the absolute value sign. Then we take the absolute value of that number.

Example 3. Compute each of the following.

a) $|-9 + 4|$ b) $|-9| + |4|$ c) $|8| + |-7|$ d) $|8 + (-7)|$

Solution: a) The absolute value sign is also a pair of parentheses. We perform the addition $-9 + 4$ inside, and get -5 . Then we take the absolute value of -5 .

$$|-9 + 4| = |-5| = 5$$

b) In this example we take the absolute values and then add.

$$|-9| + |4| = 9 + 4 = 13$$

c) We take the absolute values and then add.

$$|8| + |-7| = 8 + 7 = 15$$

d) We first perform the addition inside and then take the absolute value.

$$|8 + (-7)| = |1| = 1$$

Subtraction of Integers: the following statement is always true, and is often extremely useful.

To subtract is to add the opposite.

Of course, we don't always use this fact. In the subtraction $10 - 3$, we would only complicate things by applying this fact. It would still get us the right result. Instead of subtracting positive 3, we add its opposite, negative 3.

$$10 - 3 = 7 \quad \text{and also,} \quad 10 + (-3) = 7$$

Consider the subtraction $100 - (-20)$. We are asked to subtract negative 20. To subtract is to add the opposite. So, instead of subtracting negative 20, we will add its opposite, positive 20.

$$100 - (-20) = 100 + 20 = 120$$

Example 4. Compute each of the following.

a) $-7 - 8$ b) $-9 - (-5)$ c) $1 - 7$ d) $6 - (-3)$

Solution: a) First, the negative sign in front of the 7 cannot denote subtraction. We are asked to subtract positive 8 from negative 7. To subtract is to add the opposite. Instead of subtracting positive 8, we will add its opposite, negative 8.

$$-7 - 8 = -7 + (-8) = -15$$

b) To subtract is to add the opposite. Instead of subtracting negative 5, we will add its opposite, positive 5.

$$-9 - (-5) = -9 + 5 = -4$$

c) To subtract is to add the opposite. Instead of subtracting positive 7, we will add its opposite, negative 7.

$$1 - 7 = 1 + (-7) = -6$$

d) To subtract is to add the opposite. Instead of subtracting negative 3, we will add its opposite, positive 3.

$$6 - (-3) = 6 + 3 = 9$$

Why is $100 - (-20) = 100 + 20$? Even if we understand how to compute this, it would be nice to understand

why this is correct. So here is one way to think about this.

Imagine that we have both a bank account a credit card with a bank. Suppose that at the moment, we have 150 dollars in the bank but we also owe 50 dollars to the bank on the credit card. So our net worth is 100 dollars.

Money in bank	Debt on credit card	Total Net worth
150	50	100

Suppose now that we have collected enough bonus points on the credit card to earn rewards. So the bank reduces our credit card debt by 20 dollars. (i.e. subtracts 20 debt, i.e. subtracts negative 20). We still have our 150 in cash, but now our debt is reduced to 30 dollars. So our net worth is now 120 dollars.

	Money in bank	Debt on credit card	Total Net worth
before	150	50	100
after	150	30	120

That is 20 dollars more than before. After all, reducing our debt by 20 dollars is almost the same as if someone gave us 20 dollars so that we can pay off some of our debts.

Multiplication of Integers: Multiply the absolute values. If two integers have the same sign, their product is positive. If two integers have different signs, their product is negative. If any of the factors is zero, the product is zero.

Example 5. Compute each of the following.

- a) $-3 \cdot 5$ b) $-4(-5)$ c) $10(-2)$ d) $0(-3)$ e) $-1(8)$

Solution: a) The product of a negative and a positive number is negative.

$$-3 \cdot 5 = -15$$

b) The product of two negative numbers is positive.

$$-4(-5) = 20$$

c) The product of a positive and a negative number is negative.

$$10(-2) = -20$$

d) If any of the factors is zero, the product is zero.

$$0(-3) = 0$$

e) The product of a negative and a positive number is negative.

$$-1(8) = -8$$

Notice that if we multiply any integer by -1 , the result is the opposite of that integer. This will be very useful later.

Why do these rules work this way? Here is one possible explanation. Multiplication is defined as repeated addition. For example, $4 \cdot 7$ means that we add 7 to itself, 4 times.

$$4 \cdot 7 = 7 + 7 + 7 + 7 = 28$$

Consider now $4 \cdot (-7)$. This means that we add -7 to itself, 4 times

$$4 \cdot (-7) = -7 + (-7) + (-7) + (-7) = -28$$

The logic becomes a bit tortured, but it also works with the first factor being negative. Consider now the product

$-5 \cdot 8$. We can interpret the first negative sign as repeated subtraction. So, we are subtracting 8 repeatedly, 5 times. If we feel that we don't have anything to subtract the first 8 from, we can fix that by inserting a zero. We know that adding zero will not change any value.

$$\begin{aligned} -5 \cdot 8 &= -8 - 8 - 8 - 8 - 8 \\ &= 0 - 8 - 8 - 8 - 8 - 8 && \text{to subtract is to add the opposite} \\ &= 0 + (-8) + (-8) + (-8) + (-8) + (-8) \\ &= -40 \end{aligned}$$

The most interesting case is probably when we are multiplying two negative numbers. Consider the product $-4 \cdot (-10)$. The first negative sign is interpreted as repeated subtraction, the second one is that we are subtracting negative numbers. So we are subtracting negative 10 repeatedly, 4 times. If we need something to subtract the first negative 10 from, we will just insert a zero at the beginning.

$$\begin{aligned} -4 \cdot (-10) &= 0 - (-10) - (-10) - (-10) - (-10) && \text{to subtract is to add the opposite} \\ &= 0 + 10 + 10 + 10 + 10 = 40 \end{aligned}$$

Division of integers: We will deal with zero later. For the quotient of any two non-zero integers, the rules are very simple and similar to those of multiplication. Divide the absolute values. If the the integers have the same sign, the quotient is positive. If they have different signs, the quotient is negative.

Example 6. Compute each of the following.

$$\text{a) } 14 \div (-2) \qquad \text{b) } -24 \div (-6) \qquad \text{c) } -10 \div 5$$

Solution: a) Divide the absolute values. The quotient of a positive and a negative number is negative.

$$14 \div (-2) = -7$$

b) Divide the absolute values. The quotient of two negative numbers is positive.

$$-24 \div (-6) = 4$$

c) Divide the absolute values. The quotient of a negative and a positive number is negative.

$$-10 \div 5 = -2$$

Division is often denoted with a horizontal bar. The same computations can also be written as

$$\frac{14}{-2} = -7 \text{ and } \frac{-24}{-6} = 4 \text{ and } \frac{-10}{5} = -2$$

Why do these rules work this way? Division is defined in terms of multiplication backward. In other words,

$$\frac{20}{4} = 5 \text{ is true because } 4 \cdot 5 = 20$$

Let us apply this idea. What is the result of $14 \div (-2)$?

$$\frac{14}{-2} = \boxed{?} \text{ would be true because } -2 \cdot \boxed{?} = 14$$

Since $-2 \cdot 7$ would result in -14 , we can only choose -7 to make the multiplication backward work. $-2(-7) = 14$, therefore $\frac{14}{-2} = -7$.

$$\text{Similarly, } \frac{-24}{-6} = \boxed{?} \text{ would be true because } -6 \cdot \boxed{?} = -24$$

We need to multiply -6 by a positive number to get a negative product. Only positive 4 will work, and so $\frac{-24}{-6} = 4$.

Division by Zero: Now that we understand that division is defined in terms of multiplication backward, we can easily deal with zero. The expressions $\frac{0}{3}$ and $\frac{3}{0}$ look very similar, and yet they are very different.

$$\frac{0}{3} = \boxed{?} \text{ would be true because } 3 \cdot \boxed{?} = 0$$

In this case, we can only use zero to make the multiplication backward work. Let us investigate the other case.

$$\frac{3}{0} = \boxed{?} \text{ would be true because } 0 \cdot \boxed{?} = 3$$

Now we are in trouble. If we multiply any number by zero, the product is zero. Therefore, we can not meaningfully complete this division, and so we say that $\frac{3}{0}$ is undefined. In written notation, $\frac{3}{0} = \text{undefined}$.

We have established that we cannot divide a non-zero number by zero. What about $\frac{0}{0}$?

$$\frac{0}{0} = \boxed{?} \text{ would be true because } 0 \cdot \boxed{?} = 0$$

Now the problem is that *every* number would work, because any number times zero is zero. Mathematicians prefer one clean answer as a result of an operation. We do not like an operation that results in several numbers, let alone every number! So, one fundamental rule of mathematics is that division by zero is not allowed. Indeed, division by zero is not just an error: it is one of the worst errors.

The first commandment of mathematics: *Thou shall not divide by zero. Ever...*

Changes in Notation

With the introduction of negative numbers, our notation will have to be modified. It is a widely accepted convention that if there are several signs (operations or negative) between two numbers, a pair of parentheses must separate them.

$-2 + -6$	$-5 - -3$	$-3 \cdot -4$	$-30 \div -5$
+- is not allowed	-- is not allowed	· - is not allowed	÷ - is not allowed

For this reason, until a few decades ago, we used to put a pair of parentheses around *every negative number*.

$(-2) + (-6)$	$(-5) - (-3)$	$(-3) \cdot (-4)$	$(-30) \div (-5)$
old style	old style	old style	old style

The development of mathematical notation is an ongoing process. The most important goal in notation is clarity. As long as clarity is not jeopardized, mathematicians are in the habit of omitting things. A few decades ago we stopped putting the parentheses around the first negative number in the line or inside a parentheses, because there was no risk that we would read the sign incorrectly as subtraction. Also, there is rarely an operation sign in front of the first number.

$-2 + (-6)$	$-5 - (-3)$	$-3 \cdot (-4)$	$-30 \div (-5)$
more modern	more modern	more modern	more modern

In the case of multiplication, we can omit one more thing. Recall that multiplication is the default operation; if we see two numbers with *nothing* between them, that indicates multiplication. For example, there is no operation sign or parentheses in $2x$ or ab and yet it is clear that the operation is multiplication. Now that most negative

numbers must be placed in parentheses, we can often omit the dot indicating multiplication.

$-2 + (-6)$	$-5 - (-3)$	$-3(-4)$	$-30 \div (-5)$
cannot omit anything	cannot omit anything	we can omit the dot	cannot omit anything

The most common modern style is minimalistic, omitting as much as possible, as long as confusion is avoided. This can lead to apparent irregularities in notation. For example, our notation will be $2(-3)$, but when we swap the two factors, it will be $-3 \cdot 2$.

Our Larger Number System

The set of all integers is closed under addition, subtraction, and multiplication. It is not closed under division.

As a matter of fact, the set of all integers ($\mathbb{Z}$) is the smallest set that contains the set of all natural numbers ($\mathbb{N}$) and is closed under subtraction. Another way to state this is that $\mathbb{Z}$ is the closure of $\mathbb{N}$ under subtraction.

In the future, we will further enlarge our number system to obtain closure under division.



Practice Problems

1. Label each of the following statements as true or false.

- | | | |
|---------------------------|--------------------------------------|---|
| a) $-3 \in \mathbb{Z}$ | c) $\mathbb{Z} \subseteq \mathbb{N}$ | e) $- -2 = -2$ |
| b) $-3 \notin \mathbb{N}$ | d) $-3 \geq -3$ | f) For every integer x , $ x \geq x$ |

2. Label each of the following statements as true or false.

- | | |
|--|---|
| a) Every integer is a natural number. | e) Zero is also called the additive identity. |
| b) Every natural number is an integer. | f) $5 \leq 5$ and $- -8 = -8$ |
| c) $3 < -5$ or $-2 > -8$ | g) $-2 < -2$ or $ -2 > -10 $ |
| d) $3 < -5$ and $-2 > -8$ | |

3. Place an inequality sign between the given numbers to make the statement true.

- | | | | | |
|-------------|---------------|-------------|--------------|--------------|
| a) 5 -7 | b) -12 -4 | c) 0 -8 | d) -1 -4 | e) -7 -7 |
|-------------|---------------|-------------|--------------|--------------|

4. Simplify each of the following.

- | | | | | | | |
|-----------|----------|-----------|------------|----------|--------------|----------------|
| a) $ -5 $ | b) $ 5 $ | c) $- 5 $ | d) $- -5 $ | e) $ 0 $ | f) $ -12+9 $ | g) $ -12 + 9 $ |
|-----------|----------|-----------|------------|----------|--------------|----------------|

5. Perform the indicated operations.

- | | | | | |
|-------------------|-----------------|-------------------|-------------------|------------------|
| a) $-2+7$ | e) $-8 \cdot 0$ | i) $-4 \cdot 7$ | m) $-3-0$ | p) $0 \div (-1)$ |
| b) $-7 - (-4)$ | f) $-3 - (-10)$ | j) $-6 - -7 $ | n) $0(-4)$ | q) $9 + -1 $ |
| c) $12 \div (-2)$ | g) $-20 \div 0$ | k) $-3 \div (-3)$ | o) $\frac{-5}{0}$ | r) $ 9 + (-1) $ |
| d) $5(-3)$ | h) $-12 \div 3$ | l) $ 9 + (-1)$ | | |

3.2 Order of Operations on Integers

The good news is that order of operations on the set of all integers ($\mathbb{Z}$) is the same as on the set of all natural numbers ($\mathbb{N}$). This is not by chance: when mathematicians enlarge our number system, they make certain that the new numbers and operations defined are in harmony with properties of the smaller system. We will refer to this consistent effort as **the expansion principle**. The bad news is that notation became more complex with the introduction of zero and the negative numbers.

The Negative Sign

The negative sign has a dual role: it denotes subtraction or it describes the number after it. This ambiguity never leads to confusion, otherwise it wouldn't be allowed. It is extremely important that we correctly interpret these roles before starting a computation. Here is a neat trick: when facing a negative sign, we should ask ourselves if it could denote subtraction. If it can denote subtraction, then it *does* denote subtraction. If it cannot, then it is a sign to be read as '*negative*' or '*the opposite of*'. A negative sign denoting subtraction is one that is loose between two numbers.

-3	$-(-3)$	$8-3$	$8(-3)$	$8-(-3)$	$8(-(-3))$
negative 3	the opposite of negative 3	subtraction	negative 3	subtract negative 3	8 times the opposite of negative 3

Example 1. Simplify each of the given expressions

a) $20 - 3(-5)$ b) $7 - 2(-3)$

Solution: a) The first negative sign indicates subtraction, the second one describes 5 as being negative. We start with the multiplication. However, we do not include the subtraction sign in the multiplication. $3(-5) = -15$, and that is what we will subtract.

$$\begin{aligned}
 20 - 3(-5) &= 20 - (-15) && \text{to subtract is to add the opposite} \\
 &= 20 + 15 && \text{addition} \\
 &= \boxed{35}
 \end{aligned}$$

b) $7 - 2(-3)$ [Solution - Youtube link](#)

Example 2. Consider the expression $20 - 3(-4 - (-2)5)$.

- Pair the opening and closing parentheses. Classify them as grouping symbols or clarifying parentheses.
- How many operations are indicated?
- Perform the indicated operations. For each step, write a separate line.

Solution: a) There are two pairs of parentheses, nested inside each other. The outside pair is a grouping symbol, the one enclosing -2 is a clarifying parentheses.

- We scan the expression left to right and count the operations. At this point we must be clear on the role of each negative sign. The negative signs before 4 and 2 do not denote subtraction.
 - subtraction between 20 and 3 (although it is not going to be 3 what we are subtracting)
 - multiplication between 3 and the expression in parentheses
 - subtraction between -4 and (-2) (although it is not going to be -2 what we are subtracting)
 - multiplication between -2 and 5

So there are four operations.

- c) We will write a separate line for each step. There are two advantages of this. First, at every point we just need to find the *one* operation to execute. Second, each line we write is a new and easier problem we need to solve. Within the parentheses, there is a subtraction and a multiplication. We start with the multiplication $-2 \cdot 5$.

$$\begin{aligned}
 20 - 3(-4 - (-2)5) &= 20 - 3(-4 - (-10)) && \text{to subtract is to add the opposite} \\
 &= 20 - 3(-4 + 10) && \text{addition (also, drop parentheses)} \\
 &= 20 - 3 \cdot 6 && \text{multiplication} \\
 &= 20 - 18 && \text{subtraction} \\
 &= \boxed{2}
 \end{aligned}$$

The Trouble with Exponents - It's an Order of Operations Thing

Recall that a negative sign in front of anything can be interpreted as '*the opposite of*', which can also be interpreted as multiplication by -1 . We can interpret -3 as $-1 \cdot 3$, and so we can re-interpret the original question from comparing -3^2 and $(-3)^2$ to a question comparing $-1 \cdot 3^2$ and $(-1 \cdot 3)^2$. The rest is really just an order of operations problem.

Recall that in our order of operations agreement, exponentiation superseeds multiplication. So, when presented by multiplication and exponentiation, we first execute the exponentiation and then the multiplication.

If there is no parentheses, we have	If we have parentheses, then
$ \begin{aligned} -3^2 &= -1 \cdot 3^2 && \text{exponentiation first} \\ &= -1 \cdot 9 && \text{multiplication} \\ &= -9 \end{aligned} $	$ \begin{aligned} (-3)^2 &= (-1 \cdot 3)^2 && \text{multiplication first} \\ &= (-3)^2 && \text{square the number } -3 \\ &= 9 \end{aligned} $

The difference between -3^2 and $(-3)^2$ is truly an order of operations thing: we are talking about taking the opposite and squaring, but in different orders.

-3^2 is the opposite of the square of 3

$(-3)^2$ is the square of the opposite of 3

Example 3. Simplify each of the given expressions.

a) -2^4 b) $(-2)^4$ c) -1^3 d) $(-1)^3$ e) $-(-2)^2$ f) $-(-2^2)$

Solution: a) $-2^4 = -1 \cdot 2^4 = -1 \cdot (2 \cdot 2 \cdot 2 \cdot 2) = -1 \cdot 16 = -16$
 -2^4 can be read as the opposite of 2^4 .

b) $(-2)^4 = (-1 \cdot 2)^4 = (-1 \cdot 2)(-1 \cdot 2)(-1 \cdot 2)(-1 \cdot 2) = (-2)(-2)(-2)(-2) = 16$
 $(-2)^4$ can be read as the fourth power of -2 .

c) $-1^3 = -1 \cdot 1^3 = -1 \cdot 1 \cdot 1 \cdot 1 = -1$

d) $(-1)^3 = (-1)(-1)(-1) = -1$

e) $-(-2)^2 = -1 \cdot ((-1 \cdot 2)^2) = -1 \cdot ((-2)^2) = -1 \cdot 4 = -4$

- f) Careful! The exponent is inside the parentheses. This is squaring 2 and then taking the opposite of the result twice.

$$-(-2^2) = -1 \cdot (-1 \cdot 2^2) = -1 \cdot (-1 \cdot 4) = -1(-4) = 4$$

Example 3v. Simplify each of the given expressions.

a) -3^3 b) $(-3)^2$ c) $-(-3)^2$ [Solution - Youtube link](#)

Example 4. Evaluate each of the given expressions.

a) $5(-2)^2$ b) $5(-2^2)$

Solution: We need to pay attention to the subtle difference between the two expressions: the exponent is inside the parentheses in part b. In that second example, the parentheses is not necessary for the exponentiation, without it we would be looking at subtraction.

a) $5(-2)^2 = 5 \cdot 4 = 20$ b) $5(-2^2) = 5(-4) = -20$



Discussion: Explain why in the expression $-(-5)^2$, the two negatives do not cancel out to a positive.

Example 5v. Simplify each of the given expressions.

a) $-5 - 7(-4 - 6)$ e) $-(5 - 7(-4) - 6)$ i) $-(5 - 7) - 4(-6)$
 b) $-(5 - 7(-4 - 6))$ f) $-(5 - 7 - (4 - 6))$ j) $-5(-7) - 4(-6)$
 c) $-(5 - 7(-4)) - 6$ g) $-5 - 7 - (4 - 6)$ [Solution - Youtube link](#)
 d) $(-5 - 7)(-4 - 6)$ h) $-(5 - 7)(-4 - 6)$

Example 6. Simplify the given expression. Write a separate line for each step. $-3^2 - 2(3(10 - (-2)^2) - 4^2)^2$

Solution: We start with the exponentiation in the innermost parentheses.

$$\begin{aligned} & -3^2 - 2(3(10 - (-2)^2) - 4^2)^2 = \\ & = -3^2 - 2(3(10 - 4) - 4^2)^2 && \text{subtraction in innermost parentheses} \\ & = -3^2 - 2(3 \cdot 6 - 4^2)^2 && \text{exponentiation in parentheses} \\ & = -3^2 - 2(3 \cdot 6 - 16)^2 && \text{multiplication in parentheses} \\ & = -3^2 - 2(18 - 16)^2 && \text{subtraction in parentheses} \\ & = -3^2 - 2 \cdot 2^2 && \text{first exponentiation left to right; } -3^2 = -9 \\ & = -9 - 2 \cdot 2^2 && \text{exponentiation} \\ & = -9 - 2 \cdot 4 && \text{multiplication} \\ & = -9 - 8 && \text{subtraction} \\ & = \boxed{-17} \end{aligned}$$

Example 7v. Perform the indicated operations.

a) $-((-3) + (5 - 1)) - 7^2$ [Solution - youtube link](#)

b) $\frac{5^2 - 3^2}{14 - 3 \cdot 2^2 - 2}$ [Solution - Youtube link](#)

c) $-2(-1 + 10(7 - 5)) - (-7)^2$ [Solution - Youtube link](#)

The Trouble with the Absolute Value Sign

Absolute value signs are more difficult to pair off because their shapes do not tell us whether they are opening or closing. The following order of operations problems illustrate that very similarly looking problems can actually be quite different. The key to solving these types of is to first pair off the absolute value signs.

Example 8. Evaluate each of the following expressions.

a) $|-5 - 3| - |7 + 2|$

c) $-5 - ||3 - 7| + 2|$

b) $||-5 - 3| - 7| + 2$

d) $-5|-3 - |7 + 2||$

Solution: a) Consider the expression $|-5 - 3| - |7 + 2|$.

We will not be able to get far in solving this problem without pairing the absolute values signs. Naturally, the first vertical bar can only be an opening sign. The second one can not be an another opening sign, because $|-|$ does not make any sense. Therefore, we have two pairs, one after the other. We should indicate the pairing using different colors or different sizes for the two pairs of absolute values signs. Keep in mind, they also serve as grouping symbols. After we paired up the signs, the problem becomes quite easy.

a) $|-5 - 3| - |7 + 2| =$

$$\begin{aligned}
 &= \left| -5 - 3 \right| - |7 + 2| && \text{subtraction in first parentheses} \\
 &= \left| -8 \right| - |7 + 2| && \text{find the absolute value of } -8 \\
 &= 8 - |7 + 2| && \text{addition in parentheses} \\
 &= 8 - |9| && \text{find the absolute value of 9} \\
 &= 8 - 9 && \text{subtraction} \\
 &= \boxed{-1}
 \end{aligned}$$

b) Consider the expression $||-5-3|-7|+2$.

It is easy to see that the first and second absolute value signs can not be a pair, because $||$ does not make sense. Therefore, since they are at the beginning, they are both opening. This means that we have one pair of absolute value signs nested inside the other. The one that opens last has to close first. After we have established that, the problem becomes much easier.

$$\begin{aligned}
 ||-5-3|-7|+2 &= \left| |-5-3|-7 \right| + 2 && \text{subtraction in innermost parentheses} \\
 &= \left| |-8|-7 \right| + 2 && \text{evaluate the absolute value of } -8 \\
 &= |8-7| + 2 && \text{subtraction in parentheses} \\
 &= |1| + 2 && \text{evaluate the absolute value of 1} \\
 &= 1 + 2 = \boxed{3} && \text{addition}
 \end{aligned}$$

c) Consider the expression $-5-||3-7|+2|$.

We again see that $||$ in front of 3 can not be a pair. Thus they are both opening, and so we have one pair inside the other.

$$\begin{aligned}
 -5-||3-7|+2| &= -5-\left| |3-7|+2 \right| && \text{subtraction in innermost parentheses} \\
 &= -5-\left| |-4|+2 \right| && \text{evaluate the absolute value of } -4 \\
 &= -5-|4+2| && \text{addition in parentheses} \\
 &= -5-|6| && \text{evaluate the absolute value of 6} \\
 &= -5-6 = \boxed{-11} && \text{subtraction}
 \end{aligned}$$

d) Consider the expression $-5|-3-|7+2||$. We again see that $||$ after 2 can not be a pair. Thus they are both closing, and so we have one pair inside the other.

$$\begin{aligned}
 -5|-3-|7+2|| &= -5\left| -3-|7+2| \right| && \text{addition in innermost parentheses} \\
 &= -5\left| -3-|9| \right| && \text{evaluate the absolute value of 9} \\
 &= -5|-3-9| && \text{subtraction in parentheses} \\
 &= -5|-12| && \text{evaluate the absolute value of } -12 \\
 &= -5 \cdot 12 = \boxed{-60} && \text{multiplication}
 \end{aligned}$$

Example 8v. Simplify each of the given expressions.

a) $|-3-8|-|11+9|$

c) $|-3-8|-11+9||$

e) $|-3|-8-11+9||$

b) $|-3-8-|11+9||$

d) $|-3-|8-11+9||$

[Solution - Youtube link](#)



Practice Problems

Simplify each of the given expressions. Write a separate line for each step.

1. $-2 - 3(1 - 4^2)$

6. $\frac{-7 - (-2)^3 - (-1)^4}{-2(5 - (-5))}$

10. $13 - 5(7 - 10)$

2. $-2 - 3(1 - 4)^2$

7. $\frac{-24 \div 2(-3)}{5 - (-1)^2}$

11. $13 - 5|7 - 10|$

3. $(-2 - 3(1 - 4))^2$

8. $(2 - 5)(12 - (-4)^2)$

12. $\frac{12 - 3(8 - 5)}{2^4 - (-4)^2}$

4. $(-2 - 3)(1 - 4)^2$

9. $2 - 5(12 - (-4)^2)$

13. $-3^2 - ((-2)^2 - 3(-1)^3)$

5. $-2(-3(1 - 4))^2$

14. $(-3)^2 - 2(3 - 4(5 - 2^3))$

15. $\left(\left(\left(5 - (-2)^2 \right)^2 - 3 \right)^2 + 1 \right)^2 - 1$

20. $\frac{5(-2) - 3|-5 + (-1)^2|}{-2(3 - (-2)) - (-1)^2}$

16. $(-2)^2 - (-2)^3 - (-2)^4$

21. $\frac{10 - 2(-3 - (-1)^3)}{(-2)^3 + (-1)^4}$

17. $-(-5)^2 - 2(3 - 5^2)$

22. $\frac{12 - 2(3(4(5 - 7) + 5) - 3)}{-3 + 2(7 - (-3)^2) - (-1)^3}$

18. $-(-5)^2 - 2(3 - 5)^2$

19. $-((-5)^2 - 2(3 - 5)^2)$

23. $4 - (6 - (8 - (10 - (12 - 4^2))))$

Absolute value signs are more difficult to pair off because their shapes do not tell us whether they are opening or closing. The following order of operations problems can be solved by first pairing off the absolute value signs.

24. $5 - 2||-4 + 7| - 10|$

27. $5|-2 - 4 + |7 - 10||$

30. $|5 - 2| - |4 + 7 - 10|$

25. $5|-2|-4 + 7|| - 10$

28. $5 - |2 - |4 + 7| - 10|$

31. $||5 - 2| - 4| + 7 - 10$

26. $5 - 2|-4 + |7 - 10||$

29. $|5 - |2 - 4|| + 7 - 10$

32. $||5 - 2| - 4 + 7| - 10$



Enrichment

1. We would hope that our order of operations agreement is void of ambiguities. This is not true. In case of the absolute value sign, we cannot tell whether it is opening or closing. Because of this, it is possible to find expressions that can be interpreted correctly in several ways, and the results are different! Evaluate each of the given expressions correctly in two different ways. What are your results?

a) $|5 - 2| - 4 + 7| - 10|$

b) $2|-1 - 5| - 3|-4|$

2. Enter pairs of parentheses or absolute value signs into the expression on the left-hand side to make the following statements true.

a) $5 - 2 - 3^2 - 4 - 7^2 + 2 = 9$

b) $5 - 2 - 3^2 - 4 - 7^2 + 2 = -9$

3.3 Perimeter and Area of a Right Triangle

Part 1 – Standard Labeling

The perimeter of any triangle is simply the sum of the lengths of its three sides. We will have to learn much more for a discussion of the area of right triangles. But before we do that, let us agree first on a method of notation that avoids confusion and lengthy explanations in geometry problems. This agreement is called *standard labeling*, and it establishes a connection between the labels of sides, vertices, and angles in triangles. Every triangle has three of the following three components.

1. **vertices** (singular: vertex)

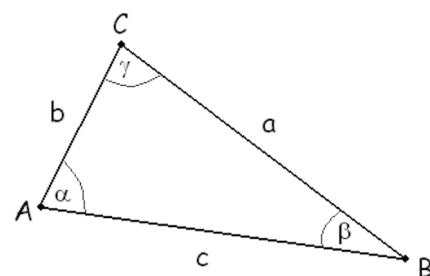
Points are usually denoted by uppercase letters. In case of triangles, we often use A , B , and C .

2. **angles**

Angles are usually denoted by lowercase Greek letters. In case of triangles, we often use α (alpha), β (beta), and γ (gamma).

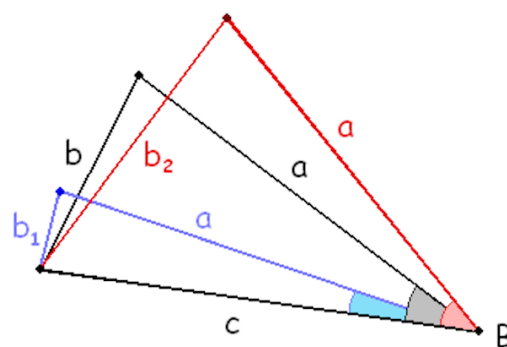
3. **sides**

Lines and line segments are usually denoted by lowercase letters. In case of triangles, we often use a , b , and c .



In case of standard labeling, we automatically associate sides, vertices, and angles. A vertex is associated with the angle located at that vertex. These two are associated with the side opposite these. For example, angle α is always assumed to be located at point A , and side a is always assumed to be the side opposite to point A and angle α . Point B , angle β , and side b are similarly grouped. Unless otherwise indicated, we should always assume standard labeling when presented with data that uses these letters.

Standard labeling is a smart approach to triangles, because there is a natural connection between an angle in a triangle and the side opposite that angle. Consider, for example, the triangle shown above with standard labeling. What if we fixed sides a and c and only modified angle β ? Imagine that we have two rods in the lengths of a and c attached to each other at one end and we can freely change the angle between them. If we increase the angle between sides a and c (see the red lines), the side opposite will also increase. If we decrease the angle between sides a and c (see the blue lines), the side opposite will also decrease. So, there seems to be a natural correspondence between side b and angle β .



Theorem: In any triangle ABC , there is a correspondence between the length of a side and the measure of the angle opposite that side:
 The longest side is opposite the greatest angle, and vica versa: the greatest angle is opposite the longest side. The shortest side is opposite the smallest angle, and vica versa: the smallest angle is opposite the shortest side.

So, the order between the three sides is the same as the order between the corresponding angles, and vica versa.

We recommend that sides in triangles are tracked by their corresponding sides. This is because we can perceive the difference in angles much better than in side lengths.

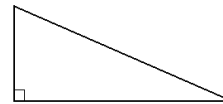
Example 1. Suppose that ABC is a triangle with $\alpha = 82^\circ$ and $\gamma = 39^\circ$. List the length of the sides of the triangle in an increasing order.

Solution: Recall that the three angles in a triangle add up to 180° . This means that if two angles are given, we can compute the third one. $\beta = 180^\circ - (82^\circ + 39^\circ) = 180^\circ - 121^\circ = 59^\circ$. Now we can see the order between the angles. γ is the smallest angle, β is in the middle, and α is the greatest angle. In short: $\gamma < \beta < \alpha$. The order between the lengths of the sides is the same: c is the shortest side, b is in the middle, and a is the longest side. In short: $c < b < a$.

There is an easy but important consequence of this property.

Theorem: In any triangle ABC , if two angles have equal measure, then the sides opposite them have equal length. If two sides are equally long, then the angles opposite those sides have equal measures. Such a triangle is called **isosceles**.

Definition: A **right triangle** is a triangle with a right angle. A right angle measures 90° .



We should never assume that a triangle is right, unless it is formally stated either in the problem, or on a picture. Just because an angle appears to be a right angle, it could have a measure of 89.5° , and that is not right. The measure must be exactly 90° .

Theorem: In a right triangle, there can only be one right angle and it is the greatest angle in the triangle.



Discussion:

1. Find an algebraic and geometric argument for the theorem stated above. Why is it true?
2. What does that mean for the sides of a right triangle?

Definition: In any right triangle, there is a single longest side, and it is opposite of the right angle. This side is called the **hypotenuse** of the right triangle. The shorter sides are called **legs**.

The hypotenuse of a triangle is often denoted by c , but this is just a convenient habit and not a rule! We should never assume that the right angle in a triangle is γ . For all we now, it could be any of α , β , or γ .

We took two pages to establish that there is a longest side in a right triangle. The perimeter and area formulas will be very easy.

Part 2 - Perimeter and Area Formulas

Recall that the perimeter is just the length of the boundary. This means that the perimeter of any triangle can be computed by adding the lengths of its three sides.

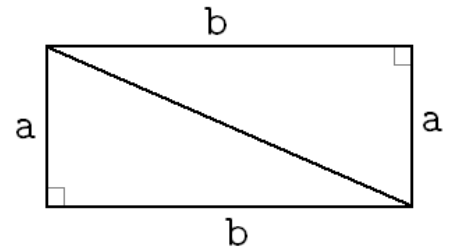
Theorem: The perimeter of a triangle ABC can be computed as $P = a + b + c$.

When we reviewed the area formula for rectangles, we have mentioned that that was a very difficult formula to prove. It is often the case in mathematics that, once we worked very hard for a formula, we use that result over and over. When it comes to area, *all* we know are rectangles. In other words, every area formula was derived from the area formula of the rectangle.

Every right triangle is half of a rectangle. In other words, given any right triangle, we can use two identical copies of it to form a rectangle.

We know how to find the area of the rectangle: $A = ab$.

Because the rectangle consists of two identical (also called congruent) right triangles, it naturally follows that each takes up half of the area of the rectangle. Thus the area of the right triangle is $A = \frac{ab}{2}$.



Theorem: The area of a right triangle ABC , where c denotes the hypotenuse, is $A = \frac{ab}{2}$.

What is unusual about this formula is that we don't need the length of the hypotenuse, only the lengths of the other two sides. This means that given the three sides of a right triangle, we need to know to only use the lengths of the two shorter sides.

Example 2. Compute the perimeter and area of a right triangle with sides 10ft, 24ft, and 26ft.

Solution: The perimeter is just the sum of all three sides.

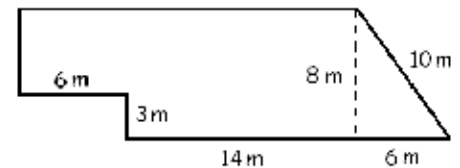
$$P = a + b + c = 10\text{ft} + 24\text{ft} + 26\text{ft} = \boxed{60\text{ft}}$$

For the area, we only need the two shorter sides.

$$A = \frac{ab}{2} = \frac{10\text{ft} \cdot 24\text{ft}}{2} = \frac{240\text{ft}^2}{2} = \boxed{120\text{ft}^2}.$$

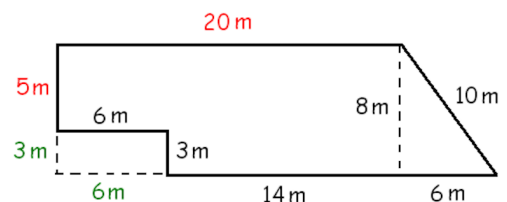
Although we will derive formulas for the area of more complicated shapes, we can often avoid those by cutting our objects into rectangles and right triangles.

Example 3. Compute the perimeter and area of the figure shown on the picture. Angles that look like right angles are right angles. Units are in meters.



Solution: For the perimeter, we need to find two missing sides.

The vertical side is 5m long since the opposite sides of rectangles are equally long. Similarly, the missing horizontal side is 20m long. We can now compute the perimeter. Notice that the vertical line in the inside is not part of the boundary. We will add all the sides, starting with the long horizontal side.



$$P = 20\text{m} + 10\text{m} + 6\text{m} + 14\text{m} + 3\text{m} + 6\text{m} + 5\text{m} = \boxed{64\text{m}}$$

Now for the area: We will compute the area of the big rectangle and from it we will subtract the area of the smaller rectangle.

$$A_1 = 8\text{m} \cdot 20\text{m} - 3\text{m} \cdot 6\text{m} = 160\text{m}^2 - 18\text{m}^2 = 142\text{m}^2$$

To this we add the area of the right triangle.

$$A_2 = \frac{6\text{m} \cdot 8\text{m}}{2} = \frac{48\text{m}^2}{2} = 24\text{m}^2$$

The area of the entire figure is the sum of the two areas:

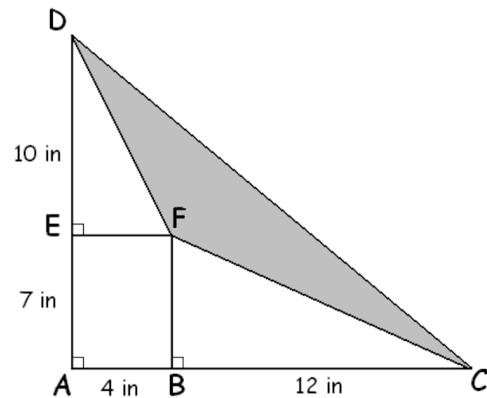
$$A = A_1 + A_2 = 142\text{m}^2 + 24\text{m}^2 = \boxed{166\text{m}^2}$$

Example 4. Compute area of the shaded region shown on the picture. Units are in inches.

Solution: First we will compute the area of right triangle ACD . $AC = 16\text{in}$ and $AD = 17\text{in}$.

$$A_{ACD} = \frac{16\text{in} \cdot 17\text{in}}{2} = \frac{272\text{in}^2}{2} = 136\text{in}^2$$

To get the area of the shaded region, we will subtract the areas of the rectangle $ABFE$ and right triangles DEF and BCF .



$$A_{ABFE} = 7\text{in} \cdot 4\text{in} = 28\text{in}^2$$

$$A_{DEF} = \frac{10\text{in} \cdot 4\text{in}}{2} = 20\text{in}^2$$

$$A_{BCF} = \frac{7\text{in} \cdot 12\text{in}}{2} = \frac{84\text{in}^2}{2} = 42\text{in}^2$$

So the shaded area is when we subtract the white areas from the big right triangle.

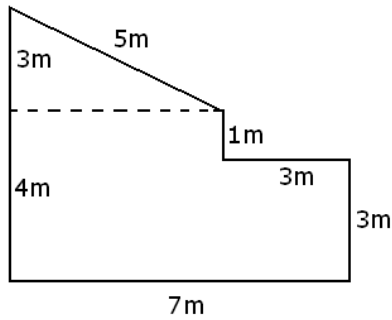
$$\begin{aligned} A &= A_{ACD} - (A_{ABFE} + A_{DEF} + A_{BCF}) \\ &= 136\text{in}^2 - (28\text{in}^2 + 20\text{in}^2 + 42\text{in}^2) = 136\text{in}^2 - 90\text{in}^2 = \boxed{46\text{in}^2} \end{aligned}$$



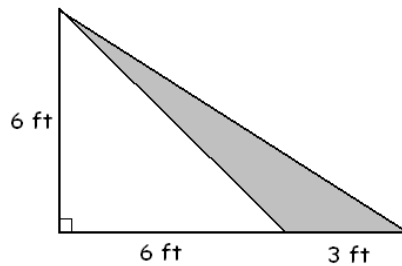
Practice Problems

1. Compute the perimeter and area of a right triangle with sides 7ft, 25ft, and 24ft. Include units in your computation and answer.

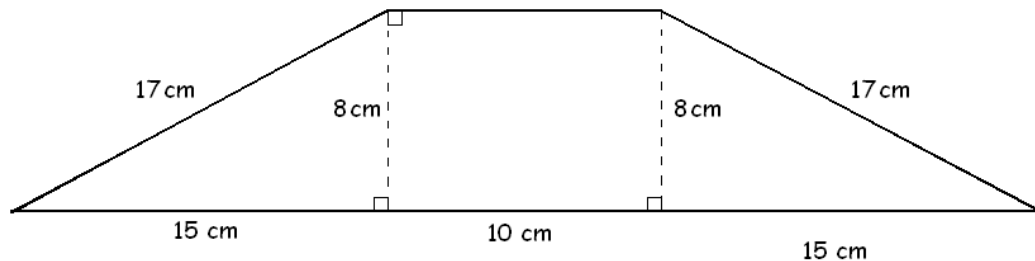
2. Compute the perimeter and area of the figure shown on the picture. Angles that look like right angles are right angles. Units are in meters. Include units in your computation and answer.



3. Compute the area of the shaded region shown on the picture. Include units in your computation and answer.

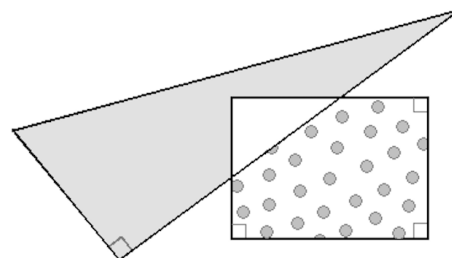


4. Compute the perimeter and area of the figure shown on the picture below. Units are in meters. Include units in your computation and answer.



Enrichment

Consider the figure shown on the picture. The shorter sides of the right triangle are 6cm and 10cm long. The sides of the rectangle are 5cm and 6cm long. Which region has the greater area, the shaded or the dotted?



3.4 Square Root of an Integer

Caution! Currently, square roots are defined differently in different textbooks. In conversations about mathematics, approach the concept with caution and first make sure that everyone in the conversation uses the same definition of square roots.

The square of an integer, such as 0, 1, 4, 9, 16, 25, ... is called a **perfect square**. In this section we will only discuss the square root of perfect squares (and their opposites). The square root of other numbers such as 2, 3, or 10 also exist and is studied later in intermediate algebra. For now, we will be dealing with perfect squares only.

The main idea of square roots is simple. Consider the number 36. What number's square equals to 36? However, there is a complication here. Both 6 and -6 , when squared, result in 36. This causes two issues. First, since both 6 and -6 square to 36, no real number, when squared, results in -36 . Thus $\sqrt{-36}$ is undefined. This is our second example for undefined result, the first being division by zero. The second issue is that mathematicians wanted square root to be an operation, with a unique result. So they defined the square root of 36 to be the *positive* number that when squared, we get 36. So, we define $\sqrt{36}$ to be 6. If we wanted to denote -6 , that is not the square root of 36, but rather its opposite, so we will write $-\sqrt{36}$.

Definition: Let N be a non-negative number. Then the **square root of N** (notation: $\sqrt{N}$) is the non-negative number that, if we square, the result is N . If N is negative, then $\sqrt{N}$ is undefined.

For example, $\sqrt{25} = 5$. On the other hand, $\sqrt{-25}$ is undefined. The definition uses the expression non-negative because $\sqrt{0}$ exists and is 0.

Example 1. Evaluate each of the given numerical expressions.

a) $\sqrt{49}$ b) $-\sqrt{49}$ c) $\sqrt{-49}$ d) $-\sqrt{-49}$

Solution: a) $\sqrt{49} = \boxed{7}$

b) $-\sqrt{49} = \boxed{-7}$

c) $\sqrt{-49} = \boxed{\text{undefined}}$

d) $-\sqrt{-49} = \boxed{\text{undefined}}$

How do square roots fit into the order of operations agreement? With this new operation, we need to revisit this old topic and see how square roots fit into it. The answer is, **taking the square root is equally strong with exponentiation**, so we perform it before all multiplications, divisions, additions and subtractions. When there are both exponents and square roots in an expression, or several square roots, we perform them left to right.

P
E $\sqrt{}$
M D
A S

Example 2. Evaluate each of the given numerical expressions.

$$\text{a) } \frac{2\sqrt{49} - (-1)^3}{\sqrt{9}} \qquad \text{b) } \sqrt{100} - 2 \left(\sqrt{36} - 3 \left(\sqrt{25} - \sqrt{16} \right) \right)$$

Solution: a) If we re-write the expression using the division sign instead of the horizontal bar, we get

$(2\sqrt{49} - (-1)^3) \div \sqrt{9}$. Thus we will work out the dividend first. There, we have a multiplication, an exponentiation, and a subtraction. Square root and exponentiation are equally strong, so will perform them left to right.

$$\begin{aligned} \frac{2\sqrt{49} - (-1)^3}{\sqrt{9}} &= && \text{square root upstairs} \\ &= \frac{2 \cdot 7 - (-1)^3}{\sqrt{9}} && \text{exponentiation} \\ &= \frac{2 \cdot 7 - (-1)}{\sqrt{9}} && \text{multiplication} \\ &= \frac{14 - (-1)}{\sqrt{9}} && \text{subtraction} \\ &= \frac{15}{\sqrt{9}} && \text{square root} \\ &= \frac{15}{3} && \text{division} \\ &= \boxed{5} \end{aligned}$$

b) There are several pairs of parentheses. We will start with the innermost one.

$$\begin{aligned} \sqrt{100} - 2 \left(\sqrt{36} - 3 \left(\sqrt{25} - \sqrt{16} \right) \right) &&& \text{square roots, left to right} \\ &= \sqrt{100} - 2 \left(\sqrt{36} - 3 \left(5 - \sqrt{16} \right) \right) \\ &= \sqrt{100} - 2 \left(\sqrt{36} - 3(5 - 4) \right) && \text{subtraction} \\ &= \sqrt{100} - 2 \left(\sqrt{36} - 3 \cdot 1 \right) && \text{square root} \\ &= \sqrt{100} - 2(6 - 3 \cdot 1) && \text{multiplication} \\ &= \sqrt{100} - 2(6 - 3) && \text{subtraction} \\ &= \sqrt{100} - 2 \cdot 3 && \text{square root} \\ &= 10 - 2 \cdot 3 && \text{multiplication} \\ &= 10 - 6 && \text{subtraction} \\ &= \boxed{4} \end{aligned}$$

Square roots, when stretched over entire expressions, also serve as grouping symbols. This is what we called an *invisible parentheses*.

Example 3. Evaluate each of the following expressions.

$$\begin{array}{lll} \text{a) } \sqrt{25} - \sqrt{16} & \text{c) } \sqrt{16} + 9 & \text{e) } \sqrt{-\sqrt{100} \div (-2) + \sqrt{(-6)^2 - 5\sqrt{16}}} \\ \text{b) } \sqrt{25 - 16} & \text{d) } \sqrt{16 + 9} & \end{array}$$

Solution: a) $\sqrt{25} - \sqrt{16} = 5 - 4 = \boxed{1}$

b) $\sqrt{25 - 16} = \sqrt{9} = \boxed{3}$

These two examples illustrate that the order here matters, so we must be careful not to confuse these two types of expressions.

c) $\sqrt{16} + 9 = 4 + 9 = \boxed{13}$

d) $\sqrt{16 + 9} = \sqrt{25} = \boxed{5}$

These two examples illustrate that we have to be precise with our notation and extend the square root sign over the entire expression as in d) to avoid confusing the two types of expressions.

e) The entire expression is enclosed in a square root, so that will be the last operation to perform. Inside, there is another square root stretched over an entire expression. That serves as parentheses, so we start there. Inside, we start with exponentiations and square roots, left to right.

$$\begin{aligned} \sqrt{-\sqrt{100} \div (-2) + \sqrt{(-6)^2 - 5\sqrt{16}}} &= && \text{exponentiation} \\ &= \sqrt{-\sqrt{100} \div (-2) + \sqrt{36 - 5\sqrt{16}}} && \text{square root} \\ &= \sqrt{-\sqrt{100} \div (-2) + \sqrt{36 - 5 \cdot 4}} && \text{multiplication} \\ &= \sqrt{-\sqrt{100} \div (-2) + \sqrt{36 - 20}} && \text{subtraction} \\ &= \sqrt{-\sqrt{100} \div (-2) + \sqrt{16}} && \text{square roots, left to right} \\ &= \sqrt{-10 \div (-2) + \sqrt{16}} \\ &= \sqrt{-10 \div (-2) + 4} && \text{division} \\ &= \sqrt{5 + 4} && \text{addition} \\ &= \sqrt{9} && \text{square root} \\ &= \boxed{3} \end{aligned}$$



Practice Problems

1. Evaluate each of the following expressions.

a) $\sqrt{100}$ b) $\sqrt{-100}$ c) $-\sqrt{100}$ d) $-\sqrt{49}$ e) $-\sqrt{-49}$ d) $\sqrt{1}$ e) $\sqrt{0}$

2. Evaluate each of the following expressions.

a) $\sqrt{\sqrt{81} + \sqrt{49}}$ b) $\sqrt{\sqrt{64} + 1}$ c) $\sqrt{\sqrt{11 - \sqrt{4}} + \sqrt{1}}$ d) $\sqrt{\sqrt{\sqrt{100} - 1} + 1}$

3. Evaluate each of the following expressions.

a) $\sqrt{25} - 2\sqrt{9 - (-3)^3} = -7$ b) $\sqrt{\sqrt{36} + 5\sqrt{9} - \frac{\sqrt{100}}{2}}$ c) $\sqrt{\sqrt{4\sqrt{64} - \sqrt{49}} - \sqrt{1}}$

4. Evaluate each of the following expressions.

a) $\sqrt{16 - 2\sqrt{-4^2 - \sqrt{1} + 2 \cdot 3^2 \div \sqrt{4} \cdot 6 - 1}}$ b) $\sqrt{3\sqrt{25} - (2\sqrt{100} - 5\sqrt{16} - \sqrt{36 \div 6})}$

3.5 Factors of a Number

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Definition: Suppose that N and m are any two integers. If there exists an integer k such that $N = mk$ then we say that m is a **factor** or **divisor** of N . We also say that N is a **multiple** of m or that N is **divisible** by m .

Notation: $m|N$

For example, 3 is a factor of 15 because there exists another integer (namely 5) so that $3 \cdot 5 = 15$. Notation: $3|15$.

Example 1. Label each of the following statements as true or false.

- a) 2 is a factor of 10 c) 14 is a factor of 7 e) every integer is divisible by 1
b) 3 is divisible by 3 d) 0 is a multiple of 5 f) every integer n is divisible by n

Solution: a) $10 = 2 \cdot 5$ and so 2 is a factor of 10. This statement is true.

b) $3 = 3 \cdot 1$ and so 3 is divisible by 3. This statement is true. c) $14 = 7 \cdot 2$ and so 14 is a multiple of 7, not a factor. Can we find an integer k so that $7 = 14 \cdot k$? This is not possible. $k = \frac{1}{2}$ would work, but $\frac{1}{2}$ is not an integer. This statement is false.

d) Since $0 = 5 \cdot 0$, it is indeed true that 0 is a multiple of 5. This statement is true.

e) For any integer n , $n = n \cdot 1$ and so every integer n is divisible by 1. This statement is true.

f) For any integer n , $n = 1 \cdot n$ and so every integer n is divisible by n . This statement is true.

We will later prove all of the following statements. They will cut down on the work as we look for divisors of a number.

Theorem (Divisibility Tests)

A number n is divisible by 2 if its last digit is 0, 2, 4, 6, or 8.

A number n is divisible by 3 if the sum of its all digits is divisible by 3.

A number n is divisible by 4 if the two-digit number formed by its last two digits is divisible by 4.

A number n is divisible by 5 if its last digit is 0, or 5.

A number n is divisible by 6 if it is divisible by 2 and by 3.

A number n is divisible by 9 if the sum of its all digits is divisible by 9.

A number n is divisible by 10 if its last digit is 0.

A number n is divisible by 11 if the difference of the sum of digits at odd places and the sum of its digits at even places, is divisible by 11.

A number n is divisible by 12 if it is divisible by 3 and by 4.

Example 2. Consider the numbers 2016, 183 422, 606 060, 123 321, and 38 115. Find all numbers from this list that are divisible by a) 3 b) 9 c) 4 d) 5 e) 11

Solution: a) We need to add all digits. If the sum of the digits is divisible by 3, so is the number.

$2 + 0 + 1 + 6 = 9$. Since 9 is divisible by 3, so is 2016.

$1 + 8 + 3 + 4 + 2 + 2 = 20$. Since 20 is not divisible by 3, neither is 183 422.

$6 + 0 + 6 + 0 + 6 + 0 = 18$. Since 18 is divisible by 3, so is 606 060.

$1 + 2 + 3 + 3 + 2 + 1 = 12$. Since 12 is divisible by 3, so is 123 321.

$3 + 8 + 1 + 1 + 5 = 18$. Since 18 is divisible by 3, so is 38 115.

Therefore, the numbers divisible by 3 are: 2016, 606 060, 123 321, 38 115.

b) The divisibility test for 9 also requires the sum of all digits. We just worked those out.

The sum of digits of 2016 is 9. Since 9 is divisible by 9, so is 2016.

The sum of digits of 183 422 is 20. Since 20 is not divisible by 9, neither is 183 422.

The sum of digits of 606 060 is 18. Since 18 is divisible by 9, so is 606 060.

The sum of digits of 123 321 is 12. Since 12 is not divisible by 9

The sum of digits of 38 115 is 18. Since 18 is divisible by 9, so is 38 115.

Therefore, the numbers divisible by 9 are: 2016, 606 060, 38 115.

c) The divisibility test for 4 requires to look at the two-digit number formed from the last two digits. If that two-digit number is divisible by 4, so is the number.

2016 ends in 16. Since 16 is divisible by 4, so is 2016.

183 422 ends in 22. Since 22 is not divisible by 4, neither is 183 422.

606 060 ends in 60. Since 60 is divisible by 4, so is 606 060.

123 321 ends in 21. Since 21 is not divisible by 4, neither is 123 321.

38 115 ends in 15. Since 15 is not divisible by 4, neither is 2016.

Therefore, the numbers divisible by 4 are: 2016, 606 060.

d) The divisibility test for 5 requires to look at the last digit. If it is 0 or 5, the number is divisible by 5.

Among the numbers 2016, 183 422, 606 060, 123 321, and 38 115, only 606 060 and 38 115 ends in 0 or 5.

Therefore, the numbers divisible by 5 are: 606 060, 38 115.

e) The divisibility test for 11 requires to split the digits into two groups, selecting every second one into one group. We add the numbers in each group. If the difference of the two sums is divisible by 11, so is the number.

We split the digits into two groups: 2016. One group is formed from the circled digits, the other group is the rest. The sums in the two groups are $2 + 1 = 3$ and $0 + 6 = 6$. We subtract one from the other. $6 - 3 = 3$. Since 3 is not divisible by 11, neither is 2016.

We split the digits into two groups: 183422. The sums in the two groups are $1 + 3 + 2 = 6$ and $8 + 4 + 2 = 14$. We subtract one from the other. $14 - 6 = 8$. Since 8 is not divisible by 11, neither is 183 422.

We split the digits into two groups: 603060. The sums in the two groups are $6 + 6 + 6 = 18$ and $0 + 0 + 0 = 0$. We subtract one from the other. $18 - 0 = 18$. Since 18 is not divisible by 11, neither is 606 060.

We split the digits into two groups: $\boxed{1}2\boxed{3}3\boxed{2}1$. The sums in the two groups are $1 + 3 + 2 = 6$ and $2 + 3 + 1 = 6$. We subtract one from the other. $6 - 6 = 0$. Since 0 is divisible by 11, so is 123321.

We split the digits into two groups: $\boxed{3}8\boxed{1}1\boxed{5}$. The sums in the two groups are $3 + 1 + 5 = 9$ and $8 + 1 = 9$. We subtract one from the other. $9 - 9 = 0$. Since 0 is divisible by 11, so is 38115.

Therefore, the numbers divisible by 11 are: $\boxed{123321, 38115}$.

Note that in case of divisibility by 11, the difference between the two groups does not need to be zero. Consider, for example 902. The sum of one group is 11, the sum of the other is 0. $11 - 0 = 11$. Since 11 is divisible by 11, so is 902. Indeed, $902 = 11 \cdot 82$.

Example 2v. Consider the numbers 159, 289, 292, 444, 591, 50020. Find all numbers from the list that are divisible by a) 3 b) 4 c) 12 [Solution - Youtube link](#)

Example 3. List all positive factors of the number 28.

Solution: We start counting, starting at 1.

Is 1 a divisor of 28? Yes, because
 $28 = 1 \cdot 28$.

We note both factors we found.

	28	
1		28

We continue counting. Is 2 a divisor of 28?

Yes, because $28 = 2 \cdot 14$.

We note both factors we found.

	28	
1		28
2		14

We continue counting. Is 3 a divisor of 28?

No. We can divide 28 by 3 and the answer is not an integer.

We continue counting. Is 4 a divisor of 28?

Yes, because $28 = 4 \cdot 7$. We note both factors we found.

	28	
1		28
2		14
4		7

We continue counting. Is 5 a divisor of 28?

No. (We can check with the calculator.) Is 6 a divisor of 28? No. Now we arrive to 7, a number that is already listed as a factor.

That's our signal that we have found all of the divisors of 28. We list the divisors in order:

factors of 28: $\boxed{1, 2, 4, 7, 14, 28}$

Example 3v. List all factors of 75. [Solution - Youtube link](#)



Discussion: Discussion: What do you think about the argument shown below?

3 is a divisor of 21 because there exists another integer, namely 7, so that $21 = 3 \cdot 7$. As we established that 3 is a divisor of 21, we also found that 7 is also a divisor of 21. In other words, divisors always come in pairs. For example, 28 has six divisors that we found in three pairs: 1 with 28, 2 with 14, and 4 with 7. Consequently, every positive integer has an even number of positive divisors.

Definition: An integer is a **prime number** if it has exactly two divisors: 1 and itself.

For example, 37 is a prime number.



Practice Problems

1. Consider the following numbers. 128, 80, 75, 270, 64
 - a) Find all numbers on the list that are divisible by 5.
 - b) Find all numbers on the list that are divisible by 3.
 - c) Find all numbers on the list that are divisible by 4.
2. Consider the following numbers: 4181, 9800, 1296, 420, 55050
 - a) Find all numbers from the list that are divisible by 3.
 - b) Find all numbers from the list that are divisible by 10.
 - c) Find all numbers from the list that are divisible by 6.
3. Consider the following numbers: 28072, 67808, 60610, 1296, and 7620.
 - a) Find the numbers from the list that are divisible by 4.
 - b) Find the numbers from the list that are divisible by 5.
 - c) Find the numbers from the list that are divisible by 9.
 - d) Find the numbers from the list that are divisible by 11.
4. List all the factors of 48.
5. Which of the following is NOT a prime number? 53, 73, 91, 101, 139



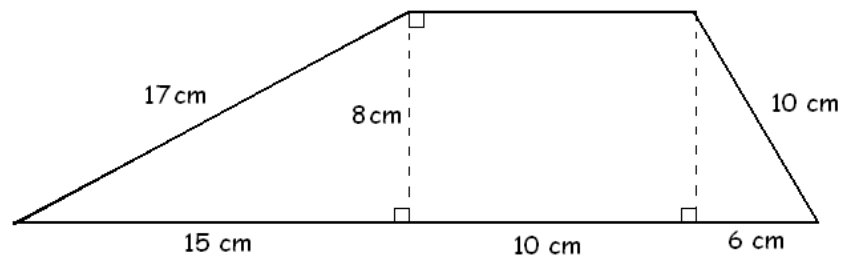
Enrichment

1. Two mathematicians are having a conversation. Mathematician A asks B about his kids. B answers: "I have three children, the product of their ages is 36." A says: "I still don't know the ages of your children." Then B tells A the sum of his three kids' ages. A answers: "I still don't know how old they are. Then B adds: "The youngest one has red hair." Now A knows the ages of all three children. Do you?
2. A king has his birthday. So he decides to let go some of his prisoners. He actually has 100 prisoners at the moment. They are each in a separate cell, numbered from 1 to 100. Well, he is a high tech king. He can close or open any prison door by a single click on the cell's number on his royal laptop. When he clicks at a locked door, it opens. When he clicks at an open door, it locks. At the beginning, every door is locked. First the king clicks on every number from 1 to 100 (therefore opening every door). Then he clicks on every second number from 1 to 100, (i.e. 2, 4, 6, 8, 10,...). Then he clicks on every third number. And so on. Finally, he only clicks on the number 100. Then he orders that the prisoners who find their door open may go free. Who gets to go and who has to stay?

Problem Set 3

- Recall that $\mathbb{N}$ denotes the set of all natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}$ denotes the set of all integers, $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$. Label each of the following statements as true or false.
 - $4 \in \mathbb{Z}$
 - $-3 \in \mathbb{N}$
 - $\mathbb{N} \subseteq \mathbb{Z}$
 - $5 \notin \emptyset$
 - $4 \in \mathbb{N}$
 - $\mathbb{Z} \subseteq \mathbb{N}$
 - $\mathbb{N} \subseteq \emptyset$
 - $\emptyset \subseteq \mathbb{Z}$
- Suppose that A , B , and C are sets. Label each of the following statements as true or false.
 - $A \subseteq A$
 - If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
 - If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- Consider the numbers 678678, 5019, 983305, 46228, 49740. Find all numbers from this list that are divisible by
 - 3
 - 4
 - 5
 - 11
- List all factors of 40.
- Suppose that A is the set of all integers divisible by 2, and B is the set of all integers divisible by 3. Label each of the following statements as true or false.
 - If an integer $n \in A$ and $n \in B$, then n is divisible by 6.
 - If an integer $n \in A$ or $n \in B$, then n is divisible by 6.
 - If C is the set of all integers divisible by 8, then $A \subseteq C$
 - If C is the set of all integers divisible by 8, then $C \subseteq A$
 - $0 \in B$
 - $A \subseteq \mathbb{N}$
 - $B \subseteq \mathbb{Z}$
- Evaluate each of the following expressions.
 - $7 - 10$
 - -3^2
 - -1^4
 - $\frac{20-8}{2}$
 - $\sqrt{25-16}$
 - $(3-7)^2$
 - $7(-10)$
 - $(-3)^2$
 - $20 - \frac{8}{2}$
 - $\sqrt{25} - \sqrt{16}$
 - $3^2 - 7^2$
 - $(3^2 - 7)^2$
- Evaluate each of the following expressions.
 - $-3 - 5$
 - $-3(-5)$
 - $-(3-5)$
 - $(-3) - 5$
- Evaluate each of the following expressions.
 - $8 - 2 - 7$
 - $8(-2-7)$
 - $8(-2)(-7)$
 - $8 - (2-7)$
 - $8-2(-7)$
 - $8(-(2-7))$
 - $(8-2)-7$
 - $(8-2)(-7)$
 - $8(-2)-7$
- Evaluate each of the following expressions.
 - $12 - 2(7+3(1-5))$
 - $\sqrt{3\sqrt{16} + \sqrt{9} + \sqrt{1}}$
 - $(-2)^1 + (-2)^2 + (-2)^3 + (-2)^4$
 - $((3-7)^2 - 13)^2$
 - $\frac{-3^2 - (-3)^2}{10-5+1}$
 - $\frac{-1^2 + (-3)^2}{4^2 - (-2)^4}$
 - $\sqrt{(-1)^2 - 3(-2^2 - 1)}$
 - $\frac{24 \div 3 \cdot 2}{9-2+1}$
- Compute the perimeter and area of the right triangle with sides 3ft, 5ft, and 4ft long. Include units in your computation and answer.

11. Compute the perimeter and area of the object shown on the picture. Include units in your computation and answer.



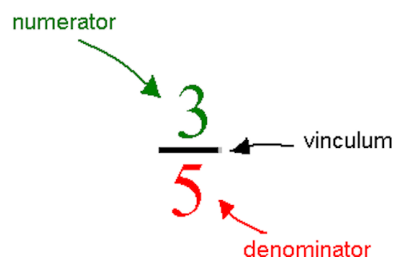
12. What is the smallest positive integer that is divisible by 2, 3, 4, 5, and 6?

Chapter 4

4.1 Fractions – Part 1: The Definition

A fraction has three components as shown on the picture. The important parts are the **numerator** above and the **denominator** below the little line that is called the vinculum.

At first let us not even consider a fraction alone. We will just define a **fraction of something**.



Definition: $\frac{3}{5}$ of a quantity can be obtained as follows.

Step 1. We first divide the quantity into 5 equal shares.

Step 2. Let us take 3 such shares. That is $\frac{3}{5}$ of our quantity.

So the numerator tells us how many shares we have. The denominator tells us how big each share is.

Example 1. Find $\frac{3}{5}$ of 100 dollars.

Solution: Step 1. Divide 100 dollars into five equal shares. This means that we exchange a 100 dollar bill into five twenty-dollar bills. In other words, $\frac{1}{5}$ of 100 dollars is 20 dollars.

Step 2. To obtain $\frac{3}{5}$, we take three such shares. In this case this means taking three twenty-dollar bills. That is 60 dollars. In other words, $\frac{3}{5}$ of 100 dollars is 60 dollars.

Example 2. Compute $\frac{4}{7}$ of 42.

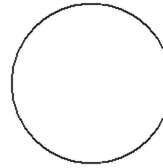
Solution: Step 1. We divide 42 into seven equal shares. $\frac{1}{7}$ of 42 is 6.

Step 2. We take four such shares. Thus $\frac{4}{7}$ of 42 is $4 \cdot 6 = 24$. The answer is 24.

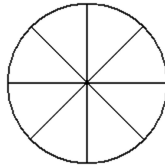
Example 2v. Find each of the following.

- a) $\frac{1}{6}$ of 18 b) $\frac{5}{6}$ of 18 [Solution - Youtube link](#)

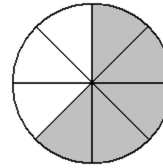
Example 3. Shade the region on the picture that corresponds to $\frac{5}{8}$ of the circle.



Solution: Step 1. Divide the circle into 8 equal parts.

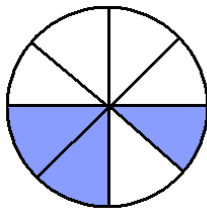


Step 2. We shade five such parts.

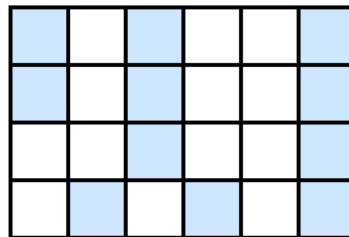


Example 3v. Express the shaded area as a fraction.

a) [Solution - Youtube link](#)



b) [Solution - Youtube link](#)



Example 4. A cake was sliced into equal slices. Amy ate 2 slices and Betsy ate 3 slices. If 2 slices were remaining, what fraction of the cake was eaten?

Solution: We need to first figure out how many slices made up the cake. If 2 were eaten by Amy and 3 by Betsy and 2 more were left, then there were all together $2 + 3 + 2 = 7$ slices. 5 slices were eaten which were

$\frac{5}{7}$ of the cake. So the answer is that $\frac{5}{7}$ of the cake was eaten.

Example 4v. There are 800 students enrolled in a school. If $\frac{3}{4}$ of the students are boys, how many students are girls? [Solution - Youtube link](#)

Example 5. Compute $\frac{8}{100}$ of 2000 dollars.

Solution: We divide 2000 dollars into 100 equal shares. Then each share is 20 dollars. (We divide 2000 by 100). Then we take 8 such shares, that is $8 \cdot 20 \text{ dollars} = 160 \text{ dollars}$. Thus $\frac{8}{100}$ of 2000 dollars is 160 dollars .

Definition: We often use fractions with 100 in the denominator. Such a fraction is also called a **percent** and is denoted by %.

For example, computing 9% of a quantity is exactly the same as computing $\frac{9}{100}$ of it.

Example 5v. Find 9% of 200. [Solution - Youtube link](#)

Example 6. In 2012, there were approximately 235 000 000 people eligible to vote in the USA. If 59% of them voted in 2012, how many people voted?

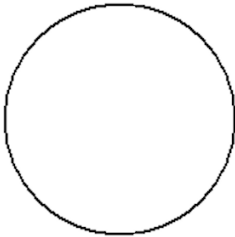
Solution: We need to compute 59% of 235 000 000. That is the same as computing $\frac{59}{100}$ of 235 000 000.

We first divide 235 000 000 by 100. Division by 100 is very easy in this case: just cut off the last two zeroes. So, 1% or $\frac{1}{100}$ of the number is 2 350 000. Now we take 59 of that, we multiply 2 350 000 by 59. The result is 138 650 000. Thus, approximately 138 650 000 people voted in 2012.

Example 6v. A \$500 TV went on a 6% off sale. What is the sale price? [Solution - Youtube link](#)



Practice Problems

1. Compute $\frac{4}{9}$ of 63.
 2. Compute $\frac{5}{6}$ of 24.
 3. Shade $\frac{5}{9}$ of the circle given.
- 
4. The price of a TV is \$400. We want to raise the price by 5%. What is the new price?
 5. The price of a couch is \$700. Next week, it will go on a 15% off sale. What is the new price? (A 15% sale means that the price of the item is lowered by 15%.)
 6. Find $\frac{3}{4}$ of 56.
 7. We placed \$2000 into a bank account with 6% yearly interest rate. How much money do we have in the bank after one year?
 8. This problem is about a method of comparing fractions.
 - a) Compute $\frac{3}{7}$ of 420.
 - b) Compute $\frac{4}{10}$ of 420.
 - c) Based on the results of parts a) and b), which fraction is larger, $\frac{3}{7}$ or $\frac{4}{10}$?
 9. Bert has made \$54 000 last year. If he has to pay 32% of his income in taxes, how much taxes does he owe and how much of his income will he keep?
 10. Sally used to make \$2400 per month, but now she got a 3% raise. How much is her monthly salary now?
 11. Mr. X won \$600 000 in the lottery two yers ago. By now he has spent some of the money. When he was asked what happened, he said the following. *"I didn't spend it all. I spent $\frac{1}{3}$ of the money by taking a luxury yacht trip around the world. Then I put the rest in the bank. Later I decided to buy a house I really liked. So I took $\frac{3}{4}$ of the money out of the bank and baught the house. For half of what was left, I purchased stocks that completely lost their value. Finally, I gave $\frac{2}{5}$ of what's left to my niece for her college education."*
How much money is left from the winnings?

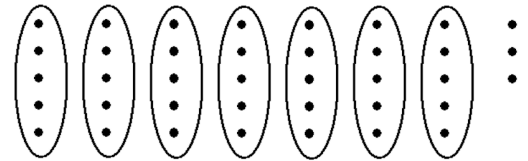
4.2 Division with Remainder

Theorem: (Division with Remainder) For every integer N and positive integer m , there exist unique integers q and r such that

$$N = mq + r \quad \text{where } 0 \leq r \text{ and } r < m$$

For example, if $N = 38$ and $m = 5$, then the quotient is $q = 7$ and the remainder is $r = 3$. The picture illustrates the division $38 \div 5 = 7 \text{ R } 3$, or, in other form: $38 = 5 \cdot 7 + 3$. Note that there are two different ways to represent this division:

$$38 \div 5 = 7 \text{ R } 3 \quad \text{and} \quad \frac{38}{5} = 7\frac{3}{5}.$$



Example 1. Perform the given division with remainder. $6071 \div 17$

Solution: One way would be to perform long division. Another is to utilize our calculator. First we enter the division into the calculator and see the result as a decimal: 357.1176470588. Note that this is NOT division with remainder. However, we can quickly adjust this. Since the result is between 357 and 358, we already know that the quotient is 357 and now we need to only find the remainder.

$$6071 \div 17 = 357 \text{ R } ?$$

If the quotient is 357, then we have already accounted for $17 \cdot 357 = 6069$. How much is missing from 6071? Clearly 2. Thus the answer is: $6071 \div 17 = 357 \text{ R } 2$.

How can we check? We should look for two things: did we account for all 6071? Indeed, $17 \cdot 357 + 2 = 6071$. The other question is: Is the remainder less than the divisor? Indeed, $2 < 17$ and so our solution is correct.

Example 2. Perform the given division with remainder. $8271 \div 45$

Solution: First we enter the division into the calculator and see the result as a decimal: 183.8. Note that this is NOT division with remainder. However, we can quickly adjust this. Since the result is between 183 and 184, we already know that the quotient is 183 and now we need to only find the remainder. If the quotient is 183, then we have already accounted for $183 \cdot 45 = 8235$. How much is missing from 8271? $8271 - 8235 = 36$. Thus the answer is: $8271 \div 45 = 183 \text{ R } 36$.

It is always a good idea to check: $183 \cdot 45 + 36 = 8271$, and $36 < 45$ and so the remainder is less than the divisor. So our solution is correct.

Example 2b. Perform the given divisions with remainder.

a) $137 \div 4$ [Solution - Youtube link](#)

b) $325 \div 7$ [Solution - Youtube link](#)

An Application: Orbits

Example 3. What is the last digit of 2^{99} ? [Solution - Youtube link](#)

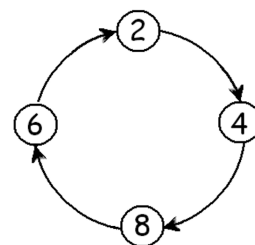
Solution: Let us first investigate what the last digits of smaller 2-powers are.

$2^1 = 2$	$2^5 = 3\boxed{2}$	$2^9 = 51\boxed{2}$
$2^2 = 4$	$2^6 = 6\boxed{4}$	$2^{10} = 102\boxed{4}$
$2^3 = 8$	$2^7 = 12\boxed{8}$	$2^{11} = 204\boxed{8}$
$2^4 = 1\boxed{6}$	$2^8 = 25\boxed{6}$	$2^{12} = 409\boxed{6}$

Before the 2-powers become uncomfortably large, we can observe a repeating pattern: the last digits being

$$2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6, \dots$$

There are only 4 numbers appearing as last digits, following each other in a repeating pattern, we also call an **orbit**. The question only remains: where does 99 land in this pattern?



One way to determine that is to apply division with remainder. Consider the last row:

$$2^4 = 1\boxed{6}, \quad 2^8 = 25\boxed{6}, \quad 2^{12} = 409\boxed{6},$$

We can see that if the exponent is divisible by 4, then the last digit is 6. Consider now the first row:

$$2^1 = \boxed{2}, \quad 2^5 = 3\boxed{2}, \quad 2^9 = 51\boxed{2},$$

These exponents, 1, 5, 9, 13, ... are right after a number divisible by 4. These are numbers that result in a remainder 1 when divided by 4. In the next row,

$$2^2 = \boxed{4}, \quad 2^6 = 6\boxed{4}, \quad 2^{10} = 102\boxed{4},$$

the exponents are even but not divisible by 4. These can also be expressed as numbers that result in a remainder 2 when divided by 4. In the last row,

$$2^3 = \boxed{8}, \quad 2^7 = 12\boxed{8}, \quad 2^{11} = 204\boxed{8},$$

the exponents have a remainder 3 when divided by 4. We amend our results with these labels.

$2^1 = 2$	$2^5 = 3\boxed{2}$	$2^9 = 51\boxed{2}$	remainder: 1	last digit: 2
$2^2 = 4$	$2^6 = 6\boxed{4}$	$2^{10} = 102\boxed{4}$	remainder: 2	last digit: 4
$2^3 = 8$	$2^7 = 12\boxed{8}$	$2^{11} = 204\boxed{8}$	remainder: 3	last digit: 8
$2^4 = 1\boxed{6}$	$2^8 = 25\boxed{6}$	$2^{12} = 409\boxed{6}$	remainder: 0	last digit: 6

We can find where 99 lands in this pattern if we divide it by 4 and focus on the remainder.

$$99 \div 4 = 24 \text{ R } 3$$

The remainder is 3. Therefore, 99 lands in the second to last row with exponents 3, 7, 11, 15, ..., and so the last digit of 2^{99} is 8.



Practice Problems

1. Perform the indicated divisions with remainders.

a) $132 \div 7$ b) $1145 \div 12$ c) $918 \div 8$ d) $201 \div 12$

2. Find the last digit of each of the following numbers.

a) 7^{2017} b) 9^{120} c) 2^{1001} d) 3^{286}

4.3 Algebraic Expressions and Statements

Part 1 – Algebraic Expressions

We will now start studying the expressions and statements frequently made in algebra.

Definition: A **numerical expression** is an expression that combines numbers and operations. To **evaluate** a numerical expression is to compute its value.

For example, $3 \cdot 5^2$ is a numerical expression. So are $-\frac{12}{3+1}$ and $5^2 - 2^2$ and $-|-5|$. We can evaluate numerical expressions by performing the operations indicated in the expression. Naturally, we must correctly apply the order of operations agreement. For that, we must clearly understand notation. It is a fundamental principle that notation can be objectively understood in only *one* possible way.

Example 1. Evaluate each of the given numerical expressions.

a) $3 \cdot 5^2$ b) $-\frac{12}{3+1}$ c) $3^2 + 2^2$ d) $(3+2)^2$ e) -3^2 f) $(-3)^2$ g) $-|-5|$

Solution: a) Between exponentiation and multiplication, we first perform the exponentiation. $3 \cdot 5^2 = 3 \cdot 25 = \boxed{75}$

b) The addition in the denominator must be performed before we divide. (Why?)

$$-\frac{12}{3+1} = -\frac{12}{4} = \boxed{-6}$$

c) $3^2 + 2^2 = 9 + 4 = \boxed{13}$

d) $(3+2)^2 = 5^2 = \boxed{25}$

Note: The error of confusing $3^2 + 2^2$ with $(3+2)^2$ is called the "Freshman's Dream Error".

e) $-3^2 = \boxed{-9}$

f) $(-3)^2 = \boxed{9}$

Note: In looking at -3^2 and $(-3)^2$, we can interpret the minus sign in front of 3 as 'the opposite of'. That is the same as multiplication by -1 . Now we can apply order of operations, and exponentiation comes before multiplication.

$$-3^2 = -1 \cdot 3^2 = -1 \cdot 9 = -9 \quad \text{but} \quad (-3)^2 = (-3)(-3) = 9$$

In the case of -3^2 , we take the opposite of the square of three.

In the case of $(-3)^2$, we square the opposite of three.

g) $-|-5| = \boxed{-5}$ This is a perfect example that two minuses don't always make a plus. What happens here?

Definition: An **algebraic expression** is an expression that combines numbers, operations, and variables.

Variables are letters that represent numbers. They are subjects to the same rules as numbers. We use variables for different reasons. Often because the variable represents a number we do not know but would like to know. However, there are other reasons. Sometimes we use variables because we would like to discuss general statements. Consider the following statement as an example. "*For every numbers x and y , $x + y = y + x$* ". This statement is not about the numbers x and y ; it is rather about the operation addition. No matter what two numbers we add, the order of the numbers in the addition does not matter. We express this property of addition by saying that addition is commutative. In this case, we use variables because we would like to talk about all the numbers at the same time.

In our modern language, we prefer to use x as a variable, especially if it is the type of unknown we want to find. A fundamental principle is that within the context of a problem, a variable has one fixed meaning. Suppose that we are solving a word problem that involves the number of books and the number of pencils. In this case, we can not label both of them x , unless we are certain that the number of books is the same as the number of pencils. **To denote quantities that may be different, we must use different letters.** For example, we can denote the number of books by x and the number of pencils by y . We might find out later in the problem that the values of x and y might be equal. That is perfectly fine. We must use different variables for quantities that *could* be different.

For example, $3x^2 - 1$ is an algebraic expression. So are $-x + 3$ and $2a - b$ and $5y + 3$. If the values of all variables in an expression are given, we can evaluate it. To evaluate an algebraic expression, we substitute the given values of the variables into the expression, and evaluate the resulting numerical expression.

When substituting a number into an algebraic expression, it is our responsibility to preserve the indicated operations and their order correctly. Consider the expression $2x$. If $x = 5$, we can not write 25 instead of $2x$, because our notation would indicate a two-digit number with no operation. If $x = -5$, we can not write $2 - 5$ instead of $2x$, because we would incorrectly indicate subtraction instead of the multiplication.

$$2x \text{ when } x = 5 \implies 2 \cdot 5 \text{ or } 2(5) \qquad 2x \text{ when } x = -5 \implies 2(-5) \text{ or } 2 \cdot (-5)$$

To evaluate an algebraic expression, we can perform the following steps.

- Step 1. Copy the entire expression with one modification: replace each variable by an empty pair of parentheses.
- Step 2. Insert the values into the parentheses. Now the problem became an order of operations problem.
- Step 3. Drop the unnecessary parentheses and work out the order of operations problem. (It may appear awkward first to create all these parentheses but they are extremely helpful.)

Example 2. Evaluate the algebraic expression $3x^2 - x + 5$ given the values of x .

a) $x = -2$ b) $x = 3$

Solution: a) We first copy the entire expression, replacing the letter x by little pairs of parentheses.

$$3x^2 - x + 5 = 3()^2 - () + 5$$

Then we insert the number -2 into each pair of parentheses.

$$3x^2 - x + 5 = 3(-2)^2 - (-2) + 5$$

Now the problem became an order of operations problem. We start with the exponent.

$$\begin{aligned} 3(-2)^2 - (-2) + 5 &= 3 \cdot 4 - (-2) + 5 && \text{perform multiplication} \\ &= 12 - (-2) + 5 && \text{subtraction: } 12 - (-2) = 12 + 2 = 14 \\ &= 14 + 5 && \text{addition} \\ &= \boxed{19} \end{aligned}$$

Notice that because we substituted a negative value for x , all little parentheses proved to be necessary.

Students' work should look like this:

$$\begin{aligned} 3x^2 - x + 5 &= 3(-2)^2 - (-2) + 5 \\ &= 3 \cdot 4 + 2 + 5 \\ &= 12 + 2 + 5 = \boxed{19} \end{aligned}$$

b) Evaluate $3x^2 - x + 5$ when $x = 3$.

We first copy the entire expression, replacing the letter x by little pairs of parentheses.

$$3x^2 - x + 5 = 3()^2 - () + 5$$

Then we insert the number 3 into each pair of parentheses.

$$3x^2 - x + 5 = 3(3)^2 - (3) + 5$$

Because we substituted a positive number, most parentheses are unnecessary. We will drop them:

$$3x^2 - x + 5 = 3(3)^2 - (3) + 5 = 3 \cdot 3^2 - 3 + 5$$

Then we solve the resulting order of operations problem. We start with the exponent.

$$\begin{aligned} 3 \cdot 3^2 - 3 + 5 &= 3 \cdot 9 - 3 + 5 && \text{perform multiplication} \\ &= 27 - 3 + 5 && \text{subtraction} \\ &= 24 + 5 && \text{addition} \\ &= \boxed{29} \end{aligned}$$

Example 2v. Evaluate $-x^2 + 3x - 5$ if $x = -8$ [Solution - Youtube link](#)

Example 3. We ejected a small object upward from the top of a 720ft tall building and started measuring time in seconds. We find that t seconds after launching, the vertical position of the object is $-16t^2 + 64t + 720$ feet.

- Evaluate the given expression with $t = 0$. What does your result mean?
- Where is the object 5 seconds after launch?
- Where is the object 9 seconds after launch?

Solution: a) We evaluate the expression with $t = 0$.

$$\begin{aligned} -16t^2 + 64t + 720 &= -16 \cdot 0^2 + 64 \cdot 0 + 720 \\ &= -16 \cdot 0 + 64 \cdot 0 + 720 = \boxed{720} \end{aligned}$$

This means that at the time of the launch of the object, it is at a height of 720 feet.

- b) To find out where the object is after 5 seconds, we evaluate the expression with $t = 5$.

$$\begin{aligned} -16t^2 + 64t + 720 &= -16 \cdot 5^2 + 64 \cdot 5 + 720 \\ &= -16 \cdot 25 + 64 \cdot 5 + 720 \\ &= -400 + 320 + 720 = -80 + 720 = 640 \end{aligned}$$

So the object is at a height of $\boxed{640 \text{ feet}}$ 5 seconds after launch.

- c) To find out where the object is after 9 seconds, we evaluate the expression with $t = 9$.

$$\begin{aligned} -16t^2 + 64t + 720 &= -16 \cdot 9^2 + 64 \cdot 9 + 720 \\ &= -16 \cdot 81 + 64 \cdot 9 + 720 \\ &= -1296 + 576 + 720 = -720 + 720 = 0 \end{aligned}$$

When we started at the top of the building, the location was 720. This means that a height of zero indicates that the object is at the ground. So the object is at a height of $\boxed{0 \text{ feet}}$ 9 seconds after launch.

Every time we enlarge our language, we have to return to old concepts and discuss them in the light of the new concepts. This is the case with the negative sign. When learning about the order of operations, we discussed that if a negative sign can denote subtraction, then it does denote subtraction. If not, then it is a sign describing the thing after it as *negative*, or *opposite of*. This might have seemed silly until now. Why would we say "*the opposite of three*" instead of "*negative three*"? With the introduction of variables, this is the moment when this interpretation becomes useful.

It is a common mistake to assume that the negative sign in $-x$ means that $-x$ is negative. This is incorrect. We are much better off thinking of $-x$ as "*the opposite of x* ".

Example 4. Evaluate $-x$ with the given values of x .

- a) when $x = 5$ b) when $x = -4$

Solution: a) We can carefully use notation: $-x = -(5) = \boxed{-5}$.

We can also use language: $-x$ is the opposite of x . If x is 5, its opposite is -5 .

b) We can carefully use notation: $-x = -(-4) = \boxed{4}$.

We can also use language: $-x$ is the opposite of x . If x is -4 , its opposite is 4.

Example 5. Evaluate each of the given expressions with the given values of x .

- a) $-x^2$ when $x = 5$ c) $(-x)^2$ when $x = 2$ e) $-(-x)^2$ when $x = 6$
 b) $-x^2$ when $x = -5$ d) $(-x)^2$ when $x = -2$ f) $-(-x)^2$ when $x = -6$

Solution: a) $-x^2$ is the opposite of the square of x . We square 5 and then take the opposite.

$$-x^2 = -(5)^2 = -1 \cdot 5^2 = \boxed{-25}$$

b) We square -5 and then take the opposite.

$$-x^2 = -(-5)^2 = -1 \cdot (-5)^2 = -1 \cdot 25 = \boxed{-25}$$

If we think about this, it is not so surprising that we got the same answer. The square of a number and its opposite is the same. Then if we take the opposite of both, we still have the same result.

c) $(-x)^2$ means the square of the opposite of x . When we carefully substitute $x = 2$, we will have two pairs of parentheses. One came with the expression, the other with the substitution.

$$(-x)^2 = (- (2))^2 = (-2)^2 = \boxed{4}$$

d) $(-x)^2$ means the square of the opposite of x . When we carefully substitute $x = -2$, we will have two pairs of parentheses. One came with the expression, the other with the substitution.

$$(-x)^2 = (- (-2))^2 = 2^2 = \boxed{4}$$

e) $-(-x)^2$ means that we take the opposite of x , we square it, and then take the opposite again. At this point, notation might be the safest way.

$$-(-x)^2 = -(- (6))^2 = -(-6)^2 = -1 \cdot (-6)^2 = -1 \cdot 36 = \boxed{-36}$$

f) $-(-x)^2$ means that we take the opposite of x , we square it, and then take the opposite again.

$$-(-x)^2 = -(- (-6))^2 = -(6)^2 = -1 \cdot 6^2 = -1 \cdot 36 = \boxed{-36}$$



Discussion: Evaluate each of the following algebraic expressions with $x = 2$ and $x = -2$. How are these results similar to or different from the results in Example 4? Can you explain why?

- a) $-x^3$ b) $(-x)^3$ c) $-(-x)^3$

Part 2 – Algebraic Statements

The most frequently made statements in algebra are equations and inequalities.

Definition: An **equation** is a pair of numerical or algebraic expressions connected with an equal sign.

For example, $2 + 3 = 9$ and $1 + 2a = 5b$ are equations. So are $x + y = y + x$, and $x^3 + 4 = x^2 + 4x$. Some equations have variables in them, some don't.

Definition: A **solution of an equation** is a value (or ordered pair of values) of the unknown(s) that, when substituted into both sides of the equation, makes the statement of equality true.

Example 6. Consider the equation $x^3 + 4 = x^2 + 4x$. Evaluate both sides of the equation with the given value of x to determine whether it is a solution of the equation or not.

- a) $x = 3$ b) $x = -2$ c) $x = -1$ d) $x = 2$

Solution: a) We substitute $x = 3$ into both sides of the equation $x^3 + 4 = x^2 + 4x$ and compare the values. We will denote the left-hand side by LHS and the right-hand side by RHS.

If $x = 3$, then

$\begin{aligned}\text{LHS} &= 3^3 + 4 \\ &= 27 + 4 = 31\end{aligned}$	$\begin{aligned}\text{RHS} &= 3^2 + 4 \cdot 3 \\ &= 9 + 12 = 21\end{aligned}$
---	---

The left-hand side is 31, and the right-hand side is 21. Since these are not equal, $x = 3$ is not a solution of the given equation. In short, $\text{LHS} = 31 \neq 21 = \text{RHS}$, so 3 is not a solution.

- b) We substitute $x = -2$ into both sides of the equation $x^3 + 4 = x^2 + 4x$ and compare the values.

If $x = -2$, then

$\begin{aligned}\text{LHS} &= (-2)^3 + 4 \\ &= -8 + 4 = -4\end{aligned}$	$\begin{aligned}\text{RHS} &= (-2)^2 + 4(-2) \\ &= 4 + (-8) = -4\end{aligned}$
--	--

Since the two sides are equal, -2 is a solution of the equation.

- c) We substitute $x = -1$ into both sides of the equation $x^3 + 4 = x^2 + 4x$.

If $x = -1$, then

$\begin{aligned}\text{LHS} &= (-1)^3 + 4 \\ &= -1 + 4 = 3\end{aligned}$	$\begin{aligned}\text{RHS} &= (-1)^2 + 4(-1) \\ &= 1 + (-4) = -3\end{aligned}$
---	--

Since the two sides are not equal, -1 is not a solution of the equation.

d) We substitute $x = 2$ into both sides of the equation $x^3 + 4 = x^2 + 4x$.

If $x = 2$, then

$$\begin{aligned}\text{LHS} &= 2^3 + 4 \\ &= 8 + 4 = 12\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 2^2 + 4 \cdot 2 \\ &= 4 + 8 = 12\end{aligned}$$

Since the two sides are equal, 21 is a solution of the equation.

We have seen that both 2 and -2 are solutions of the equation $x^3 + 4 = x^2 + 4x$. We will leave it for the reader to verify that $x = 1$ is also a solution. So, it is not true that equations can have only one solution! To solve an equation means to find *all* solutions of it.

Example 6v. Given the values $-4, -2$, and 6 , find all numbers listed that are solution of the equation $x^3 - x^2 - 27x + 2 = x^2 - 3x + 2$. [Solution - Youtube link](#)

Example 7. Consider the equation $1 + 2a = 5b$. Evaluate both sides of the equation with the given ordered pairs of values to determine whether they are a solution of the equation or not.

a) $a = 7$ and $b = 3$ b) $a = 3$ and $b = 7$

Solution: a) The pair $a = 7$ and $b = 3$ is often denoted by $(7, 3)$. We substitute these numbers into both sides of the equation $1 + 2a = 5b$.

$$\text{LHS} = 1 + 2 \cdot 7 = 1 + 14 = 15 \quad \text{and} \quad \text{RHS} = 5 \cdot 3 = 15$$

The left-hand side and the right-hand side are both 15. Thus the ordered pair $(7, 3)$ is a solution of the equation.

b) We substitute $a = 3$ and $b = 7$ into both sides of the equation.

$$\text{LHS} = 1 + 2 \cdot 3 = 1 + 6 = 7 \quad \text{and} \quad \text{RHS} = 5 \cdot 7 = 35$$

The values of the right-hand side and the left-hand side are different, and so the ordered pair $(3, 7)$ is not a solution of the equation. Notice that $(7, 3)$ is a solution, but $(3, 7)$ is not. This is why we call such pairs *ordered* pairs.

Definition: An **inequality** is a pair of numerical or algebraic expressions connected with an inequality sign.

For example, $3x - 1 \geq 5x - 7$, and $2x < 4y - 5$, and $2x^2 \leq 7x - 4$ are all inequalities.

Definition: A **solution of an inequality** is a value (or ordered pair of values) of the unknown(s) that, when substituted into both sides of the inequality, makes the statement of inequality true.

Example 8. Consider the equation $3x - 1 \geq 5x - 7$. Evaluate both sides of the equation with the given value of x to determine whether it is a solution of the inequality or not.

- a) $x = 5$ b) $x = -5$ c) $x = 3$ d) $x = -3$

Solution: a) We evaluate both sides of the inequality $3x - 1 \geq 5x - 7$ with $x = 5$ to see whether the inequality statement is true.

If $x = 5$, then

LHS = $3 \cdot 5 - 1$ $= 15 - 1 = 14$	RHS = $5 \cdot 5 - 7$ $= 25 - 7 = 18$
--	--

This means that if $x = 5$, then $3x - 1 \geq 5x - 7$ becomes

$$14 \geq 18$$

This statement is false, therefore 5 is not a solution of the inequality.

b) We evaluate both sides of the inequality with $x = -5$.

If $x = -5$, then

LHS = $3(-5) - 1$ $= -15 - 1 = -16$	RHS = $5(-5) - 7$ $= -25 - 7 = -32$
--	--

This means that if $x = -5$, then $3x - 1 \geq 5x - 7$ becomes

$$-16 \geq -32$$

This statement is true, therefore -5 is a solution of the inequality.

c) We evaluate both sides of the inequality with $x = 3$.

If $x = 3$, then

LHS = $3 \cdot 3 - 1$ $= 9 - 1 = 8$	RHS = $5 \cdot 3 - 7$ $= 15 - 7 = 8$
--	---

This means that if $x = 3$, then $3x - 1 \geq 5x - 7$ becomes $8 \geq 8$. This statement is true, therefore 3 is a solution of the inequality.

d) We evaluate both sides of the inequality with $x = -3$.

If $x = -3$, then

LHS = $3(-3) - 1$ $= -9 - 1 = -10$	RHS = $5(-3) - 7$ $= -15 - 7 = -22$
---------------------------------------	--

This means that if $x = -3$, then $3x - 1 \geq 5x - 7$ becomes $-10 \geq -22$. This statement is true, therefore -3 is a solution of the inequality.

Example 8v. Is the statement $9x^2 + 9x + 3 < 2x + 104$ true or false when $x = 3$? [Solution - Youtube link](#)

Part 3 – Translations

We will often need to translate English sentences into algebraic statements. While this might seem intimidating first, we just need to practice this skill. Here are a few basic statements for a start. Suppose our number is x .

y is twice as great as x .	$\implies y = 2x$	y is two greater than x .	$\implies y = x + 2$
y is three times as great as x .	$y = 3x$	y is three more than x .	$y = x + 3$
y is half of x .	$y = \frac{x}{2}$	y is five less than x .	$y = x - 5$
y is one-fifth of x .	$y = \frac{x}{5}$		

Recall a few additional expressions we might also need.

the sum of a and b	$\implies a + b$	the order of a and b does not matter
the product of a and b	ab	the order of a and b does not matter
the difference of a and b	$a - b$	the order of a and b does matter (first mentioned, first written)
the quotient of a and b	$\frac{a}{b}$	the order of a and b does matter (first mentioned is upstairs)
the opposite of a	$-a$	

Notice that the statements " y is the sum of x and three" and " y is three greater than x " will result in the same equation: $y = x + 3$. Similarly, " P is four less than Q " and " P is the difference of Q and four" will both result in $P = Q - 4$. We will see all of these expressions frequently.

Just like in English, we can create many statements by combining these basic expressions and basic statements.

Example 9. Translate each of the following statements into an equation.

- a) M is the sum of A and three times B .
- b) M is three times the sum of A and B .
- c) y is three less than twice x .
- d) y is twice as much as the sum of three and x .
- e) A is three more than the product of B and C .
- f) The opposite of m is five less than one-third of n .

Solution: a) M is the sum of A and three times B .

We can first translate "three times B " to $3B$. Then we have: " M is the sum of A and $3B$ ". Then we can translate "the sum of A and $3B$ " to $A + 3B$. Thus the translation is $M = A + 3B$.

b) M is three times the sum of A and B .

We can first translate the sum of A and B into $A + B$. So now we have: M is three times $A + B$. However, the translation $M = 3 \cdot A + B$ would be incorrect. This way we would multiply only A by 3 and not the entire sum. $3 \cdot A + B$ is the sum of three times A and B . If we want the sum to be multiplied by 3, we would have to force the multiplication to be performed after the addition. We can easily do that with a pair of parentheses; the correct answer is $M = 3(A + B)$.

c) Given: y is three less than twice x .

We can translate "twice x " as $2x$. Then we have: y is three less than $2x$. This is one of the basic statements: $y = 2x - 3$.

- d) *y is twice as much as the sum of three and x.*

We can first translate "*the sum of three and x*" into $3 + x$. Then we have: *y is twice as much as* $3 + x$. This means that we get y if we multiply $3 + x$ by 2. However, $y = 2 \cdot 3 + x$ would be incorrect. According to order of operations, we would not multiply the sum by 2, only three. So the correct way to express this is $y = 2(3 + x)$. The parentheses overwrites the usual order of operations, so we really multiply the entire sum and not just parts of it. So the translation is $y = 2(3 + x)$.

- e) *A is three more than the product of B and C.*

The product of B and C is simply BC . So now we have that *A is three more than BC*. This is again one of the basic ones: $A = BC + 3$.

- f) *The opposite of m is five less than one-third of n.*

We can translate "*the opposite of m*" to $-m$, and "*one-third of n*" to $\frac{n}{3}$. Then we have: "*-m is five less than* $\frac{n}{3}$ ", which can be translated to $-m = \frac{n}{3} - 5$.

Example 9v. Translate the given expression.

The sum of a number increased by one and the number decreased by six [Solution - Youtube link](#)

Example 10v. Translate the given sentence to an equation using x for the number.

- a) The sum of three times a number and six is nine. [Solution - youtube link](#)
- b) The sum of a number and four is ten. [Solution - youtube link](#)
- c) The product of three more than a number and eight is 48. [Solution - youtube link](#)

Sometimes the same expressions show up "disguised". One such frequently occurring expression is *consecutive integers*. How can we translate three consecutive integers?

Consecutive means that they come right after the other, like 3, 4, and 5. If we denote the smallest number by x , then the other two would be $x + 1$ and $x + 2$. If we denote the largest number by x , then the three numbers would be expressed as x , $x - 1$, and $x - 2$. In word problems we often have the freedom of selecting which of the three numbers should we denote by x , and the best choice often depends on the particular problem.

How about four consecutive even numbers? First let us look at an example, say 6, 8, 10, and 12. We see that each one is two greater than the one before. So if we denote the smallest even number by x , then these numbers can be labeled as x , $x + 2$, $x + 4$, and $x + 6$.



Sample Problems

- Evaluate each of the following numerical expressions.
 a) $2 - 5(3 - 7)$ b) $24 - 10 + 2$ c) -4^2 d) $(-4)^2$ e) $|3| - |8|$ f) $|3 - 8|$
- Evaluate each of the algebraic expressions when $p = -7$ and $q = 3$.
 a) $15 - p$ d) $\frac{q^2 - p}{2q + p + 1}$ g) $2q^2$ j) $(p + q)^2 - (5q + 2p)^4$
 b) $pq - |p - 2|$ e) $p^2 - q^2$ h) $(2q)^2$ k) $-p^2 - p + 8$
 c) $4p - q^3$ f) $(p - q)^2$ i) $15 - \frac{p + q}{|1 - p|}$
- Evaluate the expression $3x^2 - x + 5$ with the given values of x .
 a) $x = 0$ b) $x = -1$
- We ejected a small object upward from the top of a 720ft tall building and started measuring time in seconds. We find that t seconds after launching, the vertical position of the object is $-16t^2 + 64t + 720$ feet.
 a) Where is the object 2 seconds after launch?
 b) Where is the object 8 seconds after launch?
- Let $a = -4$, $b = 2$, and $x = -3$. Evaluate each of the following expressions.
 a) $a^2 - b^2$ b) $(a - b)^2$ c) $a^b - 2bx - x^2 - 2x$ d) $\frac{-x^2 + (x + 2)^2}{(x - 1)}$ e) $\frac{x - 1}{x + 3}$
- Consider the equation $2x^2 + x + 34 = 21x - 8$. In case of each number given, determine whether it is a solution of the equation or not.
 a) $x = 1$ b) $x = 3$ c) $x = 4$ d) $x = 7$
- Consider the equation $x^2 - 10x + x^3 - 4 = 4(x + 5)$. In case of each number given, determine whether it is a solution of the equation or not.
 a) $x = 0$ b) $x = -2$ c) $x = -3$ d) $x = 2$
- Consider the equation $3a - 2b - 1 = (a - b)^2 + 4$. In case of each pair of numbers given, determine whether it is a solution of the equation or not.
 a) $a = 8$ and $b = 5$ or, as an ordered pair, $(8, 5)$
 b) $a = 10$ and $b = 7$ or, as an ordered pair, $(10, 7)$
- Consider the inequality $3(2y - 1) + 1 \leq 5y - 7$. In case of each number given, determine whether it is a solution of the inequality or not.
 a) $y = -10$ b) $y = 3$ c) $y = -5$ d) $y = 0$
- Consider the inequality $\frac{2x + 1}{3} + 5 < \frac{3x - 1}{2}$. In case of each number given, determine whether it is a solution of the inequality or not.
 a) $x = 1$ b) $x = 13$ c) $x = 7$ d) $x = -5$

11. Translate each of the following statements to an algebraic statement.
- y is three less than four times x
 - The opposite of A is one greater than the difference of B and three times C .
 - Twice M is five less than the product of N and the opposite of M .
12. The longer side of a rectangle is three units shorter than five times the shorter side. If we label the shorter side by x , how can we express the longer side in terms of x ?
13. Suppose we have three consecutive integers.
- Express them in terms of x if x denotes the smallest number.
 - Express them in terms of y if y denotes the number in the middle.
 - Express them in terms of L if L denotes the greatest number.



Practice Problems Practice Problems

1. Evaluate each of the following numerical expressions.
- $24 - 5 + 1$
 - $24 \div 3 \cdot 2$
 - -1^2
 - $(-1)^2$
 - $-|4| - |7|$
 - $-|4 - 7|$
 - $6^2 - 4^2$
 - $(6 - 4)^2$
2. Evaluate each of the algebraic expressions when $x = 6$ and $y = 8$.
- $19 - y + x$
 - $19 - (y + x)$
 - $2x^2 - 5y + 3$
 - $x^2 + y^2$
 - $(x + y)^2$
 - $3y - x$
 - $3(y - x)$
 - $5x - \frac{y}{2}$
 - $\frac{5x - y}{2}$
 - $\frac{x^2 - 5x + 4}{y - 3}$
3. Consider the expression $\frac{6x - 3y - xy + 2x^2}{2x - y} - 3$. Evaluate this expression if
- $x = -1$ and $y = 2$ or the ordered pair $(-1, 2)$
 - $x = -3$ and $y = -6$ or the ordered pair $(-3, -6)$
 - $x = 3$ and $y = -2$ or the ordered pair $(3, -2)$
 - $x = -7$ and $y = 4$ or the ordered pair, $(-7, 4)$
4. Evaluate $-m^2 - m$ if
- $m = 2$
 - $m = -2$
 - $m = 0$
 - $m = 5$
 - $m = -5$
5. Evaluate $\frac{8x + x^2 - 33}{x + 11}$ if
- $x = 0$
 - $x = 7$
 - $x = -4$
 - $x = -11$
 - $x = -1$
6. a) It is a common mistake to think that the expressions $2x - 3$ and $2x + 3$ are opposites. They are not. Evaluate these expressions for the values given below to fill out the table below.

	$x = 2$	$x = 5$	$x = 6$	$x = 10$	$x = -1$	$x = -5$	$x = -8$
$2x - 3$	1						
$2x + 3$	7						

b) the opposite of $2x - 3$ is actually $-2x + 3$. Evaluate these expressions for the values given below to fill out the table below.

	$x = 2$	$x = 5$	$x = 6$	$x = 10$	$x = -1$	$x = -5$	$x = -8$
$2x - 3$							
$-2x + 3$							

7. Evaluate $\frac{x-2}{2-x}$ if
- a) $x = 0$ b) $x = 10$ c) $x = 2$ d) $x = -13$
8. Evaluate each of the following algebraic expressions with the value(s) given.
- a) $3x^2 - x + 7$ if $x = -1$ b) $-a + 5b$ if $a = 3$ and $b = -2$ c) $\frac{x^x - 1}{x - 1}$ if $x = 2$
9. The absolute value of a number is its distance from zero on the number line. (Recall that distances can never be negative.) If we wanted to know the distance between two numbers on the number line, the absolute value is very helpful. Consider the expression $|a - b|$. Evaluate this expression for each of the pairs of values given, and see whether $|a - b|$ really results in the distance between a and b on the number line.
- a) $a = 8$ and $b = 3$ b) $a = 2$ and $b = 10$ c) $a = -2$ and $b = -9$ d) $a = 2$ and $b = -1$
10. Consider the equation $\frac{2x^2 - 11x - 21}{2x + 3} = 3x - (2x + 7)$. In case of each number given, determine whether it is a solution of the equation or not.
- a) $x = 8$ b) $x = 13$ c) $x = 10$
11. Consider the equation $-x^2 - 2x(3 - x^2) = -x + 2$. In case of each number given, determine whether it is a solution of the equation or not.
- a) $x = 0$ b) $x = 1$ c) $x = -1$ d) $x = 2$ e) $x = -2$
12. Consider the equation $y = \frac{5x - 3}{2}$. In case of each pair of numbers given, determine whether it is a solution of the equation or not.
- a) $x = 1$ and $y = 1$ or, as an ordered pair, $(1, 1)$ c) $x = 3$ and $y = 6$ or, as an ordered pair, $(3, 6)$
b) $x = 9$ and $y = 4$ or, as an ordered pair, $(9, 4)$ d) $x = 17$ and $y = 41$ or, as an ordered pair, $(17, 41)$
13. Consider the equation $(p - q)^2 + \frac{3p - 1}{6 - q} = 4(p + 1)$. In case of each pair of numbers given, determine whether it is a solution of the equation or not.
- a) $p = 8$ and $q = 5$ b) $p = 7$ and $q = 1$
14. Consider the inequality $-x + 2 < -x^2 + 2(x + 6)$. In case of each number given, determine whether it is a solution of the inequality or not.
- a) $x = -5$ b) $x = -2$ c) $x = 0$ d) $x = 3$ e) $x = 7$
15. Consider the inequality $\frac{x}{3} + 1 \geq \frac{x + 1}{2} - 1$. In case of each number given, determine whether it is a solution of the inequality or not.
- a) $x = -9$ b) $x = -3$ c) $x = 27$ d) $x = 15$ e) $x = -15$

16. Translate each of the given statements to algebraic statements.
- a) The difference of A and B is four less than the product of A and the opposite of B .
 - b) If we square x , the result is eight less than five times the opposite of x .
 - c) If we subtract ten from P , we get a number that is five less than the sum of Q and twice R .
 - d) Four times a number y is one greater than twice the sum of y and seven.
 - e) The number x is ten greater than its own opposite.
 - f) The square of the sum of x and y is ten greater than the difference of the square of x and square of y .
17. The longer side of a rectangle is seven units longer than three times its shorter side. If we label the shorter side by x , express the longer side in terms of x .
18. Express four consecutive even numbers if
- a) we denote the smallest number by x
 - b) we denote the greatest number by x

Problem Set 4

1. Label each of the following statements as true or false.

- | | |
|---|--|
| a) 3 is a multiple of 3. | i) If a number n is divisible by 4 and 6, then it is also divisible by 24. |
| b) $4 < 4$ | j) Every number divisible by 12 is also divisible by 6. |
| c) $5 \geq 5$ | k) Every number divisible by 6 is also divisible by 12. |
| d) 1 is a prime number. | l) For all sets A , $\emptyset \subseteq A$. |
| e) 2 is a prime number. | m) $\mathbb{Z} \subseteq \mathbb{N}$ |
| f) 14 is a multiple of 4 or 7 is a prime number. | |
| g) 14 is a multiple of 4 and 7 is a prime number. | |
| h) If a number n is divisible by 2 and 3, then it is also divisible by 6. | |

2. List all factors of 90.

3. Consider the given numbers. 101010, 1189188, 35530, 1234321, 20172017. List all numbers from the list that are divisible by: a) 4 b) 6 c) 9 d) 11

4. Which of the given numbers are primes? 501, 737, 91, 101, 2017, 407

5. List the first five prime numbers.

6. Perform the given division with remainder. $2018 \div 7$

7. Suppose $A = \{2, 3, 4, 5, 7, 8\}$, and $B = \{1, 2, 4, 6, 8, 10\}$. Draw a Venn-diagram depicting A and B .

8. Suppose that $U = \{0, 1, 2, 3, \dots, 19, 20\}$. Find each of the following sets.

- | | |
|--|---|
| a) $A = \{x \in U : x \text{ is divisible by } 3\}$ | f) $F = \{x \in U : x < 4 \text{ or } x < 8\}$ |
| b) $B = \{x \in U : x \text{ is divisible by } 5 \text{ or } x < 8\}$ | g) $G = \{x \in U : x < 4 \text{ and } x < 8\}$ |
| c) $C = \{x \in U : x \text{ is divisible by } 5 \text{ and } x < 8\}$ | h) $H = \{x \in U : x \text{ is divisible by } 4\}$ |
| d) $D = \{x \in U : x < 12 \text{ or } x \geq 7\}$ | i) $I = \{x \in U : x \text{ is divisible by } 3 \text{ or } x \text{ is divisible by } 4\}$ |
| e) $E = \{x \in U : x < 12 \text{ and } x \geq 7\}$ | j) $J = \{x \in U : x \text{ is divisible by } 3 \text{ and } x \text{ is divisible by } 4\}$ |

9. Recall the following definitions. A **rectangle** is a four-sided polygon with four right angles. A **square** is a rectangle with four equal sides. Let R be the set of all rectangles and S the set of all squares.

a) Label each of the following statements as true or false.

- | | |
|----------------------------------|----------------------|
| i) Every square is a rectangle. | iii) $R \subseteq S$ |
| ii) Every rectangle is a square. | iv) $S \subseteq R$ |

b) Describe x if we know that $x \in R$ and $x \notin S$.

10. Find each of the following sets and if possible, present them by listing their elements.

- | | |
|---|---|
| a) $A = \{a \in \mathbb{N} \mid a < 6\}$ | c) $C = \{c \in \mathbb{N} \mid c < 7 \text{ or } c > 3\}$ |
| b) $B = \{b \in \mathbb{N} \mid b < 7 \text{ and } b > 3\}$ | d) $D = \{x \in \mathbb{N} \mid x \leq 10 \text{ and } x \text{ is even}\}$ |

11. Let $A = \{n \in \mathbb{N} : n \text{ is divisible by } 2\}$, $B = \{n \in \mathbb{N} : n \text{ is divisible by } 6\}$. Which (if any) of the following is true? $A \subseteq B$ $B \subseteq A$

12. Compute each of the following.

a) $\frac{2}{5}$ of 40 b) $\frac{3}{8}$ of 40 c) 15% of 700 (15% is the same as $\frac{15}{100}$)

13. Evaluate each of the given numerical expressions.

a) $-2 - 3$ e) $(-2)^2$ i) -2^2 m) $3^2 - 7^2$ q) $10 - 3(-8)$ t) $(10 - 3)(-8)$
 b) $-2(-3)$ f) $(-2)^3$ j) -2^3 n) $(3 - 7)^2$ r) $10 - (3 - 8)$ u) $10(-3 - 8)$
 c) $-(2 - 3)$ g) $(-2)^4$ k) -2^4 o) $3^2 + 7^2$ s) $10(-3 - 8)$
 d) $(-2) - 3$ h) $(-2)^5$ l) -2^5 p) $(3 + 7)^2$

14. Evaluate each of the given numerical expressions.

a) $(-2)^2$ e) $5(-2)^2$ i) $2^2 + 5^2$ m) $(2 - 5)^2$ q) $(2^2 - 5)^2$ u) $(2 - (-5)^2)$
 b) -2^2 f) $5 - 2^2$ j) $(2 + 5)^2$ n) $(2 - 5^2)$ r) $2^2 - (-5)^2$ v) $2^2(-(-5^2))$
 c) (-2^2) g) $5(-2^2)$ k) $(2 + 5^2)$ o) $2^2(-5)^2$ s) $2^2 - (-5^2)$ w) $2^2(-(-5)^2)$
 d) $-(2)^2$ h) $5 - (2)^2$ l) $2^2 - 5^2$ p) $2^2(-5^2)$ t) $(2 - (-5))^2$ x) $2^2(-(-5))^2$

15. Simplify each of the following expressions by applying the order of operations agreement. **Show all steps. Perform only one operation in each step.**

a) $7 \cdot 3^2 - (3 - 2^2 \cdot 5 - 1) \div 2$ f) $\left(\left((8 - 5)^2 - 7\right)^2 - 2\right)^2 - 1$ k) $-2^2 - 3(5 - (-2)^2) - (-1)^3$
 b) $\frac{5 - 1 + 2}{-1^2 + (-1)^2}$ g) $\frac{4^2 + 5^2 - 6 \div 2 \cdot 3}{4^2 - 8 \cdot 2}$ l) $-2 - 5(-3^2 - 2(-7))$
 c) $\frac{(-2)^3 - 5(-3) - (-1)^4 + (-3)^2}{-2^2 - (-1)}$ h) $3 + 2(5 + 3(15 - 2^3) - 2^2 - 1)$ m) $|-10 - 7| - |1 - 4|$
 d) $|3 - 8| - (|3| - |8|)$ i) $4(3(2(2^2 - 1) - 1) - 1) + 5$ n) $|-10 - 7 - |1 - 4||$
 e) $2^3 - 2(5 - (-3)^2)^2$ j) $-3^2 - (-24) \div (5 - (-1)^3) \cdot 2$ o) $|-10 - 7|1 - 4||$
 q) $|-10| - 7 - 1 - 4||$

16. Perform the indicated operations. Show all steps.

a) $8 - 2(7 - 3^2 + 1) + 5$ b) $\left((3^2 - 13)^2 - 11\right)^2 - 1^3$ c) $\sqrt{19 - \sqrt{2\sqrt{25} - 1}}$

17. Let $p = 4$, $q = -3$, and $s = 1$. Evaluate each of the following expressions.

a) $-q^2 - pq$ b) $\frac{2p - q}{p - (s - q)}$ c) $p^2 - 2s^2$ d) $p^2 - (2s)^2$ e) $\sqrt{|p - q| - 3s}$ f) $p - |q - 3s|$

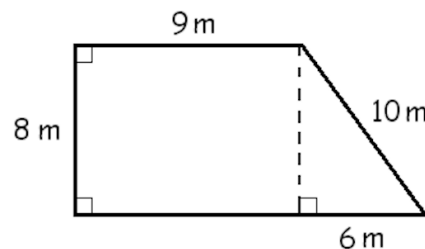
18. Suppose that $x = 4$ and $y = -3$. Evaluate each of the algebraic expressions.

a) $2x - y + 1$ b) $-y^2 - 3x^2y$ c) $(-x)^2 - 5y$ d) $5x - 2y + 2x + y$ e) $\left|\frac{x^2 - y^2}{y^2 - x^2}\right|$

19. Consider the equation $x^3 - x^2 + 7 = x^2 + 5x + 1$. Which of the given numbers are solutions of the equation? The given numbers: 0, 1, -1, 2, and -2

20. Consider the inequality $x^2 + 3x \leq x + 24$. Which of the given numbers are solutions of the inequality? The given numbers: 5, 6, 0, -10, 3, and 4

21. Compute the perimeter and area of the figure shown. Include units in your computation and answer.



22. Translate each of the following sentences into algebraic statements.

- | | |
|--|--|
| a) X is three less than twice Y . | g) Suppose that a cab-fare for one person in Chicago is 2.50 dollars for the first mile and 1.5 dollars for each additional mile. Write an algebraic expression for the price of a cab ride in Chicago of a distance of x miles. |
| b) The opposite of x is five greater than the sum of y and half of z . | h) We are thinking of three consecutive numbers. Express them in terms of x if x represents the smallest number. |
| c) The product of a and b is fourteen less than three times the sum of a and b . | i) The longer side of a rectangle is three feet shorter than four times the shorter side. If the shorter side of the rectangle is denoted by x , express the area of the rectangle in terms of x . |
| d) The sum of m and twice n is one greater than the quotient of m and n . | j) Suppose that Ann has A dollars and Beatrix has B dollars. The girls made a bet that Ann lost, so she must pay Beatrix 30 dollars. Express how much money do the girls have after Ann paid. |
| e) Three times the difference of x and y is one less than the product of x and the opposite of y . | |
| f) Suppose that Peter's age is ten less than twice the age of his younger brother, David. If Peter's age is denoted by P and David's age is denoted by D , write an algebraic statement expressing P in terms of D . | |

23. What is the last digit of 7^{2019} ?

24. Compute each of the following.

- a) $-100 + (-99) + (-98) + \dots + 98 + 99 + 100 + 101 + 102$
 b) $-100(-99)(-98)\dots \cdot 98 \cdot 99 \cdot 100 \cdot 101 \cdot 102$

- 25*. (Enrichment) There are thirty students in our Math 99 class. Twenty of them also takes English 101, fifteen of them also takes Speech 101, and ten of them takes both English 101 and Speech 101. How many of the students in Math 99 are taking neither English 101 nor Speech 101? (Hint: draw a Venn Diagram!)
- 26*. (Enrichment) Two mathematicians are having a conversation. Mathematician A asks B about his kids. B answers: "I have three children, the product of their ages is 36." A says: "I still don't know how old your children are." Then B tells A the sum of his three kids' ages. A answers: "I still don't know how old they are. Then B adds: "The youngest one has red hair." Now A knows how old the kids are. Do you?

Chapter 5

5.1 Set Operations

We have previously studied sets. At this point, we can define and compare sets. We will now start studying operations on sets.

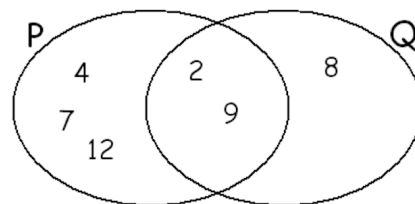
Definition: If A and B are sets, then the **intersection** of A and B , denoted by $A \cap B$, is the set such that for all x , $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

The intersection of two sets is the set of all elements that belong to *both* sets.

Example 1. Suppose that $P = \{2, 4, 7, 9, 12\}$ and $Q = \{2, 8, 9\}$. Find $P \cap Q$.

Solution: The intersection of P and Q is the set containing those elements that are in **both** P and Q . Since P and Q are small sets, we check from element to element, and collect those that belong to both. We can see that $P \cap Q = \{2, 9\}$.

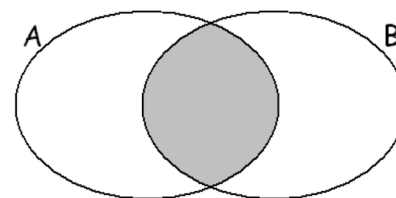
A picture such as this one is called a **Venn diagram**. Venn diagrams often provide useful visual tools to solve set theory problems. We can depict the intersection using Venn Diagrams.



Example 2. Suppose that $A = \{1, 3, 5, 7, 9\}$ and $B = \{5, 6, 7, 8, 9, 10\}$. Find $A \cap B$.

Solution: The intersection of A and B is the set containing those elements that are in both A and B . Since A and B are small sets, we check from element to element, and collect those that belong to both. We can see that $A \cap B = \{5, 7, 9\}$.

If we use a Venn diagram, the intersection of the two sets is the 'overlap' between the two sets as shown.



The shaded region is $A \cap B$

Example 3. Suppose that $T = \{3, 4, 7, 10\}$ and $Q = \{1, 6, 8\}$. Find $T \cap Q$.

Solution: As we look for elements in common, we find none. Thus $T \cap Q = \emptyset$. When this happens, we say that the two sets are **disjoint**.

We want the intersection of two sets to always be a set. In other words, we want the set of sets (!) to be closed under intersection. This is why it was important for us to define $\emptyset$, the empty set.

Example 4. Find $\mathbb{N} \cap \mathbb{Z}$.

Solution: If we start with the natural numbers, we notice that they are automatically in $\mathbb{Z}$. Indeed, $\mathbb{N}$ is a subset of $\mathbb{Z}$, and so every element in $\mathbb{N}$ is also in $\mathbb{Z}$ and thus in both sets. However, with the negative integers and zero we find that they are not in both sets because they are not in $\mathbb{N}$. Thus the intersection of the two sets is $\mathbb{N}$. In short, $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$.

Example 5. Let $S = \{3, 8, 14\}$. Find $S \cap \emptyset$.

Solution: The intersection of two sets is the set of elements in *both* sets. Since there is nothing in the empty set, there cannot be anything in the intersection. Thus $S \cap \emptyset = \emptyset$.

Definition: If A and B are sets, then the **union** of A and B , denoted by $A \cup B$, is the set such that for all x ,
 $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

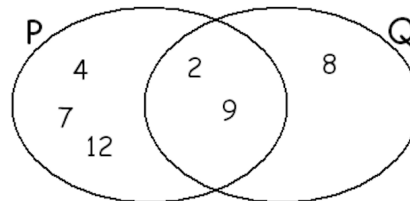
The word 'or' is used in the strict mathematical sense. $x \in A$ or $x \in B$ is true if either x is in A only, or x is in B only, or if x is in both. So, x is in the union of A and B if it is in A , in B , or in both A and B .

The union of two sets is the set of all elements from one set, put together with the set of all elements of the other. Imagine we throw the elements of both sets together and then we list them as elements of a single set, ignoring repetitions.

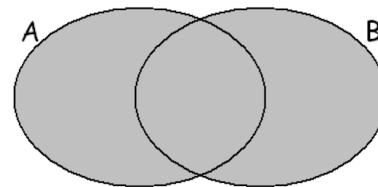
Example 6. Suppose that $P = \{2, 4, 7, 9, 12\}$ and $Q = \{2, 8, 9\}$. Find $P \cup Q$.

Solution: The union of P and Q is the set containing those elements that are in P or in Q . Since P and Q are small sets, we check from element to element, and collect those that belong to either sets or to both. Another way of visualizing the union is to throw together P and Q and list the resulting set without repetition. We can see that $P \cup Q = \{2, 4, 7, 8, 9, 12\}$.

A Venn diagram might help again. For the union, we collect every element from the three separate regions.



If we use a Venn diagram, the union of the two sets is the collection of those three regions as shown.



The shaded region is $A \cup B$.

Example 7. Find $\mathbb{N} \cup \mathbb{Z}$.

Solution: Let us start with the integers this time. All integers are in the union since they are in $\mathbb{Z}$. Now we look at the other set, $\mathbb{N}$, and notice that all natural numbers are already listed in the union because they are automatically in $\mathbb{Z}$. Indeed, $\mathbb{N}$ is a subset of $\mathbb{Z}$, and so every element in $\mathbb{N}$ is also in $\mathbb{Z}$. Therefore, $\mathbb{N}$ does not bring anything new to the union, and so $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$.

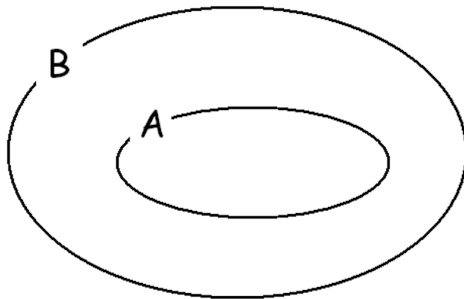
Example 8. Suppose $T = \{2, 10, 12, 30\}$. Find $T \cup \emptyset$.

Solution: The union obviously contains all four elements of T . Now we need to add the elements that are not in T but are in the empty set. Since there isn't anything in the empty set, it does not bring anything new to the union, and so $T \cup \emptyset = \{2, 10, 12, 30\}$. We can also state the answer as $T \cup \emptyset = T$.

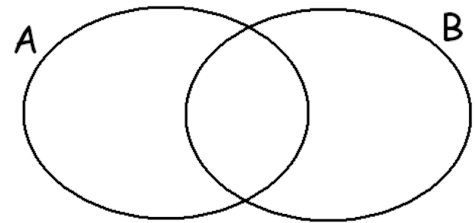


Practice Problems

- Suppose that $P = \{1, 4, 6, 9\}$ and $Q = \{1, 2, 3, 4, 5\}$. Find each of the following.
 - $P \cap Q$
 - $P \cup Q$
 - $P \cap \emptyset$
 - $Q \cup \emptyset$
- Let $A = \{1, 2, 5, 8, 9\}$ and $B = \{2, 4, 6, 8\}$.
 - Draw a Venn diagram depicting these sets.
 - Find each of the following.
 - $A \cap B$
 - $A \cup B$
 - $B \cup (A \cap B)$
 - Label each of the following statements as true or false.
 - $A \subseteq A \cap B$
 - $B \subseteq A \cup B$
 - $A \cap B \subseteq A \cup B$
- Let P denote the set of all students taking physics at Truman College. Let M denote the set of all students taking mathematics at Truman College.
 - describe the set $P \cap M$
 - describe the set $P \cup M$
- Label each of the following statements as true or false.
 - $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$
 - $\mathbb{N} \cap \mathbb{Z} = \mathbb{Z}$
 - $\mathbb{N} \cup \mathbb{Z} = \mathbb{N}$
 - $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$
- Label each of the following statements as true or false. (Hint: make up suitable examples for yourself and then investigate!)
 - If $A \subseteq B$, then $A \cap B = A$
 - If $A \subseteq B$, then $A \cup B = B$
 - For all sets A and B , $A \cap B \subseteq A$
 - For all sets A and B , $B \subseteq A \cup B$
- Recall that a visual representation of subset is to draw one set inside the other. However, this is not a Venn diagram. In case of a Venn diagram, we must have the three distinct regions.



This is not a Venn diagram.



This is a Venn diagram.

Given a Venn diagram depicting sets A and B , how does it show up that A is a subset of B ?



Enrichment

Suppose that A and B are sets such that $A \cap B = \{1, 2, 5\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$. How many different sets are possible for A ?

5.2 Introduction to Number Theory

Definition: Suppose that N and m are any two integers. If there exists an integer k such that $N = mk$, then we say that m is a **factor** or **divisor** of N . We also say that N is a **multiple** of m or that N is **divisible** by m . Notation: $m|N$

For example, 3 is a factor of 15 because there exists another integer (namely 5) so that $3 \cdot 5 = 15$. Notation: $3|15$.

Example 1. Label each of the following statements as true or false.

- a) 2 is a factor of 10 b) 3 is divisible by 3 c) 14 is a factor of 7 d) 0 is a multiple of 5
e) every integer is divisible by 1 f) every integer n is divisible by n

Solution: a) $10 = 2 \cdot 5$ and so 2 is a factor of 10. This statement is true.

b) $3 = 3 \cdot 1$ and so 3 is divisible by 3. This statement is true.

c) $14 = 7 \cdot 2$ and so 14 is a multiple of 7, not a factor. Can we find an integer k so that $7 = 14 \cdot k$? This is not possible. $k = \frac{1}{2}$ would work, but $\frac{1}{2}$ is not an integer. This statement is false.

d) Since $0 = 5 \cdot 0$, it is indeed true that 0 is a multiple of 5. This statement is true.

e) For any integer n , $n = n \cdot 1$ and so every integer n is divisible by 1. This statement is true.

f) For any integer n , $n = 1 \cdot n$ and so every integer n is divisible by n . This statement is true.



Discussion: What do you think about the argument shown below?

3 is a divisor of 21 because there exists another integer, namely 7 so that $21 = 3 \cdot 7$. As we established that 3 is a divisor of 21, we also found that 7 is also a divisor of 21. In other words, divisors always come in pairs. For example, 28 has six divisors that we found in three pairs: 1 with 28, 2 with 14, and 4 with 7. Consequently, every positive integer has an even number of positive divisors.

Definition: An integer is a **prime number** if it has exactly two divisors: 1 and itself.

For example, 37 is a prime number. Prime numbers are a fascinating study within mathematics. Let us first recall the definition. Given a number n , we can find all of its divisors. For example, $n = 20$ has six divisors: 1, 2, 4, 5, 10, and 20. Prime numbers have a very short list of divisors: only the trivial divisors, 1 and the number itself. For example, 7 is a prime number. 6 is not a prime number since it has divisors other than 1 and 6.

It is important to notice that 1 is not a prime number. The first few prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, ... With respect to multiplication, prime numbers are the basic building blocks of numbers.

Theorem: (*Fundamental Theorem of Arithmetic*) Every integer greater than 1 can be written as a product of prime numbers, and this decomposition is unique up to order of factors.

For example, $20 = 2 \cdot 2 \cdot 5$. According to the fundamental theorem of arithmetic, there is no other way to write 20 as a product of prime numbers. We usually use exponential notation: $20 = 2^2 \cdot 5$.

Example 2. Find the prime factorization of 180.

Solution: We start with the smallest prime number, 2 and ask: is 180 divisible by 2? If the answer is yes, we write down and divide 180 by 2.

$$\begin{array}{r|l} 180 & 2 \\ 90 & \end{array}$$

Now we ask: is 90 divisible by 2? The answer is yes. So we write 2 next to 90 and divide by 2.

$$\begin{array}{r|l} 180 & 2 \\ 90 & 2 \\ 45 & \end{array}$$

Now we ask: is 45 divisible by 2? The answer is no. So, we are done with the prime factor 2 and move on to the next prime number, 3. Since 45 is divisible by 3, we write it next to 45 and divide by 3.

$$\begin{array}{r|l} 180 & 2 \\ 90 & 2 \\ 45 & 3 \\ 15 & \end{array}$$

Once we wrote 1, we are done. The prime factorization of 180 is therefore

$$180 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \text{ or } 180 = 2^2 \cdot 3^2 \cdot 5$$

Now we ask: is 15 divisible by 3? The answer is yes. So we write 3 next to 15 and divide by 3.

$$\begin{array}{r|l} 180 & 2 \\ 90 & 2 \\ 45 & 3 \\ 15 & 3 \\ 5 & \end{array}$$

The only prime divisor of 5 is 5 itself. We write 5 next to 5 and then divide.

$$\begin{array}{r|l} 180 & 2 \\ 90 & 2 \\ 45 & 3 \\ 15 & 3 \\ 5 & 5 \\ 1 & \end{array}$$

Example 3. a) Find the prime factorization of 300.

b) Find the prime factorization of 2250. [Solution - Youtube link](#)

Solution: We start with the first prime number, 2. Is our number, 300 divisible by 2? If yes, we divide 300 by 2. Since $300 = 2 \cdot 150$, we now have one prime factor, 2 and we must find the prime factorization of 150.

We ask next: is the number 150 divisible by 2? If yes, we divide 150 by 2. So now $300 = 2 \cdot 2 \cdot 75$ and we are looking for the prime factorization of 75.

$$\begin{array}{r|l} 300 & 2 \\ 150 & 2 \\ 75 & 3 \\ 25 & 5 \\ 5 & 5 \\ 1 & \end{array}$$

We ask next: is the number 75 divisible by 2? This time, the answer is no. We have exhausted the prime factor 2. So we roll up to 3 and ask: is 75 divisible by 3? If yes, we divide 75 by 3. Since $75 \div 3 = 25$, so now $300 = 2 \cdot 2 \cdot 3 \cdot 25$ and we are looking for the prime factorization of 25. Although we know the final answer now, we continue the process. Every time we find a factor, we write it down and divide. The quotient is written down under the pair in the first column. Once that column reaches 1, the second column is the prime factorization of our number. Thus $300 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 = 2^2 \cdot 3 \cdot 5^2$

Prime Numbers

Suppose we are given a number N and asked to determine if the number is a prime number or not. In theory, this is an easy question to answer. We just take the numbers, $1, 2, 3, \dots, N$ and check each whether they are a divisor of N or not. If we checked all these numbers and find that the only divisors are the trivial divisors 1 and N , the number is a prime number. If we find a non-trivial divisor, the number is not a prime.

One interesting thing we can notice is that the level of difficulty depends on the answer. For example, we can use the divisibility rule for 3 to find that the number 10300800050703 is divisible by 3 (A number is divisible by 3 if and only if the sum of its digits is divisible by 3). So 3 is a non-trivial divisor and so 10300800050703 is not a prime number. Consider now the number 10300800050719. If this is a prime number, that means that as we check all numbers from 2 to 10300800050711, none of them would be a divisor. So we would have to keep going and check millions and millions of numbers. So whether 10300800050703 is a prime or not is an easy question, while the same question for 10300800050719 (assuming it is a prime) is a very difficult question.

While in theory we just need to check all numbers between 1 and the given number, this can be very difficult in practice. Even in the age of computers, if a number is great enough, in practice it is nearly impossible to answer this question. *Cryptography* is a branch of mathematics that studies secret codes. To this day, the safest, most difficult to break codes are based on very large prime numbers.

If we are given a number N and we need to determine whether it is a prime number or not, there are two important ways we can reduce our work.

Theorem: Given an integer $N \geq 2$. If N is not a prime number, then the smallest non-trivial divisor is a prime number.

This means that *if we check the numbers, $2, 3, 4, \dots, N$ in this order*, then we can skip all composite numbers and check only the prime numbers.

Suppose we are given a number N and we need to see if it is a prime number. Let us also suppose that we are looking for divisors starting at 2, then 3, and so on, in an increasing order.

Suppose that our number N is divisible by 77. Then there exists an integer k , such that $N = 77k$. We can express 77 as $7 \cdot 11$. Then

$$N = 77k = 7 \cdot 11 \cdot k = 7(11k)$$

Therefore, if N is divisible by 77, then it is also divisible by 7.

We are not asked to list all divisors, we only want to know if our number is a prime number. Therefore, we can stop searching the moment we find just one non-trivial divisor. If we are checking for divisors in an increasing order, reaching 77 means that all previous numbers, including 7 and 11 failed to be divisors. So, if we didn't find a divisor until 77, we can skip 77 because divisibility by 77 means that N is also divisible by 7 - but then we would have stopped right there.

So, when checking for divisors of N among $2, 3, \dots, N$, in an increasing order, we can skip all composite numbers and check for prime divisors only.

Theorem: Given an integer $N \geq 2$. Suppose further that x is an integer such that $x^2 > N$. Then if we did not find a divisor between 2 and x , the number is a prime number.

Recall that divisors come in pairs. Let us consider the number 36. The pair of divisors are $1 \cdot 36, 2 \cdot 18, 3 \cdot 12, 4 \cdot 9$ and $6 \cdot 6$. In every pair of divisors, (except 6 with itself), the smaller one is less than 6, the greater one is

greater than 6.

Suppose a and b are integers, both greater than 6. Then ab is greater than 36. So if we have a pair of divisors and one is greater than 6, then the other one must be less than 6.

Suppose we would like to determine if 307 is a prime number or not. If we enter $\sqrt{307}$ into the calculator, we get a decimal $\sqrt{307} = 17.52141\dots$. This means that 307 is between 17^2 and 18^2 . Indeed, $17^2 = 289$ and $18^2 = 324$. Let us round the number 17.52141... up, we get 18. Since $18^2 = 324$, (greater than 301), in any pair of divisors, one of them must be 18 or less. This is because if $a, b > 18$, then $ab > 324$.

Suppose we would like to decide whether 307 is a prime number. If we don't find a non-trivial divisor until 18, then our number, 307, is a prime number.

Example 4. For each of the numbers given, determine whether it is a prime number or not.

a) 307

b) 57 c) 59 [Solution - Youtube link](#)

Solution: Given the two facts established above, 307 is a prime number if none of the prime numbers between 2 and 18 is a divisor of 307. These primes are: 2, 3, 5, 7, 11, 13, and 17. We can use the calculator or long division. For example, when we check 7, and enter $307 \div 7$, the result is 43.8571... This is not an integer, therefore 7 is not a divisor of 307. When we check all of 2, 3, 5, 7, 11, 13, and 17, none of them is a divisor of 307. Therefore, 307 is a prime number.

Example 5. Determine whether 401 is a prime number or not.

Solution: We enter $\sqrt{401}$ into the calculator. The result is $\sqrt{401} = 20.02498\dots$. This means that 21 is the smallest number such that its square is greater than 400. Therefore, 401 is a prime number if none of 2, 3, 5, 7, 11, 13, 17, and 19 is a divisor of 401. We check all these numbers and find that none of them is a divisor. Therefore, 401 is a prime number.

Example 6. For each of the numbers given, determine whether it is a prime number or not.

a) 1037 b) 1139 [Solution - Youtube link](#)

Solution: We enter $\sqrt{1037}$ into the calculator. The result is $\sqrt{1037} = 32.202484\dots$. This means that 33 is the smallest number such that its square is greater than 1037. Therefore, 1037 is a prime number if none of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31 is a divisor of 1037. We check all these numbers and find $1037 \div 17 = 61$. Therefore, 1037 is not a prime number since it is divisible by 17, so it has more than two divisors.

Example 7. Find all prime numbers from 21, 37, 77, 89, 93. [Solution - Youtube link](#)



Practice Problems

1. Consider the numbers 64, 75, 80, 128, 270. From this list, find all numbers that are divisible by
a) 5 b) 3 c) 4.
2. List all factors of 48.
3. Which of 53, 73, 91, 101, 139 is NOT a prime number?
4. Find the prime factorization for each of the following numbers.
a) 600 b) 5500 c) 2016 d) 2015
5. Given the numbers, determine about each of them whether it is a prime number or not.
a) 2017 c) 2021 e) 2027 g) 4003
b) 2019 d) 2023 f) 4001 h) 4009
6. What is the smallest prime number greater than
a) 100 b) 1000?



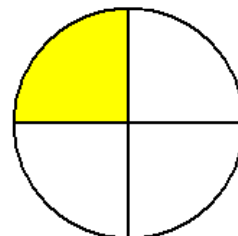
Enrichment

1. Magic: think of a three digit number. Enter a six-digit number into your calculator by repeating your three-digit number twice. For example, if you thought of the three-digit number 275, then enter 275275 into your calculator.
Done? No matter what number you used to start, the number in your calculator is divisible by 7. Divide by 7. The number in your calculator now is still divisible by 11. Divide it by 11. The number in your calculator is still divisible by 13. Divide it by 13. What do you see? Can you explain it?
2. Two mathematicians are having a conversation. Mathematician A asks B about his kids. B answers: "I have three children, the product of their ages is 36." A says: "I still don't know the ages of your children." Then B tells A the sum of his three kids' ages. A answers: "I still don't know how old they are. Then B adds: "The youngest one has red hair." Now A knows the ages of all three children. Do you?
3. A king has his birthday. So he decides to let go some of his prisoners. He actually has 100 prisoners at the moment. They are each in a separate cell, numbered from 1 to 100. Well, he is a high tech king. He can close or open any prison door by a single click on the cell's number on his royal laptop. When he clicks at a locked door, it opens. When he clicks at an open door, it locks. At the beginning, every door is locked. First the king clicks on every number from 1 to 100 (therefore opening every door). Then he clicks on every second number from 1 to 100, (i.e. 2, 4, 6, 8, 10,...). Then he clicks on every third number. And so on. Finally, he only clicks on the number 100. Then he orders that the prisoners who find their door open may go free. Who gets to go and who has to stay?

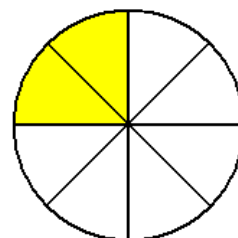
5.3 Fractions 2 – Equivalent Fractions

So far we didn't look at a standalone fraction on its own. We only defined a fraction as expressing part of something. We will continue to do this. A warning to the reader: do not read this section while hungry. We will be talking a lot about cakes and pizzas.

Suppose that Ann, Bethany, Cecile and Desire are about to share a pizza. They cut the pizza into four equal slices. Ann happily looks at her slice (the shaded region). What fraction can express this slice? The answer is $\frac{1}{4}$. We are talking about one slice, so big so that four equal slices make up the whole.



Just before they start to eat, four other friends arrive. They decide to share, so everyone cuts their own slice in two equal pieces. Before she gives her friend half of her food, Ann looks at her plate. Her share now looks like this. What fraction could express this? Since we cut each slice in two, the four slices became eight slices. So, we are now looking at $\frac{2}{8}$.



The fractions $\frac{1}{4}$ and $\frac{2}{8}$ express the same part of a given quantity. Such fractions are called **equivalent** to each other. Given any fraction, we can create infinitely many equivalent fractions by multiplying both numerator and denominator by the same non-zero number. For example, if we start with $\frac{2}{3}$, the following fractions are all equivalent:

$$\frac{2}{3} = \frac{4}{6}$$

multiply both numerator
and denominator by 2

$$\frac{2}{3} = \frac{6}{9}$$

multiply both numerator
and denominator by 3

$$\frac{2}{3} = \frac{8}{12}$$

multiply both numerator
and denominator by 4

$$\frac{2}{3} = \frac{10}{15}$$

multiply both numerator
and denominator by 5

Theorem: (also called the Fundamental Property of Fractions) Given a fraction $\frac{a}{b}$ where a and b are integers, $b \neq 0$, and any non-zero number c , $\frac{a}{b}$ and $\frac{ac}{bc}$ are equivalent fractions, i.e.

$$\frac{a}{b} = \frac{ac}{bc}$$

Another way to express this is as follows: One thing we can always do to a fraction without changing its value is to multiply both numerator and denominator by the same number. Equivalent fractions are essentially different representations of the same number.

Example 1. Re-write $\frac{4}{5}$ with a denominator of 20.

Solution: The only thing we can do is to multiply both numerator and denominator by the same number. If we want a new denominator of 20, that means that we have to multiply the old denominator, 5, by 4. Then we have no other option than multiplying the numerator by the same number. $4 \cdot 4 = 16$ and $5 \cdot 4 = 20$. Thus the equivalent fraction we are looking for is $\boxed{\frac{16}{20}}$.

Example 2. Re-write $\frac{4}{5}$ as a percent.

Solution: The only thing we can do is to multiply both numerator and denominator by the same number. Since we must write $\frac{4}{5}$ as a percent, that means that we want a new denominator of 100. What number should we multiply by 5 to get to 100? This is a division problem, $100 \div 5 = 20$. This means that we will multiply both numerator and denominator by 20.

$$\frac{4}{5} = \frac{4 \cdot 20}{5 \cdot 20} = \frac{80}{100}$$

and $\frac{80}{100}$ is the same as $\boxed{80\%}$.

Now that we know that each fraction can be represented in infinitely many forms, we should find an "official" one, hopefully the simplest.

We now know that a fraction's value does not change if we multiply both numerator and denominator by the same non-zero number.

$$\frac{a}{b} = \frac{ac}{bc} \quad b, c \neq 0$$

Reading the same equality from right to left as $\frac{ac}{bc} = \frac{a}{b}$ tells us that we are also allowed to *divide* both numerator and denominator by the same non-zero number, and we did not change the value of the fraction. This gives us the concept of the simplest form of a fraction.

Definition: Given a fraction $\frac{a}{b}$ where a and b are integers, $b \neq 0$, the fraction is **reduced** or **in lowest terms** if a and b have no common divisor greater than 1. If this is the case, we also say that a and b are **relatively prime**.

It feels true that every fraction has a unique reduced form. Since every fraction can be reduced, and the reduced form is clearly its simplest form, **we always present fractions in lowest terms as final answers.**

Example 3. Reduce $\frac{12}{28}$ to lowest terms.

Solution: We need to look for a common factor between 12 and 28. The greatest common divisor is 4. Indeed, when we divide both numerator and denominator by 4, we get that $\frac{12}{28} = \frac{3}{7}$.

Note that we do not have to get it right for the first time. Suppose we only notice the common factor

2. Then we have $\frac{12}{28} = \frac{6}{14}$. This is not reduced yet, because both 6 and 14 are even. So we go again:

we divide both both numerator and denominator by 2 again: $\frac{12}{28} = \frac{6}{14} = \frac{3}{7}$. Now the fraction is in

lowest terms, so the answer is $\boxed{\frac{3}{7}}$.

Example 4. Re-write 35% as a fraction in lowest terms.

Solution: First, $35\% = \frac{35}{100}$. Clearly both numerator and denominator are divisible by 5, so let's get rid of that 5 by dividing both numerator and denominator by 5. $\frac{35}{100} = \frac{7}{20}$. This is now in lowest form since 7 and 20 do not share any divisor besides the obvious one, 1. So 35% as a reduced fraction is $\boxed{\frac{7}{20}}$.

Until now, we only looked at fractions that were a part of a whole. If we cut a pizza into six equal slices and take all six slices, how can we express this as a fraction? Based on what we learned so far, $\frac{6}{6}$ is the fraction expressing

this. Dividing both numerator and denominator by 6, we get $\frac{6}{6} = \frac{1}{1}$ or simply 1. 1 whole pizza.

So, the reduced form of fractions such as $\frac{3}{3}$ or $\frac{7}{7}$ or $\frac{12}{12}$ is simply 1.

What if we buy two pizzas, we slice them both into 6 equal slices and take 10 such slices? That could be expressed as $\frac{10}{6}$. And if we take all 12 slices? That would be $\frac{12}{6} = \frac{2}{1} = 2$, or two 2 whole pizzas.

This means that every positive integer can be expressed as a fraction, with infinitely many equivalent forms. For example, $5 = \frac{5}{1} = \frac{30}{6}$. Indeed, if we buy 5 whole pizzas and cut each into 6 equal slices, we will have 30 slices, each so big so that 6 of them makes an entire pizza.

Definition: If the numerator of a fraction is less than its denominator, we call such a fraction **proper**. A proper fraction always expresses less than a unit of a given quantity. Examples of proper fractions are $\frac{2}{3}$, $\frac{3}{10}$, $\frac{6}{15}$, or 55%.

If the numerator of a fraction is greater than or equal to its denominator, we call the fraction **improper**. An improper fraction always expresses a whole unit or more of a quantity.

Examples of improper fractions are $\frac{5}{5}$, $\frac{12}{10}$, $\frac{40}{2}$, or 120%.

In spite of the suggested value judgement, we don't see anything improper about an improper fraction. Improper fractions can be presented as final values, as long as they are in lowest terms.

Example 5. Reduce $\frac{30}{12}$ to lowest terms.

Solution: Both 30 and 12 are divisible by 6. So we divide both numerator and denominator by 6 and we obtain that $\frac{30}{12} = \frac{5}{2}$. Since 5 and 2 are obviously relatively primes (i.e. share no divisor greater than 1), this

is the final answer. So the reduced form of $\frac{30}{12}$ is $\boxed{\frac{5}{2}}$.

Example 6. Re-write 160% as a fraction in lowest terms.

Solution: $160\% = \frac{160}{100}$. We can easily divide both numerator and denominator by 10, that is just chopping off the last zero. $160\% = \frac{160}{100} = \frac{16}{10}$. This is still not reduced because both 16 and 10 are even. So we divide both numerator and denominator by 2 and get $160\% = \frac{160}{100} = \frac{16}{10} = \frac{8}{5}$. Since 5 and 8 are obviously relatively primes (i.e. share no divisor greater than 1), this is the final answer. The reduced fraction form of 160% is $\boxed{\frac{8}{5}}$.

Consider now the expression $\frac{15}{5}$. Are we looking at two integers with an operation, namely division between them, or are we looking at a single fraction, expressing 15 slices, where the slices are so big that five of them makes up a whole? There is a tremendous duality here: we can read the same expression in two fundamentally different way. This is only allowed in mathematics if the duality never causes conflict or confusion. That is, every question has the same answer, no matter which of the two interpretations are we using. Instead of confusion, this duality is really a source of power. We can always look at a fraction as a division between integers, and we can always look at a division between integers as if it was a fraction.

Before the time of calculators, people used a lot of mental techniques to help in computation. The following is just an example of such a technique using the duality described above.

Example 7. Perform the division $\frac{140}{5}$ without the help of a calculator.

Solution: Even though this is a division between two integers, we will look at it for a while as if it was a fraction. If so, we are allowed to multiply numerator and denominator by the same number, and our chosen number is 2. This is because division by 10 is very easy: just chop off a zero at the end.

$$\frac{140}{5} = \frac{140 \cdot 2}{5 \cdot 2} = \frac{280}{10} = 28$$



Practice Problems

1. a) Re-write $\frac{2}{9}$ with a denominator of 45, i.e. find x so that $\frac{2}{9} = \frac{x}{45}$.
b) Verify your result by taking $\frac{2}{9}$ of 90 and $\frac{x}{45}$ of 90. If you get the same result, your answer is probably correct.
2. a) Re-write $\frac{5}{6}$ with a numerator of 30.
b) Re-write $\frac{5}{6}$ with a denominator of 30.
3. Re-write each of the given percents as a reduced fraction.
a) 60% b) 25% c) 100% d) 50% e) 150% f) 220%
4. Re-write each of the given fractions as a percent.
a) $\frac{3}{10}$ b) $\frac{17}{20}$ c) $\frac{60}{25}$ d) $\frac{3}{50}$ e) 5
5. Use the duality between division of integers and standalone fractions to perform the given divisions without the aid of a calculator.
a) $\frac{120}{5}$ b) $\frac{2050}{50}$ c) $\frac{1200}{25}$
6. a) Bring the fractions $\frac{7}{10}$ and $\frac{2}{3}$ to a common denominator. Compare the numerators.
Which fraction is greater, $\frac{7}{10}$ or $\frac{2}{3}$?
b) Compute both $\frac{7}{10}$ and $\frac{2}{3}$ of 60. Is your answer in conflict with part a)?

5.4 One- and Two-Step Equations

Part 1 – One-Step Equations

Equations are a fundamental concept and tool in mathematics.

Definition: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign.

For example, $3x^2 - x = 4x + 28$ is an equation. So is $x^2 + 5y = -y^2 + x + 2$.

Definition: A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality true.

- Example 1.** a) Verify that -2 is not a solution of the equation $3x^2 - x = 4x + 28$.
 b) Verify that 4 is a solution of the equation $3x^2 - x = 4x + 28$.
 c) [Given the values \$-4, -2\$, and \$6\$, find all numbers listed that are solution of the equation \$x^3 - x^2 - 27x + 2 = x^2 - 3x + 2\$. Solution - Youtube link](#)

Solution: a) Consider the equation $3x^2 - x = 4x + 28$ with $x = -2$. We substitute $x = -2$ into both sides of the equation and evaluate the expressions.

If $x = -2$, the left-hand side of the equation is

$$\begin{aligned}\text{LHS} &= 3x^2 - x \\ &= 3(-2)^2 - (-2) \\ &= 3 \cdot 4 + 2 = 12 + 2 = 14\end{aligned}$$

If $x = -2$, the right-hand side of the equation is

$$\begin{aligned}\text{RHS} &= 4x + 28 \\ &= 4(-2) + 28 \\ &= -8 + 28 = 20\end{aligned}$$

Since the two sides are not equal, $14 \neq 20$, the number -2 is not a solution of this equation.

- b) Consider the equation $3x^2 - x = 4x + 28$ with $x = 4$. We evaluate both sides of the equation after substituting 4 into x .

If $x = 4$, the left-hand side of the equation is

$$\begin{aligned}\text{LHS} &= 3x^2 - x \\ &= 3 \cdot 4^2 - 4 \\ &= 3 \cdot 16 - 4 = 48 - 4 = 44\end{aligned}$$

If $x = 4$, the right-hand side of the equation is

$$\begin{aligned}\text{RHS} &= 4x + 28 \\ &= 4 \cdot 4 + 28 \\ &= 16 + 28 = 44\end{aligned}$$

Since the two sides are equal, $x = 4$ is a solution of this equation.

Definition: To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Caution! Finding one solution for an equation is not the same as solving it. For example, the number 2 is a solution of the equation $x^3 = 4x$. However, -2 is also a solution of this equation.

If we think about it a little, trial and error could rarely be a legitimate method because there is no way for us to guarantee that there are no other solutions. It is impossible for us to try all real numbers because there are infinitely many of them, and we have finite lives.

So we will need to develop systematic methods to solve equations. We will start with the easiest group of equations, linear equations. There are several types of linear equations, and we will start with the easiest type that is called one-step equations.

To solve a linear equation, we isolate the unknown by applying the same operation(s) to both sides. Consider, for example, Ann and Dewitt who has the same monthly salary. This month they both get a 40 dollar raise. Who is making more money now? It is clear that if we start with two equal quantities and we add the same amount to them, they will still stay equal. This is the underlying principle of solving equations. We always apply the same operations to both sides in an effort to bring the equation in a simple form such as $x = -2$. The following equations are **one-step equations** because there is only one operation that separates us from the desired form.

Example 2. Solve each of the given equations. Make sure to check your solutions.

$$\text{a) } x - 8 = 10 \quad \text{b) } 3y = -12 \quad \text{c) } \frac{x}{-3} = 8 \quad \text{d) } m + 10 = -5$$

Equations like these are called **one-step equations** because they can be solved in only one step. We need to isolate the unknown on one side. In order to do that, we perform the inverse operation. (The inverse operation of addition is subtraction and vice versa. The inverse operation of multiplication is division and vice versa.)

Solution: a) In order to isolate the unknown, we add 8 to both sides.

$$\begin{aligned} x - 8 &= 10 && \text{add 8} \\ x &= 18 \end{aligned}$$

So the only solution of this equation is 18. We can also say that the solution set is $\{18\}$. We should check; if $x = 18$, the left-hand side is

$$\text{LHS} = x - 8 = 18 - 8 = 10 = \text{RHS} \checkmark$$

So our solution, $x = 18$ is correct.

b) In order to isolate the unknown, we divide both sides by 3.

$$\begin{aligned} 3y &= -12 && \text{divide by 3} \\ y &= -4 \end{aligned}$$

So the only solution of this equation is -4 . We check; if $y = -4$, then

$$\text{LHS} = 3y = 3(-4) = -12 = \text{RHS} \checkmark$$

So our solution, $y = -4$ is correct.

c) In order to isolate the unknown, we multiply both sides by -3 .

$$\begin{aligned} \frac{x}{-3} &= 8 && \text{multiply by } -3 \\ x &= -24 \end{aligned}$$

So the only solution of this equation is -24 . We check; if $x = -24$, then

$$\text{LHS} = \frac{-24}{-3} = 8 = \text{RHS} \checkmark$$

So our solution, $x = -24$ is correct.

d) In order to isolate the unknown, we subtract 10 from both sides.

$$\begin{aligned} m + 10 &= -5 && \text{subtract 10} \\ m &= -15 \end{aligned}$$

So the only solution of this equation is -15 . We check; if $m = -15$, then

$$\text{LHS} = m + 10 = -15 + 10 = -5 = \text{RHS}$$

So our solution, $m = -15$ is correct.



Discussion: Solve each of the following equations. How are these unusual?

a) $5x = 5$ b) $5x = 0$ c) $x - 4 = -4$ d) $\frac{x}{3} = 0$

Example 3. One side of a rectangle is 12 feet long. Find the length of the other side if the area of the rectangle is 60 square-feet.

Solution: Let us denote the missing side by x . We will write and solve an equation expressing the area of the rectangle.

$$\begin{aligned} 12x &= 60 && \text{divide by 12} \\ x &= 5 \end{aligned}$$

Thus the other side is 5 feet long. Note that if we carry the units in the computation, they will work out perfectly.

$$\begin{aligned} (12\text{ft})x &= 60\text{ft}^2 && \text{divide by 12ft} && \text{margin work: } \frac{60\text{ft}^2}{12\text{ft}} = 5\frac{\text{ft} \cdot \text{ft}}{\text{ft}} = 5\text{ft} \\ x &= 5\text{ft} \end{aligned}$$

Sometimes we will solve equations in more abstract forms. The following examples are called **formulas** or **literal equations**. While they might be intimidating for students, the ideas and techniques are the same.

Example 4. Solve each of the given equations for the specified variable.

- a) Solve $A + B = C$ for A b) $IR = V$ for I

Solution: a) All of A , B , and C are unknown, but the instructions identify A as the unknown in which we are interested. We pretend that we know the values of B and C , we just don't care about them. So, first we ask: *what happened to our unknown?* On the left-hand side, we see $A + B$. This means that someone came along and added B to the unknown A . In order to isolate the unknown A , we need to 'undo' this operation, that is, we will subtract B from both sides. Although we do not know the value of B , we still are subtracting the same amount when subtracting B .

$$A + B = C \quad \text{subtract } B$$

$$A = C - B$$

What is somewhat unsettling is that the expression $C - B$ does not collapse to a number because their values are not known. Either way, the solution is $A = C - B$. We can still check, if $A = C - B$, then the left-hand side is

$$\text{LHS} = A + B = \underbrace{C - B + B}_A = C - B + B = C = \text{RHS} \quad \checkmark$$

So our solution, $A = C - B$ is correct.

- b) Consider now the equation $IR = V$. This equation is from physics, it is called Ohm's law. If we connect a lightbulb to a battery, the electric current created depends on properties of the light bulb and the battery. Ohm's law expresses the connection between resistance of the lightbulb (denoted by R), the potential or voltage of the battery (denoted by V), and the electric current (denoted by I). Our unknown is I . The unknown was multiplied by R . In order to isolate the unknown, we will divide both sides by R .

$$IR = V \quad \text{divide by } R$$

$$I = \frac{V}{R}$$

So the only solution of this equation is $I = \frac{V}{R}$.

If we can compute with formulas in the abstract, one formula becomes many. We can solve $V = IR$ for I and get $I = \frac{V}{R}$ and also, solve $V = IR$ for R and get $R = \frac{V}{I}$.

There is an application of one-step equations that helps us with a tricky algebraic expression that comes up often. Suppose we want to express that two numbers add up to 10. If label one number by x , how can we label the other number?

Example 5. Suppose that x represents a number. Let y be another number such that the sum of x and y is 10. Express y in terms of x .

Solution: Our numbers are labeled x and y . We state that their sum is 10 and solve the equation for y in terms of x .

$$x + y = 10 \quad \text{subtract } x$$

$$y = 10 - x$$

So y can be expressed in terms of x as $\boxed{10 - x}$.

Caution! $x - 10$ and $10 - x$ look similar but they are very different. $x - 10$ is a number ten less than x , while $10 - x$ is the number, that, when added to x , results in 10. If we confuse the two, we can quickly check which is which by evaluating the expressions using a few numbers for x . We come up with a few values for x , say -10 , -5 , 1 , 6 , 10 , and 20 . Then we evaluate $x - 10$ and $10 - x$ using these values for x .

x	-10	-5	1	6	10	20		and	x	-10	-5	1	6	10	20
$x - 10$	-20	-15	-9	-4	0	10			$10 - x$	20	15	9	4	0	-10

We can now easily tell which table's columns add up to 10 and which table has columns in which the second number is ten less than the first one. We needed a few values because sometimes we can get unlucky: notice that if x is 10, then both $10 - x$ and $x - 10$ give us the same zero.

Part 2 – Two-Step Equations

Suppose we decide to hide a small object, say a coin. We put the coin on the table, then place an envelope over it, and then, just to be sure, we place a hat on top of the envelope. Let us find the coin! To do that, what do we need to remove, and in what order? We would first remove the hat and then the envelope, right?

This is the basis of solving two-step equations. To isolate the unknown, we will perform the inverse operations, in the reverse order. What happened last can be undone first

Example 6. Solve each of the given equations. Make sure to check your solutions.

$$\text{a) } 10 = 3x - 11 \quad \text{b) } 3x + 8 = -7 \quad \text{c) } \frac{t-7}{2} = -8 \quad \text{d) } \frac{x}{-3} + 4 = 15$$

Solution: a) The equation $10 = 3x - 11$ looks unusual in the sense that two-step equations often contain the unknown on the left-hand side. We are always allowed to swap two sides of an equation. If $A = B$, then clearly, also $B = A$. We will do this first. This is an optional step that is always available.

$$\begin{aligned} 10 &= 3x - 11 && \text{we swap the two sides} \\ 3x - 11 &= 10 \end{aligned}$$

We now look at the side that contains x and ask: *What happened to the unknown?* The answer is: *Multiplication by 3 and then subtraction of 11.* We need to apply the inverse operations, in reverse order. In this case, this means that we will add 11 to both sides and then divide both sides by 3.

$$\begin{aligned} 3x - 11 &= 10 && \text{add 11} \\ 3x &= 21 && \text{divide by 3} \\ x &= 7 \end{aligned}$$

So the only solution of this equation is 7. We check: if $x = 7$, then

$$\text{LHS} = 3x - 11 = 3 \cdot 7 - 11 = 21 - 11 = 10 = \text{RHS} \checkmark$$

So our solution, $\boxed{x = 7}$ is correct.

- b) As we look at the equation $3x + 8 = -7$, we ask: *What happened to the unknown?* The answer is: *Multiplication by 3 and then addition of 8.* We need to apply the inverse operations, in a reverse order. In this case, this means that we will subtract 8 from both sides and then divide both sides by 3.

$$\begin{array}{rcl} 3x + 8 & = & -7 \quad \text{subtract 8} \\ 3x & = & -15 \quad \text{divide by 3} \\ x & = & -5 \end{array}$$

So the only solution of this equation is -5 . We check: if $x = -5$, then

$$\text{LHS} = 3x + 8 = 3(-5) + 8 = -15 + 8 = -7 = \text{RHS} \checkmark$$

So our solution, $x = -5$ is correct.

- c) What happened to the unknown? On the left-hand side, there was a subtraction of 7 and then a division by 2. To reverse that, we will multiply both sides by 2 and then add 7 to both sides.

$$\begin{array}{rcl} \frac{t-7}{2} & = & -8 \quad \text{multiply by 2} \\ t-7 & = & -16 \quad \text{add 7} \\ t & = & -9 \end{array}$$

So the only solution of this equation is -9 . We check: if $t = -9$, then

$$\text{LHS} = \frac{t-7}{2} = \frac{-9-7}{2} = \frac{-16}{2} = -8 = \text{RHS} \checkmark$$

So our solution, $t = -9$ is correct.

- d) What happened to the unknown? On the left-hand side, there was a division by -3 and then an addition of 4. To reverse that, we will subtract 4 from both sides by and then multiply both sides by -3 .

$$\begin{array}{rcl} \frac{x}{-3} + 4 & = & 15 \quad \text{subtract 4} \\ \frac{x}{-3} & = & 11 \quad \text{multiply by } -3 \\ x & = & -33 \end{array}$$

So the only solution of this equation is -33 . We check: if $x = -33$, then

$$\text{LHS} = \frac{x}{-3} + 4 = \frac{-33}{-3} + 4 = 11 + 4 = 15 = \text{RHS} \checkmark$$

So our solution, $x = -33$ is correct.

Example 6v. Solve each of the given equations. Make sure to check your solutions.

a) $3x + 8 = -7$

[Solution - Youtube link](#)

c) $\frac{x+20}{3} = 2$

[Solution - Youtube link](#)

e) $\frac{x-2}{5} = -1$

[Solution - Youtube link](#)

b) $-5x + 3 = 38$

[Solution - Youtube link](#)

d) $\frac{x+3}{-5} = 2$

[Solution - Youtube link](#)

f) $\frac{x}{-4} + 5 = 8$

[Solution - Youtube link](#)

Example 7. The sum of three times a number and seven is -5 . Find this number.

Solution: Let us denote our mystery number by x . The equation will be just the first sentence, translated to algebra. The sum of three times the number and seven is $3x + 7$. So our equation is $3x + 7 = -5$. We will know the number if we solve this equation.

$$\begin{array}{rcl} 3x + 7 & = & -5 \quad \text{subtract 7} \\ 3x & = & -12 \quad \text{divide by 3} \\ x & = & -4 \end{array}$$

Good news! We do not need to check if -4 is indeed the solution of the equation. What if we correctly solved the *wrong* equation? Recall that *we* came up with the equation, it was not given. Instead of checking the number against the equation, we should check if our solution satisfies the conditions stated in the problem. Is it true the sum of three times -4 and seven is -5 ? Indeed, $3(-4) + 7 = -12 + 7 = -5$. Thus our solution, $\boxed{-4}$, is correct.

Example 7v. Translate the given sentence to an equation using x for the number. Then solve the equation for x .

a) The sum of a number and four is ten. Find this number. [Solution - Youtube link](#)

b) The sum of three times a number and six is nine. Find this number. [Solution - Youtube link](#)

c) The product of three more than a number and eight is 48. Find this number. [Solution - Youtube link](#)

Example 9. Solve each of the given equations for the unknown indicated.

a) $A = 3B - C$ for B b) $A = 3(B - C)$ for B

Solution: a) We are to solve the equation $A = 3B - C$ for B . Two things happened to the unknown: first a multiplication by 3 and then C was subtracted. To isolate B , we will reverse those operations in the reverse order. This means that we will first add C and then divide by 3.

$$\begin{array}{rcl} A & = & 3B - C \quad \text{add } C \\ A + C & = & 3B \quad \text{divide by 3} \\ \frac{A+C}{3} & = & B \quad \text{and so } B = \frac{A+C}{3}. \end{array}$$

b) The equation $A = 3(B - C)$ is very similar to the previous one, because it involves the same two operations; only the order is different. We first subtract C and then multiply by 3. So we will divide by 3 first and then add C .

$$\begin{array}{rcl} A & = & 3(B - C) \quad \text{divide by 3} \\ \frac{A}{3} & = & B - C \quad \text{add } C \\ \frac{A}{3} + C & = & B \end{array} \quad \text{Thus our solution is } B = \frac{A}{3} + C.$$

Often a presented method is not the only one possible. We can solve this equation differently, by first distributing 3 and then basically solving an equation very similar to the previous example.

$$\begin{array}{ll}
 A &= 3(B - C) && \text{distribute 3} \\
 A &= 3B - 3C && \text{add } 3C \\
 A + 3C &= 3B && \text{divide by 3} \\
 \frac{A + 3C}{3} &= B
 \end{array}$$

Both methods are correct, and the two results are the same, although they might appear different at first. Once we have more algebra skills under our belt, we will be able to verify that the two expressions are really the same.

The next example is not a two-step equation but it can be solved using the same method. We will see many steps. We will perform the inverse operations, in the reverse order. It is sort of like an onion we take apart. We can always peel off the outermost layer.

Example 8. Solve the given equations

$$\begin{array}{ll}
 \text{a) } \frac{\frac{\frac{3x-1}{5}+2}{-2}-8}{4} = -2 & \text{b) } \frac{\frac{x-5}{-4}+5}{4} - 1 = 4 \quad \text{Solution - Youtube link}
 \end{array}$$

Solution: a) The unknown is on the left-hand side, and lots of operations were done to it. In order, there were: multiplication by 3, subtracting 1, division by 5, adding 2, division by -2 , subtraction of 8 and division by 4. We will undo them in the reversed order. This means: first multiply by 4, then add 8, then multiply by -2 , then subtract 2, then multiply by 5, then add 1, and finally divide by 3. So, that's the plan.

$$\begin{array}{llll}
 \frac{\frac{\frac{3x-1}{5}+2}{-2}-8}{4} = -2 & \text{multiply by 4} & \frac{3x-1}{5} = -2 & \text{multiply by 5} \\
 \frac{\frac{3x-1}{5}+2}{-2} - 8 = -8 & \text{add 8} & 3x-1 = -10 & \text{add 1} \\
 \frac{\frac{3x-1}{5}+2}{-2} = 0 & \text{multiply by } -2 & 3x = -9 & \text{divide by 3} \\
 \frac{3x-1}{5} + 2 = 0 & \text{subtract 2} & x = -3 &
 \end{array}$$

We check: if $x = -3$, then the left-hand side is

$$\begin{aligned}
 \text{LHS} &= \frac{\frac{\frac{3(-3)-1}{5}+2}{-2}-8}{4} = \frac{\frac{\frac{-9-1}{5}+2}{-2}-8}{4} = \frac{\frac{\frac{-10}{5}+2}{-2}-8}{4} = \frac{\frac{-2+2}{-2}-8}{4} = \frac{\frac{0}{-2}-8}{4} \\
 &= \frac{0-8}{4} = \frac{-8}{4} = -2 = \text{RHS } \checkmark
 \end{aligned}$$

Thus our solution, $\boxed{x = -3}$ is correct.

Example 10v. Susan is asked about her age. She answers as follows. "My age is 8 years less than twice the age of my brother." How old is her brother if Susan is 24 years old? [Solution - Youtube link](#)

Example 11v. The cost of a taxi ride is \$1.55 for the first mile and \$1.35 for each additional mile or part thereof. Find the distance we covered if we paid \$24.50. [Solution- Youtube link](#)



Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x - 5 = 17$

4. $\frac{t-5}{12} = 4$

2. $\frac{a-10}{5} = -3$

5. $2x - 7 = -3$

3. $\frac{t}{4} - 10 = -4$

6. $\frac{x+8}{3} = -2$

7. $\frac{x}{3} + 8 = -2$

10. $3x - 10 = -10$

8. $-2x + 3 = 3$

11. $-4x + 6 = -18$

9. $3(x+7) = 36$

12. $\frac{\frac{5x-1}{7} + 3}{5} - 10 = -4$

Solve each of the following application problems.

13. Paul invested his money on the stock market. First he bet on a risky stock and lost half of his money. Then he became a bit more careful and invested money in more conservative stocks that involved less risk but also less profit. His investments made him 80 dollars. If he has 250 dollars in the stock market today, with how much money did he start investing?

14. In a hotel, the first night costs 45 dollars, and all additional nights cost 35 dollars. How long did Mr. Williams stay in the hotel if his bill was 325 dollars?

15. Solve each of the following formulas.

a) $3x - 4y = z$ for x

b) $3x - 4y = z$ for y

c) $AB + PQ = S$ for Q



Practice Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x - 3 = -11$

5. $\frac{x}{7} - 3 = -1$

9. $\frac{x}{7} - 1 = -3$

13. $\frac{x-8}{7} = -2$

2. $-2x - 3 = 7$

6. $-4x - 3 = 13$

10. $-x + 5 = -7$

14. $3b + 13 = -5$

3. $5x - 3 = 17$

7. $\frac{a+1}{4} = -9$

11. $\frac{2x-1}{7} = -3$

15. $\frac{x}{3} - 7 = 7$

4. $\frac{x-3}{7} = -2$

8. $5x - 6 = -6$

12. $5(x-2) = -20$

16. $\frac{\frac{3x-1}{2} + 4}{-5} - 6 + 7 = 2$

17. $\frac{5\left(\frac{5x+2}{-7} - 7\right) + 11}{2} + 12 = 2$

Solve each of the following application problems.

18. Three times the difference of x and 7 is -15 . Find x .

19. Ann and Bonnie are discussing their financial situation. Ann said: *If you take 50 bucks from me and then doubled what is left, I would have \$300.* Bonnie answers: *That's funny. If you doubled my money first and then took \$50, then I would have \$300!* How much do they each have?

20. Susan was asked about her age. She answered as follows: My big brother's age is six less than three times my age. How old is Susan if her big brother is 21 years old?

21. Solve each of the formulas.

a) $\frac{a+b}{2} = c$ for a

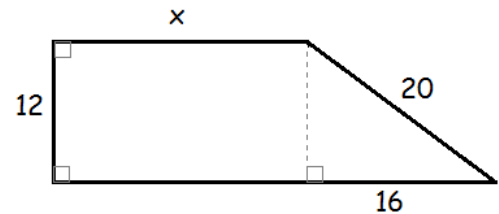
b) $5x + 3y = 15$ for y

c) $M(F - S) = C$ for F

Problem Set 5

- Suppose that $A = \{3, 4, 8\}$ and $B = \{1, 3, 5, 8, 10\}$. Find each of the following.
 - $A \cup B$
 - $A \cap B$
 - Draw a Venn diagram about A and B .
- Suppose that $P = \{1, 2, 3, 6, 7, 9\}$, $Q = \{1, 4, 8, 9\}$, and $S = \{1, 3, 5, 6, 9, 10\}$. Find each of the following.
 - $P \cup Q$
 - $P \cap Q$
 - $(P \cup Q) \cap S$
 - $P \cup (Q \cap S)$
 - $(P \cap Q) \cap S$
- Suppose that F is the set of all integers divisible by five, S is the set of all integers divisible by six, and T is the set of all integers divisible by thirty. Label each of the following statements as true or false.
 - $F \subseteq \mathbb{Z}$
 - $F \subseteq S$
 - $S \subseteq T$
 - $T \subseteq F$
 - $T \subseteq F \cup S$
 - $T \subseteq F \cap S$
 - $T = F \cap S$
- Find the prime factorization of each of the following.
 - 120
 - 480
 - 2016
- List all factors of 150.
- Which of the following numbers is NOT a prime number? 103, 73, 53, 201, 37
- Compute each of the following.
 - $\frac{5}{12}$ of 60
 - $\frac{3}{4}$ of 60
 - 8% of 500
- This problem is about the fraction $\frac{48}{60}$.
 - Simplify the fraction.
 - Re-write this fraction with a numerator of
 - 16
 - 28
 - Re-write this fraction with a denominator of
 - 15
 - 25
 - Re-write this fraction as a percent.
- Compute each of the following.
 - $-20 \div 2(-5) - 3 + 8$
 - $\sqrt{-4^2 - 5(\sqrt{36} - \sqrt{100})}$
 - $\frac{18 - 5 + 2}{2^2 + 1^2}$
- Evaluate each of the given expressions if $x = -2$ and $y = 3$
 - $4x + 5y - 2$
 - $\frac{x^2 + 5x - 4}{y - x}$
 - $\sqrt{x^2 - 10x + 25}$
- Consider the equation $x^3 - 4 = 5x^2 - x^3 + 11x$. Find all numbers from the given list that are solutions of the equation. 2, -3, 4, -1, 0
- Consider the inequality $2x(x - 2) \leq x^2 + 12$. Find all numbers from the given list that are solutions of the inequality. 5, -2, -5, 0, 7
- Solve each of the given equations.
 - $3x + 7 = 1$
 - $\frac{x}{5} - 2 = 6$
 - $\frac{x - 2}{5} = 6$
 - $-2x + 3 = 3$
- Solve each of the following.
 - $\frac{7x - 1}{2} - 1 + 8 = 11$
 - $\frac{\frac{x + 20}{3} - 1}{2} + 5 = 2$
- Solve each of the following.
 - $A = 3B - M$ for B
 - $P = \frac{Q}{R} + T$ for Q

16. 22 is one more than three times a number. Find this number.
17. Pat is asked about her age. Her answer was: *'my age is five less than three times Raul's age'*. How old is Raul if Pat is 31 years old?
18. If we add 24 to twice the opposite of a number, the result is 6. Find this number.
19. The longer side of a rectangle is two feet shorter than five times the shorter side. How long is the shorter side if the longer side is 18 feet long?
20. Ann said that If we doubled her money and added \$20, she would have exactly \$300. Beatrix said, that if we added \$20 and then doubled her money, she would also have \$300. How much money do they each have?
21. The motel charged \$40 for the first night and \$25 for all additional nights. How long did we stay there if the bill was \$265?
22. Find the value of x if the area of the trapezoid on the picture is 348 unit^2



Chapter 6

6.1 Simplifying Algebraic Expressions

We will now continue studying algebraic expressions. In this section, we will learn how to simplify complicated algebraic expressions. We already know how to evaluate algebraic expressions. However, that is not always desirable or even possible. One frequently occurring example for this is when we cannot evaluate an algebraic expression because we do not know the value of the unknown. In such cases, we often need to **simplify** algebraic expressions. Consider, for example, the algebraic expression $2x + 3x$. We can simplify $2x + 3x$ and just write $5x$ instead. This is also called **combining like terms**.

Example 1. Simplify each of the following by combining like terms.

- a) $3x - 2 - 10x + x + 7$ b) $3m - 2n - 4 - 8m + n + 1$ c) $ab - a + b$ d) $p + q + p - q$

Solution: a) $3x$, $-10x$, and x are like terms, and -2 and 7 are like terms.

To combine like terms, we add the numbers (sign included!) that are multiplying the variable(s). Such a number is called the **coefficient**. When combining like terms, we add the coefficients.

To combine $3x$, $-10x$, and x , we add the coefficients: $3x - 10x + x = (3 - 10 + 1)x = -6x$

Caution! It is a common mistake to misinterpret $3 - 10 + 1$ as $3 - 11$. Not so!

To combine -2 and 7 , we just add and so we get 5 . The entire process can be done mentally, so our computation will look like this:

$$3x - 2 - 10x + x + 7 = \boxed{-6x + 5}$$

- b) $3m - 2n - 4 - 8m + n + 1 = 3m - 8m - 2n + n - 4 + 1 = \boxed{-5m - n - 3}$
 -3 , $-n$, and $-5m$ are unlike terms and so this expression can not be further simplified.

- c) $\boxed{ab - a + b}$ since all three terms are unlike, so this expression can not be simplified.

- d) $p + q + p - q = p + p + q - q = \boxed{2p}$

In the last example, as we simplified the expression, one of the variables disappeared. This special and often celebrated case of combining like terms is what we call **cancellation**.

To add two or more algebraic expressions, we drop parentheses and combine like terms.

Example 2. Add the algebraic expressions as indicated.

$$\text{a) } (3a - 5b) + (2a - b) \quad \text{b) } (-m + 3n - 4) + (5m - n - 4) \quad \text{c) } (3y + 5) + (3y - 5)$$

Solution: a) $(3a - 5b) + (2a - b) = 3a - 5b + 2a - b = 3a + 2a - 5b - b = \boxed{5a - 6b}$

b) $(-m + 3n - 4) + (5m - n - 4) = -m + 5m + 3n - n - 4 - 4 = \boxed{4m + 2n - 8}$

c) $(3y + 5) + (3y - 5) = 3y + 5 + 3y - 5 = 3y + 3y + 5 - 5 = \boxed{6y}$

To multiply an algebraic expression by a number or a one-term expression, we apply the distributive law:

$$a(b + c) = ab + ac \quad \text{for all numbers } a, b, \text{ and } c.$$

Example 3. Expand the products as indicated.

$$\text{a) } 3(5a - b + 1) \quad \text{b) } -1(-x^2 + 3x - 4) \quad \text{c) } 5x(2a - x) \quad \text{d) } -ab(3a - 5b - 1) \quad \text{e) } -4(3m - 5)$$

Solution: a) $3(5a - b + 1) = \boxed{15a - 3b + 3}$

b) $-1(-x + 3y - 4) = \boxed{x - 3y + 4}$

Note that multiplication by -1 means we change the sign in front of each term.

c) $5x(2a - 3) = 5x \cdot 2a - 5x \cdot 3 = 10ax - 15x = \boxed{10ax - 15x}$

d) $-ab(3a - 5b - 1) = -ab \cdot 3a - ab(-5b) - ab(-1) = \boxed{-3a^2b + 5ab^2 + ab}$

e) $-4(3m - 5) = \boxed{-12m + 20}$

To subtract an algebraic expression, we add the opposite.

Example 4. Perform the subtractions between algebraic expressions as indicated.

$$\text{a) } (3a - 5b) - (2a - b) \quad \text{b) } (3y + 5) - (3y - 5)$$

Solution: a) We apply one fundamental fact of algebra: *To subtract is to add the opposite.* The opposite is always obtained by multiplication by -1 . And this multiplication by -1 means we need to distribute -1 . We subtract the entire expression, not just its first term. Here is the argument, broken down to logical steps.

$$(3a - 5b) - (2a - b) =$$

to subtract is to add the opposite

$$= (3a - 5b) + (-1)(2a - b) \quad \text{the opposite is obtained by multiplying by } -1$$

(careful with the distributive law)

$$= (3a - 5b) + (-2a + b) \quad \text{we add the algebraic expressions by dropping the parentheses}$$

$$= 3a - 5b - 2a + b \quad \text{and combine like terms}$$

$$= 3a - 2a - 5b + b = \boxed{a - 4b}$$

But this is way too much writing. While the idea is the same, our computation usually looks like this:

$$(3a - 5b) - (2a - b) = 3a - 5b - 2a + b = a - 4b$$

Careful! When computing on paper, we advise *not to subtract mentally*. To subtract is to add the opposite. Take the time of writing down the opposite, and then *add* mentally.

$$\text{b) } (3y + 5) - (3y - 5) = (3y + 5) + (-3y + 5) = \boxed{10}$$

We can now combine more complicated expressions.

Example 5. Simplify each of the following expressions.

$$\text{a) } 4(a - 2b + 1) - 5(2a - b - 1) \quad \text{b) } 6(2y + 1) - 5(3y - 5) \quad \text{c) } 3(5x - 2) - 5(3x + 1)$$

Solution: a) We apply the distributive law and then combine like terms.

$$4(a - 2b + 1) - 5(2a - b - 1) = 4a - 8b + 4 - 10a + 5b + 5 = \boxed{-6a - 3b + 9}$$

$$\text{b) } 6(2y + 1) - 5(3y - 5) = 12y + 6 - 15y + 25 = \boxed{-3y + 31}$$

$$\text{c) } 3(5x - 2) - 5(3x + 1) = 15x - 6 - 15x - 5 = \boxed{-11}$$

The last example illustrates the benefits of algebra. If we were asked to evaluate the expression $3(5x - 2) - 5(3x + 1)$ when $x = 8, -20$, or -97 , or any other number, we might be computing for minutes, but the result will always be -11 . In other words, this expression is **equivalent** to -11 . It is a natural instinct and *widely accepted convention to always present expressions in their simplest possible form.*



Discussion: Explain how $2x + 3x = 5x$ can be explained in terms of the distributive law.

Note: The equation $3a + 2a = 5a$ is an **identity**. An identity is an equation for which all numbers are solutions. We only check two values for a :

$$\text{If } a = 5, \text{ then } 3 \cdot 5 + 2 \cdot 5 = 15 + 10 = 25 = 5 \cdot 5$$

$$\text{If } a = -2, \text{ then } 3(-2) + 2(-2) = -6 + (-4) = -10 = 5(-2)$$

When we simplify algebraic expressions, equations such as $5x - 2x = 3x$ are always identities.

Example 6. The longer side of a rectangle is one foot shorter than three times the shorter side. Find the sides of the rectangle if we also know that its perimeter is 54 feet.

There is no denying that this problem is our first more complex real word problem (also called application problem). The general steps for solving such a problem are as follows.

1. Read the text of the problem. Make a decision about what quantity to label by x . Write this decision down.
2. Read the text of the problem. Label all other quantities in the problem in terms of x . This usually involves translations from English to algebraic expressions using x . Organize all this using a table or a picture.
3. Read the text of the problem. There should be one more piece of information that wasn't used yet. Based on that information, write an equation in x .
4. Solve the equation for x .
5. Now that we know the value of x , all other quantities in the problem can be found, because they were labeled in terms of x .
6. Read the text of the problem. Formulate your answer.
7. Check whether your solution satisfies all conditions stated in the problem.

At first, do not expect to solve the problem on the first attempt. We often have to try to call several things x before we can successfully solve the problem.

Solution: Let us denote the shorter side by x . Then the longer side can be expressed as $3x - 1$. Before we move on, it might be useful to organize our data using a table or a picture.



We will write and solve an equation expressing the perimeter of the rectangle. The perimeter of a rectangle with sides a and b is $P = 2a + 2b$.

$$2 \cdot \underbrace{x}_{\text{shorter side}} + 2 \cdot \underbrace{(3x-1)}_{\text{longer side}} = 54$$

We will simplify the left-hand side by applying the distributive law and then combining like terms.

$$2x + 2(3x - 1) = 54 \quad \text{distribute 2}$$

$$2x + 6x - 2 = 54 \quad \text{combine like terms}$$

$$8x - 2 = 54$$

This is now a two-step equation we can easily solve for x .

$$8x - 2 = 54 \quad \text{add 2}$$

$$8x = 56 \quad \text{divide by 8}$$

$$x = 7$$

Now that we found x , we suddenly know everything, because x is 7 everywhere.

$$\text{shorter side: } x = 7$$

$$\text{longer side: } 3x - 1 = 3 \cdot 7 - 1 = 20$$

The shorter side was labeled x . We know now that x is 7. The longer side was labeled $3x - 1$. Since x is still 7, that means that the longer side is $3 \cdot 7 - 1 = 20$. Thus the rectangle has sides 7 feet and 20 feet long.

Good news! We do not need to check if 7 is indeed the solution of the equation. What if we correctly solved the wrong equation? Recall that we came up with the equation, it wasn't given. Instead, we should check if our solution satisfies the conditions stated in the problem. Is the longer side indeed one foot shorter than three times the shorter side? Indeed, $3 \cdot 7 - 1 = 20$ ✓ Is the perimeter 54 feet? Indeed,

$$2(7\text{ft}) + 2(20\text{ft}) = 14\text{ft} + 40\text{ft} = 54\text{ft}. \quad \checkmark \text{ So the sides are indeed } \boxed{7 \text{ feet and } 20 \text{ feet}} \text{ long.}$$

Example 7. When Candice took over the register, it contained 30 bills, all five-dollar and ten-dollar bills. How many of each were there if the value of all of these bills together was 210 dollars?

Solution: Suppose that we label the number of ten-dollar bills by x . Then we have $30 - x$ many five-dollar bills. (see Example 4.) We will set up the equation to express the total value of the bills.

	ten-dollar bills	five-dollar bills
number of bills	x	$30 - x$
value in dollars	$10x$	$5(30 - x)$

The equation will express the value of the bills. We solve the equation.

$$\begin{array}{ll}
 10x + 5(30 - x) = 210 & \text{distribute 5} \\
 10x + 150 - 5x = 210 & \text{combine like terms} \\
 5x + 150 = 210 & \text{subtract 150} \\
 5x = 60 & \text{divide by 5} \\
 x = 12 &
 \end{array}$$

Since we labeled the number of ten-dollar bills by x , we have 12 ten-dollar bills. The number of five-dollar bills was labeled $30 - x$, thus there are $30 - 12 = 18$ five-dollar bills. Our answer is:

12 ten-dollar bills and 18 five-dollar bills. We check: $12 + 18 = 30$ so we indeed have 30 bills. The value of all bills is $12 \cdot 10 + 18 \cdot 5 = 120 + 90 = 210$ dollars. Thus our answer is correct.

Sometimes expressions are more complicated than the ones we just saw. One complication can be several pairs of parentheses nested inside each other. Just like in case of order of operations, we start with the innermost parentheses.

Example 8. Simplify the expression $-3(2(3x - 1) + 5 - 7x) + 10$

Solution: We will eliminate parentheses by applying the distributive law, combine like terms, and repeat. Note that if we take the time to combine like terms after expanding a multiplication, that will cut down on the computation in the next step.

$$\begin{array}{ll}
 -3(2(3x - 1) + 5 - 7x) + 10 = & \text{distribute in innermost parentheses} \\
 = -3((6x - 2) + 5 - 7x) + 10 & \text{drop parentheses, combine like terms} \\
 = -3(-x + 3) + 10 & \text{apply the distributive law} \\
 = 3x - 9 + 10 & \text{combine like terms} \\
 = \boxed{3x + 1} &
 \end{array}$$



Enrichment

Think of an integer. Add six to it. Double the sum. Take its opposite. Add eight. Multiply by five. Add ten times your original number. Add 25. Double again. No matter what number you had in mind, the final result is 10. Try this with a few numbers. Can you explain why this happens?



Sample Problems

1. Evaluate each of the following algebraic expressions with the value(s) given.
 - a) $-x^2 - 5x + 2$ if $x = -2$
 - b) $-16t^2 + 32t + 240$ if $t = 3$
2. Add the algebraic expressions as indicated.
 - a) $(3x - 5y) + (2x + 4y)$
 - b) $(2a - 5b + 3) + (-a - 8b + 3)$
3. Multiply the algebraic expressions by a number as indicated.
 - a) $3(2x + 4y - 5)$
 - b) $-5(2a - b + 8)$
 - c) $-1(-p + 3q - 8m + 6)$
 - d) $-(-a + 3b - 7)$
4. Subtract the algebraic expressions as indicated.
 - a) $(2a + b) - (a - b)$
 - b) $(3x - 5y) - (2x + 4y)$
 - c) $(2m - 5n + 3) - (-m - 8n + 3)$
 - d) $(2a - 2b) - (b - a)$
 - e) $(3a - 2) - (1 - 4a)$
5. Simplify each of the following.
 - a) $3(x - 5) - 5(x - 1)$
 - b) $4(2a - b) - 3(5a - 2b)$
 - c) $2(a - 2b) + 3(5b - 2a) - 4(2b - a)$
 - d) $-(3a - 2) - (1 - 4a)$
6. Simplify the given expression. $3(-2(-(6x - 1) + 3(2x - 5) - 5x + 7) - 8x - 10)$
7. Solve each of the given equations. Make sure to check your solutions.
 - a) $5(3x - 8) - 2(8x - 1) = -33$
 - b) $4(5x - 3(2x - 1) + 2x) = 52$
 - c) $2(3(4(5x - 1) - 17x + 2) - 3x + 1) = 14$
8. Solve each of the following application problems.
 - a) Ann and Betty dine together. The total bill is \$38. Ann paid \$2 more than Betty. How much did Betty pay?
 - b) 55 people showed up on the party. There were 3 less women than men. How many men were there?
 - c) One side of a rectangle is 7 cm shorter than five times the other side. Find the length of the sides if the perimeter of the rectangle is 118 cm.
 - d) One side of a rectangle is 3 cm longer than four times the other side. Find the sides if the perimeter of the rectangle is 216 cm.
 - e) The sum of two consecutive even integers is -170 . Find these numbers.
 - f) The sum of three consecutive odd integers is 57. Find these numbers.
 - g) Small ones weigh 3 lb, big ones weigh 4 lb. The number of small ones is 3 more than twice the number of big ones. All together, they weigh 79 lb. How many small ones are there?
 - h) We have a jar of coins, all quarters and dimes. All together, they are worth \$17.60 We have 13 more quarters than dimes. How many quarters, how many dimes?
 - i) Red pens cost \$2 each, blue ones cost \$3 each. We bought some pens. The number of red pens is 7 less than five times the number of blue pens. How many of each did we buy if we paid \$116?



Practice Problems

1. Evaluate each of the following algebraic expressions with the value(s) given.

a) $3x^2 - x + 7$ if $x = -1$ b) $-a + 5b$ if $a = 3$ and $b = -2$ c) $\frac{x^x - 1}{x - 1}$ if $x = 2$

2. Add the algebraic expressions as indicated.

a) $(2a - 7y + 1) + (2a - 7y - 1)$ c) $(4p - 5q + 1) + (5p - 4q - 2)$ e) $(-5y + 8) + (5y - 8)$
 b) $(-2a - 5) + (3a + 5)$ d) $(2x - 3y + 8) + (-2x - 3y - 8)$

3. Multiply the algebraic expressions by a number as indicated.

a) $5(-3a + b - 5)$ b) $0(2x - 7y + 8z)$ c) $-6(n - 3m - 8)$ d) $-(p - 5q + 3r - 1)$

4. Subtract the algebraic expressions as indicated.

a) $(2a - 7y) - (2a - 7y)$ c) $(2x - 5y - 1) - (-y + 1)$
 b) $(-2a - 5b - 1) - (3a + 5b - 1)$ d) $(2m - 3n + 8) - (-2m - 3n - 8)$

5. Simplify each of the following.

a) $(x + 1) - (x - 1)$ c) $-2 + x - 3(x - 1) - (1 - 2x)$
 b) $2(x - 1) - 3(x - 7)$ d) $-2(a - 3b + c) - 3(a - b + 4c)$

6. Simplify each of the given expressions.

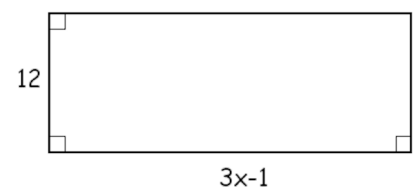
a) $-2((3x - 8)(-2x + 1) - 3x) - 5x + 2$
 b) $3(5(-2(-(2x - 1) + 1) - 2) + 3) - 7$
 c) $2((5x - 1) - 3((8x + 1) + 4(x - 5) - x + 1) + 3x - 3)$

7. Solve each of the following equations. Make sure to check your solutions.

a) $3(x + 2) - 2(5x - 1) = 1 - 2(x + 4)$
 b) $4(2(3(x - 5) + 7 - 2x) - x + 8) = -3(x - 8)$
 c) $5(3(3x - 1) - 2(4x + 3)) = -7(2x + 9) - 1$
 d) $-2(5(2x - 1) - 2(5x + 1) - 3x) + 3 = 2(x - 4) + 1$
 e) $5x - 4(2x - 1) - 3(x - 2) = 5(2(3x - 1) - 4(x - 1))$
 f) $7x - 2((3x + 5) - 3((4x + 1) - 5(x - 2)) - x) = -4$

8. Consider the rectangle shown on the picture. Express the perimeter and area of the rectangle in terms of x . Simplify the expressions.

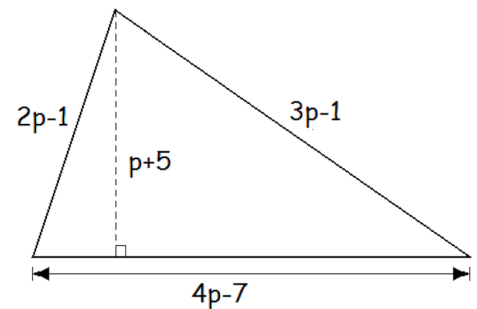
Units are not given. This means that the answer can be presented either without units, or perimeter in unit and area in unit². (For example, $P = 14$ unit and $A = 24$ unit²).



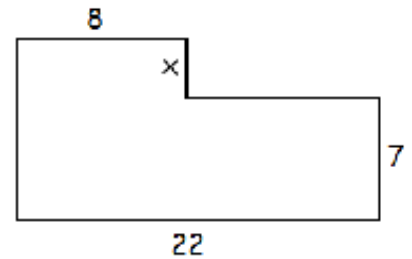
9. Solve each of the following application problems.

- a) One side of a rectangle is 4ft shorter than three times the other side. Find the sides of the rectangle if its perimeter is 64ft.
- b) A bank teller has some five-dollar bills and ten-dollar bills. The number of five-dollar bills is ten less than twice the number of ten-dollar bills. The total value of the money is \$610. How much of each denomination of bill does he have?
- c) Sally worked 50 hours last week and made \$660 for the week. For every hour worked over 40 her job pays time and a half. What is Sally's regular hourly pay rate?
- d) One number is five greater than six times another number. Find these numbers if their sum is 33.
- e) Amy's age is three less than five times her son's age. How old are they if the sum of their ages is 33?
- f) Eight times a the sum of a number and six is 34 greater than the number. Find this number.

- g) Find the value of p if the triangle shown on the picture has a perimeter 54 units. Given the value of p you found, compute the area of the triangle.
- h) Three times a number is one less than twice the difference of the number and three. Find this number.



- i) There were a lot of coins in that jar, all quarters and dimes. The number of dimes was two less than five times the number of quarters. How many of each coins were there if all the coins in the jar were worth 8 dollars and 80 cents?
- j) Consider the figure shown on the picture. Angles that look like right angles are right angles. Find the value of x if we know that the area of this object is 178 unit^2 .
- k) The tickets for the field trip were purchased yesterday for both students and instructors. Children tickets cost \$12, adult tickets cost \$19. The number of children ticket purchased was three more than four times the number of adults tickets purchased. How many of each were purchased if all of the tickets cost a total of \$304 dollars?



- l) The sum of three consecutive multiples of 5 is 750. Find these numbers.
- m) Stan has five more textbooks than fiction books. How many of each does he have if all together he has 93 books?

6.2 The Greatest Common Factor and Least Common Multiple

Suppose that two natural numbers are given. The greatest common factor and least common multiple of two numbers are important quantities that can be obtained using the prime factorization them. In what follows, we will see how these can be computed.

Definition: Suppose that n and m are any two positive integers. The **greatest common factor** (GCF) of the two numbers is the greatest integer that is a factor of both numbers.

Common factors always exist, because 1 is a common factor of all integers. The greatest common factor (or greatest common divisor) of n and m is denoted by $\gcd(n, m)$.

Example 1. Find the greatest common factor of 60 and 96.

Solution: We can list the factors of 60 and 96.

factors of 60 :	1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 60
factors of 96 :	1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96

We can list the common factors:

common factors: 1, 2, 3, 4, 6, 12

Thus the greatest common factor of 60 and 96 is 12.

While the method presented above works, it will be increasingly difficult to apply it for larger numbers. There is a more efficient method that uses the prime factorization.

Solution 2: We will use the prime factorization of these numbers.

$$60 = 2^2 \cdot 3 \cdot 5 \quad \text{and} \quad 96 = 2^5 \cdot 3$$

If d is a divisor of 60, then d can not have a prime divisor other than 2 or 3 or 5. If d is a divisor of 96, then d can not have a prime divisor other than 2 and 3. So, common factors can only have 2 and 3 in their prime factorization.

Now we just need to find the exponents of those prime factors. Consider first the prime factor 2. 60 has 2^2 in its prime factorization. This means that 60 is divisible by 4 but not by 8. So, any factor of 60 can not have more than two 2-factors. 96 has 2^5 in its prime factorization. This means that 96 is divisible by 32 but not by 64. So, any factor of 96 can not have more than five 2-factors. In order to be a factor of both, the exponent can be at most 2. We look at the exponents of 3 and in both prime factorizations, there is exactly one 3-factor. Thus the greatest common factor is $2^2 \cdot 3 =$ 12.

Example 2. Find the greatest common factor of 240 and 135.

Solution: We will use the prime factorization of these numbers.

$$240 = 2^4 \cdot 3 \cdot 5 \quad \text{and} \quad 135 = 3^3 \cdot 5$$

If d is a divisor of 240, then d can not have a prime divisor other than 2 or 3 or 5. If d is a divisor of 135, then d can not have a prime divisor other than 3 and 5. So, common factors can only have 3 and 5 in their prime factorization. We now just need to find the exponents of those prime factors. Consider first the prime factor 3. 240 has one 3–factor in its prime factorization. This means that 240 is divisible by 3 but not by 9. So, any factor of 240 can have at most one 3–factor. 135 has 3^3 in its prime factorization. This means that 135 is divisible by 27 but not by 81. So, any factor of 135 can have at most three 3–factors. In order to be a factor of both, the exponent can be at most 1. We look at the exponents of 5 and in both prime factorizations, there is exactly one 5–factor. Thus the greatest common factor is $3 \cdot 5 = \boxed{15}$.

Example 3. Find the greatest common factor of 40 and 63.

Solution: We will use the prime factorization of these numbers.

$$40 = 2^3 \cdot 5 \quad \text{and} \quad 63 = 3^2 \cdot 7$$

If d is a divisor of 40, then d can not have a prime divisor other than 2 or 5. If d is a divisor of 63, then d can not have a prime divisor other than 3 and 7. So, the common factors can not have any of these in their prime factorization. This means that the two numbers share no divisors besides 1. The greatest common factor of these numbers is $\boxed{1}$. When the greatest common factor of two numbers is 1, we say that the numbers are **relatively prime**.

To compute the greatest common factor of two or more numbers, we list all prime factors that are common to the prime factorization of all numbers. With each such prime number, we use the lowest occurring exponent from the prime factorizations.

Definition: Suppose that n and m are any two positive integers. The **least common multiple (LCM)** of the two numbers is the least positive integer that is a multiple of both numbers.

Common multiples always exist, because the product mn is a common multiple of both integers. The least common multiple of n and m is denoted by $\text{lcm}(n, m)$.

Example 4. Find the greatest common factor of 60 and 96.

Solution: We will use the prime factorization of these numbers.

$$60 = 2^2 \cdot 3 \cdot 5 \quad \text{and} \quad 96 = 2^5 \cdot 3$$

If T is a multiple of 60, then T must have the prime divisors 2, 3, and 5 in its prime factorization. If T is a multiple of 96, then T must have the prime divisors 2 and 3 in its prime factorization. So, common multiples all contain 2, 3, and 5 in their prime factorization. We now just need to find the exponents of those prime factors. Consider first the prime factor 2. 60 has 2^2 in its prime factorization. This means that 60 is divisible by 4 but not by 8. So, any multiple of 60 must have at least two 2–factors. 96 has 2^5 in its prime factorization. This means that 96 is divisible by 32. So, any multiple of 96 must have at least five 2–factors. In order to be a multiple of both, the exponent can be at least 5. We look at the exponents of 3 and in both prime factorizations, there is exactly one 3–factor. Because 60 is divisible by 5, we need a factor of 5 as well. Thus the least common multiple is $2^5 \cdot 3 \cdot 5 = \boxed{480}$.

Example 5. Find the least common multiple of 240 and 135.

Solution: We will use the prime factorization of these numbers.

$$240 = 2^4 \cdot 3 \cdot 5 \quad \text{and} \quad 135 = 3^3 \cdot 5$$

If T is a multiple of 240, then T must have 2, 3, and 5 in its prime factorization. If T is a multiple of 135, then T must have 3 and 5 in its prime factorization. So, common multiples must have 2, 3, and 5 in their prime factorization. We now just need to find the exponents of the least common multiple. Furthermore, in order to be a multiple of 240, T must have at least four 2-factors in its prime factorization. Similarly, in order to be a multiple of 135, T must have at least three 3-factors in its prime factorization. Thus the least common multiple is $2^4 \cdot 3^3 \cdot 5 = \boxed{2160}$.

Example 6. Find the greatest common factor of 40 and 63.

Solution: We will use the prime factorization of these numbers.

$$40 = 2^3 \cdot 5 \quad \text{and} \quad 63 = 3^2 \cdot 7$$

If T is a multiple of 40, then T must have 2 and 5 in its prime factorization. If T is a multiple of 63, then T must have 3 and 7 in its prime factorization. So, the common multiples must have 2, 3, 5, and 7 in its prime factorization. For the exponents: T must have at least three 2-factors if it is a multiple of 40 and T must have at least two 3-factors if it is a multiple of 63. Thus the least common multiple is $2^3 \cdot 3^2 \cdot 5 \cdot 7 = \boxed{2520}$.

Example 7v. Find the greatest common factor and least common multiple of the given pairs of numbers.

- a) 36 and 45. [Solution - Youtube link](#)
 b) 189 and 126 [Solution - Youtube link](#)



Discussion:

Consider our results for examples 1-6. In each case, find the product of the two numbers and the product of their least common multiple and greatest common factor. For example, in case of 60 and 96, $\text{lcm}(60, 96) = 480$ and $\text{gcd}(60, 96) = 12$. So, $60 \cdot 96 = 5760$ and $480 \cdot 12 = 5760$. Is it a coincidence that the products are the same?



Practice Problems

- Find the greatest common factor and least common multiple of each of the given pairs of numbers.
 - 72 and 300
 - 240 and 3600
 - 100 and 420
- Is it possible that the greatest common factor and least common multiple of two numbers are equal to each other?
- Find all possible positive integer values of n if we know that $\text{gcd}(6, 120, n) = 6$ and $\text{lcm}(6, 120, n) = 120$.

6.3 Fractions – Part 3: Improper Fractions and Mixed Numbers

Recall the definition of an improper fraction.

Definition: If the numerator of a fraction is less than its denominator, we call it a **proper fraction**. A proper fraction always expresses less than a unit of a given quantity. Examples of proper fractions are $\frac{2}{3}$, $\frac{3}{10}$, $\frac{6}{15}$, or 55%.

If the numerator of a fraction is greater than or equal to its denominator, we call it an **improper fraction**. An improper fraction always expresses a whole unit or more of a quantity. Examples of improper fractions are $\frac{5}{5}$, $\frac{12}{10}$, $\frac{40}{2}$, or 120%.

Also recall that if we multiply both numerator and denominator of a fraction by the same non-zero number, the resulting fraction is equivalent to the original fraction. Equivalent fractions express the same amount. We can also write any integer as a fraction. For example, 1 can be written as $\frac{1}{1}$, $\frac{3}{3}$, $\frac{5}{5}$, $\frac{100}{100}$ or 100%. The integer 5 can be written as $\frac{5}{1}$, $\frac{10}{2}$, $\frac{15}{3}$, or 500%. We can also divide both numerator and denominator by the same number. When a fraction's numerator and denominator share no divisor greater than 1, the fraction is in **lowest terms**.

With the introduction of improper fractions, our notation includes a new (and huge) duality. We can look at the expression $\frac{20}{4}$ as a division between two integers, i.e. two objects and an operation. We can also interpret the expression $\frac{20}{4}$ as an improper fraction, i.e. a single object. This ambiguity is only allowed because every question we can ever ask has the same answer, no matter which interpretations we used. As final result, fractions must be presented in their simplest form. For example, the fraction $\frac{15}{18}$ must be reduced to lowest terms and presented as $\frac{5}{6}$. The fraction $\frac{20}{4}$ must be presented as the integer 5 because it is a much simpler presentation than $\frac{5}{1}$. But how do we simplify (if we even can) the improper fraction $\frac{7}{3}$?

The name improper fraction already suggests that there is something wrong with such a fraction. We strongly disagree with this notion. However, this might have not been the general opinion when the concept of mixed numbers was developed.

A dime is a common nickname for the silver colored, small ten-cent coin. 10 dimes are worth a dollar, so one dime can be represented as $\frac{1}{10}$ of a dollar. Suppose we have 42 dimes. How can we express this fact? As an improper fraction, we can of course write $\frac{42}{10}$.

Suppose we go to the bank and change all dimes we can for dollar bills. In this case, we could exchange 40 dimes for four dollar bills. Using this idea, we can write $\frac{42}{10}$ as a mixed number as $4\frac{2}{10}$. The integer part expresses the bills, the fraction part expresses the coins. Of course we can also bring the fraction part to lowest terms and get $4\frac{1}{5}$. For some, this is the only way to present the fraction $\frac{42}{10}$ in its simplest form.

Definition: A **mixed number** is an alternative representation for improper fractions that can not be simplified as integers. A mixed number has an integer part and a fraction part. For example, $2\frac{1}{3}$ is a mixed number where the integer part is 2 and the fraction part is $\frac{1}{3}$.

Not every country uses mixed number notation. In countries that don't use mixed numbers, $2\frac{1}{3}$ would appear as $2 + \frac{1}{3}$. In the USA, mixed numbers are fairly common, but their usefulness is debated. Let us also note that this notation is an example where two objects are written next to each other with no operation between them, and it does *not* represent multiplication.

Example 1. Re-write the improper fraction $\frac{87}{10}$ as a mixed number.

Solution: We can think of this as a person with 87 dimes in their pocket. How much money can be exchanged to dollar bills? Since 10 dimes are worth a dollar, we can exchange 80 dimes for eight dollars in paper money, and are left with seven dimes. This can be expressed as $8\frac{7}{10}$.

Example 2. Re-write the improper fraction $\frac{513}{100}$ as a mixed number.

Solution: We can think of this as a person with 403 pennies in their pocket. How much money can be exchanged to dollar bills? Since 100 pennies are worth a dollar, we can exchange 500 pennies for five dollars in paper money, and are left with thirteen pennies. This can be expressed as $5\frac{13}{100}$.

Notice that in converting an improper fraction to a mixed number, we apply division with remainder. The division $87 \div 10 = 8 \text{ R } 7$ is behind the conversion $\frac{87}{10} = 8\frac{7}{10}$ and the division $513 \div 100 = 5 \text{ R } 13$ is behind the conversion $\frac{513}{100} = 5\frac{13}{100}$. The idea behind mixed numbers is simplifying, which means that the integer part needs to be reduced to lowest terms.

Example 3. Re-write the improper fraction $\frac{28}{10}$ as a mixed number.

Solution: We can think of this as a person with 28 dimes in their pocket. We perform the division with remainder:

$28 \div 10 = 2 \text{ R } 8$. This is the same as saying that $\frac{28}{10} = 2\frac{8}{10}$, just as in the previous examples. However, this mixed number needs to be simplified where we bring the fraction part to lowest terms. Clearly $\frac{8}{10} = \frac{4}{5}$, so $\frac{28}{10} = 2\frac{4}{5}$.

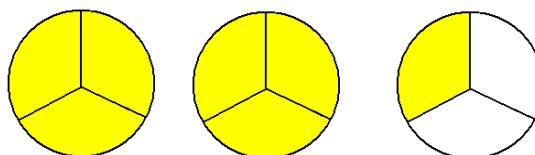
Let us see an example beyond coins. Suppose that we work in a pizza place where each pizza is cut into 6 slices. So, one slice of pizza can be represented as $\frac{1}{6}$. A whole pizza can be represented as 1 (imagine we are not cutting it into slices) or $\frac{6}{6}$ (if we do cut it into slices). How can we express $\frac{25}{6}$ as a mixed number? We perform the division with remainder: $25 \div 6 = 4 \text{ R } 1$ and we convert $\frac{25}{6}$ to the mixed number $4\frac{1}{6}$. We can interpret this as 4 whole pizzas, and one additional slice. This is correct, 4 whole pizzas can be represented as $4 = \frac{4}{1} = \frac{24}{6}$ that is, 24 slices.

Example 4. Re-write the improper fraction $\frac{17}{6}$ as a mixed number.

Solution: We perform the division with remainder: $17 \div 6 = 2 \text{ R } 5$. Thus $\frac{17}{6} = 2\frac{5}{6}$.

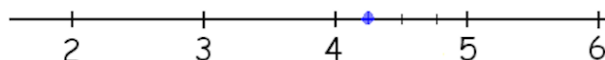
Example 5. Draw a picture representing the improper fraction $\frac{7}{3}$. Use your picture to convert $\frac{7}{3}$ to a mixed number.

Solution: To represent a fraction with denominator three means that we will draw circles and slice them into three equal part. Because this is an improper fraction, we will need more than one circle. We keep starting new units until we have seven slices. Our picture shows that the mixed number corresponding to $\frac{7}{3}$ is $2\frac{1}{3}$.



Example 6. Re-write the improper fraction $\frac{17}{4}$ as a mixed number. Use your result to plot $\frac{17}{4}$ on the number line.

Solution: We perform the division with remainder: $17 \div 4 = 4 \text{ R } 1$. Therefore, $\frac{17}{4} = 4\frac{1}{4}$. When we plot $4\frac{1}{4}$, we first find 4 and 5. We split the line segment between 4 and 5 into four equal part, and count one from 4.



Example 7. Re-write the mixed number $3\frac{2}{5}$ as an improper fraction.

Solution: We can imagine a pizza place where each pizza is cut into five slices. Then the question is: if someone orders three full pizzas and two more slices, how many slices were ordered? Three full pizzas will account for 15 slices, so all together we have 17 slices. Algebraically, $3 = \frac{3}{1} = \frac{15}{5}$ and so $3\frac{2}{5} = \frac{17}{5}$.

Example 8. Re-write the mixed number $2\frac{5}{8}$ as an improper fraction.

Solution: We can imagine a pizza place where each pizza is cut into eight slices. Then the question is: if someone orders two full pizzas and five more slices, how many slices were ordered? Two full pizzas will account for 16 slices, so all together we have $16 + 5 = 21$ slices. Algebraically, $2 = \frac{2}{1} = \frac{16}{8}$ and so $2\frac{5}{8} = \frac{21}{8}$.

In this course, we will always have to present fractions in their simplest possible form as final answers. **However, we will not view mixed numbers as more simplified than improper fractions.** This means that improper fractions can always be presented in their improper form as a final answer, as long as it is in lowest terms. For example, $\frac{42}{10}$ is not acceptable as final answer, but $\frac{21}{5}$ is. The mixed number $4\frac{2}{10}$ is not simplified, $4\frac{1}{5}$ is. However, $4\frac{1}{5}$ is not considered more simplified than $\frac{21}{5}$. They are equally acceptable, and, from an algebraic

point of view, $\frac{21}{5}$ is actually preferred. We will see that with just a very few exceptions, improper fractions have nicer properties than mixed numbers.



Practice Problems

1. Re-write each of the following improper fractions as mixed numbers. Bring the fraction part to lowest terms.

a) $\frac{10}{3}$ b) 150% c) $\frac{35}{10}$ d) $\frac{120}{7}$ e) $\frac{42}{8}$ f) $\frac{715}{100}$ g) 320%

2. Re-write each of the mixed numbers as improper fractions in lowest terms.

a) $3\frac{4}{5}$ b) $1\frac{3}{8}$ c) $5\frac{6}{7}$ d) $3\frac{4}{10}$ e) $10\frac{5}{7}$ f) $5\frac{1}{4}$

3. Draw a picture to represent each of the given fractions.

a) $\frac{13}{4}$ b) $2\frac{3}{5}$ c) $\frac{7}{2}$ d) $\frac{10}{7}$

4. Plot each of the given fractions on the number line.

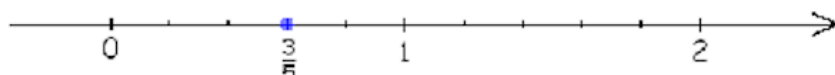
a) $\frac{8}{3}$ b) $3\frac{2}{5}$ c) $\frac{9}{2}$ d) $\frac{7}{4}$

6.4 Fractions – Part 4: Adding and Subtracting

Part 1 - Fractions as Part of Our Number System

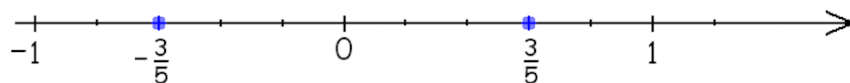
Until now, we have only looked at fractions as expressing a part of a quantity. Before we define the four basic operations on fractions, it is best to start looking at them as part of our number system.

We have defined what it means $\frac{3}{5}$ of something. What about $\frac{3}{5}$ alone, as a number? We understand $\frac{3}{5}$ alone as $\frac{3}{5}$ of the number 1. So, we can actually place fractions on the number line. In case of $\frac{3}{5}$, we divide the unit between zero and one into five equal parts, and place $\frac{3}{5}$ after three such line segments.



We can also interpret all integers as fractions. For example, 4 is the same as $\frac{8}{2}$ or $\frac{4}{1}$. The expression $\frac{4}{1}$ could mean division between the integers 4 and 1, and also, the fraction with numerator 4 and denominator 1. Thus duality is fundamental, and only allowed because all answers are always the same when worked with the two different meaning.

How about negative numbers? Negative fractions? We can clearly think of $-\frac{3}{5}$ as the opposite of $\frac{3}{5}$.



Definition: A *simplified* or *reduced* fraction cannot have more than one negative sign.

Let us consider the fraction $\frac{-2}{-7}$. Recall the fundamental property of fractions: that we can multiply both numerator and denominator by the same non-zero number without changing the value of the fraction. Let us proceed with -1 .

$$\frac{-2}{-7} = \frac{-2(-1)}{-7(-1)} = \frac{2}{7}$$

So, a fraction with a negative number in the numerator and denominator can be simplified, and the result is simply a positive fraction. Notice that the rule is the same as with division between two integers.

The negative sign of a fraction can be located in three places. All three of the fractions shown below are equivalent.

$$-\frac{2}{3} \quad \text{or} \quad \frac{-2}{3} \quad \text{or} \quad \frac{2}{-3}$$

The first form, $-\frac{2}{3}$ (the opposite of $\frac{2}{3}$) is considered simplified, and it is an elegant way to present a fraction as final answer. However, it is not the recommended form for computing with signed fractions.

The second form, $-\frac{2}{3}$ (the numerator is -2 , the denominator is 3) is both simplified and highly recommended for computations. Indeed, when performing operations with signed fractions, we recommend that the negative sign is interpreted as belonging to the numerator.

The third form, $\frac{2}{-3}$ (the numerator is 2 , the denominator is -3) is not considered simplified, and not recommended for computations either. Thus we recommend that when faced with such a fraction, we immediately multiply both numerator and denominator by -1 , thereby switching to the second form, $-\frac{2}{3}$. **Fractions with a negative sign in their denominators are not acceptable as final answers.**

Example 1. Simplify each of the given fractions.

a) $\frac{10}{-28}$ b) $\frac{-12}{1}$ c) $\frac{-3}{-18}$ d) $\frac{4}{-7}$

Solution: a) The fraction itself can be simplified by dividing both numerator and denominator by 2 . But then $\frac{5}{-14}$ is not simplified because the negative sign is in the wrong place. Multiplying both numerator and denominator by -1 will give us the answer in the correct form, $-\frac{5}{14}$. We can also present this as $-\frac{5}{14}$.

$$\frac{10}{-28} = \frac{5}{-14} = \boxed{-\frac{5}{14} \text{ or } -\frac{5}{14}}$$

Eventually we will be able to speed up the process by dividing numerator and denominator of $\frac{10}{-28}$ by -2 .

b) The fraction $\frac{-12}{1}$ is not simplified, because it can be simplified to -12 . Integers should never be presented as fractions. Clearly -12 is simpler than $\frac{-12}{1}$. So, the correct answer is $\boxed{-12}$.

c) We can simplify $\frac{-3}{-18}$ by dividing numerator and denominator by -3 .

$$\frac{-3}{-18} = \boxed{\frac{1}{6}}$$

d) The fraction $\frac{4}{-7}$ is not simplified only because the negative sign is in the denominator. We can fix that by multiplying numerator and denominator by -1 .

$$\frac{4}{-7} = \frac{4 \cdot (-1)}{-7 \cdot (-1)} = \boxed{-\frac{4}{7} \text{ or } -\frac{4}{7}}$$

Example 2. Re-write the mixed number $-4\frac{5}{8}$ as an improper fraction.

Solution: We simply ignore the negative sign, convert the mixed number to an improper fraction, and then put the negative sign back.

$$-4\frac{5}{8} = -\left(\frac{4 \cdot 8 + 5}{8}\right) = -\frac{37}{8}$$

So the correct answer is $\boxed{-\frac{37}{8} \text{ or } -\frac{37}{8}}$.

Part 2 - Adding and Subtracting Fractions with the Same Denominator

Recall that the numerator and denominator play very different roles. The numerator counts how many slices we have. The denominator only describes how large each slice is. For example, in the fraction $\frac{3}{8}$, we are dealing with three slices. Each slice is so large so that eight of them makes up a whole. So, $\frac{3}{8}$ and $\frac{3}{5}$ both express three slices, only the slices in $\frac{3}{5}$ are bigger than those in $\frac{3}{8}$.

To add or subtract fractions with the same denominator, we write a single denominator and express the addition or subtraction of the signed numbers in the numerator.

For example, the addition $\frac{2}{7} + \frac{3}{7}$ is simple: we are talking about the same size of slices, and we add three slices to two slices. Thus $\frac{2}{7} + \frac{3}{7} = \frac{2+3}{7} = \frac{5}{7}$.

Consider now the addition $\frac{4}{5} + \frac{2}{5}$. Again, the same denominator indicates that the slices are of the same size, so we simply add two slices to the four. In short, $\frac{4}{5} + \frac{2}{5} = \frac{4+2}{5} = \frac{6}{5}$.

The answer is an improper fraction. However, we do not see anything improper about it, as long as it is in lowest terms. We could present the final answer as $1\frac{1}{5}$ but $\frac{6}{5}$ is correct, simplified, and is actually preferred.

In case of signed numbers, the negative sign should always be kept in the numerator. Then the addition or subtraction of signed fractions will be reduced to addition or subtraction of (signed) integers in the numerator.

Example 3. Perform the addition or subtraction of the fractions as indicated.

a) $\frac{3}{8} - \frac{7}{8}$ b) $\frac{2}{5} - \frac{4}{5} - \left(-\frac{3}{5}\right)$ c) $\frac{3}{7} - \frac{9}{7}$ d) $\frac{3}{10} - \frac{7}{10} + \left(-\frac{1}{10}\right)$ e) $2\frac{1}{3} - 5\frac{2}{3}$

Solution: a) We re-write the subtraction as one in the numerator. Then, if we can, we simplify the answer.

$$\frac{3}{8} - \frac{7}{8} = \frac{3-7}{8} = \frac{-4}{8} = \boxed{\frac{-1}{2} \text{ or } -\frac{1}{2}}$$

b) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{2}{5} - \frac{4}{5} - \left(-\frac{3}{5}\right) = \frac{2-4-(-3)}{5} = \frac{-2-(-3)}{5} = \frac{-2+3}{5} = \boxed{\frac{1}{5}}$$

c) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{3}{7} - \frac{9}{7} = \frac{3-9}{7} = \boxed{\frac{-6}{7} \text{ or } -\frac{6}{7}}$$

d) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{3}{10} - \frac{7}{10} + \left(-\frac{1}{10}\right) = \frac{3-7+(-1)}{10} = \frac{-4+(-1)}{10} = \frac{-5}{10} = \boxed{\frac{-1}{2} \text{ or } -\frac{1}{2}}$$

e) In case of adding or subtracting mixed numbers, we recommend the use of improper fractions.

$$2\frac{1}{3} - 5\frac{2}{3} = \frac{7}{3} - \frac{17}{3} = \frac{7-17}{3} = \boxed{\frac{-10}{3} \text{ or } -\frac{10}{3}}$$

There are other ways of handling this, but we recommend the method shown above. However, there is a way to do this subtraction by separating integer and fraction parts. We need to be careful though to correctly interpret the negative signs and subtraction signs to both integer and fraction parts.

$$2\frac{1}{3} - 5\frac{2}{3} = \underset{\text{integer part}}{2-5} + \underset{\text{fraction part}}{\frac{1}{3} - \frac{2}{3}} = -3 + \left(-\frac{1}{3}\right) = -3\frac{1}{3}$$

We got lucky with the last problem because both integer and fraction part ended up with the same sign in the result. What about a subtraction such as $4\frac{2}{5} - 1\frac{4}{5}$?

$$4\frac{2}{5} - 1\frac{4}{5} = \underset{\text{integer part}}{4-1} + \underset{\text{fraction part}}{\frac{2}{5} - \frac{4}{5}} = 3 + \left(-\frac{2}{5}\right)$$

We can not present a mixed number with mixed signs. The sign of the entire mixed number is determined by the integer part, because it has more value than the fraction part. This answer should be positive. How do we express 3 and $-\frac{2}{5}$ as a mixed number? We borrow 1 from the integer part, express it as $\frac{5}{5}$ and throw it in with the integer part.

$$3 + \left(-\frac{2}{5}\right) = 2 + 1 + \left(-\frac{2}{5}\right) = 2 + \frac{5}{5} + \left(-\frac{2}{5}\right) = 2 + \frac{5+(-2)}{5} = 2 + \frac{3}{5} = 2\frac{3}{5}$$

This can get quite complicated, which is why we recommend to use improper fractions instead of mixed numbers.

$$4\frac{2}{5} - 1\frac{4}{5} = \frac{22}{5} - \frac{9}{5} = \frac{22-9}{5} = \frac{13}{5} \quad \left(\text{same as } 2\frac{3}{5}\right)$$

Part 3 - Adding and Subtracting Fractions with Different Denominators

How do we add the fractions $\frac{5}{6}$ and $\frac{3}{4}$? It is true that we are adding five slices and three slices, but they do not have the same size. So, how do we add fractions with different denominators? Well, we just don't. Instead, we use the fundamental property of fractions to bring the fractions to a common denominator.

The fundamental property of fractions is that we can multiply both numerator and denominator by the same non-zero number without changing the value of the fraction. Therefore, we can find fractions equivalent to $\frac{5}{6}$ with all denominators that are multiples of 6. For example, $\frac{5}{6} = \frac{10}{12} = \frac{15}{18} = \frac{20}{24} = \frac{30}{36} = \frac{50}{60}$ and so on.

Similarly, we can find fractions equivalent to $\frac{3}{4}$ with all denominators that are multiples of 4. For example, $\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{20} = \frac{15}{20} = \frac{18}{24}$, and so on.

If we multiply 6 and 4, we get a common multiple, 24. However, we should always attempt to work with **the least common multiple**, and that is 12 in this case. To bring $\frac{5}{6}$ to a denominator 12, we multiply numerator and denominator by 2. To bring $\frac{3}{4}$ to a denominator 12, we multiply numerator and denominator by 3.

$$\frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12} \quad \text{and} \quad \frac{3}{4} = \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12} \quad \text{Thus we have that} \quad \frac{5}{6} + \frac{3}{4} = \frac{10}{12} + \frac{9}{12} = \frac{10+9}{12} = \frac{19}{12}$$

This method also works with adding fractions with integers, because we can always re-write an integer as a fraction with denominator 1.

Example 4. Perform the addition or subtraction of the fractions as indicated.

$$\text{a) } \frac{3}{10} - \frac{4}{5} \quad \text{b) } 4 - \frac{2}{3} - \left(-\frac{1}{2}\right) \quad \text{c) } \frac{2}{5} - 3 \quad \text{d) } \frac{2}{3} - \frac{5}{6} - \left(-\frac{1}{2}\right) \quad \text{e) } 4\frac{1}{6} - 3\frac{2}{3}$$

Solution: a) The least common denominator between 10 and 5 is 10. We re-write $\frac{4}{5}$ as a fraction with denominator 10. by multiplying both numerator and denominator by 2. $\frac{4}{5} = \frac{4 \cdot 2}{5 \cdot 2} = \frac{8}{10}$

$$\frac{3}{10} - \frac{4}{5} = \frac{3}{10} - \frac{8}{10} = \frac{3-8}{10} = \frac{-5}{10} = \boxed{\frac{-1}{2} \text{ or } -\frac{1}{2}}$$

b) The integer 4 is not a fraction, but that can be easily fixed, by re-writing it as $\frac{4}{1}$. We are now looking for the least common multiple of 1, 3, and 2. Clearly, 6 will be good as the least common multiple. Then $\frac{4}{1}$ can be re-written as $\frac{4}{1} = \frac{4 \cdot 6}{1 \cdot 6} = \frac{24}{6}$. Similarly, $\frac{2}{3} = \frac{2 \cdot 2}{3 \cdot 2} = \frac{4}{6}$ and $\frac{-1}{2} = \frac{-1 \cdot 3}{2 \cdot 3} = \frac{-3}{6}$. Once the fractions have the same denominator, we perform the additions and subtractions in the numerator. If there is a negative sign, we keep it in the numerator.

$$4 - \frac{2}{3} - \left(-\frac{1}{2}\right) = \frac{4}{1} - \frac{2}{3} - \left(\frac{-1}{2}\right) = \frac{24}{6} - \frac{4}{6} - \left(\frac{-3}{6}\right) = \frac{24-4-(-3)}{6} = \frac{20-(-3)}{6} = \boxed{\frac{23}{6}}$$

c) We re-write 3 as $\frac{3}{1}$. The least common multiple of 1 and 5 is 5. To re-write $\frac{3}{1}$ with a denominator 5, we multiply both numerator and denominator by 5. $\frac{3}{1} = \frac{3 \cdot 5}{1 \cdot 5} = \frac{15}{5}$

$$\frac{2}{5} - 3 = \frac{2}{5} - \frac{3}{1} = \frac{2}{5} - \frac{15}{5} = \frac{2-15}{5} = \boxed{\frac{-13}{5} \text{ or } -\frac{13}{5}}$$

We can express the answer as a mixed number, but there is no need, unless we are specifically asked for it.

d) The least common multiple of 3, 6, and 2 is 6. So we will re-write each fraction with a denominator of 6. If there is a negative sign, we will keep it in the numerator.

$$\text{So, } \frac{2}{3} = \frac{2 \cdot 2}{3 \cdot 2} = \frac{4}{6} \quad \text{and} \quad \frac{5}{6} = \frac{5}{6} \quad \text{and} \quad -\frac{1}{2} = \frac{-1}{2} = \frac{-1 \cdot 3}{2 \cdot 3} = \frac{-3}{6}$$

We can now perform the subtractions, left to right.

$$\frac{2}{3} - \frac{5}{6} - \left(-\frac{1}{2}\right) = \frac{4}{6} - \frac{5}{6} - \left(\frac{-3}{6}\right) = \frac{4-5-(-3)}{6} = \frac{-1-(-3)}{6} = \frac{-1+3}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

e) In case of mixed numbers, we again recommend to switch to improper fractions. We re-write $4\frac{1}{6}$ as $\frac{25}{6}$. Notice that the addition $4 + \frac{1}{6}$ gives us exactly the same result. $3\frac{2}{3}$ can be re-written as $\frac{11}{3}$. The least common denominator between 6 and 3 is 6.

$$4\frac{1}{6} - 3\frac{2}{3} = \frac{25}{6} - \frac{11}{3} = \frac{25}{6} - \frac{22}{6} = \frac{25-22}{6} = \frac{3}{6} = \boxed{\frac{1}{2}}$$



Practice Problems

Perform the indicated operations. Present your answer as an integer or a reduced fraction. You do not need to convert improper numbers to mixed numbers.

1. $\frac{2}{3} + \frac{5}{6}$

5. $\frac{3}{4} - \left(-\frac{5}{8}\right)$

9. $2\frac{4}{5} - 3\frac{1}{2}$

13. $\frac{5}{12} - \frac{3}{4}$

2. $\frac{2}{3} - \frac{5}{6}$

6. $\frac{5}{4} - \frac{1}{6}$

10. $-3\frac{1}{4} - 2\frac{5}{6}$

14. $\frac{19}{3} - 2\frac{5}{6}$

3. $-2 - \frac{2}{3}$

7. $2 - \frac{2}{3}$

11. $\frac{1}{3} - \frac{3}{4} + \frac{5}{12}$

4. $-\frac{2}{5} - \left(-\frac{5}{2}\right)$

8. $-2 + \frac{5}{8}$

12. $-\frac{3}{4} - \frac{5}{6}$

15. $\frac{5}{2} - \frac{7}{10} + \frac{6}{5}$

Problem Set 6

1. Label each of the following statements as true or false.

- | | |
|---|--|
| a) If $A \subseteq B$, then $A \cap B = A$ | h) There is no prime number divisible by 5. |
| b) If $A \subseteq B$, then $A \cup B = B$ | i) Every integer has at least two positive divisors. |
| c) 1 is a prime number | j) For all sets A , $A \cap \emptyset = \emptyset$ |
| d) If a number m is divisible by 4 and by 6, then it is also divisible by 24. | k) For all sets A , $A \cup \emptyset = \emptyset$ |
| e) If a number m is divisible by 3 and by 8, then it is also divisible by 24. | l) $ x - 3 $ can be simplified as $x + 3$. (Hint: try a few values for x !) |
| f) If a number n is divisible by 3, then its square, n^2 is divisible by 9. | m) $\mathbb{Z} \cap \mathbb{N} = \mathbb{Z}$ |
| g) For all sets A and B , $A \cap B \subseteq A \cup B$. | |

2. Suppose that $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 4, 9\}$, and $B = \{2, 4, 6, 7\}$. Find each of the following.

- a) $A \cap B$ b) $A \cup B$ c) $P = \{x \in U : x > 4 \text{ or } x \leq 7\}$ d) $Q = \{x \in U : x > 4 \text{ and } x \leq 7\}$

3. Suppose that S is the set of all squares and R is the set of all rectangles. Label each of the following statements as true or false.

- | | | | |
|--------------------|----------------------------|-------------------|-------------------|
| a) $S \subseteq R$ | c) $R \subseteq S$ | e) $R \cup S = S$ | g) $R \cap S = S$ |
| b) $R \subseteq S$ | d) $\emptyset \subseteq S$ | f) $R \cup S = R$ | h) $R \cap S = R$ |

4. Suppose that F is the set of all integers divisible by four, S is the set of all integers divisible by six, and T is the set of all integers divisible by three. Label each of the following statements as true or false.

- a) $S \subseteq T$ b) $F \subseteq S$ c) $F \cap T = S$ d) $F \cap T \subseteq S$ e) $F \subseteq S \cup T$

5. We know that addition is associative, i.e, for every real numbers x , y , and z , $(x + y) + z = x + (y + z)$. This is the reason we are allowed to write something such as $x + y + z$. Are the two set operations we know also associative?

Suppose that $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 7, 9, 10\}$, and $C = \{1, 3, 4, 7, 8, 10\}$. Find each of the following.

- | | | | |
|------------------------|------------------------|------------------------|------------------------|
| a) $A \cap B$ | c) $B \cap C$ | e) $A \cup B$ | g) $B \cup C$ |
| b) $(A \cap B) \cap C$ | d) $A \cap (B \cap C)$ | f) $(A \cup B) \cup C$ | h) $A \cup (B \cup C)$ |

i) Based on your results, what is your suspicion about union and intersection being associative? (If we suspect something to be true in mathematics, but we haven't proved it yet, the statement is called a **conjecture**.)

6. Suppose that $P = \{2, 5, 6, 8\}$, $T = \{1, 2, 4, 7, 8, 9\}$, and $S = \{1, 3, 4, 5, 8, 10\}$. Draw a Venn diagram depicting these sets.

7. Suppose that $P = \{2, 5, 6, 8\}$, $T = \{1, 2, 4, 7, 8, 9\}$, and $S = \{1, 3, 4, 5, 8, 10\}$. Find each of the following.

- a) $P \cap T$ b) $T \cup S$ c) $(P \cap T) \cup S$ d) $P \cap (T \cup S)$ e) $(P \cup S) \cap T$ f) $P \cup (S \cap T)$

8. Perform each of the divisions with remainder.

a) $2017 \div 31$ b) $99 \div 7$ c) $1355 \div 24$

9. List all factors of 60.

10. Which of the following numbers is a prime number? 39, 91, 71, 45, 81, 201

11. Consider the following numbers: 72, 99, 40, 150, 135, 190, 360

List all the numbers from the given list that are divisible by a) 2 b) 3 c) 5 d) 9

12. Find the least common multiple and greatest common factor of 120 and 80.

13. Compute each of the following. a) $\frac{5}{8}$ of 48 b) $\frac{5}{6}$ of 48

14. This problem is about the fraction $\frac{24}{42}$.

a) Simplify the fraction.

c) Re-write this fraction with

b) Re-write this fraction with

a numerator of i) 20 ii) 28

a denominator of i) 14 ii) 35

15. Draw a picture illustrating $3\frac{1}{4} = \frac{13}{4}$.

16. a) Convert $\frac{28}{5}$ to a mixed number.

c) Convert $5\frac{3}{8}$ to an improper fraction.

b) Convert $\frac{11}{25}$ to a percent.

17. Compute each of the following. a) $\frac{3}{4} - \frac{5}{6}$ b) $\frac{3}{4} - \frac{5}{8}$

18. Perform the indicated operations and simplify.

a) $\frac{-2^2 - (-2)^3 - (-2)^4}{1 - 3(4 - 5(-3^2 - 2(-5)))}$

e) $|2 - 3| - 8||$

k) $\frac{\sqrt{16}}{4}$

b) $\sqrt{4} - \frac{30}{1 - \frac{4}{-6^2 - 5(-7)}}$

f) $|2 - |3 - 8||$

g) $||2 - 3| - 8|$

l) $\sqrt{7 + \sqrt{3(-2)^2 - \sqrt{100} + \sqrt{4}}}$

h) $\sqrt{25 - 16}$

m) $3(-2^2)$

c) $\frac{(-5^2 - 3(-7))^2 - 2(-7)}{(-3)^2 - (-1)^3}$

i) $\sqrt{25} - \sqrt{16}$

n) $-\sqrt{-9}$

d) $(-1)^{200} + (-1)^{201} + (-1)^{400}$

j) $\frac{\sqrt{16}}{\sqrt{4}}$

19. Simplify each of the following.

a) $(4a - b) + (-3a + 2b)$

f) $(3x - 1) + (2x - 1)$

k) $5 \cdot 3x - 1 - 7(2x - 1)$

b) $(4a - b) - (-3a + 2b)$

g) $(3x - 1) - (2x - 1)$

l) $(2m + 5) + (2m - 5)$

c) $5(4a - b) - 2(-3a + 2b)$

h) $3(x - (1 - (2x - 1)))$

m) $(2m + 5) - (2m - 5)$

d) $5(4a - b) + 2(-3a + 2b)$

i) $5(3x - 1) - 7(2x - 1)$

e) $5(4a - b + 2(-3a + 2b))$

j) $5((3x - 1) - 7(2x - 1))$

20. Solve each of the given equations. Make sure to check your solutions.

a) $3x - 8 = 10$

j) $3(2y - 5) - (y - 8) = -7$

b) $\frac{x+7}{2} = -3$

k) $8x - 3 - 5(2x - 7) = 0$

c) $\frac{x}{2} + 7 = -3$

l) $4((-2a + 5) - 3((5a + 1) - 3(2a - 1))) = -2$

d) $5a - 8 = -8$

m) $a - 3((10a - 3) - 6(2a - 1) - 7) = -2$

e) $5(x + 3) - 2(x - 1) = 2$

$$\frac{\frac{x-1}{3} + 2}{5} - 7$$

f) $5((x + 3) - 2(x - 1)) = 0$

n) $\frac{5}{2} + 7 = 4$

g) $5(x + 3(-2x - 1)) = -15$

h) $4(8a + 5) - 3(5a + 1) - 7(2a - 1) = 6$

$$4\left(\frac{3x-1}{2} + 5\right) - 2$$

i) $4(2(3m + 5) - 2(3(5m + 1) - 7(2m - 1))) = 40$

o) $\frac{\frac{5}{3} + 7}{5} - 6 = -3$

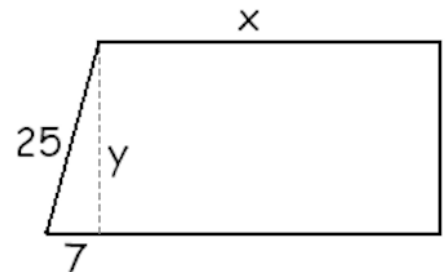
21. Solve each of the following application problems.

- The difference between two numbers is 17. Find the numbers if their sum is 73.
- The sum of four consecutive even numbers is 84. Find these numbers.
- Kendall's age is fifteen years less than twice the age of his sister, Meloney. How old are they if the sum of their age is 69?
- The length of one side of a rectangle is seven meters less than four times another side. Find the dimensions of the rectangle if we also know that its perimeter is 106m.
- Three times the difference of x and 7 is -15 . Find x .
- Ann and Bonnie are discussing their financial situation. Ann said: *If you take 50 bucks from me and then doubled what is left, I would have \$300.* Bonnie answers: *That's funny. If you doubled my money first and then took \$50, then I would have \$300!* How much do they each have?
- The sum of three times a number and 7 is -20 . Find this number.
- We have some five-dollar bills and ten-dollar bills in the register. The number of five dollar bills is eleven less than six times the number of ten-dollar bills. How many of each bills do we have if the total value of all bills is \$265?
- The greatest angle in a triangle 20° less than three times the smallest angle. The third angle is 20° more than the smallest angle. Find the three angles.
- We have some five-dollar bills and ten-dollar bills in the register. All together, we have 32 bills. How many of each bills do we have if the total value of all bills is \$265?

22. Consider the figure shown on the picture. Angles that look like right angles are right angles.

- Find the value of y , given that the area of the right triangle is 84 unit^2 .
- Find the value of x , given that the perimeter of the entire figure is 156 unit.

23. What is the last digit of 7^{2018} ?



Chapter 7

7.1 Fractions — Part 5: Multiplying and Dividing

Part 1 - Multiplying Fractions

Theorem: To multiply two fractions, we multiply one numerator by the other and one denominator by the other.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

We will refer to multiplying numerator by numerator and denominator by denominator as multiplying straight across.

Example 1. Perform each of the multiplications as indicated.

$$\text{a) } \frac{3}{4} \cdot \frac{5}{12} \quad \text{b) } -2 \cdot \frac{5}{4} \quad \text{c) } -3\frac{1}{3} \cdot 1\frac{1}{2} \quad \text{d) } \left(\frac{2}{5}\right)^2$$

Solution: a) We multiply straight across: numerator by numerator and denominator by denominator. Then we reduce the fraction to lowest terms before presenting our final answer.

$$\frac{3}{4} \cdot \frac{5}{12} = \frac{3 \cdot 5}{4 \cdot 12} = \frac{15}{48}$$

The result is not in lowest terms. We reduce it before presenting it as our answer by dividing both numerator and denominator by 3.

$$\frac{15}{48} = \boxed{\frac{5}{16}}$$

We often save time and effort by simplifying the fraction before performing the multiplications. Consider the problem we just solved example. We were able to divide both numerator and denominator by 3, because the numerator in $\frac{3}{4}$ and the denominator in $\frac{5}{12}$ are both divisible by 3. So, we can divide out by that 3 before we perform the multiplication. We just re-write 12 as $3 \cdot 4$ and cross out a factor of 3 from both numerator and denominator.

$$\frac{3}{4} \cdot \frac{5}{12} = \frac{3 \cdot 5}{4 \cdot 12} = \frac{\cancel{3} \cdot 5}{4 \cdot \cancel{3} \cdot 4} = \frac{5}{16}$$

Crossing out the same factor from the products in both numerator and denominator is the same as dividing both by that factor. This is often called cancellation.

- b) We can interpret integers as fractions with denominator 1. If there is a negative sign, we keep it with the numerator for all computations. Therefore, we will re-write -2 as $\frac{-2}{1}$.

$$-2 \cdot \frac{5}{4} = \frac{-2}{1} \cdot \frac{5}{4} = \frac{-2 \cdot 5}{1 \cdot 4} = \frac{-10}{4} = \boxed{-\frac{5}{2}}$$

- c) While it is possible to multiply mixed numbers, it takes a lot of work and it will not be shown here. We can simply re-write mixed numbers as improper fractions. If there is a negative sign, we keep it in the numerator during computations.

$$-3\frac{1}{3} \cdot 1\frac{1}{2} = \frac{-10}{3} \cdot \frac{3}{2} = \frac{-10}{2} = \boxed{-5}$$

- d) If we square a fraction, we must use a pair of parentheses. Other wise, $\frac{2^2}{5}$ can be interpreted as division between integers 2^2 and 5. The parentheses guarantees that we all understand that the entire fraction $\frac{2}{5}$ is being squared, and not just its denominator.

$$\left(\frac{2}{5}\right)^2 = \frac{2}{5} \cdot \frac{2}{5} = \boxed{\frac{4}{25}}$$

As we are defining operations on fractions, let us note that the notation and order of operations agreement remain the same with fractions as they were with integers. Some issues, such as the difference between -3^2 and $(-3)^2$ will also remain.

Example 2. Perform the indicated operations. $-\frac{2}{3} - \frac{4}{5} \left(-2\frac{6}{7}\right)$

Solution: We see a multiplication and a subtraction. We start with the multiplication, but first re-write the mixed number as an improper fraction. We keep the negative sign in the numerator. $2\frac{6}{7} = \frac{20}{7}$ and so $-2\frac{6}{7} = \frac{-20}{7}$.

$$-\frac{2}{3} - \frac{4}{5} \left(-2\frac{6}{7}\right) = -\frac{2}{3} - \frac{4}{5} \left(\frac{-20}{7}\right) = -\frac{2}{3} - \frac{4(-20)}{5 \cdot 7} = -\frac{2}{3} - \frac{4(-4) \cdot 5}{5 \cdot 7} = -\frac{2}{3} - \frac{-16}{7}$$

Next we bring the fractions to the least common denominator and perform the subtraction of signed numbers in the numerator. The least common denominator is 21.

$$-\frac{2}{3} - \frac{-16}{7} = -\frac{14}{21} - \frac{-48}{21} = \frac{-14 - (-48)}{21} = \frac{-14 + 48}{21} = \boxed{\frac{34}{21}}$$

Part 2 - Multiplying by One

It is a frequently occurring task in algebra to re-write an expression without changing its value. There are just about two operations that guarantee this in case of any quantity: we can always *add zero* to it or *multiply it by one*. (Actually, we can also subtract zero and divide by one.) In abstract algebra, an element that does not make any change under the operation to any quantity, is called an identity element. Zero is called the **additive identity**, as it does nothing in addition. The number 1 is called the **multiplicative identity** because it does nothing in multiplication.

For every real number x , $x + 0 = x$. This is the **additive identity** property of zero.

For every real number x , $x \cdot 1 = x$. This is the **multiplicative identity** property of one.

Many algebraic techniques are based zero or multiplication (or division) by 1. With fractions, multiplication by 1 is extremely useful. We can even explain the fundamental property of fractions in terms of multiplication by 1. Recall the fundamental property of fractions: we can multiply both numerator and denominator by the same non-zero number and the value of the fraction would remain the same. In the addition $\frac{2}{3} + \frac{1}{2}$, we would need to re-write $\frac{2}{3}$ with a denominator of 6.

$$\frac{2}{3} = \frac{2}{3} \cdot 1 = \frac{2}{3} \cdot \frac{2}{2} = \frac{2 \cdot 2}{3 \cdot 2} = \frac{4}{6}$$

Theorem: The fractions $-\frac{2}{3}$, $\frac{-2}{3}$, and $\frac{2}{-3}$ all have the same value.

Proof. We already knew that $\frac{-2}{3}$ and $\frac{2}{-3}$ are equivalent fractions. We can get from one to the other one by multiplying both numerator and denominator by -1 . As a general habit, we should rarely tolerate a negative denominator.

$$\frac{2}{-3} = \frac{2 \cdot (-1)}{-3 \cdot (-1)} = \frac{-2}{3}$$

But only now can we connect $\frac{-2}{3}$ to $-\frac{2}{3}$.

$-\frac{2}{3}$ is the opposite of $\frac{2}{3}$. That means multiplication by -1 , and we will express -1 as $\frac{-1}{1}$.

$$-\frac{2}{3} = -1 \cdot \frac{2}{3} = \frac{-1}{1} \cdot \frac{2}{3} = \frac{-1 \cdot 2}{1 \cdot 3} = \frac{-2}{3} \quad \blacksquare \text{ (End of Proof)}$$

The negative sign is not the only thing that can freely move in a fraction. Other factors can only move between being a factor in the numerator to being a multiplier of a fraction.

Theorem: The algebraic expressions $\frac{3}{5}x$ and $\frac{3x}{5}$ are equivalent.

Proof. The key here is that we can re-write x as $\frac{x}{1}$. Division by one never changes the value of a number.

$$\frac{3}{5}x = \frac{3}{5} \cdot x = \frac{3}{5} \cdot \frac{x}{1} = \frac{3x}{5} \quad \blacksquare \text{ (End of Proof)}$$

Units behave the same way as x does. Kilogramm is a measure of mass, 1 kilogram is about 2.2 pounds. If we measure mass in kilograms, (denoted by kg), we can have the unit either in the numerator, or after the number. For example, $\frac{2}{3}$ kg is the same as $\frac{2\text{kg}}{3}$. Although these fractions express slightly different things, they are equivalent, and we can freely move between the two forms. Inside computations we often prefer $\frac{2\text{kg}}{3}$ and we would present the final answer as $\frac{2}{3}\text{kg}$.

Part 3 - The Reciprocal

Example 3. Perform the multiplication $\frac{3}{8} \cdot \left(2\frac{2}{3}\right)$.

Solution: Before multiplying straight across, we re-write the mixed number as an improper fraction.

$$\frac{3}{8} \cdot \left(2\frac{2}{3}\right) = \frac{3}{8} \cdot \frac{8}{3} = \frac{24}{24} = \boxed{1}$$

Definition: If the product of two numbers is 1, we call such a pair of numbers **reciprocals** of each other.

For example, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, and the reciprocal of $\frac{3}{2}$ is $\frac{2}{3}$. The reciprocal is also called the **multiplicative inverse**.

There is an obvious symmetry between $\frac{2}{3}$ and $\frac{3}{2}$. It is easy to see why those two fractions would multiply to 1. This is because the product of them is

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1.$$

So, we simply flip a fraction upside down to get to its reciprocal. However, we should always remember that the definition of the reciprocal of a number x is another number so that their product is 1.

Example 4. Find the reciprocal for each of the following.

a) $\frac{2}{5}$ b) $-\frac{3}{7}$ c) 2 d) x e) $1\frac{3}{5}$

Solution: a) To find the reciprocal of $\frac{2}{5}$, we need to flip it upside down. So the reciprocal of $\frac{2}{5}$ is $\boxed{\frac{5}{2}}$. Indeed,

the product of these two fractions is $\frac{2}{5} \cdot \frac{5}{2} = \frac{10}{10} = 1$.

b) How does the flipping work with the negative sign? The short story is that the reciprocal of $-\frac{3}{7}$ is $\boxed{-\frac{7}{3}}$.

The long story is that we can re-write $-\frac{3}{7}$ as $\frac{-3}{7}$. Then we flip for the reciprocal: we get from $\frac{-3}{7}$ to $\frac{7}{-3}$. Negative signs are not acceptable in the denominator, so we immediately lift it up to the numerator. (Just multiply numerator and denominator by -1 .) So the reciprocal of $\frac{-3}{7}$ is $\frac{7}{-3}$,

which is $-\frac{7}{3}$. That's the same as $-\frac{7}{3}$.

The reciprocal of $-\frac{3}{7}$ is the flip of $\frac{-3}{7}$. That is $\frac{7}{-3} = \frac{-7}{3} = \boxed{-\frac{7}{3}}$

This algebraic gymnastics with the negative sign works, but we can also think of reciprocals in terms of the definition. A number and its reciprocal multiply to 1. That's a positive product. Therefore, the reciprocal of a negative number is negative so that the product can be positive. For the reciprocal of $-\frac{3}{7}$, we do two things: flip the fraction $\frac{3}{7}$ and decide that the reciprocal of $-\frac{3}{7}$ must be negative. The reciprocal is then $-\frac{7}{3}$.

c) How do we take the reciprocal of something that is not necessarily a fraction? We can re-write 2 as $\frac{2}{1}$ and then we can flip. The reciprocal of 2 is the same as the reciprocal of $\frac{2}{1}$, which is $\boxed{\frac{1}{2}}$.

d) Let x be any non-zero number. We re-write it as $\frac{x}{1}$ and flip. So the reciprocal of x is $\boxed{\frac{1}{x}}$. This expression is meaningful as long as x is not zero. Therefore, every non-zero number has a unique reciprocal and is denoted by $\frac{1}{x}$. To check if x and $\frac{1}{x}$ are really reciprocals, we use the definition. The product of a number and its reciprocal is 1. Indeed,

$$x \cdot \frac{1}{x} = \frac{x}{1} \cdot \frac{1}{x} = \frac{x}{x} = 1 \quad \text{This is true as long as } x \text{ is not zero.}$$

e) We can only find the reciprocal of a mixed number by converting it to an improper fraction and then flip it. $1\frac{3}{5} = \frac{8}{5}$, so

the reciprocal of $1\frac{3}{5}$ is the reciprocal of $\frac{8}{5}$, which is $\boxed{\frac{5}{8}}$.

Notice that separately flipping the integer- and fraction parts in a mixed number does not produce the right results: $1\frac{3}{5}$ and $1\frac{5}{3}$ are not reciprocals. Their product is

$$1\frac{3}{5} \cdot 1\frac{5}{3} = \frac{8}{5} \cdot \frac{8}{3} = \frac{64}{15} \text{ and not } 1.$$

To get the reciprocal, we *have to* convert mixed numbers to improper fractions.

Part 4 - Dividing Fractions

As it turns out, we never divide fractions. Every non-zero number has a reciprocal. We interpret division as multiplication by the reciprocal.

Theorem: To divide is to multiply by the reciprocal.

Proof. Let x be any non-zero number. Let us multiply y by $\frac{1}{x}$.

$$y \cdot \frac{1}{x} = \frac{y}{1} \cdot \frac{1}{x} = \frac{y \cdot 1}{1 \cdot x} = \frac{y}{x} \text{ is the same as division by } x. \blacksquare$$

Example 5. Perform the divisions of the fractions as indicated.

a) $\frac{5}{6} \div \frac{2}{3}$ b) $\frac{2}{5} \div (-4)$ c) $4\frac{1}{5} \div 1\frac{1}{6}$ d) $-2 \div \left(-\frac{3}{8}\right)$

Solution: a) To divide is to multiply by the reciprocal. Instead of dividing by $\frac{2}{3}$, we will multiply by its reciprocal, $\frac{3}{2}$.

$$\frac{5}{6} \div \frac{2}{3} = \frac{5}{6} \cdot \frac{3}{2} = \frac{5 \cdot 3}{6 \cdot 2} = \frac{5 \cdot \cancel{3}}{2 \cdot \cancel{3} \cdot 2} = \boxed{\frac{5}{4}}$$

b) To divide is to multiply by the reciprocal. Instead of dividing by -4 , we multiply by its reciprocal.

$-4 = \frac{-4}{1}$, therefore, its reciprocal is $\frac{1}{-4}$. However, we immediately lift the negative sign and as always, we keep it in the numerator. Thus, the reciprocal of -4 is $-\frac{1}{4}$ and we will write it as $\frac{-1}{4}$.

$$\frac{2}{5} \div (-4) = \frac{2}{5} \cdot \frac{-1}{4} = \frac{-2}{20} = \boxed{-\frac{1}{10}}$$

c) We re-write the subtraction with one big fraction bar and a subtraction of integers the same numerator. Then, if we can, we simplify the answer.

$$\frac{3}{7} - \frac{9}{7} = \frac{3-9}{7} = \boxed{\frac{-6}{7} \text{ or } -\frac{6}{7}}$$

d) We can not take the reciprocal of a mixed number. Therefore, we immediately convert each mixed number to an improper fraction, and instead of division, we multiply by the reciprocal.

$$4\frac{1}{5} \div 1\frac{1}{6} = \frac{21}{5} \div \frac{7}{6} = \frac{21}{5} \cdot \frac{6}{7} = \frac{(\cancel{7} \cdot 3) \cdot 6}{5 \cdot \cancel{7}} = \boxed{\frac{18}{5}}$$

Example 6. a) Perform the division $10 \div \frac{1}{4}$ b) How many quarters are there in a 10-dollar roll of quarters?

Solution: a) To divide is to multiply by the reciprocal. The reciprocal of $\frac{1}{4}$ is $\frac{4}{1}$ or 4.

$$10 \div \frac{1}{4} = 10 \cdot 4 = \boxed{40}$$

b) Each dollar can be exchanged for four quarters, so ten dollars would be the same as ten times four, or forty quarters.

From this last example we see that division still has its same old meaning: how many times can we fit $\frac{1}{4}$ into 10?

Example 7. Simplify the given expression: $\frac{\frac{2}{3} + \frac{3}{4}}{\frac{5}{6} - \frac{1}{3}}$

Solution: Recall that if a fraction bar (or division bar) is stretching under entire expressions, then it also serves as (invisible) parentheses. Thus

$$\frac{\frac{2}{3} + \frac{3}{4}}{\frac{5}{6} - \frac{1}{3}} \text{ is the same as } \left(\frac{2}{3} + \frac{3}{4}\right) \div \left(\frac{5}{6} - \frac{1}{3}\right)$$

Therefore, we will first perform the addition in the numerator and the subtraction in the denominator. We will finally divide. For additions and subtractions, we need to use a common denominator.

$$\frac{\frac{2}{3} + \frac{3}{4}}{\frac{5}{6} - \frac{1}{3}} = \frac{\frac{8}{12} + \frac{9}{12}}{\frac{5}{6} - \frac{2}{6}} = \frac{\frac{8+9}{12}}{\frac{5-2}{6}} = \frac{\frac{17}{12}}{\frac{3}{6}} = \frac{17}{12} \cdot \frac{6}{3} = \frac{17 \cdot \cancel{6}}{(\cancel{6} \cdot 2) \cdot 3} = \frac{17}{2 \cdot 3} = \boxed{\frac{17}{6}}$$

Example 8. Perform each of the given divisions. a) $\frac{\frac{2}{3}}{5}$ b) $\frac{2}{\frac{3}{5}}$

Solution: a) If needed, we can always re-write an integer with a denominator of 1. $\frac{\frac{2}{3}}{5} = \frac{\frac{2}{3}}{\frac{5}{1}} = \frac{2}{3} \cdot \frac{1}{5} = \boxed{\frac{2}{15}}$

b) This time we will need to re-write 2 as a fraction. $\frac{2}{\frac{3}{5}} = \frac{\frac{2}{1}}{\frac{3}{5}} = \frac{2}{1} \cdot \frac{5}{3} = \boxed{\frac{10}{3}}$

As the previous example shows, $\frac{\frac{a}{b}}{c}$ and $\frac{a}{\frac{b}{c}}$ are different expressions, with different values. Because of this, we have to be careful with expressions with more than one fraction bar (or division bar). Order of operations must be clearly indicated by the different sizes of the fraction bars. A fraction or division problem with more than one fraction bar is called a **complex fraction**. Complex fractions can always be simplified to a form with just one fraction bar.

Example 9. Simplify each of the given expressions. a) $\frac{\frac{a}{b}}{c}$ b) $\frac{a}{\frac{b}{c}}$

Solution: a) In this case, we need to re-write c as $\frac{c}{1}$. Then we multiply by the reciprocal.

$$\frac{\frac{a}{b}}{c} = \frac{\frac{a}{b}}{\frac{c}{1}} = \frac{a}{b} \cdot \frac{1}{c} = \boxed{\frac{a}{bc}}$$

b) In this case, we need to re-write a as $\frac{a}{1}$. Then we multiply by the reciprocal.

$$\frac{a}{\frac{b}{c}} = \frac{\frac{a}{1}}{\frac{b}{c}} = \frac{a}{1} \cdot \frac{c}{b} = \boxed{\frac{ac}{b}}$$

Part 5 - An Application: Conversion Factors

In physics, in some applications of mathematics, and even in every day life, we dealing with numbers with units. For example, length can not be determined or communicated by a number only. 1 mile is much much longer than 1 inch. Also, 5 minutes is much shorter than 5 years.

The cancellation of factors in numerator and denominator of fractions can be used to convert a quantity from one unit to another one. Recall that we never change the value of any number if we multiply it by 1. For this reason, 1 is sometimes called the **multiplicative identity**.

Consider now the statement 1 hour = 60 minutes. In physics, hour is denoted by h and minutes by min. If we divide any non-zero quantity by itself, the result is 1. Thus, using the equality 1h = 60min, we can write two fractions, both of value 1.

$$\frac{1\text{h}}{60\text{min}} \quad \text{and} \quad \frac{60\text{min}}{1\text{h}}$$

These two fractions are called **conversion factors** or unit multipliers, and we use them to converting time measurements from minutes to hours or backward, from hours to minutes.

Example 10. Convert $\frac{7}{12}$ hours to minutes.

Solution: We will re-write $\frac{7}{12}$ hours as $\frac{7\text{h}}{12}$ and multiply it by one of the conversion factors shown above. Since we would like to get rid of *hours*, we will select the conversion factor that has *hour* in its denominator.

$$\frac{7}{12}\text{h} = \frac{7\text{h}}{12} \cdot 1 = \frac{7\text{h}}{12} \cdot \frac{60\text{min}}{1\text{h}}$$

Now the *hour* unit is a factor in both numerator and denominator, so we can cancel it out, and we will be left with just minutes in the numerator.

$$\frac{7\cancel{\text{h}}}{12} \cdot \frac{60\text{min}}{1\cancel{\text{h}}} = \frac{7 \cdot 60\text{min}}{12} = \frac{7 \cdot (5 \cdot 12)\text{min}}{12} = \frac{35\text{min}}{1} = \boxed{35\text{min}}$$

Example 11. Convert 9000 minutes to hours.

Solution: We first re-write 9000 minutes as a fraction, and keep the unit with the denominator. If needed, we can always write a denominator 1. Then we multiply it by 1.

$$9000 \text{ min} = \frac{9000 \text{ min}}{1} = \frac{9000 \text{ min}}{1} \cdot 1$$

The fraction expressing 1 is going to be the conversion factor that has minutes in its denominator. The minutes unit is cancelled out, and we are left with hours instead.

$$9000 \text{ min} = \frac{9000 \cancel{\text{min}}}{1} \cdot \frac{1\text{h}}{60\cancel{\text{min}}} = \frac{9000\text{h}}{60} = \frac{150\text{h}}{1} = \boxed{150 \text{ h}}$$

Conversion factors can also be used in groups.

Example 12. Convert 10 years to seconds.

Solution: We will go from years to days, then from days to hours, then to minutes and finally to seconds. The entire computation can be done in a single line. Let us assume that a year is 365 days long.

$$10\text{y} = \frac{10\text{y}}{1} = \frac{10\text{y}}{1} \cdot \frac{365\cancel{\text{d}}}{1\text{y}} \cdot \frac{24\cancel{\text{h}}}{1\cancel{\text{d}}} \cdot \frac{60\cancel{\text{min}}}{1\cancel{\text{h}}} \cdot \frac{60\text{s}}{1\cancel{\text{min}}} = \frac{10 \cdot 365 \cdot 24 \cdot 60 \cdot 60\text{s}}{1} = \boxed{315360000\text{s}}$$



Practice Problems

1. Perform the indicated operations. Present your answer as an integer or a reduced fraction. You do not need to convert improper numbers to mixed numbers.

a) $-\frac{8}{15} \cdot \frac{3}{4}$

e) $\frac{1}{6} - \frac{8}{25} \left(-1\frac{2}{3}\right)$

i) $\frac{\frac{5}{4} - \left(\frac{1}{6}\right)^2}{\left(-1\frac{1}{3}\right) - \frac{1}{2}}$

b) $\left(-\frac{1}{2}\right)^2$

f) $\frac{2}{7} - \frac{2}{3} \div \left(1\frac{2}{5}\right)$

c) $\frac{4}{15} \cdot \left(2\frac{1}{7}\right)$

g) $-\frac{5}{6} - \left(3 - \frac{2}{5}\right) \left(-\frac{1}{2}\right)$

j) $\frac{1 - \frac{2}{3}}{2 - \frac{1}{3}}$

d) $-\left(-\frac{2}{3}\right)^2$

h) $\left(3\frac{1}{3}\right) \div \left(2\frac{1}{2}\right)$

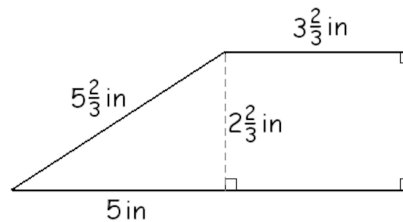
2. Find the perimeter and area of the object shown on the picture.

3. Evaluate the expression $x^2 - 6x - 1$ if

a) $x = -\frac{1}{2}$ b) $x = \frac{5}{2}$ c) $x = -\frac{2}{3}$

4. Evaluate the expression $\frac{-2x^2 + x + 3}{2x - 3}$ if

a) $x = \frac{1}{2}$ b) $x = \frac{3}{2}$ c) $x = -\frac{2}{3}$ d) $x = -\frac{5}{6}$



5. Perform each of the following conversions.

a) 6000 inches to feet, given that 12in = 1ft

b) 900000 square-inches to square-feet. (Hint: $1\text{ft}^2 = 1\text{ft} \cdot 1\text{ft}$)

c) $81 \frac{\text{mi}}{\text{h}}$ (miles per hour) to meter per second $\left(\frac{\text{m}}{\text{s}}\right)$. Hint: $1\text{mi} \approx 1600\text{m}$



Enrichment

1. (Enrichment) Contributed by Prof. Abdallah Shuaibi.

Two travelers meet a third one, who is very hungry. He offers the two travelers 8 dollars for a meal. One traveler has three pieces of bread, the other one has five. So the hungry man gives them the 8 dollars, they all sit down and eat all 8 pieces of bread together. Afterwards, the two get into an argument about how to divide up the money. The one who contributed 5 pieces of bread wants to split it to 5 and 3. The other wants to divide the money evenly, 4 and 4. They go to a wise man for advice. They tell him their story and ask him to divide the money between them. The wise man gives the man who had 3 pieces of bread 1 dollar and 7 to the man with 5 pieces of bread. Is this a just or even reasonable decision?

7.2 Rules of Exponents

Part 1 - The Definition

Exponential notation expresses repeated multiplication.

Definition: We define 2^7 to denote the factor 2 multiplied by itself repeatedly 7 times, such as

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{7 \text{ factors}} = 2^7$$

The new operation defined is called **exponentiation**. The factor (in this case 2) is called the **base**. The number written above the base, in smaller font (in this case, 7) is called the **exponent**.

Since the definition does not elegantly fit the case when the exponent is one, we also define 5^1 to be 5. One factor, so technically, no multiplication.

When we enlarge our mathematical notation by the inclusion of exponential expressions, a few things might become problematic. For example, is there a difference between -3^2 and $(-3)^2$?

Recall that a negative sign in front of anything can be interpreted as '*the opposite of*', which is the same as multiplication by -1 . We can interpret -3 as $-1 \cdot 3$, and so we can re-interpret the original question from comparing -3^2 and $(-3)^2$ to a question comparing $-1 \cdot 3^2$ and $(-1 \cdot 3)^2$. The rest is really just an order of operations problem.

Recall that in our order of operations agreement, $\begin{matrix} P \\ E \\ M \\ D \\ A \\ S \end{matrix}$, exponentiation superseeds multiplication. So, when presented by multiplication and exponentiation, we first execute the exponentiation and then the multiplication.

And so, if there is no parentheses, we have:

$$\begin{array}{ll} -1 \cdot 3^2 & \text{perform exponentiation} \\ -1 \cdot 9 & \text{perform multiplication} \\ \boxed{-9} & \end{array}$$

And if we have a parentheses, that serves to overwrite the usual order of operations:

$$\begin{array}{ll} (-1 \cdot 3)^2 & \text{whatever is in the parentheses first} \\ (-3)^2 & \text{square the number } -3 \\ \boxed{9} & \end{array}$$

The difference between -3^2 and $(-3)^2$ is truly an order of operations thing: we are talking about taking the opposite and squaring, but in different orders.

$$\begin{array}{lll} -3^2 & \text{is square 3 and then take the result's opposite} & \text{--- or the opposite of the square of 3} \\ (-3)^2 & \text{is take the opposite of 3 and then square} & \text{--- or the square of the opposite of 3} \end{array}$$

In algebra, it is important to correctly read notation. Confusing -3^2 and $(-3)^2$ is an error that commonly occurs and messes up computations.

Caution! In the expression $(-3)^2$, the base of the exponentiation is -3 . In the expression -3^2 , the base of the exponentiation is 3.

Example 1. Simplify each of the given expressions.

a) -2^4 b) $(-2)^4$ c) -1^3 d) $(-1)^3$ e) $-(-2)^2$ f) $-(-2^2)$

Solution: a) The base of the exponentiation is 2.

$$-2^4 = -1 \cdot 2^4 = -1 \cdot (2 \cdot 2 \cdot 2 \cdot 2) = -1 \cdot 16 = \boxed{-16}$$

-2^4 can be read as the opposite of 2^4 .

b) The base of the exponentiation is -2 .

$$(-2)^4 = (-1 \cdot 2)^4 = (-1 \cdot 2)(-1 \cdot 2)(-1 \cdot 2)(-1 \cdot 2) = (-2)(-2)(-2)(-2) = \boxed{16}$$

$(-2)^4$ can be read as the fourth power of -2 .

c) The base of the exponentiation is 1.

$$-1^3 = -1 \cdot 1^3 = -1 \cdot 1 \cdot 1 \cdot 1 = \boxed{-1}$$

d) The base of the exponentiation is -1 .

$$(-1)^3 = (-1)(-1)(-1) = \boxed{-1}$$

e) The base of the exponentiation is -2 .

$$-(-2)^2 = -((-2)(-2)) = -(4) = \boxed{-4}$$

f) Careful! The base of the exponentiation is 2. This is NOT squaring -2 . This is squaring 2 and then taking the opposite of the result twice.

$$-(-2^2) = -(-1 \cdot 2 \cdot 2) = -(-4) = \boxed{4}$$



Discussion: Explain why in the expression $-(-5)^2$, the two negatives do not cancel out to a positive.

Part 2 - Rules of Exponents

When mathematicians agreed to define 3^5 as $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$, that was a free choice. They could have gone with other definitions. Once this definition exists, however, certain properties are automatically true, and we have no other option but to recognize them as true. The following statements are straightforward consequences of the definition - and the mathematics we already have.

Consider the expression $2^3 \cdot 2^4$. If we re-write the expression using the definition of exponents, we quickly get that

$$2^3 \cdot 2^4 \stackrel{\text{def}}{=} (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) \stackrel{\substack{\text{mult is} \\ \text{associative}}}{=} 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \stackrel{\text{def}}{=} 2^7$$

The computation above illustrates why the following theorem is true.

Theorem 1. If a is any number and m and n are any two positive integers, then

$$a^n \cdot a^m = a^{n+m}$$

We can see that this rule follows from the definition of exponents and from the fact that multiplication is associative.

Consider now the expression $\frac{2^5}{2^3}$. If we re-write the expression using the definition of exponents, we quickly get that

$$\frac{2^5}{2^3} \stackrel{\text{def}}{=} \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} \stackrel{\text{cancellation}}{=} \frac{-2 \cdot -2 \cdot -2 \cdot 2 \cdot 2}{-2 \cdot -2 \cdot -2} \stackrel{\text{def}}{=} \frac{2^2}{1} = 2^2$$

Theorem 2. If a is any number and m and n are any two positive integers, then

$$\frac{a^n}{a^m} = a^{n-m}$$

As our computation shows, this property is a consequence of the definition of exponentials and our rules of cancellation.

Consider now $(2^3)^5$. If we re-write the expression using the definition of exponents, we quickly get that

$$\begin{aligned} (2^3)^5 &\stackrel{\text{def}}{=} (2^3) \cdot (2^3) \cdot (2^3) \cdot (2^3) \cdot (2^3) \stackrel{\text{def}}{=} (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \\ &\stackrel{\substack{\text{mult is} \\ \text{associative}}}{=} 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \stackrel{\text{def}}{=} 2^{15} \end{aligned}$$

Clearly, we have five groups of three two-factors.

Theorem 3. If a is any number and m and n are any two positive integers, then

$$(a^n)^m = a^{nm}$$

As our computation shows, this property is a consequence of the definition of exponentials and the fact that multiplication is associative.

Consider now $(2 \cdot 3)^4$. If we re-write the expression using the definition of exponents, we quickly get that

$$\begin{aligned} (2 \cdot 3)^4 &\stackrel{\text{def}}{=} (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) \stackrel{\substack{\text{mult is} \\ \text{associative}}}{=} 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \\ &\stackrel{\substack{\text{mult is} \\ \text{commutative}}}{=} 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \stackrel{\substack{\text{mult is} \\ \text{associative}}}{=} (2 \cdot 2 \cdot 2 \cdot 2) \cdot (3 \cdot 3 \cdot 3 \cdot 3) \stackrel{\text{def}}{=} 2^4 \cdot 3^4 \end{aligned}$$

Theorem 4. If a and b are any numbers and n is any positive integer, then

$$(ab)^n = a^n \cdot b^n$$

As our computation shows, this property is a consequence of the definition of exponentials and the fact that multiplication is commutative and associative.

Caution! Exponentiation denotes repeated multiplication, so it is a fundamentally multiplicative concept. Exponents will exhibit nice behavior with respect to multiplication and division, but NOT with respect to addition and subtraction. Similar-looking statements fail to be true if addition or subtraction is involved. For example, $(2 \cdot 5)^2 = 2^2 \cdot 5^2$, but $(2 + 5)^2 \neq 2^2 + 5^2$.

Caution! Another common mistake is to confuse $(ab)^n$ with ab^n . (It is an order of operations thing.) The base

of exponentiation is ab in $(ab)^n$ but only b in ab^n .

$$ab^n = a \cdot \underbrace{b \cdot b \cdot \dots \cdot b}_{n \text{ times}} \quad \text{and} \quad (ab)^n = \underbrace{(ab) \cdot (ab) \cdot \dots \cdot (ab)}_{n \text{ times}}$$

Consider now the expression $\left(\frac{3}{4}\right)^5$. If we re-write the expression using the definition of exponents, we quickly get that

$$\left(\frac{3}{4}\right)^5 \stackrel{\text{def}}{=} \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right) \stackrel{\substack{\text{rules of multiplying} \\ \text{fractions}}}{=} \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4} \stackrel{\text{def}}{=} \frac{3^5}{4^5}$$

Theorem 5. If a and b are any numbers and n is any positive integer, then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

As our computation shows, this property is a consequence of the definition of exponentials and the rule of how we multiply fractions.

Caution! Exponentiation denotes repeated multiplication, so it is a fundamentally multiplicative concept. Exponents will exhibit nice behavior with respect to multiplication and division, but NOT with respect to addition and subtraction. Similar-looking statements fail to be true if addition or subtraction is involved. For example, $\left(\frac{2}{5}\right)^2 = \frac{2^2}{5^2}$, but $(5-2)^2 \neq 5^2 - 2^2$.

To summarize what just happened: once we defined exponentiation as repeated multiplication, certain properties immediately followed from the definition. These properties are as follows.

Theorem: If a, b are any numbers and m, n are any positive integers, then

- | | |
|--------------------------------|---|
| 1. $a^n \cdot a^m = a^{n+m}$ | 4. $(ab)^n = a^n b^n$ |
| 2. $\frac{a^n}{a^m} = a^{n-m}$ | 5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ |
| 3. $(a^n)^m = a^{nm}$ | |

Example 2. Simplify each of the following expressions.

a) $x^5 \cdot x^3$ b) $(x^5)^3$ c) $(2x^5)^3$ d) $\frac{(2x)^5}{2x^3}$ e) $(-x^2)^3$ f) $(-x^3)^2$

Solution: a) In case of $x^5 \cdot x^3$, we can apply the definition or our first rule. Either way, we will end up adding the exponents.

$$x^5 \cdot x^3 = \boxed{x^8}$$

b) In case of $(x^5)^3$, we have repeated exponentiation. Applying the definition, we see three groups of factors, each group with five factors in it. So, we multiply the exponents.

$$(x^5)^3 = \boxed{x^{15}}$$

c) In case of $(2x^5)^3$, we will have to apply several rules. First, when a product is exponentiated, we exponentiate each factor, i.e. $(ab)^n = a^n b^n$.

$$(2x^5)^3 = 2^3 \cdot (x^5)^3 = 8 \cdot x^{15} = \boxed{8x^{15}}$$

d) In the expression $\frac{(2x)^5}{2x^3}$, the number 2 is part of the base in the exponentiation in the numerator, but not in the exponentiation in the denominator. Once we got rid of the parentheses in the numerator, we subtract the exponents corresponding to cancellation.

$$\frac{(2x)^5}{2x^3} = \frac{2^5 \cdot x^5}{2x^3} = \frac{32x^5}{2x^3} = 16x^{5-3} = \boxed{16x^2}.$$

e) In the expression $(-x^2)^3$, the leading negative sign brings some algebraic complications. One way to handle this, either mentally or also in writing, to interpret $-x^2$ as the opposite of x^2 , or $-1 \cdot x^2$. Notice that the -1 is not getting raised to second power, only to the third power.

$$(-x^2)^3 = (-1 \cdot x^2)^3 = (-1)^3 (x^2)^3 = -1 \cdot x^6 = \boxed{-x^6}.$$

f) This expression is very similar to the previous one. Here the negative sign gets squared only but not cubed.

$$(-x^3)^2 = (-1 \cdot x^3)^2 = (-1)^2 (x^3)^2 = 1 \cdot x^6 = \boxed{x^6}.$$

Example 3. Simplify the expression $\frac{(-2a)^4 (-ab^4)^3 ab^2}{(-2ab^2)^3 ba^2}$. Assume that a and b represent non-zero numbers.

Solution: There are several ways to solve this problem. We will take our time and apply one rule at the time. We will start with the rule $(ab)^n = a^n b^n$. Also, standalone negative signs will be replaced with a -1 multiplier.

$$\frac{(-2a)^4 (-ab^4)^3 ab^2}{(-2ab^2)^3 ba^2} = \frac{(-2a)^4 (-1ab^4)^3 ab^2}{(-2ab^2)^3 ba^2} = \frac{(-2)^4 a^4 (-1)^3 a^3 (b^4)^3 ab^2}{(-2)^3 a^3 (b^2)^3 ba^2}$$

Next, we perform the exponentiation on the numbers and bring them forward, re-ordered the variables alphabetically, and simplify expressions containing repeated exponentiation using the rule $(a^n)^m = a^{nm}$.

$$\frac{(-2)^4 a^4 (-1)^3 a^3 (b^4)^3 ab^2}{(-2)^3 a^3 (b^2)^3 ba^2} = \frac{16a^4 (-1) a^3 b^{12} ab^2}{-8a^3 b^6 ba^2} = \frac{-16a^4 a^3 ab^{12} b^2}{-8a^3 a^2 b^6 b}$$

Next, we apply our first rule, $a^n a^m = a^{n+m}$ in both numerator and denominator.

$$\frac{-16a^4 a^3 a b^{12} b^2}{-8a^3 a^2 b^6 b} = \frac{-16a^{4+3+1} b^{12+2}}{-8a^{3+2} b^{6+1}} = \frac{-16a^8 b^{14}}{-8a^5 b^7}$$

At this point, both numerator and denominator are completely simplified. We can tell because both expressions have no repetition of sign, number, or any of the variables. What is left for us to do, is to consolidate numerator and denominator. We will simplify (or perform the division) between -16 and -8 , and perform cancellation between the variables, using the rule $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{-16a^8 b^{14}}{-8a^5 b^7} = \frac{-2a^{8-5} b^{14-7}}{1} = 2a^3 b^7$$

So the simplified expression is $\boxed{2a^3 b^7}$.



Discussion: Why was it necessary in the previous example to assume that a and b represent non-negative numbers?

Explain why this step is incorrect: $2 \cdot 5^x = 10^x$.

Example 4. Consider the expression $3^x \cdot 3^x$.

- Simplify the expression using our first rule, $a^n \cdot a^m = a^{n+m}$ only.
- Simplify the expression using our fourth rule, $(ab)^n = a^n b^n$ only.
- Is there a rule of exponentiation that can be used to verify that the results from part a) and part b) are the same?

Solution: a) The base is the same, so

$$3^x \cdot 3^x = 3^{x+x} = \boxed{3^{2x}}.$$

- b) Now we will use the fact that the exponents of factors are the same, and so we will re-write $a^n \cdot b^n$ as $(ab)^n$.

$$3^x \cdot 3^x = (3 \cdot 3)^x = \boxed{9^x}$$

- c) The expressions 3^{2x} and 9^x are the same. We can use our third rule, $(a^n)^m = a^{nm}$.

$$9^x = (3^2)^x = 3^{2 \cdot x} = 3^{2x}.$$

Example 5. Find the prime-factorization of 24^{100} .

Solution: If we tried to enter 24^{100} in our calculator, we will probably find that the number is too great for it to handle. So it would be futile to compute the gigantic number and start the prime-factorization from scratch. Instead, we will use the prime-factorization of 24 and rules of exponents. The prime-factorization of $24 = 2^3 \cdot 3$.

$$24^{100} = (2^3 \cdot 3)^{100} = (2^3)^{100} \cdot 3^{100} = 2^{300} \cdot 3^{100}.$$

So the prime-factorization of 24^{100} is $\boxed{2^{300} \cdot 3^{100}}$.

Part 3 - Scientific Notation

In physics, we use units that are internationally established by scientists. The abbreviation SI stands for (Système international d'unités). In natural sciences such as chemistry and physics, we often have to communicate numbers so large that it is uncomfortable for both of our imagination and even for our notation. Consider for example, the radius of the sun. The SI unit of measuring length is meters.

The radius of the sun is 695 700 000 meters. The distance between our planet Earth and the Sun is approximately 149 600 000 000 meters. The basic particles that make up material are so tiny that a handful of some material includes a very large number of particles. In chemistry, the basic measurement of how much material we have (i.e. how many particles) is 1 mole. One mole of a substance contains approximately 602 214 076 000 000 000 000 000 particles.

Scientific notation was developed to handle such uncomfortably large numbers, using properties of exponentiation. Scientific notation expresses a single number as a product, where the first number (kind of, sort of) expresses the number, and the second part expresses the number of zeroes to place after the number. Naturally, the definition is going to be a bit more rigorous.

Example 6. Re-write $4 \cdot 10^7$ using regular notation.

Solution: If we look at ten-powers, we will notice some very nice properties.

$10^1 = 10$ and multiplying an integer by 10 results in adding a zero as its last digit. $4 \cdot 10^1 = 40$.

$10^2 = 100$ and multiplying an integer by 100 results in adding two zeroes as its last digits. $4 \cdot 10^2 = 400$.

$10^3 = 1000$ and multiplying an integer by 1000 results in adding three zeroes as its last digits. $4 \cdot 10^3 = 4000$.

$10^4 = 10000$ and multiplying an integer by 10000 results in adding four zeroes as its last digits. $4 \cdot 10^4 = 40000$.

This is indeed a very nice pattern. The exponent on 10 is the same as the number of zeroes to be added at the end. Therefore, we can interpret $4 \cdot 10^7$ as placing 7 zeroes after the digit 4.

$$4 \cdot 10^7 = \boxed{40000000}.$$

Example 7. Re-write $3.215 \cdot 10^{12}$ using regular notation.

Solution: When we are dealing with decimals, a multiplication by a 10–power means moving the decimal point.

$$3.215 \cdot 10 = 32.15, \quad 3.215 \cdot 100 = 321.5, \quad 3.215 \cdot 1000 = 3215, \quad \text{and} \quad 3.215 \cdot 10000 = 32150$$

Once we reached the end of the decimal, i.e. multiplied it by a ten-power large enough to create an integer, multiplication by the remaining 10–powers is again a matter of placing zeroes at the end.

$$3.215 \cdot 10^{12} = 3.215 \cdot 10^{3+9} = 3.215 \cdot 10^3 \cdot 10^9 = 3215 \cdot 10^9 = \boxed{3215000000000}.$$

Definition: We can write numbers in scientific notation. This means to write a number as a product of two numbers. The first number is between 1 and 10 (can be 1 but must be less than 10), and the second number is a 10–power. For example, the scientific notation for 428 600 000 000 is 4.286×10^{11} .

Please note that scientific notation uses the cross notation for multiplication. We will break with tradition and simply use the dot notation for scientific notation. Part of the problem with the cross notation is that in hand-written computations it looks too much like the letter x .

Example 8. Re-write 602 200 000 000 000 000 000 000 using scientific notation.

Solution: Let us first count the trailing zeroes at the end. There are six groups of three zeroes and then two more, so that is 20. So now we can re-write this giant number as $6022 \cdot 10^{20}$. But remember that the first factor in scientific notation must be between 1 and 10. So we will extract more ten-powers by moving the decimal point. When in doubt, check with the calculator. $6022 = 602.2 \cdot 10 = 60.22 \cdot 100 = 6.022 \cdot 1000$. Thus our number is

$$602\,200\,000\,000\,000\,000\,000\,000 = 6022 \cdot 10^{20} = (6.022 \cdot 1000) \cdot 10^{20} = 6.022 \cdot 10^{3+20} = \boxed{6.022 \cdot 10^{23}}$$

This number is famous in chemistry, it is called Avogadro's number. 1 mole of a substance contains $6.022 \cdot 10^{23}$ particles. Mole is the SI unit for the amount of a substance.

Example 9. Suppose that two numbers, A and B are given in scientific notation as follows. $A = 3 \cdot 10^8$ and $B = 6 \cdot 10^3$. Compute each of the following. Present your answer in scientific notation.

- a) $A + B$ b) $A - B$ c) AB d) $\frac{A}{B}$

Solution: a) Surprisingly, $A + B$ will look a whole lot like A . Scientific notation will serve us very well in multiplications and divisions, but we must be careful when adding or subtracting. To see what happens, we will return to regular notation. Notice also that there is no rule for how to add exponential expressions in the rules we just learned.

$$A = 3 \cdot 10^8 = 300\,000\,000 \text{ and } B = 6 \cdot 10^3 = 6000. \text{ So } A + B = 300\,000\,000 + 6000 = 300\,006\,000.$$

This can be written in scientific notation as $3.00006 \cdot 10^8$ or simply $3 \cdot 10^8$. Basically, A is so much larger than B , that the addition of B is almost negligible compared to the size of A . (Imagine that we added 1 inch to 1 mile or a penny to a million dollars) Indeed, in science, there will be agreements about the precision of results, and in most agreements, $3 \cdot 10^8 + 6 \cdot 10^3$ is simply $3 \cdot 10^8$, as we round 3.00006 down to 3. So the answer is $\boxed{300\,006\,000}$ or $\boxed{3.00006 \cdot 10^8 \approx 3 \cdot 10^8}$.

b) Very similarly, $A - B = 300\,000\,000 - 6000 = \boxed{299\,994\,000}$. Using scientific notation, the result is $\boxed{2.99994 \cdot 10^8 \approx 3 \cdot 10^8}$. We can imagine that we subtracted an inch from a mile or a penny from a million dollars.

c) Scientific notation will be awesome for multiplication and division! Consider $AB = (3 \cdot 10^8)(6 \cdot 10^3)$. We will group the first factors and the ten-powers. $3 \cdot 6 = 18$ and $10^8 \cdot 10^3 = 10^{3+8} = 10^{11}$.

$$AB = (3 \cdot 10^8)(6 \cdot 10^3) = (3 \cdot 6) \cdot (10^8 \cdot 10^3) = 18 \cdot 10^{11}$$

This is not in scientific notation. Recall that in scientific notation, the first factor must be less than 10. So we have to re-write 18 as $1.8 \cdot 10$.

$$AB = 18 \cdot 10^{11} = (1.8 \cdot 10) \cdot 10^{11} = 1.8 \cdot 10^{12}. \text{ So the answer is } \boxed{1.8 \cdot 10^{12} = 1800\,000\,000\,000}.$$

d) In order to help the division, we will re-write A from $3 \cdot 10^8$ to $30 \cdot 10^7$. Then the rules of exponents will work with the notation very nicely.

$$\frac{A}{B} = \frac{3 \cdot 10^8}{6 \cdot 10^3} = \frac{30 \cdot 10^7}{6 \cdot 10^3}$$

We simply divide 30 by 6. As for the ten-powers, then cancellation can be expressed via the rule

$$\frac{a^n}{a^m} = a^{n-m}.$$

$$\frac{A}{B} = \frac{30 \cdot 10^7}{6 \cdot 10^3} = \frac{5 \cdot 10^{7-3}}{1} = \boxed{5 \cdot 10^4 \text{ or } 50\,000}.$$



Sample Problems

1. Simplify each of the following.

a) $(2x^5)(x^4)$

c) $(2x^5)^4$

e) $-2a^3(-2a^4)^2$

g) $\frac{(-2x^5)^2 y^3}{2x^3 y^2}$

b) $(2x)^5(x^4)$

d) $(-xy)^2(-xy^2)^3$

f) $2a^3(-2ab^2)^3 ab^2$

h) $\frac{(2ab)^3(-3a^2b)^2}{-a(6ab^2)^2}$

2. Write each of the following expressions in terms of a fixed number or a single exponential expression.

a) $\frac{3^{2x+1}}{9^{x-1}}$

b) $\frac{(8^{b-2})(2^{b+1})}{4^{2b-3}}$

c) $5^{2x-1} \cdot 25^{3-x}$

3. Let us denote 3^{100} by M . Express each of the following in terms of M .

a) 3^{101}

b) $3^{100} - 2 \cdot 3^{101} + 3^{102}$

c) 3^{99}

d) 9^{100}

4. Find the prime-factorization for each of the following numbers.

a) 10^{2018}

b) 18^{1000}

c) 360^{50}

d) $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

5. Re-write each of the given numbers using scientific notation.

a) 3800000000

b) 625000000000

6. Suppose that $A = 5 \cdot 10^{18}$, and $B = 8 \cdot 10^7$. Compute each of the following. Present your answer using scientific notation.

a) AB

b) A^2

c) $2A$

d) $\frac{4A}{B}$



Practice Problems

1. Simplify each of the following.

$$\begin{array}{lllll}
 \text{a) } -3^2 & \text{c) } (-2a^3)^4 & \text{e) } (2a^2b)^3 & \text{g) } (2a)^2b^3 & \text{i) } (3p^2q^5)(2pq^3) \\
 \text{b) } (-3)^2 & \text{d) } (-2a^4)^3 & \text{f) } ((2a)^2b)^3 & \text{h) } \frac{m^4m^5}{m^3} & \text{j) } \frac{(a^2)^6a^3}{(-a^3)^2} \\
 \text{k) } \frac{(-5ts^3t)^3(4s^2t)^2}{(10st^3)^2} & \text{m) } \frac{(3ab^2)^2(-2a^3b)^4}{(-2ab)^3} & \text{n) } \frac{(-2x^2y^3)^4xy^3(2x^2y)^2}{(2x)^2y^9(2x^2y)^4} \\
 \text{l) } (-2xy^3)^2xy^5x^2 & \text{o) } \left(\frac{6a^3b^5}{-3ab^2}\right)^2\left(\frac{12ab^3}{-6b^2}\right)^3
 \end{array}$$

2. Write each of the following expressions in terms of a fixed number or a single exponential expression.

$$\text{a) } \frac{2^{2x-1}}{4^{x-2}} \quad \text{b) } \frac{100^{x+1}}{2^{2x+1} \cdot 5^{x-1}} \quad \text{c) } \frac{9^x \cdot 4^{x+2}}{6^{2x-1}}$$

3. Let P denote 5^{2015} . Express each of the following in terms of P .

$$\text{a) } 5^{2016} \quad \text{b) } 5^{2017} \quad \text{c) } 5^{2014} \quad \text{d) } 25^{2015} \quad \text{e) } 5^{2015} - 3 \cdot 5^{2016} + 5^{2017}$$

4. Find the prime-factorization for each of the given numbers.

$$\text{a) } 20^{100} \quad \text{b) } 18^{2000} \quad \text{c) } 120^{120} \quad \text{d) } (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^3$$

5. Re-write each of the given numbers in scientific notation.

$$\text{a) } 21\,000\,000\,000 \quad \text{b) } 300\,000\,000\,000 \quad \text{c) } 325\,000\,000$$

6. Suppose that $X = 3 \cdot 10^{10}$, and $Y = 6 \cdot 10^4$. Compute each of the following. Present your answer using scientific notation.

$$\text{a) } X^2 \quad \text{b) } XY \quad \text{c) } \frac{X}{Y} \quad \text{d) } \frac{3X}{Y^2}$$

7.3 The n th root of a Number

Caution! Currently, square roots are defined differently in different textbooks. In conversations about mathematics, approach the concept with caution and first make sure that everyone in the conversation uses the same definition of square roots.

Definition: Let A be a non-negative number. Then the **square root of A** (notation: $\sqrt{A}$) is the non-negative number that, if squared, the result is A . If A is negative, then $\sqrt{A}$ is undefined.

For example, consider $\sqrt{25}$. The square-root of 25 is a number, that, when we square, the result is 25. There are two such numbers, 5 and -5 . The square root is defined to be the non-negative such number, and so $\sqrt{25}$ is 5. On the other hand, $\sqrt{-25}$ is undefined, because there is no real number whose square, is negative.

Example 1. Evaluate each of the given numerical expressions.

a) $\sqrt{49}$ b) $-\sqrt{49}$ c) $\sqrt{-49}$ d) $-\sqrt{-49}$

Solution: a) $\sqrt{49} = \boxed{7}$ b) $-\sqrt{49} = -1 \cdot \sqrt{49} = -1 \cdot 7 = \boxed{-7}$ d) $-\sqrt{-49} = -1 \cdot \sqrt{-49} = \boxed{\text{undefined}}$
 c) $\sqrt{-49} = \boxed{\text{undefined}}$

Square roots, when stretched over entire expressions, also function as grouping symbols.

Example 2. Evaluate each of the following expressions.

a) $\sqrt{25} - \sqrt{16}$ b) $\sqrt{25 - 16}$ c) $\sqrt{144 + 25}$ d) $\sqrt{144} + \sqrt{25}$

Solution: a) $\sqrt{25} - \sqrt{16} = 5 - 4 = \boxed{1}$ c) $\sqrt{144 + 25} = \sqrt{169} = \boxed{13}$
 b) $\sqrt{25 - 16} = \sqrt{9} = \boxed{3}$ d) $\sqrt{144} + \sqrt{25} = 12 + 5 = \boxed{17}$

Definition: Let A be any real number. Then the **third root of A** (notation: $\sqrt[3]{A}$) is the number that, if raised to the third power, the result is A . (We also refer to $\sqrt[3]{A}$ as cube root of A).

Notice that this definition is much simpler than the previous one. If we square 3 and -3 , we get 9 in both cases. For this reason, there are two candidates for $\sqrt{9}$ and no candidate for $\sqrt{-9}$. Third roots behave more pleasantly. If we cube (same as raise to the third power) 3 and -3 , we get 27 and -27 . Thus, cube roots exist of both positive and negative numbers, and there is no ambiguity on the choice of the cube root. Simply $\sqrt[3]{8}$ is 2 and $\sqrt[3]{-8}$ is -2 .

Example 3. Evaluate each of the given numerical expressions.

a) $\sqrt[3]{125}$ b) $-\sqrt[3]{8}$ c) $\sqrt[3]{-27}$ d) $-\sqrt[3]{-64}$

Solution: a) $\sqrt[3]{125} = \boxed{5}$ c) $\sqrt[3]{-27} = \boxed{-3}$
 b) $-\sqrt[3]{8} = -1 \cdot \sqrt[3]{8} = -1 \cdot 2 = \boxed{-2}$ d) $-\sqrt[3]{-64} = -1 \cdot \sqrt[3]{-64} = -1(-4) = \boxed{4}$

Just as with square roots, third roots, can also serve as grouping symbols.

Example 4. Evaluate each of the following expressions.

a) $\sqrt[3]{10^2 + 5^2}$ b) $\sqrt[3]{36 - 100}$

Solution: a) $\sqrt[3]{10^2 + 5^2} = \sqrt[3]{125} = \boxed{5}$

b) $\sqrt[3]{36 - 100} = \sqrt[3]{-64} = \boxed{-4}$

Definition: Let n be any positive integer, greater than 1, and let A be any real number.

Suppose that n is an even number. If A is negative, then the n th root of A , denoted by $\sqrt[n]{A}$, is undefined. If A is non-negative, then $\sqrt[n]{A}$ is the non-negative number that, when raised to the n th power, the result is A .

Suppose that n is an odd number. Then the n th root of A , denoted by $\sqrt[n]{A}$, is the number that, when raised to the n th power, the result is A .

Example 5. Simplify each of the given expressions.

a) $(\sqrt[5]{3})^5$ b) $(\sqrt[4]{7})^8$ c) $(\sqrt[3]{2})^{12}$ d) $(\sqrt[5]{10})^{15}$

Solution: a) By definition, $\sqrt[5]{3}$ is the number that, when raised to the 5th power, the result is 3. So, the answer is $\boxed{3}$.

b) By definition, $\sqrt[4]{7}$ is the number that, when raised to the 4th power, the result is 7. Also recall the rule of exponents about repeated exponentiation: $(a^n)^m = a^{nm}$.

$$(\sqrt[4]{7})^8 = (\sqrt[4]{7})^{4 \cdot 2} = \left[(\sqrt[4]{7})^4 \right]^2 = 7^2 = \boxed{49}$$

c) By definition, $\sqrt[3]{2}$ is the number that, when raised to the 3rd power, the result is 2. We will again use the rule of exponents about repeated exponentiation: $(a^n)^m = a^{nm}$. $(\sqrt[3]{2})^{12} =$

$$(\sqrt[3]{2})^{3 \cdot 4} = \left[(\sqrt[3]{2})^3 \right]^4 = 2^4 = \boxed{16}$$

d) By definition, $\sqrt[5]{10}$ is the number that, when raised to the 5th power, the result is 10. We will use the same rule of exponents.

$$(\sqrt[5]{10})^{15} = (\sqrt[5]{10})^{5 \cdot 3} = \left[(\sqrt[5]{10})^5 \right]^3 = 10^3 = \boxed{1000}$$



Practice Problems

Simplify each of the given expressions.

1. $\sqrt[3]{-8}$

7. $\sqrt[3]{-64}$

13. $\sqrt{-100}$

19. $\sqrt[100]{-1}$

25. $\left(\sqrt[5]{2}\right)^{20}$

2. $\sqrt[5]{32}$

8. $\sqrt[5]{-0}$

14. $\sqrt[6]{-64}$

20. $-\sqrt[5]{-32}$

26. $\left(\sqrt[4]{-2}\right)^{20}$

3. $\sqrt[4]{1}$

9. $\sqrt{16}$

15. $\sqrt{0}$

21. $-\sqrt[4]{-16}$

27. $\left(\sqrt[5]{-2}\right)^{20}$

4. $\sqrt{0}$

10. $\sqrt[4]{16}$

16. $\sqrt[4]{-1}$

22. $\left(\sqrt[3]{-12}\right)^3$

28. $\left(\sqrt[6]{x}\right)^{24}$

5. $\sqrt{-64}$

11. $\sqrt[7]{-1}$

17. $\sqrt[5]{-1}$

23. $\left(\sqrt[3]{-12}\right)^6$

6. $-\sqrt[3]{-27}$

12. $\sqrt[5]{-243}$

18. $\sqrt[99]{-1}$

24. $\left(\sqrt[4]{2}\right)^{20}$

29. $\sqrt[3]{2 \cdot 5 - 37}$

31. $\sqrt[6]{3^2 - 2^3}$

33. $\sqrt[4]{(-2)^4 - 4^2}$

30. $\sqrt[4]{1} - \sqrt[5]{-36 + 4}$

32. $\sqrt[7]{3 \cdot 5 - (-4)^2}$

34. $\sqrt[3]{\sqrt{3(-4)^2 + 1} - \sqrt[5]{-1}}$

7.4 Linear Equations 2

Part 1 - The Same Old Equations with Fractions

We will further study solving linear equations. Let us first recall a few definitions.

Definition: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign. A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality of the equation true. To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

For example, the equation $-x^2 + 3 = 4x - 2$ is an equation with two solutions, -5 and 1 . We leave it to the reader to verify that these numbers are indeed solution. We will have to deploy systematic methods to find all solutions. The methods we will use usually depends on the type of equation. We have been studying the simplest equations, linear equations.

We have seen one- and two-step equations. First we will revisit those again, now that we have a greater number set. The most important thing to realize here is that the appearance of fractions does not change the fundamental methods of solving equations; rather, it makes each of the steps a bit more laborious. However, we should not let ourselves be intimidated by fractions.

Example 1. Solve each of the following equations.

$$\text{a) } 2x - 3 = 15 \quad \text{b) } \frac{2}{3}x - \frac{1}{2} = -\frac{1}{3} \quad \text{c) } \frac{x-4}{3} = -6 \quad \text{d) } \frac{x + \frac{1}{2}}{\frac{3}{5}} = \frac{1}{3}$$

Solution: a) There is nothing new or unusual about this equation. The right-hand side is just a number. The unknown only appears on the left-hand side. There, we see two operations: the unknown was first multiplied by 2 and then 3 was subtracted. To isolate the unknown on the left-hand side, we will perform the inverse operations to both sides, in a reverse order. We will first add 3 and then we will divide by 2.

$$\begin{array}{rcll} 2x - 3 & = & 15 & \text{add 3} \\ 2x & = & 18 & \text{divide by 2} \\ x & = & 9 & \end{array}$$

We check: If $x = 9$, then the left-hand side is:

$$\text{LHS} = 2x - 3 = 2 \cdot 9 - 3 = 18 - 3 = 15 = \text{RHS} \quad \checkmark$$

Thus our solution, $x = \boxed{9}$ is correct.

b) Consider the equation $\frac{2}{3}x - \frac{1}{2} = -\frac{1}{3}$. There is nothing new or unusual about this equation either. If we could solve $2x - 3 = 15$, then we can solve this equation using the same steps. On the left-hand side, the unknown was multiplied by $\frac{2}{3}$ and then $\frac{1}{2}$ was subtracted. To isolate the unknown, we will perform to both sides the inverse operations, in a reverse order. This means that we will add $\frac{1}{2}$ and then divide by $\frac{2}{3}$. The main computation should be clean; we should just record the result

of each step. We will perform computations on the margin.

$$\begin{array}{llll} \frac{2}{3}x - \frac{1}{2} = -\frac{1}{3} & \text{add } \frac{1}{2} & \text{margin work: } -\frac{1}{3} + \frac{1}{2} = \frac{-2}{6} + \frac{3}{6} = \frac{1}{6} \\ \frac{2}{3}x = \frac{1}{6} & \text{divide by } \frac{2}{3} & \frac{1}{6} \div \frac{2}{3} = \frac{1}{6} \cdot \frac{3}{2} = \frac{1}{2 \cdot 3} \cdot \frac{-3}{2} = \frac{1}{4} \\ x = \frac{1}{4} \end{array}$$

We check: If $x = \frac{1}{4}$, then the left-hand side is

$$\text{LHS} = \frac{2}{3} \left(\frac{1}{4} \right) - \frac{1}{2} = \frac{-2}{3} \cdot \frac{1}{-2 \cdot 2} - \frac{1}{2} = \frac{1}{6} - \frac{1}{2} = \frac{1}{6} - \frac{3}{6} = \frac{-2}{6} = -\frac{1}{3} = \text{RHS} \checkmark$$

Thus our solution, $x = \boxed{\frac{1}{4}}$ is correct.

- c) Consider now the equation $\frac{x-4}{3} = -6$. This is a simple two-step equation, we only have it here to serve as an analogous example for part d. On the left-hand side we have first a subtraction of 4 and then a division by 3. To isolate the unknown, we will multiply by 3 and then add 4. As always, we will perform all operations to both sides.

$$\begin{array}{llll} \frac{x-4}{3} = -6 & \text{multiply by 3} \\ x-4 = -18 & \text{add 4} \\ x = -14 \end{array}$$

We check: If $x = -14$, then the left-hand side is:

$$\text{LHS} = \frac{x-4}{3} = \frac{-14-4}{3} = \frac{-18}{3} = -6 = \text{RHS} \checkmark$$

Thus our solution, $x = \boxed{-14}$ is correct.

- d) Consider now the equation $\frac{x + \frac{1}{2}}{\frac{3}{5}} = \frac{1}{3}$. If we could solve $\frac{x-4}{3} = -6$, then we can solve this

equation using the same steps. On the left-hand side, there was an addition of $\frac{1}{2}$, and then a division by $\frac{3}{5}$. To isolate the unknown, we will perform to both sides the inverse operations, in a

reverse order. This means that we will multiply by $\frac{3}{5}$ and then subtract $\frac{1}{2}$. The main computation

should be clean; we should just record the result of each step. We will perform computations on the margin.

$$\frac{x + \frac{1}{2}}{\frac{3}{5}} = \frac{1}{3} \quad \text{multiply by } \frac{3}{5} \quad \text{margin work: } \frac{1}{3} \cdot \frac{3}{5} = \frac{1}{5}$$

$$x + \frac{1}{2} = \frac{1}{5} \quad \text{subtract } \frac{1}{2} \quad \frac{1}{5} - \frac{1}{2} = \frac{2-5}{10} = -\frac{3}{10}$$

$$x = -\frac{3}{10}$$

We check: If $x = -\frac{3}{10}$, then the left-hand side is

$$\text{LHS} = \frac{x + \frac{1}{2}}{\frac{3}{5}} = \frac{-\frac{3}{10} + \frac{1}{2}}{\frac{3}{5}} = \frac{\frac{-3+5}{10}}{\frac{3}{5}} = \frac{\frac{2}{10}}{\frac{3}{5}} = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{5} \cdot \frac{5}{3} = \frac{1}{3} = \text{RHS}$$

Thus our solution, $x = \boxed{-\frac{3}{10}}$ is correct.

Part 2 - The Unknown is On Both Sides

We will now consider slightly more complex situations, in which the unknown appears on both sides of the equation.

Example 2. Solve the equation $2x - 8 = 5x + 10$. Make sure to check your solution.

Solution: Recall that in an algebraic expression, the numbers multiplying the unknown are called **coefficients**. In this equation, the coefficients of x are 2 on the left-hand side, and 5 on the right-hand side. We can easily reduce this equation to a two-step equation by subtracting $2x$ from both sides.

$$\begin{array}{rcll} 2x - 8 & = & 5x + 10 & \text{subtract } 2x \\ -8 & = & 3x + 10 & \text{subtract } 10 \\ -18 & = & 3x & \text{divide by } 3 \\ -6 & = & x & \end{array}$$

The only solution of this equation is -6 . We check; if $x = -6$, then the left-hand side (LHS) is

$$\text{LHS} = 2(-6) - 8 = -12 - 8 = -20$$

and the right-hand side (RHS) is

$$\text{RHS} = 5(-6) + 10 = -30 + 10 = -20$$

So our solution, $x = \boxed{-6}$ is correct.

Note that there is another way to reduce this equation to a two-step equation. Instead of subtracting $2x$, we could also subtract $5x$. Both methods lead to a correct solution. This is a new situation for us. We have a choice between two methods. Although both methods are equally correct, one is better than the other one because it makes the rest of the computation easier. These kind of strategic decisions will become more and more important as we advance in mathematics.

Let us see now, what would happen if we chose to subtract $5x$.

$$\begin{array}{rcl}
 2x - 8 & = & 5x + 10 & \text{subtract } 5x \\
 -3x - 8 & = & 10 & \text{add } 8 \\
 -3x & = & 18 & \text{divide by } -3 \\
 x & = & -6 &
 \end{array}$$

As we can see, now we had to divide by a negative number in the last step. This can be always avoided if we address the side on which the unknown has the smaller coefficient. Keep in mind, coefficients include the sign.

Example 3. Solve the equation $7a - 12 = -a + 20$. Make sure to check your solution.

Solution: Let us first compare the coefficients. On the left-hand side, the coefficient of a is 7. On the right-hand side, the coefficient of a is -1 . Since -1 is less than 7, we will eliminate $-a$ from the right-hand side. We reduce this equation to a two-step equation by adding a to both sides.

$$\begin{array}{rcl}
 7a - 12 & = & -a + 20 & \text{add } a \\
 8a - 12 & = & 20 & \text{add } 12 \\
 8a & = & 32 & \text{divide by } 8 \\
 a & = & 4 &
 \end{array}$$

So the only solution of this equation is 4. We check; if $a = 4$,

$$\text{LHS} = 7 \cdot 4 - 12 = 28 - 12 = 16 \text{ and } \text{RHS} = -4 + 20 = 16 \implies \text{LHS} = \text{RHS} \checkmark$$

So our solution, $\boxed{a = 4}$ is correct.

Example 4. Solve the equation $-4x + 2 = -x + 17$. Make sure to check your solutions.

Solution: -4 is less than -1 . Therefore, we will eliminate $-4x$ from the left-hand side.

$$\begin{array}{rcl}
 -4x + 2 & = & -x + 17 & \text{add } 4x \\
 2 & = & 3x + 17 & \text{subtract } 17 \\
 -15 & = & 3x & \text{divide by } 3 \\
 -5 & = & x &
 \end{array}$$

We check; if $x = -5$, then

$$\text{LHS} = -4(-5) + 2 = 20 + 2 = 22 \text{ and } \text{RHS} = -(-5) + 17 = 5 + 17 = 22 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{x = -5}$ is correct.

Example 5. Solve the equation $\frac{1}{2}m - 1 = \frac{5}{4}m - \frac{1}{4}$. Make sure to check your solutions.

Solution:

$$\begin{array}{lll}
 \frac{1}{2}m - 1 = \frac{5}{4}m - \frac{1}{4} & \text{subtract } \frac{1}{2}m & \text{margin work: } \frac{5}{4} - \frac{1}{2} = \frac{5}{4} - \frac{2}{4} = \frac{3}{4} \\
 -1 = \frac{3}{4}m - \frac{1}{4} & \text{add } \frac{1}{4} & -1 + \frac{1}{4} = \frac{-4}{4} + \frac{1}{4} = -\frac{3}{4} \\
 -\frac{3}{4} = \frac{3}{4}m & \text{divide by } \frac{3}{4} & -\frac{3}{4} \div \frac{3}{4} = -\frac{3}{4} \cdot \frac{4}{3} = -1 \\
 -1 = m & &
 \end{array}$$

So the only solution of this equation is -1 . We check; if $m = -1$,

$$\begin{aligned}
 \text{LHS} &= \frac{1}{2}(-1) - 1 = -\frac{1}{2} - 1 = -\frac{1}{2} - \frac{2}{2} = -\frac{3}{2} \text{ and} \\
 \text{RHS} &= \frac{5}{4}(-1) - \frac{1}{4} = -\frac{5}{4} - \frac{1}{4} = -\frac{6}{4} = -\frac{3}{2} \quad \implies \quad \text{LHS} = \text{RHS} \checkmark
 \end{aligned}$$

So our solution, $\boxed{m = -1}$ is correct.

Again, the steps were the same as before, they only took a bit more work to perform because of the appearance of fractions.

Once the unknown appears on both sides, we might face new situations that were not possible in two-step equations. Consider each of the following.

Example 6. Solve each of the following equations. Make sure to check your solutions.

a) $5y + 16 = 5y + 16$ b) $-6x + 5 = -6x + 9$

Solution: a) $5y + 16 = 5y + 16$

This equation looks different from all the others because the two sides are identical. Logically, the value of two sides will be equal no matter what number we substitute into the equation. Computation will confirm this idea.

$$\begin{array}{ll}
 5y + 16 &= 5y + 16 \\
 16 &= 16
 \end{array}
 \quad \text{subtract } 5y$$

The statement $16 = 16$ is true, no matter what the value of y is. Such a statement is called an **unconditionally true statement** or **identity**. All numbers are solution of this equation.

b) $-6x + 5 = -6x + 9$

$$\begin{array}{ll}
 -6x + 5 &= -6x + 9 \\
 5 &= 9
 \end{array}
 \quad \text{add } 6x$$

When we tried to eliminate the unknown from one side, it disappeared from both sides. We are left with the statement $5 = 9$. No matter what the value of the unknown is, this statement can not be made true. Indeed, our last line is an **unconditionally false statement**. This means that there is no number that could make this statement true, and so this equation **has no solution**. An equation like this is called a **contradiction**.

These strange situations happen when the unknown has the same coefficient on both sides. Otherwise, there is exactly one solution. Based on their solution sets, linear equations can be classified as follows.

1. If the last line is of the form $x = 5$, the equation is called **conditional**.
This is because the truth value of the statement depends on the value of x . True if x is 5, false otherwise. A conditional equation has exactly **one solution**.
2. If the last line is of the form $1 = 1$, the equation is unconditionally true. Such an equation is called an **identity** and **all numbers are solutions** of it.
3. If the last line is of the form $3 = 8$, the equation is unconditionally false. Such an equation is called a **contradiction** and its solution set is **the empty set**.



Discussion: Classify each of the given equations as a conditional equation, an identity, or a contradiction.

- a) $3x + 1 = 3x - 1$ c) $x - 4 = 4 - x$ d)
b) $2x - 4 = 7x - 4$ $x - 1 = -1 + x$

Part 3 - Using the Distributive Law

Linear equations might be more complicated. Most often we will be dealing with the distributive law to eliminate parentheses. Then we combine like terms on both sides. After that, these equations will be reduced to the type we just saw.

Example 7. Solve each of the given equations. Make sure to check your solutions.

- a) $3(2x - 1) + 5(-3x + 2) = 70$ c) $2(8x - 5(2x + 1)) - (x + 1) = 49$
b) $4(3x + 1) - (x + 3) = 10x - 4$ d) $-5(x + 2(x - 2(-4x + 3))) = 60(-5x + 1)$

Solution: a) We simplify the expression on the right-hand side. After that, the problem is reduced to a two-step equation.

$$\begin{array}{rcll}
 3(2x - 1) + 5(-3x + 2) & = & 70 & \text{apply distributive law to open parentheses} \\
 6x - 3 - 15x + 10 & = & 70 & \text{combine like terms} \\
 -9x + 7 & = & 70 & \text{subtract 7} \\
 -9x & = & 63 & \text{divide by } -9 \\
 x & = & -7 &
 \end{array}$$

We check: if $x = -7$, then

$$\text{LHS} = 3(2(-7) - 1) + 5(-3(-7) + 2) = 3(-15) + 5(23) = -45 + 115 = 70 = \text{RHS} \quad \checkmark$$

So $x = \boxed{-7}$ is our solution.

- b) $4(3x + 1) - (x + 3) = 10x - 4$ [Solution - YouTube Link](#)

- c) As before, we simplify the expression on the right-hand side. In this case, this might take a little bit more steps. After that, the problem is reduced to a two-step equation.

$$\begin{aligned}
 2(8x - 5(2x + 1)) - (x + 1) &= 49 && \text{apply distributive law to open the inner parentheses} \\
 2(8x - 10x - 5) - (x + 1) &= 49 && \text{combine like terms} \\
 2(-2x - 5) - (x + 1) &= 49 && \text{apply distributive law again} \\
 -4x - 10 - x - 1 &= 49 && \text{combine like terms again} \\
 -5x - 11 &= 49 && \text{add 11} \\
 -5x &= 60 && \text{divide by } -5 \\
 x &= -12
 \end{aligned}$$

We check: if $x = -12$, then

$$\begin{aligned}
 \text{LHS} &= 2(8(-12) - 5(2(-12) + 1)) - ((-12) + 1) = 2(-96 - 5(-24 + 1)) - (-11) \\
 &= 2(-96 - 5(-23)) + 11 = 2(-96 + 115) + 11 = 2(19) + 11 = 38 + 11 = 49 = \text{RHS} \quad \checkmark
 \end{aligned}$$

So $x = \boxed{-12}$ is our solution.

$$d) -5(x + 2(x - 2(-4x + 3))) = 60(-5x + 1) \quad \text{Solution - YouTube Link}$$

Example 8. Ann was asked about her age. Her answer was: "My age is 10 years less than three times my brother's age. The sum of our ages is 18 years". How old are the siblings?

Solution: Let us denote the brother's age by x . Then Ann's age can be expressed as $3x - 10$. The equation will express the sum of these two quantities.

$$\begin{aligned}
 x + 3x - 10 &= 18 && \text{combine like terms} \\
 4x - 10 &= 18 && \text{add 10} \\
 4x &= 28 && \text{divide by 4} \\
 x &= 7
 \end{aligned}$$

So, if the age of Ann's brother is 7 years, then Ann's age is $3 \cdot 7 - 10 = 11$ years old. So the siblings are $\boxed{7 \text{ and } 11 \text{ years}}$ old. We check: the ages add up to 18, and 11 is ten less than three times 7.

Example 8b. Wendy is asked about her age. She answers as follows. "My age is seven years less than twice the age of my son. The sum of our ages is 61 years. How old is Wendy?" [Solution - YouTube Link](#)

Example 9. Three times the sum of a number and two is fifteen greater than the sum of one and the number. Find this number.

Solution: Let us denote the number by x . Then three times the sum of this number and two can be translated as $3(x + 2)$. The sum of one and the number is $1 + x$. The equation will express the comparison between the two.

$$\begin{aligned}
 3(x + 2) &= 1 + x + 15 && \text{distribute 3 and combine like terms} \\
 3x + 6 &= x + 16 && \text{subtract } x \\
 2x + 6 &= 16 && \text{subtract 6} \\
 2x &= 10 && \text{divide by 2} \\
 x &= 5
 \end{aligned}$$

Therefore, this number is $\boxed{5}$. We check: Three times the sum of our number and two is $3 \cdot 7 = 21$, and the sum of one and the number is 6. Indeed, 21 is fifteen greater than 6, so our solution is correct.

Example 9b. Five times the sum of four and a number is six less than seven times the same number.

[Solution - YouTube Link](#)

Example 10. Children's tickets cost 12 dollars, and adult tickets cost 20 dollars. The number of children's tickets purchased was five more than three times the number of adult tickets purchased. How many of each tickets did we buy if we paid 900 dollars for all these tickets?

Solution: Let us denote the number of adult tickets purchased by x . Then the number of children's ticket can be expressed as $3x + 5$. We obtain an equation expressing the value of all the tickets.

We paid 20 dollars for each of the x many adult tickets. So the adult tickets cost $20x$ dollars. Similarly, each children's ticket cost 12 dollars, and we purchased $3x + 5$ many children's tickets. Therefore, we paid $12(3x + 5)$ for the children's tickets. We will now set up an equation to express the total value of all tickets:

$$\begin{aligned} 20x + 12(3x + 5) &= 900 && \text{apply the distributive law} \\ 20x + 36x + 60 &= 900 && \text{combine like terms} \\ 56x + 60 &= 900 && \text{subtract 60} \\ 56x &= 840 && \text{divide by 56} \\ x &= 15 \end{aligned}$$

Since x represents the number of adult tickets, our result is that 15 adult tickets were purchased. Then the children's ticket, denoted by $3x + 5$, must be $3 \cdot 15 + 5 = 50$. So it appears that the correct answer is 15 adults and 50 children tickets.

We check: $3 \cdot 15 + 5 = 50$ and so the number of children's ticket is indeed 5 more than three times the number of adult tickets. As for the total cost, $12(50) + 15(20) = 600 + 300 = 900$. This means that our solution is correct.

Example 10b. The tickets for the field trip were purchased yesterday for both students and instructors. Children tickets cost 11 dollars, adult tickets cost 14 dollars. The number of children tickets purchased was five less than twice the number of adults tickets purchased. How many of each were purchased if all of the tickets cost a total of 465 dollars? [Solution - YouTube Link](#)

Part 4 - Three Types of Equations

Example 11. Solve each of the given equations. Make sure to check your solutions.

a) $3x - 2(4 - x) = 3(3x - 1) - (x - 7)$ c) $\frac{2}{3}x - 4 - \frac{1}{6}(x + 6) = \frac{1}{2}(x - 10)$

b) $4(y - 2) - 6(3y - 5) = 5 - 2(7y + 1)$

Solution: a) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} 3x - 2(4 - x) &= 3(3x - 1) - (x - 7) && \text{eliminate parentheses} && \text{Caution! } -2(-x) = 2x \\ 3x - 8 + 2x &= 9x - 3 - x + 7 && \text{combine like terms} && \text{and } -(-7) = 7 \\ 5x - 8 &= 8x + 4 && \text{subtract } 5x \\ -8 &= 3x + 4 && \text{subtract 4} \\ -12 &= 3x && \text{divide by 3} \\ -4 &= x \end{aligned}$$

We check: if $x = -4$, then

$$\text{LHS} = 3(-4) - 2(4 - (-4)) = 3(-4) - 2 \cdot 8 = -12 - 16 = -28 \quad \text{and}$$

$$\text{RHS} = 3(3(-4) - 1) - (-4 - 7) = 3(-12 - 1) - (-11) = 3(-13) + 11 = -39 + 11 = -28 \implies \text{LHS} = \text{RHS}$$

So our solution, $x = -4$ is correct.

b) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} 4(y - 2) - 6(3y - 5) &= 5 - 2(7y + 1) && \text{eliminate parentheses} \\ 4y - 8 - 18y + 30 &= 5 - 14y - 2 && \text{combine like terms} \\ -14y + 22 &= -14y + 3 && \text{add } 14y \\ 22 &= 3 \end{aligned}$$

We are left with the statement $22 = 3$. No matter what the value of the unknown is, this statement can not be made true, this is an **unconditionally false statement**. This equation **has no solution**. An equation like this is called a **contradiction**.

c) We first eliminate the parentheses by applying the distributive law.

$$\begin{aligned} \frac{2}{3}x - 4 - \frac{1}{6}(x + 6) &= \frac{1}{2}(x - 10) && \text{eliminate parentheses} \\ \frac{2}{3}x - 4 - \frac{1}{6}x - 1 &= \frac{1}{2}x - 5 && \text{combine like terms} \quad \text{margin work: } \frac{2}{3} - \frac{1}{6} = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2} \\ \frac{1}{2}x - 5 &= \frac{1}{2}x - 5 && \text{subtract } \frac{1}{2}x \\ -5 &= -5 \end{aligned}$$

When we tried to eliminate the unknown from one side, it disappeared again from both sides. We are left with the statement $-5 = -5$. This statement is true for all values of x . Indeed, our last line is an **unconditionally true statement**. This means that every number makes make this statement true, and so the solution set of this equation is the set of all numbers. An equation like this is called an **identity**.

We often use identities in mathematics, although it seems at first that we would not need equations whose solution set is every number. Consider the following equation: $a + b = b + a$. This equation is an identity, because every pair of numbers is a solution. We use this identity to express a property of *addition*: that the sum of two numbers does not depend on the order of the two numbers.



Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x + 3 = 4x + 9$

5. $2w + 1 = 2w - 9$

9. $7(j - 5) + 9 = 2(-2j + 5) + 5j$

2. $3w - 5 = 5(w + 1)$

6. $\frac{1}{6}x - 1 = \frac{2}{3}x + 4$

10. $3(x - 5) - 5(x - 1) = -2x + 1$

3. $7x - 2 = 5x - 2$

7. $-\frac{2}{5}x + \frac{1}{3} = \frac{4}{15}x$

11. $\frac{2}{3}(x - 7) = \frac{4}{5}(x + 1)$

4. $3y - 9 = -2y + 4$

8. $4 - x = 3(x - 7)$

12. Five times the opposite of a number is one less than seven times the sum of the number and seven.



Practice Problems

Solve each of the following equations. Make sure to check your solutions.

1. $5x - 3 = x + 9$

9. $3(x - 4) = 2(x + 5)$

17. $-2x - (3x - 1) = 2(5 - 3x)$

2. $-x + 13 = 2x + 1$

10. $4(5x + 1) = 6x + 4$

18. $3(x - 4) + 5(x + 8) = 2(x - 1)$

3. $-2x + 4 = 5x - 10$

11. $a - 3 = 5(a - 1) - 2$

19. $5(x - 1) - 3(-x + 1) = -3 + 8x$

4. $5a + 1 = -3a - 1$

12. $4m - 1 = -4m + 3$

20. $\frac{3}{4}x - \frac{2}{3}\left(x + \frac{1}{2}\right) = -x$

5. $5x - 7 = 6x + 8$

13. $3y - 2 = -2y + 18$

21. $\frac{3}{8}x + 1\frac{4}{5} = \frac{1}{4}x + 1\frac{3}{10}$

6. $8x - 1 = 3x + 19$

14. $8(x - 3) - 3(5 - 2x) = x$

7. $-7x - 1 = 3x - 21$

15. $5(x - 1) - 3(x + 1) = 3x - 8$

8. $2 - 3(y - 1) = 2y - 7$

16. $\frac{2}{3}x - 1 = -\frac{2}{3}\left(x + \frac{1}{2}\right)$

22. $\frac{1}{2}\left(x + \frac{2}{3}\right) - \frac{1}{3}\left(x - \frac{1}{2}\right) = \frac{1}{6}x + \frac{1}{2}$

25. $3(2x - 1) - 5(2 - x) = 4(x - 1) + 5$

23. $2(b + 1) - 5(b - 3) = 2(b - 7) + 1$

26. $3(2x - 7) - 2(5x + 2) = -5x - 30$

24. $-3y - 2(5y - 1) = 7y - 3$

27. $3(x - 4) - 4(x - 3) = 3(x - 2) + 2(3 - x)$

28. $\frac{1}{2}(x + 1) - \frac{3}{5}(x - 1) = -\frac{3}{8}x$

29. Four times the difference of ten and a number is three greater than the sum of the number and five. Find this number.

30. If we add 24 to a number and then take half of that sum, we get a number that is 15 less than twice the number. Find this number.

7.5 Intervals

Part 1 - Definitions

Some sets are small enough for us to list their elements. With larger sets, we develop notation that helps in defining the set without having to write too much. Consider the following two sets.

$$A = \{x \in \mathbb{Z} : x > 2 \text{ and } x < 7\} \quad \text{and} \quad B = \{x \in \mathbb{R} : x > 2 \text{ and } x < 7\}$$

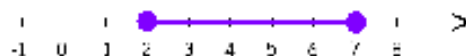
Although the definitions appear to be similar, set A is much smaller than set B . Set A contains all integers greater than 2 and less than 7. That is, A is simply the set $\{3, 4, 5, 6\}$ containing only four elements.

Set B has many more elements, because it is the set of all *real numbers* greater than 2 and less than 7. That means that B contains numbers such as 2, 5, 2.01, 3.1, 6.999999998. If we think of the numbers 2.1, 2.01, 2.001, 2.0001, and so on, these are already infinitely many numbers in B .

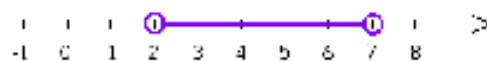
How could we describe set B with less writing? One option is to write the compound inequality $x > 2$ and $x < 7$ in a more effective form, as $2 < x < 7$. Still, we should be able to do better than $B = \{x \in \mathbb{R} : 2 < x < 7\}$.

The set of all real numbers x with $2 < x < 7$ can be also expressed using interval notation.

Definition: The set $\{x \in \mathbb{R} : 2 < x < 7\}$ is also called an **interval**, and is denoted by $(2, 7)$. The endpoints of the interval, 2 and 7 are not elements of the set. Such an interval is called an **open interval**.

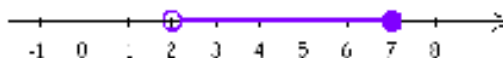


Definition: The set $\{x \in \mathbb{R} : 2 \leq x \leq 7\}$ is denoted by $[2, 7]$. The endpoints of the interval, 2 and 7 belong to the set. Such an interval is called a **closed interval**.



We will see in future courses that open and closed intervals have very different properties. For example, the closed interval $[2, 7]$ has a smallest and greatest element. At the same time, the open interval $(2, 7)$ has no smallest and greatest element. With respect to the endpoints, there are two other possibilities, as follows.

Definition: The set $\{x \in \mathbb{R} : 2 < x \leq 7\}$ is denoted by $(2, 7]$.



Definition: The set $\{x \in \mathbb{R} : 2 \leq x < 7\}$ is denoted by $[2, 7)$.



We also often use interval notation when presenting solution sets of inequalities. Interval notation can also be applied to express simple sets that can be obtained from inequalities such as $x < 3$, $x \leq 3$, $x > 3$, or $x \geq 3$.

Definition: The set $\{x \in \mathbb{R} : x > 3\}$ in interval notation is denoted by $(3, \infty)$.

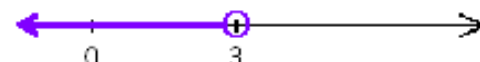


Definition: The set $\{x \in \mathbb{R} : x \geq 3\}$ in interval notation is denoted by $[3, \infty)$.

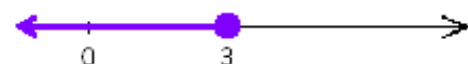


Notice that in case of both $x > 3$ and $x \geq 3$, the closing parentheses indicate that ∞ does not belong to the set. We can also consider the set $(3, \infty)$ to be an open interval. For example, this set has neither smallest nor greatest element.

Definition: The set $\{x \in \mathbb{R} : x < 3\}$ in interval notation is denoted by $(-\infty, 3)$.



Definition: The set $\{x \in \mathbb{R} : x \leq 3\}$ in interval notation is denoted by $(-\infty, 3]$.



Part 2 - Operations

Intervals are sets, and so we can perform set operations on them.

Example 1. Perform each of the following set operations on the intervals.

- a) $(2, 8) \cap (5, 10)$ c) $[1, 4] \cap (2, 5)$ e) $[-1, 2) \cap (3, 6]$ g) $(2, 8) \cap [4, 7]$
 b) $(2, 8) \cup (5, 10)$ d) $[1, 4] \cup (2, 5)$ f) $[-1, 2) \cup (3, 6]$ h) $(2, 8) \cup [4, 7]$

Solution: a) Plotting the two intervals on the same number line is extremely helpful. We will use the same picture for taking unions and intersections.



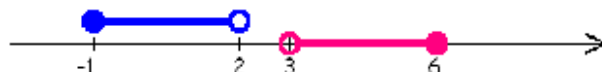
The intersection of the intervals $(2, 8)$ and $(5, 10)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. That is the line segment between 5 and 8. We consider the endpoints: 5 is not in both sets because 5 is not in $(2, 8)$. Similarly, 8 is not in both sets because it is not in $(5, 10)$. Consequently, the intersection of the two intervals is $(5, 8)$.

- b) The union of the intervals $(2, 8)$ and $(5, 10)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. That is the line segment between 2 and 10. We consider the endpoints: 2 is not in either set, so it is not in the union. Similarly, 10 is not in either set, so it is not in the union. Consequently, the union of the two intervals is $(2, 10)$.

- c) The intersection of the intervals $[1, 4]$ and $(2, 5)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. That is the line segment between 2 and 4. We consider the endpoints: 2 is not in both sets because 2 is not in $(2, 5)$. However, 4 is in both sets. Therefore, the intersection of the two intervals is $(2, 4]$.



- d) The union of the intervals $[1, 4]$ and $(2, 5)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. That is the line segment between 1 and 5. We consider the endpoints: 1 is in $[1, 4]$, so it is in the union. On the other hand, 5 is not in either set, so it is not in the union. Consequently, the union of the two intervals is $[1, 5)$.
- e) The intersection of the intervals $[-1, 2)$ and $(3, 6]$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. In this case, there is no number in both sets, and so the intersection is the empty set, $\emptyset$.



- f) The union of the intervals $[-1, 2)$ and $(3, 6]$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, the union fails to form a single interval, and so we cannot simplify the expression either. So the answer is $[-1, 2) \cup (3, 6]$.
- g) After we plotted the picture, we might notice that one interval is a subset of the other. In light of that, the answers will not be surprising. (Recall that if A and B are sets and $A \subseteq B$, then $A \cup B = B$ and $A \cap B = A$.) The intersection of the intervals $(2, 8)$ and $[4, 7]$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. The answer is $[4, 7]$.

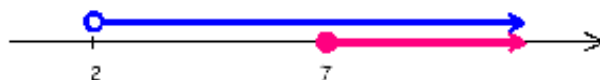


- h) The union of the intervals $(2, 8)$ and $[4, 7]$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, the union is $(2, 8)$.

Example 2. Perform each of the following set operations on the intervals.

- a) $(2, \infty) \cap [7, \infty)$ c) $[0, \infty) \cap (-\infty, 1)$ e) $(-\infty, 4] \cap [9, \infty)$
 b) $(2, \infty) \cup [7, \infty)$ d) $[0, \infty) \cup (-\infty, 1)$ f) $(-\infty, 4] \cup [9, \infty)$

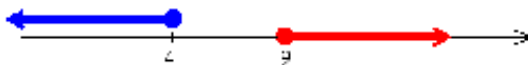
Solution: a) Notice again that one interval is a subset of the other. The intersection of the intervals $(2, \infty)$ and $[7, \infty)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. In this case, the intersection is $[7, \infty)$.



- b) The union of the intervals $(2, \infty)$ and $[7, \infty)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. That is $(2, \infty)$.
 c) The intersection of the intervals $[0, \infty)$ and $(-\infty, 1)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. That is the line segment between 0 and 1. We consider the endpoints: 0 is in both sets but 1 is not, since 1 is not in $(-\infty, 1)$. Therefore, the intersection of the two intervals is $[0, 1)$.



- d) The union of the intervals $[0, \infty)$ and $(-\infty, 1)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, this is the entire number line. Consequently, the union of the two intervals is $\mathbb{R}$, the set of all real numbers. This can be also expressed using interval notation, as $(-\infty, \infty)$.
 e) The intersection of the intervals $(-\infty, 4]$ and $[9, \infty)$ is the set of all numbers that belong to both sets. Visually, that would be the part of the number line over which we see both lines. In this case, there is no number in both sets, and so the intersection is the empty set, $\emptyset$.



- f) The union of the intervals $(-\infty, 4]$ and $[9, \infty)$ is the set of all numbers that belong to one set, or the other, or both. Visually, that would be the part of the number line over which we can see one line or both lines. In this case, the union fails to form a single interval, and so we cannot simplify the expression either. So the answer is $(-\infty, 4] \cup [9, \infty)$.



Practice Problems

1. Re-write each of the given sets using interval notation.

- | | | |
|---|--|--|
| a) $\{x \in \mathbb{R} : x \geq 3\}$ | d) $\{y \in \mathbb{R} : y < -5 \text{ or } y > 4\}$ | g) $\{x \in \mathbb{R} : x < -2 \text{ or } x > 2\}$ |
| b) $\{m \in \mathbb{R} : -2 < m \leq 9\}$ | e) $\{x \in \mathbb{R} : x < 12\}$ | h) $\{t \in \mathbb{R} : t > 0 \text{ and } t < 7\}$ |
| c) $\mathbb{R}$ | f) $\{r \in \mathbb{R} : r < 10 \text{ and } r \geq 6\}$ | i) $\{a \in \mathbb{R} : 3 \leq a \leq 4\}$ |

2. Perform each of the set operations on the intervals.

- | | | |
|-------------------------|---------------------------|---------------------------|
| a) $(1, 5) \cup (2, 7)$ | g) $(3, 8) \cup (-1, 10)$ | m) $[-2, 2] \cup (4, 7)$ |
| b) $(1, 5) \cap (2, 7)$ | h) $(3, 8) \cap (-1, 10)$ | n) $[-2, 2] \cap (4, 7)$ |
| c) $[1, 5] \cup [2, 7]$ | i) $[3, 8] \cup [-1, 10]$ | o) $[-2, 1] \cup (0, 8]$ |
| d) $[1, 5] \cap [2, 7]$ | j) $[3, 8] \cap [-1, 10]$ | p) $[-2, 1] \cap (0, 8]$ |
| e) $[1, 5] \cup (2, 7)$ | k) $[3, 8] \cup (-1, 10)$ | q) $[5, 10] \cup [7, 11)$ |
| f) $[1, 5] \cap (2, 7)$ | l) $[3, 8] \cap (-1, 10)$ | r) $[5, 10] \cap [7, 11)$ |

3. Perform each of the set operations on the intervals.

- | | | |
|-------------------------------------|------------------------------------|-------------------------------------|
| a) $(-\infty, 4) \cup (-\infty, 8)$ | g) $(-\infty, 5) \cup (3, \infty)$ | m) $(-\infty, -2) \cup (1, \infty)$ |
| b) $(-\infty, 4) \cap (-\infty, 8)$ | h) $(-\infty, 5) \cap (3, \infty)$ | n) $(-\infty, -2) \cap (1, \infty)$ |
| c) $(-\infty, 4] \cup (-\infty, 8]$ | i) $(-\infty, 5] \cup [3, \infty)$ | o) $(-\infty, -2] \cup [1, \infty)$ |
| d) $(-\infty, 4] \cap (-\infty, 8]$ | j) $(-\infty, 5] \cap [3, \infty)$ | p) $(-\infty, -2] \cap [1, \infty)$ |
| e) $(-\infty, 4] \cup (-\infty, 8)$ | k) $(-\infty, 5] \cup (3, \infty)$ | q) $(-\infty, -2] \cup (1, \infty)$ |
| f) $(-\infty, 4] \cap (-\infty, 8)$ | l) $(-\infty, 5] \cap (3, \infty)$ | r) $(-\infty, -2] \cap (1, \infty)$ |

Problem Set 7

1. Suppose that $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. Find each of the following sets.

- a) $A = \{x \in U : x \text{ is divisible by 3 or } x \text{ is odd}\}$ d) $A \cap C$
 b) $B = \{x \in U : x > 6 \text{ is divisible by 3 and } x \text{ is odd}\}$ e) $B \cup C$
 c) $C = \{x \in U : x \text{ is divisible by 4}\}$

2. Perform each of the given operations.

- a) $(-\infty, 5) \cup (-\infty, 10]$ c) $(3, 8) \cap [4, 11]$ e) $(-\infty, 9) \cup [2, \infty)$ g) $(-\infty, 1] \cap (3, \infty)$
 b) $(-\infty, 5) \cap (-\infty, 10]$ d) $(3, 8) \cup [4, 11]$ f) $(-\infty, 9) \cap [2, \infty)$ h) $(-\infty, 1] \cup (3, \infty)$

3. Label each of the following statements as true or false.

- a) If n is an integer such that n^2 is divisible by 20, then n^2 is divisible by 400.
 b) If n is an integer greater than 1, then all exponents in the prime-factorization of n^2 are even.
 c) If integer n is divisible by 4 and by 6, then it is also divisible by 24.
 d) The square of an odd number is always odd.
 e) If A and B are sets such that $A \subseteq B$, then $A \cup B = A$.
 f) If A and B are sets such that $A \subseteq B$, then $A \cup B = B$.
 g) There is no even prime number.

4. Find the prime factorization for each of the following numbers.

- a) $10!$ b) 2016^{100} c) 2009 d) 219615

Note: $10!$ (read as ten factorial is notation for $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$)

5. Compute each of the following.

- a) $(7 - (-4^2 + 10)) \cdot 3 - 8(-5) \div (-2^2)$ b) $\sqrt{(-5)^2 - 3(2 - 10)}$

6. List all factors of 80.

7. Find the prime factorization of 80.

8. Find the least common multiple and greatest common factor of 80 and 96.

9. Compute each of the following.

- a) 45% of 200 b) $\frac{4}{5}$ of 200

10. This problem is about the fraction $\frac{36}{40}$.

- a) Simplify the fraction. c) Re-write this fraction with a denominator of 70.
 b) Re-write the fraction as a percent. d) Re-write this fraction with a numerator of 18.

11. a) Convert $\frac{80}{11}$ to a mixed number.

c) Re-write $\frac{3}{20}$ as a percent.

b) Convert $3\frac{5}{7}$ to an improper fraction.

12. Compute each of the following .

a) $\frac{2}{5} - \frac{3}{10} \cdot \frac{16}{9}$ c) $\frac{3}{8} \left(2\frac{2}{3}\right)$ e) $\frac{1}{2} - \frac{6}{7} \div \frac{2}{3}$

b) $\left(-\frac{2}{3}\right)^2 - \frac{7}{20} \left(\frac{10}{21} - \frac{2}{7}\right)$ d) $\left(2\frac{5}{8}\right) \left(5\frac{1}{3}\right)$

13. Suppose that $x = 0.000000000125$ and $y = 45\,000\,000\,000\,000$.

a) Express x and y using scientific notation.

b) Compute each of the following. Present your answer using scientific notation.

i) xy ii) x^2 iii) $x + y$ iv) $\frac{y}{x}$ v) $\sqrt{\frac{y}{x}}$

14. The price of a book was increased from \$80 to \$92. What percentage of an increase does this represent?

15. a) We increased a quantity by 20%. Later, there was an additional increase of 30%. If we express the two subsequent changes as a single change, what percentage of an increase does this represent?

b) We increased a quantity by 40%. Later, there was decrease of 30%. If we express the two subsequent changes as a single change, what percentage of a change does this represent? Is it an increase or a decrease?

c) We decreased a quantity by 20%. Later, there was an additional decrease of 10%. If we express the two subsequent changes as a single change, what percentage of a decrease does this represent?

16. Simplify each of the following.

a) $\frac{(x^3)^7}{x^3 \cdot x^7}$ b) $6^{x-2} \cdot 6^{x+3}$ c) $(-x)^2(-x)^3$ d) $(-x^2)(-x^3)$ e) $\frac{(-2x^4y)^3(-xy^2)}{(-2xy^2x^3)^2}$

17. Simplify each of the following.

a) $\sqrt[3]{-27}$ c) $\sqrt[4]{81}$ e) $\sqrt[7]{-1}$ g) $(\sqrt[4]{-2})^4$
 b) $\sqrt[4]{-16}$ d) $-\sqrt[3]{-\frac{1}{8}}$ f) $-(\sqrt[3]{-2})^3$ h) $(\sqrt[3]{-2})^{12}$

18. Consider the expression $\frac{-2x^2 + 3x - 1}{-2x + 1}$. Compute the value of the expression with each of the values of x given.

a) $x = 5$ b) $x = -2$ c) $x = \frac{1}{3}$ d) $x = -\frac{1}{2}$ e) $x = \frac{1}{2}$ f) 1

19. Consider the rule $(a^n)^m = a^{nm}$. Simplify $(x^{10})^2$. Based on this result, what is $\sqrt{x^{20}}$?

20. Simplify each of the following.

a) $\sqrt{x^{36}}$ b) $\sqrt[3]{x^{36}}$ c) $\sqrt[6]{x^{36}}$ d) $\sqrt[9]{x^{36}}$

21. Recall the definition of $\sqrt[n]{a}$. If n is even, $\sqrt[n]{a}$ is the non-negative number x such that $x^n = a$. If n is odd, $\sqrt[n]{a}$ is the number x such that $x^n = a$. We can re-write $(\sqrt[5]{a})^{20}$ as $(\sqrt[5]{a})^{5 \cdot 4} = \left[(\sqrt[5]{a})^5\right]^4 = a^4$. Use this technique to simplify each of the following.

a) $(\sqrt{2})^6$ b) $(\sqrt[3]{2})^6$ c) $(\sqrt[3]{x})^{24}$ d) $(\sqrt[7]{x})^{21}$

22. Let N denote 6^{2000} . Express each of the following in terms of N .

a) 6^{2001} b) 6^{2002} c) $6^{2002} - 6^{2001}$ d) 6^{1998} e) 6^{1000} f) 6^{200}

23. Solve each of the following equations.

a) $5x - 3 = -38$

f) $\frac{3}{8} \left(x - \frac{2}{5} \right) = -\frac{3}{2}$

k) $3(2x - 4) - 2(7x + 1) = 8(-x + 1)$

b) $\frac{2}{3}x - \frac{3}{4} = -\frac{5}{12}$

g) $3(2x - 5) - 2(5x + 3) = 3x$

l) $\frac{\frac{2x-5}{3} + 1}{-2} - 1 = 5$

c) $\frac{x-3}{7} = -2$

h) $\frac{1}{2} \left(6x - \frac{2}{3} \right) - \frac{5}{6} \left(12x + \frac{1}{2} \right) = -\frac{31}{4}$

j) $5(2x - 3) = 2(3x - 7) - (-4x + 1)$

d) $\frac{x - \frac{5}{6}}{\frac{3}{-8}} = \frac{4}{9}$

i) $\frac{3}{4}x - \frac{1}{2} \left(\frac{2}{3}x - \frac{3}{5} \right) = \frac{1}{20}$

e) $3(x + 8) = -15$

j) $\frac{3}{8}x + \frac{2}{5}(x + 1) = \frac{1}{8}(2x - 1)$

24. $\frac{4}{5}$ of a number is 36. Find this number.

25. The sum of three consecutive multiples of 5 is 105. Find the value of these numbers.

26. The tickets for the field trip were purchased yesterday for both students and instructors. Children tickets cost \$6, adult tickets cost \$20. The number of children ticket purchased was three more than twice the number of adults tickets purchased. How many of each were purchased if all of the tickets cost a total of \$274 dollars?

27. One side of a rectangle is five meters shorter than seven times the other side. Find the length of the shorter side if we also know that the perimeter of the rectangle is 278 meters.

28. Find the last digit of $7^{99} + 7^{100} + 7^{101} + 7^{102}$.

29. There are 50 students in the senior class. 35 students are taking English, 28 are taking French, and 15 are taking both English and French. How many students are taking neither English, nor French?

30. We are monitoring a cell culture in a container. If the number of cells double every minute, and the container is full after 100 minutes, when is the container half full?

Chapter 8

8.1 Multiplying Algebraic Expressions

In the previous section, we learned how to add and subtract algebraic expressions. In this handout, we will multiply them.

To multiply an algebraic expression by a number or a one-term expression, we apply the distributive law.

Example 1. Expand the products as indicated.

- a) $3x(5x^2 - x + 1)$ c) $5ax(2a - x)$ e) $-3ab^3(4a - b + 2ab)$
 b) $-1(-x^2 + 3x - 4)$ d) $-ab(3a - 5b - 1)$

Solution: a) $3x(5x^2 - x + 1) = 15x^3 - 3x^2 + 3x$

b) $-1(-x^2 + 3x - 4) = x^2 - 3x + 4$

Note that multiplication by -1 means we change the sign in front of each term.

c) $5ax(2a - x) = 5ax \cdot 2a - 5ax \cdot x = 10a^2x - 5ax^2$

d) $-ab(3a - 5b - 1) = -ab \cdot 3a - ab(-5b) - ab(-1) = -3a^2b + 5ab^2 + ab$

e) $-3ab^3(4a - b + 2ab) = -3ab^3(4a) + (-3ab^3)(-b) + (-3ab^3)(2ab) = -12a^2b^3 + 3ab^4 - 6a^2b^4$

When multiplying algebraic expressions, we apply the distributive law and then combine like terms.

Example 2. Expand each of the following.

- a) $(x - 3)(-2x + 5)$ b) $(3a - 2b)(5x - y)$ c) $(2y + 1)(3y - 5)$

Solution: When we multiply two two-term expressions, FOIL is just one of the possible ways to make sure we applied the distributive law. (F - first with first; O - the outer terms; I - the inner terms; L - last with last)

$$\begin{aligned} \text{a) } (x - 3)(-2x + 5) &= \overbrace{x \cdot (-2x)}^{\text{F}} + \overbrace{x \cdot 5}^{\text{O}} + \overbrace{(-3) \cdot (-2x)}^{\text{I}} + \overbrace{(-3) \cdot 5}^{\text{L}} \\ &= -2x^2 + 5x + 6x - 15 \quad \text{5x and 6x are like terms} \\ &= -2x^2 + 11x - 15 \end{aligned}$$

$$\text{b) } (3a - 2b)(5x - y) = \overbrace{3a \cdot 5x}^{\text{F}} + \overbrace{3a \cdot (-y)}^{\text{O}} + \overbrace{(-2b) \cdot (5x)}^{\text{I}} + \overbrace{(-2b) \cdot (-y)}^{\text{L}}$$

$$\begin{aligned}
 &= 15ax - 3ay - 10bx + 2by && \text{all terms are unlike} \\
 &= \boxed{15ax - 3ay - 10bx + 2by} \\
 \text{c) } (2y+1)(3y-5) &= \overbrace{2y \cdot 3y}^{\text{F}} + \overbrace{2y \cdot (-5)}^{\text{O}} + \overbrace{1 \cdot 3y}^{\text{I}} + \overbrace{1 \cdot (-5)}^{\text{L}} \\
 &= 6y^2 - 10y + 3y - 5 && -10y \text{ and } 3y \text{ are like terms} \\
 &= \boxed{6y^2 - 7y - 5}
 \end{aligned}$$

If the multiplication is more complicated, the distributive law is applied so that each term from one expression is multiplied by each term of the other expression exactly once.

Example 3. Expand each of the following.

$$\text{a) } (x+2)(3x^2-5x-7) \quad \text{b) } (2a-b-1)(5a-4b)$$

Solution: a) We will first multiply all three terms of the second expression by x , and then by 2. We expect six little products. Finally, we combine like terms.

$$\begin{aligned}
 (x+2)(3x^2-5x-7) &= x \cdot 3x^2 + x \cdot (-5x) + x \cdot (-7) + 2 \cdot 3x^2 + 2 \cdot (-5x) + 2 \cdot (-7) \\
 &= 3x^3 - 5x^2 - 7x + 6x^2 - 10x - 14 = \boxed{3x^3 + x^2 - 17x - 14}
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } (2a-b-1)(5a-4b) &= 2a \cdot 5a + 2a \cdot (-4b) + (-b) \cdot 5a + (-b) \cdot (-4b) + (-1) \cdot 5a + (-1) \cdot (-4b) \\
 &= 10a^2 - 8ab - 5ab + 4b^2 - 5a + 4b && -8ab \text{ and } -5ab \text{ are like terms.} \\
 &= \boxed{10a^2 - 13ab + 4b^2 - 5a + 4}
 \end{aligned}$$

Example 4. Expand each of the following.

$$\begin{array}{lll}
 \text{a) } (x+1)^2 & \text{d) } (2y-1)^3 & \text{g) } (2x+1)^2 + (2x-1)^2 \\
 \text{b) } (3a-2)^2 & \text{e) } (3x-1)(3x+1) & \text{h) } (3x+5)^2 - (3x-5)^2 \\
 \text{c) } (2y-1)^2 & \text{f) } (a+b)(a-b) & \text{i) } -(5x-1)^2
 \end{array}$$

Solution: When we square an entire sum or difference, the expression is called a **complete square**.

$$\text{a) } (x+1)^2 = (x+1)(x+1) = x \cdot x + x \cdot 1 + 1 \cdot x + 1 \cdot 1 = x^2 + x + x + 1 = \boxed{x^2 + 2x + 1}$$

$$\text{b) } (3a-2)^2 = (3a-2)(3a-2) = 9a^2 - 6a - 6a + 4 = \boxed{9a^2 - 12a + 4}$$

$$\text{c) } (2y-1)^2 = (2y-1)(2y-1) = 4y^2 - 2y - 2y + 1 = \boxed{4y^2 - 4y + 1}$$

d) In solving this problem, we will apply our result from the previous problem.

$$\begin{aligned}
 (2y-1)^3 &= (2y-1)(2y-1)(2y-1) = (2y-1)(2y-1)^2 = (2y-1)(4y^2 - 4y + 1) \\
 &= 8y^3 - 8y^2 + 2y - 4y^2 + 4y - 1 = \boxed{8y^3 - 12y^2 + 6y - 1}
 \end{aligned}$$

e) The expressions $3x-1$ and $3x+1$ are **conjugates** of each other. Conjugates have very useful properties.

$$(3x-1)(3x+1) = 9x^2 + 3x - 3x - 1 = \boxed{9x^2 - 1}$$

$$\text{f) } (a+b)(a-b) = a^2 - ab + ba - b^2 = a^2 - ab + ab - b^2 = \boxed{a^2 - b^2}$$

$$\text{g) } (2x+1)^2 + (2x-1)^2 \quad \text{Solution - Youtube Link}$$

h) $(3x+5)^2 - (3x-5)^2$ [Solution - Youtube link](#)

i) $-(5x-1)^2$ [Solution - YouTube Link](#)



Discussion:

1. Explain why re-writing $2(x-3)^2$ as $(2x-6)^2$ would be an incorrect step.
2. Expand $(m-3)^2$ and $(3-m)^2$. Explain what you notice.

Example 5. Simplify the expression $(2x-3)^2 - (2x+1)(3x-7)$

Solution: If we consider the expression as one that connects algebraic expressions, then there are three operations: squaring, a multiplication, and a subtraction. We apply order of operations. We will work out the products separately.

Exponentiation first: $(2x-3)^2 = (2x-3)(2x-3) = 4x^2 - 6x - 6x + 9 = 4x^2 - 12x + 9$

then multiplication: $(2x+1)(3x-7) = 6x^2 - 14x + 3x - 7 = 6x^2 - 11x - 7$

and finally subtraction.

$(2x-3)^2 - (2x+1)(3x-7) = (4x^2 - 12x + 9) - (6x^2 - 11x - 7)$ to subtract is to add the opposite

$$= 4x^2 - 12x + 9 - 6x^2 + 11x + 7 = \boxed{-2x^2 - x + 16}$$

Caution! This last example contains a situation in which students are at risk to make an error. After we expand an expression such as $(2x+1)(3x-7)$, we usually do not need a parentheses. However, in this case, the product is subtracted and so we still need a pair of parentheses indicating that we are subtracting the entire expression and not just its first term. This is just one of the several examples in which we are tempted to violate the distributive law.

Example 5b. Simplify the expression $(2x-1)^2 - (3x+2)(4x-3)$ [Solution - Youtube Link](#)

Example 6. Solve the given equation. Make sure to check your solutions.

$$(2x-3)^2 - (x+1)(3x-5) = 11 - (x-1)(3-x)$$

Solution: We carefully expand the indicated products and combine like terms. Notice that even after we expanded $(x+1)(3x-5)$ and $(x-1)(3-x)$, we still need to keep them in parentheses because we are subtracting them. We will first work out the products.

$$(2x-3)^2 = (2x-3)(2x-3) = 4x^2 - 6x - 6x + 9 = 4x^2 - 12x + 9$$

$$(x+1)(3x-5) = 3x^2 - 5x + 3x - 5 = 3x^2 - 2x - 5$$

$$(x-1)(3-x) = 3x - x^2 - 3 + x = -x^2 + 4x - 3$$

We are now ready to begin to solve the equation.

$$\begin{aligned} (2x-3)^2 - (x+1)(3x-5) &= 11 - (x-1)(3-x) \\ 4x^2 - 12x + 9 - (3x^2 - 2x - 5) &= 11 - (-x^2 + 4x - 3) && \text{to subtract is to add the opposite} \\ 4x^2 - 12x + 9 - 3x^2 + 2x + 5 &= 11 + x^2 - 4x + 3 && \text{combine like terms} \\ x^2 - 10x + 14 &= x^2 - 4x + 14 && \text{subtract } x^2 \\ -10x + 14 &= -4x + 14 && \text{add } 10x \\ 14 &= 6x + 14 && \text{subtract } 14 \\ 0 &= 6x && \text{divide by } 6 \\ 0 &= x \end{aligned}$$

We check: if $x = 0$, then

$$\text{LHS} = (2 \cdot 0 - 3)^2 - (0 + 1)(3 \cdot 0 - 5) = (-3)^2 - 1(-5) = 9 + 5 = 14$$

$$\text{RHS} = 11 - (0 - 1)(3 - 0) = 11 - (-1)3 = 11 + 3 = 14$$

and so our solution, $x = 0$ is correct.

Example 6b. Solve the given equation. $(x+2)(2x-3) = 2(x+1)^2 - 17$ [Solution - YouTube Link](#)

Example 7. If we increase the length of each side of a square by 4cm, the area of the square increases by 64cm^2 . How long are the sides before the increase?

Solution: Let us denote the length of the original square by x . Then its area is x^2 . The side of the larger square is $x+4$. Therefore, the area of the larger square is $(x+4)^2$. The equation will express the comparison between the two areas.

$$\begin{aligned} (x+4)^2 &= x^2 + 64 && \text{expand complete square on the left-hand side} \\ x^2 + 8x + 16 &= x^2 + 64 && \text{subtract } x^2 \\ 8x + 16 &= 64 && \text{subtract } 16 \\ 8x &= 48 && \text{divide by } 8 \\ x &= 6 \end{aligned}$$

Thus the original square has sides 6cm long. The area is 36cm^2 . If we increased each side by 4cm, the new side is 10cm, and the new area 100cm^2 . Indeed, the two areas differ by $100\text{cm}^2 - 36\text{cm}^2 = 64\text{cm}^2$. Thus our solution is correct: the original square has sides that are 6cm long.

Example 7b. The area of a square would increase by 17 square-yards if we increased its sides by 1 yard.. How long is a side of the square now (before the increase)? [Solution - YouTube Link](#)

Example 8. If we square a number, the result is 29 less than the product of the numbers one greater and two greater than the original number. Find this number.

Solution: Let us denote the number by x . The numbers one and two greater are $x+1$ and $x+2$. The equation will compare the two products:

$$\begin{array}{ll}
 x^2 &= (x+1)(x+2) - 29 && \text{expand products} \\
 x^2 &= x^2 + 2x + x + 2 - 29 && \text{combine like terms} \\
 x^2 &= x^2 + 3x - 27 && \text{subtract } x^2 \\
 0 &= 3x - 27 && \text{add 27} \\
 27 &= 3x && \text{divide by 3} \\
 9 &= x
 \end{array}$$

Thus this number is 9. We check: $9^2 = 81$ and $(9+1)(9+2) = 10 \cdot 11 = 110$ and 110 is 29 greater than 81. Thus our solution is correct.



Enrichment

1. Special products. Expand each of the following.

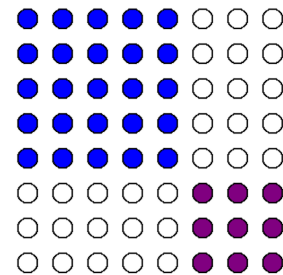
$$\begin{array}{llll}
 \text{a) } (a+b)^2 & \text{c) } (a+b)^3 & \text{e) } (x-y)(x+y) & \text{g) } (x-y)(x^3+x^2y+xy^2+y^3) \\
 \text{b) } (a-b)^2 & \text{d) } (a-b)^3 & \text{f) } (x-y)(x^2+xy+y^2) &
 \end{array}$$

2. Explain how $2x+3x=5x$ can be explained in terms of the distributive law.

3. The *Freshman's Dream Error*. We saw that

$$3^2 + 5^2 = 9 + 25 = 34 \text{ and } (3+5)^2 = 8^2 = 64.$$

To confuse $a^2 + b^2$ - a sum of two squares with the complete square $(a+b)^2$ is an error so egregious, it actually has a name: The Freshman's Dream. The Freshman's Dream is a violation of the distributive law in which we state that $(a+b)^2 = a^2 + b^2$. Use the picture given to illustrate the error and the correct formula for the expansion of $(a+b)^2$.





Sample Problems

1. Multiply the algebraic expressions as indicated.

a) $3xy(2x^2 + 4y - 5)$ b) $-5x^3(2x^2 - x + 8)$ c) $-(-x + 3y - 8z^2 + 6)$ d) $-2a(-a + 3b^2 - 2ab + 7)$

2. Multiply the algebraic expressions as indicated.

a) $(x + 3)(5x - 3)$ b) $(5 - 2x)^2$ c) $(x + 4)(1 - 2x)$ d) $-2(x - 3)^2$

3. Simplify each of the following expressions.

a) $(x - 5)^2 - (2x - 1)(x + 3)$ b) $-(m - 3)^2$ c) $-2(3x - 5) - (2x - 1)^2$

4. Solve each of the following equations. Make sure to check your solutions.

a) $(x - 3)^2 - (2x - 5)(x + 1) = 5 - (x - 1)^2$ c) $12 - (2p - 1)(p + 1) = -2(-p + 5)^2$
b) $(x + 1)^2 - (2x - 1)^2 + (3x)^2 = 6x(x - 2)$ d) $(x + 3)^2 - (x + 1)(2x - 1) = 5 - (x - 2)^2$

5. If we increase each side of a square by 3cm, its area increases by 51cm^2 . How long are the sides before the increase?

6. If we add two to a number and multiply that by one less than that number, the product is 6 less than the square of the number. Find this number.

7. One side of a rectangle is three less than twice another side. If we increase both sides by 1 unit, the area of the rectangle will increase by 10 units. Find the sides of the original rectangle.



Practice Problems

1. Multiply the algebraic expressions as indicated.

a) $5a^2b(-3ab^2 + y - 5)$ b) $0(2x^2 - 7x + 8)$ c) $-16t^3(-t^2 + 6t - 1)$ d) $-(x - 5a + 3ax - 1)$

2. Multiply the algebraic expressions as indicated.

a) $(3 - 2x)(x - 7)$ b) $(3 - 2x)(3x - 2)$ c) $(3 - 2x)(3 + 2x)$ d) $(3 - 2x)^2$ e) $2(a - 1)^2$

3. Simplify each of the following. Notice what is the same and what is different in the problems.

a) $(-2x + 5) + (3x - 8)$ b) $(-2x + 5) - (3x - 8)$ c) $3(-2x + 5) - 2(3x - 8)$ d) $(-2x + 5)(3x - 8)$

4. Simplify each of the following expressions.

a) $(x + 2)^2 - (x - 2)^2$ b) $1 - (x - 4)(3x - 1)$ c) $-3(5m - 1)^2$ d) $3x - 1 - 3x(3x - 1)$

5. Solve each of the following equations.

a) $2x(3x - 1) - x(5x - 1) = (x - 1)^2$	g) $3(a + 11) - a(8 - 3a) = 3(a - 2)^2$
b) $y^2 - (y - 1)^2 + (y - 2)^2 = (y - 3)(y - 5)$	h) $-5(2x - 1) - (4 - x)^2 = 3 - (x + 1)^2$
c) $(3x)^2 - (x + 3)(5x - 3) = (5 - 2x)^2 - 16$	i) $5(-3 - x) - 3x(x - 2) = x - 3(x + 2)(x - 5)$
d) $(w + 4)(1 - 2w) = 3w - 2(w - 3)^2$	j) $2(-m - 2)^2 - (m - 2)^2 = 8m + (m + 2)^2$
e) $(2x - 3)^2 - 3(x - 2)^2 = 10 - (x - 2)(7 - x)$	k) $(3a - 5)(2 - a) - (2a - 1)(a + 3) = -5a^2 - 7$
f) $(2 - w)^2 - (2w - 3)^2 + 7 = (w - 2)(5 - 3w)$	

6. If we increase the length of each side of a square by 2cm, the area of the square increases by 24cm^2 . How long are the sides before the increase?

7. If we square the sum of a number and three, the result is twelve less than the square of the sum of the number and one. Find this number.

8. The product of two consecutive numbers is 8 less than the square of the greater number. Find these numbers.

8.2 Basic Percent Problems

Consider the following problem: Find $\frac{3}{5}$ of 100.

Solution: $\frac{1}{5}$ of 100 can be obtained by splitting 100 into 5 equal shares. If we think of money, this is an easy task: 5 twenty-dollar bills make up a hundred dollar bill. In short, $\frac{1}{5}$ of 100 is 20.

We need $\frac{3}{5}$ of 100, which means that we need to take 3 shares out of the 5. This is $3(20) = 60$. Thus, $\frac{3}{5}$ of 100 is 60.

Notice that we obtain the same result if we simply multiply $\frac{3}{5}$ by 100.

$$\frac{3}{5} \cdot 100 = \frac{3}{5} \cdot \frac{100}{1} = \frac{300}{5} = 60$$

We will establish a language we will use to solve word problems involving fractions. Consider the statement

$$\frac{3}{5} \text{ of } 100 \text{ is } 60.$$

These three quantities are always present in word problems involving fractions. The following definitions are not very elegant but seem to be useful. The fraction $\frac{3}{5}$ will be called F for fraction. The quantity we are splitting up (in this case, 100) will be called the (of) number, because the word *of* is always near it. The result will be called the (is) number, since the word *is* is always near it. Then all word problems involving fractions can be solved using one formula,

$$\boxed{(\text{is}) = (\text{Fraction}) \cdot (\text{of})}$$

In the following problems we are given two quantities, and are to find the third one. Consequently, there are three types of word problems involving fractions. We will call a problem type 1 if we have to find the (is) number, type 2 if we have to find F (the fraction), and type 3 if we have to find the (of) number.

Example 1. Find $\frac{2}{9}$ of 450.

Solution: The formula we use is $\boxed{(\text{is}) = (\text{Fraction}) \cdot (\text{of})}$. First we identify which two quantities are given. We will denote the third one by x .

$$(\text{is}) = x$$

$$F = \frac{2}{9}$$

$$(\text{of}) = 450$$

We will substitute these into the formula and solve for x . The fraction and the (of) are given, and so

$$(\text{is}) = F \cdot (\text{of}) \quad \text{becomes} \quad x = \frac{2}{9} \cdot 450 = \frac{900}{9} = 100$$

So the answer is $\boxed{100}$.

Because the fraction and the (of) numbers were given and the (is) number was to be found, this problem is type 1.

Example 2. 75 is what fraction of 400?

Solution: The formula we use is $(\text{is}) = (\text{Fraction}) \cdot (\text{of})$. We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x . In this case,

$$(\text{is}) = 75$$

$$F = x$$

$$(\text{of}) = 400$$

We will substitute these into the formula and solve for x .

$$(\text{is}) = F \cdot (\text{of})$$

$$75 = x \cdot 400 \quad \text{Solve for } x \text{ by dividing both sides by 400}$$

$$\frac{75}{400} = x \quad \text{simplify the result}$$

$$x = \frac{3}{16}$$

The answer is $\frac{3}{16}$. We check by computing $\frac{3}{16}$ of 400. It is indeed 75, and so our solution is correct.

Because the (is) and the (of) numbers were given and the fraction was to be found, this problem is type 2.

Example 3. $\frac{4}{11}$ of a number is 36. Find this number.

Solution: The formula we use is $(\text{is}) = (\text{Fraction}) \cdot (\text{of})$. We first identify the two numbers given and call the missing number x .

$$(\text{is}) = 36$$

$$F = \frac{4}{11}$$

$$(\text{of}) = x$$

The fraction and the (is) are given, and so we label the (of) number as x .

$$(\text{is}) = (\text{Fraction}) \cdot (\text{of})$$

$$36 = \frac{4}{11} \cdot x \quad \text{Divide both sides by } \frac{4}{11}$$

$$\frac{36}{\frac{4}{11}} = x \quad \text{To divide is to multiply by the reciprocal:}$$

$$99 = x \quad \frac{36}{\frac{4}{11}} = 36 \cdot \frac{11}{4} = 99$$

The answer is 99 . We check: is it true that $\frac{4}{11}$ of 99 is 36? Since $\frac{4}{11}(99) = 36$, our solution is correct.

Because the (is) number and the fraction were given and the (of) number was to be found, this problem is type 3.

We can now solve percent problems as well, because percents are fractions with denominator 100.

Example 4. Find 15% of 400.

Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x . In this case,

(is) = x	We will substitute these into the formula and solve for x .
$F = 15\% = \frac{15}{100}$	(is) = $F \cdot (\text{of})$
(of) = 400	$x = \frac{15}{100} \cdot 400$
	$x = 60$

Thus 15% of 400 is 60. This problem is type 1.

Example 5. 21 is what percent of 350?

Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x . In this case,

(is) = 21	We will substitute these into the formula and solve for x .
$F = x$	(is) = $F \cdot (\text{of})$
(of) = 350	$21 = x \cdot 350$ divide by 350
	$\frac{21}{350} = x$

We obtained the value of x , but not as a percent. We will convert x into a percent.

$$x = \frac{21}{350} = \frac{-7 \cdot 3}{-7 \cdot 50} = \frac{3}{50} = \frac{3 \cdot 2}{50 \cdot 2} = \frac{6}{100} = 6\%$$

Thus 21 is 6% of 350. We can check by computing 6% of 350: $\frac{6}{100} \cdot 350 = 21$. Thus our solution, 6% is correct. This problem is type 2.

Example 6. 24% of what number is 72?

Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x . In this case,

(is) = 72	(is) = $F \cdot (\text{of})$
$F = 24\% = \frac{24}{100}$	$72 = \frac{24}{100} \cdot x$ divide by $\frac{24}{100}$
(of) = x	$\frac{72}{\frac{24}{100}} = x$
	$300 = x$

The computation is $72 \div \frac{24}{100} = \frac{72}{1} \cdot \frac{100}{24} = \frac{3 \cdot 24}{1} \cdot \frac{100}{24} = \frac{300}{1} = 300$. Thus 72 is 24% of 300.

We can check by computing 24% of 300: $\frac{24}{100} \cdot 300 = \frac{24}{100} \cdot \frac{300}{1} = 24 \cdot 3 = 72 = 72$. Thus our solution is correct.

We have focused on the steps of solving such equations. Computations can be simplified by reducing fractions or working with decimals instead of fractions.



Sample Problems

1. Find 16% of 3600.
2. 27 is what percent of 18?
3. 120% of what number is 150?
4. What do we get if we increase 600 by 150%?
5. We placed \$ 8000 into a bank account with an annual 7% of interest rate. How much money do we have in the account a year later?
6. The population of a town has decreased from 80000 to 68000. What percent of a decrease does this represent?
7. Tom got a 4% raise in his job. Now he makes 2496 per month. How much was he making before the raise?
8. A TV set went on a 35% sale. The sale price is \$312. Find the original price.



Practice Problems

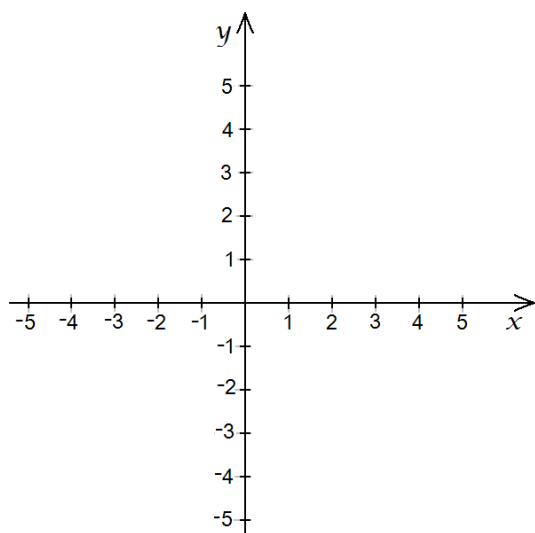
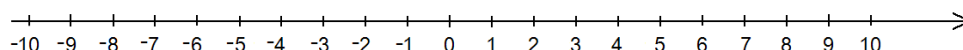
1. What fraction of $2\frac{1}{8}$ is $\frac{1}{4}$?
2. Compute 87% of 300.
3. 130% percent of what number is 78?
4. Fifteen percent of what number is 105?
5. What percent of 450 is 288?
6. What number do we get if we increase 80 by 240%?
7. What percent of 460 is 1472?
8. 347% of what number is 2429?
9. Fifteen percent of the town's population are students. If there are 1800 students living in the town, how many people live there?
10. Paul earned \$128 this week in his part time job. If this was a sixty percent increase from last week, how much money did he make last week?
11. A TV went on a 14% sale. The sale price is \$516. Find the original price of the TV.
12. Overnight, the number of bacteria increased by one hundred sixty percent. There are now 650000 bacteria. How many was there yesterday?
13. We want to increase quantity Q by 2%. Which of the following is the correct expression for that?
A) $1.2Q$ B) $1.12Q$ C) $1.02Q$ D) $3Q$ E) $2.2Q$
14. We want to increase quantity Q by 20%. Which of the following is a correct expression for that?
A) $1.2Q$ B) $1.12Q$ C) $1.02Q$ D) $3Q$ E) $2.2Q$
15. We want to increase quantity Q by 12%. Which of the following is a correct expression for that?
A) $1.2Q$ B) $1.12Q$ C) $1.02Q$ D) $3Q$ E) $2.2Q$

16. We want to increase quantity Q by 120%. Which of the following is a correct expression for that?
A) $1.2Q$ B) $1.12Q$ C) $1.02Q$ D) $3Q$ E) $2.2Q$
17. We want to increase quantity Q by 200%. Which of the following is a correct expression for that?
A) $1.2Q$ B) $1.12Q$ C) $1.02Q$ D) $3Q$ E) $2.2Q$
18. Last month, Randy was making \$1800 a month in his sales job. Based on his performance, he could get a raise at any time.
- a) Two weeks ago, he got a raise of 10%. How much was he making after this increase?
 - b) A week ago, he got a second raise of 10%. How much is he making now?
 - c) Look at Randy's current pay and compare it to the original \$1800. What percent increase does his current pay represent?
19. The population of a town has increased 20% in the 80s. During the next ten years, the town's population further grew 15%. What percent of an increase occurred during these twenty years?
20. A stock loses 60% of its value. What must the percent of increase be to recover all of its lost value? (Hint: if no value for the stock is given, make up a few different numbers.)

8.3 The Rectangular Coordinate System

In mathematics, we often deal with algebraic objects such as numbers and equations. We also deal with geometric objects such as lines, points, or triangles. The relationship between things of algebraic and geometric nature is an important recurring theme within mathematics. We will discuss how algebraic properties manifest geometrically and vica versa. The beautiful and fascinating relationship between geometry and algebra starts with the concept of the number line.

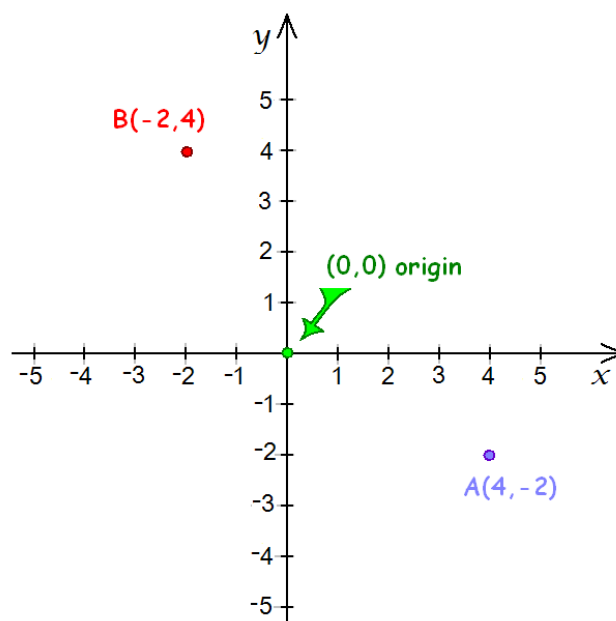
There is a one-to-one correspondance between the set of all real numbers and the set of points on a straight line. In other words, for every real number, there exists exactly one point on the line; and for every point on the line, there exists exactly one real number. We display this correspondance using the **number line**, representing the set of all real numbers.



When we use two identical copies of the real number line, we create a **rectangular coordinate system**. This type of a coordinate system is also called a **Cartesian coordinate system**, after French mathematician and philosopher René Descartes who published this idea first in 1637. The horizontal line is called the **x -axis**, and the vertical line the **y -axis**. The arrows on the axes indicate the positive direction.

Every point in the plane can be uniquely described by an **ordered pair of real numbers** such as $(4, -2)$. The first number in the pair describes the x -coordinate (or horizontal address) of the point. The second number in the pair describes the y -coordinate (or vertical address) of the point. The word *ordered* is important since $A(4, -2)$ and $B(-2, 4)$ are different points. The point $(0, 0)$, where the two axes intersect each other, is called the **origin**.

Suprising facts: the x -axis is the set of all points whose y -coordinate is zero. Similarly, the y -axis is the set of all points whose x -coordinate is zero.



Theorem: Suppose that $A(x_A, y_A)$ and $B(x_B, y_B)$ are two points given. Let M denote the midpoint of line segment AB . The coordinates of the $M(x_M, y_M)$ are the average of the corresponding coordinates of A and B .

$$x_M = \frac{x_A + x_B}{2} \quad \text{and} \quad y_M = \frac{y_A + y_B}{2}$$

Example 1. Find the midpoint of the line segment AB where points A and B are given as $A(-5, 4)$ and $B(3, -4)$.

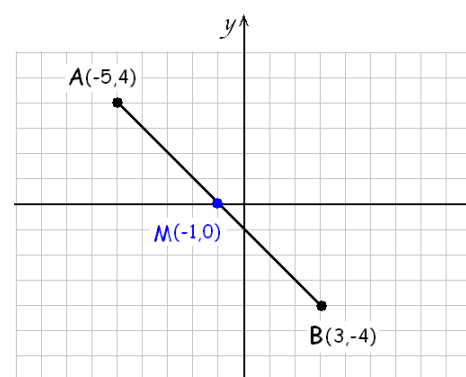
Solution: The x -coordinate of the midpoint is the average of the x -coordinates of A and B .

$$x_M = \frac{x_A + x_B}{2} = \frac{-5 + 3}{2} = \frac{-2}{2} = -1$$

Similarly, the y -coordinate of the midpoint is the average of the y -coordinates of A and B .

$$y_M = \frac{y_A + y_B}{2} = \frac{4 + (-4)}{2} = \frac{0}{2} = 0$$

Thus the midpoint is $M(-1, 0)$.



Problem Set 8

- Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{1, 5, 6, 8, 11\}$, $B = \{1, 2, 9, 11\}$, and $C = \{3, 7, 8, 9, 10, 11\}$.
 - Draw a Venn diagram depicting these sets.
 - Find $(A \cap B) \cup (A \cap C)$.
 - Find $(A \cup B) \cap (A \cup C)$.
 - List all two-element subsets of A .
 - List all subsets of B .
- List all factors of 78.
- Which of the following numbers are primes? 201, 91, 701, 2019, 901
- Find the least common multiple and greatest common factor for each of the pairs of numbers given.
 - 200 and 360
 - 72 and 35
 - 120 and 135
- Find the prime factorization of 12^{100} and 15^{50} .
 - Find the least common multiple and greatest common factor for 12^{100} and 15^{50} . Present your answers in their prime factorization form.
- True or false?
 - If n is a positive integer, greater than 1, then every exponent in the prime factorization of n^2 is even.
 - If m is divisible by 6 and by 8, then m is also divisible by 48.
- Suppose that $A = \{k \in \mathbb{Z} : k \text{ is a multiple of } 3\}$ and $B = \{k \in \mathbb{Z} : k \text{ is a multiple of } 5\}$.
 - Find $A \cap B$.
 - Find $S = \{a \in A : a > -5 \text{ and } a \leq 10\}$
 - Find $T = \{b \in B : b > -12 \text{ and } b < 12\}$
- Three vertices of a rectangle are $A(-8, -3)$, $B(2, -3)$, and $C(2, 5)$. What are the coordinates of the missing vertex?
 - Compute the perimeter and area of the rectangle.
- Evaluate each of the following.

$\sqrt{(-1)^4 - \left(2^3 - 2(5 - (-3)^2)^2\right)}$ $\left(8 - 3\left(4^2 - 2(5 - 8)^2\right)^2\right)^2$ $\frac{4^2 + 5^2 - 6 \div 2 \cdot 3}{-4^2 - 8(-2)}$ $-1^2 - 2(5 + 3(-2^4 - 2(-5) + 1))$	$3 - 8 - 2 - 3 - 10$ $3 - 8 - 2 - 3 - 10$ $3 - 8 - 2 - 3 - 10$ $3 - 8 - 2 - 3 - 10$ $3 - 8 - 2 - 3 - 10$ $3 -8 - 2 - 3 - 10$ $3 - 8 - 2 - 3 - 10$
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- Find $\frac{38}{5}$ of 10.
 - Re-write $\frac{38}{5}$ as a mixed number.
 - Re-write $\frac{38}{5}$ as a percent.
- Re-write 75% as a reduced fraction.
- A living room furniture set originally cost \$1200. What will be the sale price if the furniture set goes on a 15% off sale?

13. Consider the fractions $\frac{3}{5}$ and $\frac{5}{8}$.
- Compute $\frac{3}{5}$ of 80 and $\frac{5}{8}$ of 80 to compare the two fractions.
 - Bring the two fractions to the common denominator to compare them.
14. a) We placed \$2000 dollars in a bank account with a simple annual interest rate of 3%. This means that after one year, the bank will pay us 3% of the amount we kept in the account for a year. How much money is in the account after we received the 3% interest?
- b) Compute 103% of 2000. Can you explain the connection between your results for a) and b)?
15. Suppose that $x = 4 \cdot 10^{12}$ and $y = 5 \cdot 10^{28}$. Compute each of the following. Present your answer using scientific notation.
- $x + y$
 - xy
 - $\frac{y}{x^2}$
 - xy^2
 - $\frac{x^3}{y}$
16. Simplify each of the following expressions.
- $2(3x - 1) - 4(2x - 5)$
 - $2((3x - 1) - 4(2x - 5))$
 - $(3x - 1) - (2x - 5)$
 - $2(3((4y - 1) - 2y + 1) - (y - 10)) - y$
 - $5(n - (1 - (2(3n - 1) - 4n) + 2n) - 1)$
 - $-(3x - 1) - 2(5 - 2x)$
17. Simplify each of the following expressions.
- $(-2)^4$
 - -2^4
 - a^3a^4
 - $(a^3)^4$
 - $(2a^3)(2a)^2$
 - $\frac{8a^2}{16a^3}$
 - $(-2x)^2(-2x^2)$
 - $\frac{(8a)^2}{16a^3}$
 - $\frac{-3x^4}{(-3x)^2}$
 - $\frac{(x^3)^6}{x^2}$
 - $2^{143} \cdot 2^{143}$
 - $\frac{(2x^4)^3}{2x^3 \cdot x}$
 - $\frac{(-2x)^2x^3}{(-2x)^5}$
 - $\left(-\frac{2a^3}{3b^4}\right)^2 \left(\frac{3ab^5}{-a^2b}\right)^3$
18. Simplify each of the following expressions. Present your answer as exponential expressions with the only exponent being x .
- $\frac{2^{x+3}}{2^x}$
 - $2^x \cdot 4^x$
 - $\frac{9^x}{3^x}$
 - $\frac{25^{x+1}}{5^{x+1}}$
19. Simplify each of the following expressions.
- $(x - 1)(3x + 1)$
 - $(2x - 1)^2$
 - $(2x - 1)^3$
 - $(x - 1)^2$
 - $(x + 1)(x - 1)$
 - $(3x - 2)^2 - (x - 5)(2x + 1)$
 - $(x^3 + 4)^2 - (x^3 - 4)^2$
 - $(x^3 + 4)^2 + (x^3 - 4)^2$
 - $(a - b)(a^2 + ab + b^2)$

20. Solve each of the following equations.

a) $\frac{x-3}{-7} = -2$

b) $3(7x-1-2(5x-1))-x=-17$

c) $x-2(5x-4(2x-1))=-8$

d) $2(-x+1)-3(2x-1)=13$

e) $\frac{3x-1}{5} + 4 - 6 = -2$

f) $(4x-7)(x-3) = (2x-5)^2$

g) $(2x+1)^2 - x(3x-4) = (x+4)^2$

h) $(x+1)(x-3) = (x+5)(x-5)$

21. 42% of a number is 210. Find this number.

22. a) The population of our town has grown 12% during the past decade. If the population now is 672,000, what was it a decade ago?

b) The living room furniture went on a 15% off sale. If the sale price is \$1071, what was its regular price?

23. Solve each of the following applications problems.

a) The difference between two numbers is 37. Find these numbers if their sum is -5 .

b) We are building a fence around a rectangle-shaped garden that is located next to a lake. The side of the garden that lies along the lake is 30 feet longer than another side of the garden. Fencing along this side costs \$15 per foot. On the other sides, fencing costs \$10 per foot. Find the dimensions of the garden if the fencing costed us \$4350.

c) If we increase all sides of a square by 3 units, the area of the square increases by 75 unit². How long are the sides of the square?

d) The greatest angle in a triangle has a measure that is exactly three times the measure of the smallest angle. The third angle is 20 degrees more than the smallest angle. Find the angles in the triangle.

e) Joy said: "I need to go to the bank because my pocket is too heavy from all these coins. I have three times as many dimes as many quarters, and four times as many nickels as many quarters." How many quarters does Joy have if all the coins in her heavy pocket have a total value of 6 dollars?

f) If we double a number, add 1, and then square the sum, the result is 21 greater than four times the square of the number. Find this number.

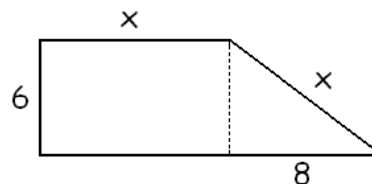
24. Find the last digit of $7^{2018} + 3^{2018}$.

25. Find the sum of the digits of the number $\frac{10^{2018}-1}{9}$. (Hint: if you don't know for 2018, try it with 1, 2, 3, and the idea might come!)

26. Consider the figure shown on the picture. Angles that look like right angles are right angles.

a) Find x if the figure has a perimeter of 44 units.

b) Compute the area of the figure.



27. *(Enrichment) Simplify the expression $2^{2018} + 2^{2018}$. (Hint: if you don't know for 2018, try it with 1, 2, 3, and the idea might come!)

28. *(Enrichment) If the greatest common factor of x and y is 6, and the least common multiple of x and y is 1440, how many different numbers are possible for the value of x ?

29. *(Enrichment) Which is greater, $2^{1000} + 2^{1001} + 2^{1002}$ or 2^{1003} ?

Chapter 9

9.1 Integer Exponents

Part 1 - The History Thus Far and the Problem

Recall what we know about exponentiation thus far. Exponential notation expresses repeated multiplication.

Definition: We define 2^7 to denote the factor 2 multiplied by itself repeatedly, such as

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{7 \text{ factors}} = 2^7$$

When mathematicians agreed to this definition, that was a free choice. They could have gone with other definitions. Once this definition exists, however, certain properties are automatically true, and we have no other option but to recognize them as true. They just fell into our laps.

Theorem 1. If a is any number and m, n are any positive integers, then $a^n \cdot a^m = a^{n+m}$

Theorem 2. If a is any non-zero number and m, n are any positive integers, then $\frac{a^n}{a^m} = a^{n-m}$

Theorem 3. If a is any number and m, n are any positive integers, then $(a^n)^m = a^{nm}$

Theorem 4. If a, b are any numbers and n is any positive integer, then $(ab)^n = a^n b^n$

Theorem 5. If a, b are any numbers, $b \neq 0$, and n is any positive integer, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Again, the definition, immediately followed by the theorems. And then there was a quiet. Another opening for a free choice.

Consider the expression 2^x . The problem is that the definition of exponentiation only allows for a positive integer value of x . The expression 2^x is meaningful for $x = 2$ or 9 or 100 , but it is not meaningful for values of x such as -3 or $\frac{3}{5}$ or 3.2 . In short, the world of exponents was just the set of all natural numbers. Mathematicians usually don't like that. The best case scenario, the ultimate hope is that the definition of exponents could be extended to any number for x . That way, 2^x would be meaningful, no matter what the value of x is.

So, one of the issues was the desire to grow our world of exponents beyond the set of all natural numbers. This will be achieved in several steps. Today, we are only focusing on enlarging the world of exponents from $\mathbb{N}$ to $\mathbb{Z}$ (i.e. from the set of all natural numbers to the set of all integers).

The other issue was that as we enlarge our world, we pay especial attention that the new definitions will not conflict with the mathematics we already have. This principle comes up often in our choices, and it is sometimes called the **expansion principle**.

Definition: In many situations, mathematicians attempt to increase, to enlarge our world. The **expansion principle** is that when we enlarge our mathematics by adding new definitions, we do so in such a way that the new definitions never create conflicts with the mathematics we already have.

Part 2 - Integer Exponents

Suppose we want to define 2^0 . The repeated multiplication definition can not be applied to zero, so we have complete freedom to define 2^0 . As it turns out, if we insist on a definition that does not conflict with Rule 2, $\frac{a^n}{a^m} = a^{n-m}$, then we do not have all that many choices for 2^0 . Let us think of zero as the result of the subtraction $3 - 3$, and that we would like to define 2^0 so that Rule 2 is still true.

$$2^0 = 2^{3-3} \stackrel{\text{rule 2}}{=} \frac{2^3}{2^3} = \frac{8}{8} = 1$$

This is an expansion principle proof. It did not prove that the value of 2^0 is or must be zero. It showed much less; that if we wanted to define 2^0 without harming Rule 2 in the example given, then the only possible value for 2^0 is 1. The reader should imagine a team of mathematicians making first sure that no part of our good old math is hurt if we define $2^0 = 1$. And as it turned out, this is exactly the case.

This computation can be repeated with many different bases. For example,

$$5^0 = 5^{2-2} \stackrel{\text{rule 2}}{=} \frac{5^2}{5^2} = \frac{25}{25} = 1 \quad \text{or} \quad (-3)^0 = (-3)^{2-2} \stackrel{\text{rule 2}}{=} \frac{(-3)^2}{(-3)^2} = \frac{9}{9} = 1$$

The only base that is problematic is 0. Indeed, division by zero is not allowed and Rule 2, $\frac{a^n}{a^m} = a^{n-m}$ does not work with $a = 0$. If we try to perform the same computation with zero, we ultimately end up in $\frac{0}{0}$ which is undefined.

Theorem 6. If a is any non-zero number, then $a^0 = 1$.

0^0 is undefined.

Please note that as we extend our world of exponents, old issues might re-surface. For example, $(-3)^0 = 1$ but $-3^0 = -1$ is an important distinction, but not a new one.

Now that we have defined zero exponent, we will similarly try to define negative integer exponents such as 2^{-3} .

Again, the original definition can not be applied. We cannot write down the factor two negative three times. So we have a freedom here to define 2^{-3} in any way we wish. In this decision, we will again use the expansion principle: that we would like to keep our old rules after having 2^{-3} defined.

We will again use Rule 2, $\frac{a^n}{a^m} = a^{n-m}$ and write -3 as a subtraction between two positive integers.

$$2^{-3} = 2^{1-4} \stackrel{\text{Rule 2}}{=} \frac{2^1}{2^4} = \frac{2}{16} = \frac{1}{8} = \frac{1}{2^3} \text{ or, more elegantly, } 2^{-3} = 2^{1-4} \stackrel{\text{Rule 2}}{=} \frac{2^1}{2^4} = \frac{-2}{-2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2^3}$$

When we discovered this rule, we saw that it was true because of cancellation. In case of a negative exponent, we have the same cancellation, it's just that we run out of factors in the numerator first. The computation can be repeated with any base except for zero.

Theorem 7. If a is any non-zero number, and n is any positive integer, then $a^{-n} = \frac{1}{a^n}$.
 0^{-n} is undefined.

Example 1. Simplify each of the following expressions. Use only positive exponents in your answer.

a) 5^{-2} b) a^{-5} c) $\frac{1}{3^{-2}}$ d) $\left(\frac{2}{3}\right)^{-3}$ e) $\frac{1}{x^{-3}}$ f) $2x^{-3}$

Solution: a) Recall our new rule, $a^{-n} = \frac{1}{a^n}$. We apply this rule: $5^{-2} = \frac{1}{5^2} = \boxed{\frac{1}{25}}$.

b) We can use the same rule again: $a^{-5} = \boxed{\frac{1}{a^5}}$.

c) In this case, the expression with the negative exponent is in the denominator.

The short story is that $\frac{1}{3^{-2}} = 3^2 = 9$. The long story is that we apply our new rule $a^{-n} = \frac{1}{a^n}$ and then we divide by multiplying by the reciprocal.

$$\frac{1}{3^{-2}} = \frac{1}{\frac{1}{3^2}} = \frac{1}{\frac{1}{9}} = \frac{1}{1} \cdot \frac{3^2}{1} = \frac{9}{1} = \boxed{9}$$

So, $\frac{1}{a^{-n}}$ can be re-written as a^n .

d) In this case, the expression with the negative exponent is already a fraction.

The short story is that $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$. The long story is that we apply our new rule $a^{-n} = \frac{1}{a^n}$ and then we divide by multiplying by the reciprocal.

$$\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\frac{8}{27}} = \frac{1}{1} \cdot \frac{27}{8} = \boxed{\frac{27}{8}}$$

This computation shows that $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.

- e) The short story is that $\frac{1}{a^{-n}}$ can be re-written as a^n . The computation below justifies this step.

$$\frac{1}{x^{-3}} = \frac{1}{\frac{1}{x^3}} = \frac{\frac{1}{1}}{\frac{1}{x^3}} = \frac{1}{1} \cdot \frac{x^3}{1} = \frac{x^3}{1} = \boxed{x^3}$$

So, $\frac{1}{a^{-n}}$ can be re-written as a^n .

- f) It is a common mistake to interpret $2x^{-3}$ as $(2x)^{-3}$. Without the parentheses, we perform the exponentiation before the multiplication. Therefore, the correct computation is

$$2x^{-3} = 2 \cdot x^{-3} = \frac{2}{1} \cdot \frac{1}{x^3} = \boxed{\frac{2}{x^3}}.$$

Theorem: The following statements are practical applications of the rule $a^{-n} = \frac{1}{a^n}$ and frequently occur in computations.

$$\frac{1}{a^{-n}} = a^n \quad \text{and} \quad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

Proof: As the computation shows, we apply the rule $a^{-n} = \frac{1}{a^n}$ and then perform the division by multiplying by the reciprocal.

$$\frac{1}{a^{-n}} = \frac{1}{\frac{1}{a^n}} = \frac{\frac{1}{1}}{\frac{1}{a^n}} = \frac{1}{1} \cdot \frac{a^n}{1} = \frac{a^n}{1} = a^n \quad \text{and} \quad \left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \frac{1}{\frac{a^n}{b^n}} = \frac{1}{1} \cdot \frac{b^n}{a^n} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n \quad \blacksquare$$

Example 2. Re-write the expression $\frac{a^3b^{-5}}{c^{-2}d^4}$ using only positive exponents.

Solution: We re-write the expressions with negative exponents using the rule $a^{-n} = \frac{1}{a^n}$.

$$\frac{a^3b^{-5}}{c^{-2}d^4} = \frac{a^3 \cdot \frac{1}{b^5}}{\frac{1}{c^2} \cdot d^4} = \frac{\frac{a^3}{1} \cdot \frac{1}{b^5}}{\frac{1}{c^2} \cdot \frac{d^4}{1}} = \frac{\frac{a^3}{b^5}}{\frac{d^4}{c^2}} = \frac{a^3}{b^5} \cdot \frac{c^2}{d^4} = \boxed{\frac{a^3c^2}{b^5d^4}}.$$

Notice the pattern here. If a factor with a negative exponent is in the numerator, we can re-write it with a positive exponent in the denominator. Also, if a factor with a negative exponent is in the denominator, we can re-write it with a positive exponent in the numerator.

Theorem: $\frac{a^{-n}b^m}{c^pd^{-q}} = \frac{b^md^q}{a^nc^p}$ where a, c, d are any non-zero numbers and n, m, p, q are positive integers.

The definitions of a^0 and a^{-n} were developed with the intention that the previous rules (1 through 5) will remain true. Keep that in mind in case of computations with more complex exponential expressions.

Example 3. Simplify each of the given expressions. Present your answer using only positive exponents.

$$\text{a) } (a^{-2})^{-5} \quad \text{b) } \frac{(-x^{-2})^{-3}}{x^{-6}(-x)^{-4}} \quad \text{c) } \frac{a^{-3}}{a^{-8}} \quad \text{d) } \frac{a^{-2}b^{-3}}{a^{-5}b^3} \quad \text{e) } \frac{(2a^{-4}b^3)^{-5}}{(3a^3b^{-2})^0}$$

Solution: a) It is much preferred to first simplify the exponent. Repeated exponentiation means multiplication in the exponent.

$$(a^{-2})^{-5} = a^{-2(-5)} = \boxed{a^{10}}$$

b) Let us re-write the solo negative signs as multiplications by -1 . Then we will use the rules of exponents to simplify the exponents. Only after that will we address negative exponents.

$$\frac{(-x^{-2})^{-3}}{x^{-6}(-x)^{-4}} = \frac{(-1 \cdot x^{-2})^{-3}}{x^{-6}(-1 \cdot x)^{-4}} = \frac{(-1)^{-3}(x^{-2})^{-3}}{x^{-6}(-1)^{-4}x^{-4}} = \frac{(-1)^{-3}x^6}{x^{-6}(-1)^{-4}x^{-4}}$$

Now we get rid of all negative exponents by moving the factors. A factor with exponent -5 in the numerator can be re-written as a factor with exponent 5 in the denominator, and vice versa.

$$\frac{(-1)^{-3}x^6}{x^{-6}(-1)^{-4}x^{-4}} = \frac{(-1)^4x^6x^4}{(-1)^3} = \frac{1 \cdot x^{16}}{-1} = \boxed{-x^{16}}$$

$$\text{c) Solution 1: apply the rule } \frac{a^n}{a^m} = a^{n-m}. \quad \frac{a^{-3}}{a^{-8}} = a^{-3-(-8)} = a^{-3+8} = \boxed{a^5}$$

Solution 2: First we get rid of negative exponents and then apply the rule $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{a^{-3}}{a^{-8}} = \frac{a^8}{a^3} = a^{8-3} = \boxed{a^5}$$

d) First we get rid of negative exponents.

$$\frac{a^{-2}b^{-3}}{a^{-5}b^3} = \frac{a^5}{a^2b^3b^3} = \frac{a^5}{a^2b^6} = \boxed{\frac{a^3}{b^6}}$$

e) We can save a lot of work by noticing that the denominator is just 1 , because any non-zero quantity raised to the power zero is 1 , and so $(3a^3b^{-2})^0 = 1$.

$$\frac{(2a^{-4}b^3)^{-5}}{(3a^3b^{-2})^0} = \frac{2^{-5}(a^{-4})^{-5}(b^3)^{-5}}{1} = \frac{2^{-5}a^{20}b^{-15}}{1} = \frac{a^{20}}{2^5b^{15}} = \boxed{\frac{a^{20}}{32b^{15}}}$$

Part 3 - Scientific Notation Revisited

When we first saw scientific notation, we learned to use it to handle uncomfortably large numbers.

Recall the definition of scientific notation:

Definition: We can write numbers in scientific notation. This means to write a number as a product of two numbers. The first number is between 1 and 10 (can be 1 but must be less than 10), and the second number is a 10–power. For example, the scientific notation for 428 600 000 000 is 4.286×10^{11} .

With negative exponents, we can also use scientific notation to handle extremely small numbers. For example, the mass of an electron is 0.00000000000000000000000091094 grams. Instead of hundreds of trailing zeroes, now we are faced with many zeroes after the decimal point. This number can be re-written as $9.1094 \cdot 10^{-28}$.

Example 4. Re-write the number 0.0000000317 using scientific notation.

Solution: The first number in scientific notation needs to be between 1 and 10. In this case, this number is 3.17. We just need to figure out the 10–power in the second part. We count how many decimal places we move the decimal from 0.0000000317 to 3.17. We count 8 decimal places. So the correct answer is $3.17 \cdot 10^{-8}$.

Example 5. Suppose that $A = 3.8 \cdot 10^{15}$ and $B = 6.5 \cdot 10^{-8}$. Perform each of the following operations. Present your answer using scientific notation.

a) B^2 b) AB^2 c) $\frac{B}{A}$

Solution: a) We will apply rules of exponents.

$$B^2 = (6.5 \cdot 10^{-8})^2 = 6.5^2 \cdot (10^{-8})^2 = 42.25 \cdot 10^{-16}$$

This number is not in scientific notation because 42.25 is too large for the first part of scientific notation. Recall that the first factor must be between 1 and 10. So we re-write 42.25 as $4.225 \cdot 10$.

$$B^2 = 42.25 \cdot 10^{-16} = 4.225 \cdot 10 \cdot 10^{-16} = 4.225 \cdot 10^{1+(-16)} = 4.225 \cdot 10^{-15}$$

b) We will apply rules of exponents.

$$\begin{aligned} AB^2 &= (3.8 \cdot 10^{15}) (6.5 \cdot 10^{-8})^2 = 3.8 \cdot 10^{15} \cdot 6.5^2 \cdot (10^{-8})^2 = 3.8 \cdot 10^{15} \cdot 42.25 \cdot 10^{-16} \\ &= (3.8 \cdot 42.25) \cdot (10^{15} \cdot 10^{-16}) = 160.55 \cdot 10^{15+(-16)} = 160.55 \cdot 10^{-1} = 16.055 \end{aligned}$$

This number is not in scientific notation because 16.055 is too large for the first part of scientific notation.

$$AB^2 = 16.055 = 1.6055 \cdot 10$$

$$\text{c) } \frac{B}{A} = \frac{6.5 \cdot 10^{-8}}{3.8 \cdot 10^{15}} = \left(\frac{6.5}{3.8} \right) \cdot 10^{-8-15} = 1.7105 \cdot 10^{-23}$$



Sample Problems

Simplify each of the following. Assume that all variables represent positive numbers. Present your answer without negative exponents.

1. 3^{-2}

2. $\frac{1}{2^{-3}}$

3. m^{-4}

4. $\frac{1}{x^{-5}}$

5. $a^8 \cdot a^{-1}$

6. $p^3(p^{-7})p^8$

7. $\frac{x^{-4}}{x^{-9}}$

8. $\frac{50a^{12}}{10a^{-3}}$

9. $\frac{t^{-3}}{t^4}$

10. x^0

11. $-x^0$

12. $(-x)^0$

13. $(b^{-5})(b^2)(b^{-1})$

14. $\frac{1}{(b^{-5})(b^2)(b^{-1})}$

15. $\frac{m^{-2}}{m^{-5}}$

16. $\frac{x^3y^{-5}}{z^{-4}}$

17. $\frac{18q^3}{6q^{-3}}$

18. $\left(\frac{2}{3}\right)^{-3}$

19. $2y^{-3}$

20. $(2y)^{-3}$

21. $\left(-\frac{3}{5}\right)^{-2}$

22. $\frac{a^3b^{-5}}{a^{-2}b^3}$

23. $(3m^3)^{-2}$

24. $(-2ab^{-3})^{-3}$

25. $\frac{(k^3)^{-3}}{(k^{-5})^2}$

26. $\left(\frac{2a^{-3}b^5}{-3a^3b^{-2}}\right)^{-2} (a^3b^{-5})^{-3}$

30. $\left(-\frac{x^3y^0x^{-5}}{y^{-3}}\right)^{-2}$

33. $\frac{(-2a^{-2})^{-2}b^3a^0(-aba^{-2}b^{-2})^{-3}}{2a^2(-2a^{-2}b)^{-2}ab^0}$

27. $(-2a^{-3})(-2a^{-2}b)^{-4}$

28. $\frac{(-3p^3q^5)^2}{(2q^0p^3)^{-1}}$

31. $\left(-\frac{x^3y^7x^{-5}}{y^{-3}}\right)^0$

34. $\left(\frac{-a^2(b^{-1}a)^{-5}}{b^7(-ab^2)^{-3}}\right)^{-2}$

29. $\left(\frac{2a^{-2}b^3}{-2^2(a^{-1}b)^{-3}}\right)^{-2}$

32. $\frac{x^{-1}+y^{-1}}{x^{-2}-y^{-2}}$

35. $\frac{(x^{-2})^{-2}y^3x^0(-2yx^0y^{-2}x^{-2})^0}{yx^5(y^{-2}x)^{-3}(2x^{-1}yx^3)^{-1}}$

36. Suppose that $x = 8.5 \cdot 10^{-12}$ and $y = 7.5 \cdot 10^7$. Perform each of the following operations. Present your answer using scientific notation.

a) xy

b) x^3

c) xy^2

d) $\frac{x}{y}$

e) $\frac{y}{x^5}$

9.2 Graph of an Equation

As we have seen more and more algebraic statements, the solution sets became increasingly more complex. A linear equation usually has a single number solution. In case of linear inequalities, we often have infinitely many solutions. To express those solution sets, we developed interval notation.

Suppose we have an equation in two variables, x and y . The equations $y = 2x - 3$ or $x^2 - y^2 = 5$ or $xy = -2$ are examples for such equations. A solution for such equations is a set of ordered pairs of numbers, (x, y) . For example, $(5, 7)$ is short for $x = 5$ and $y = 7$ and this ordered pair is a solution of the equation $y = 2x - 3$. The ordered pair $(3, -2)$ is a solution of $x^2 - y^2 = 5$, and the ordered pair $(2, -1)$ is a solution of $xy = -2$.

Equations in two variables often have infinitely many solutions, where that can no longer meaningfully be represented on a number line. We step out into two dimensions, and use a coordinate system to depict solution sets. On a coordinate system, each ordered pair (x, y) can be represented as a point.

Definition: The **graph** of an equation in x , in y , or both in x and y is the set of all points $P(x, y)$ whose coordinates are solution of the equation.

In short, the graph of an equation is a solution set of an equation in x and y . The shape of graphs depends on the type of equation. Before we started to graph equations, it is useful to know that we can do quite a lot just using the definition of graphs.

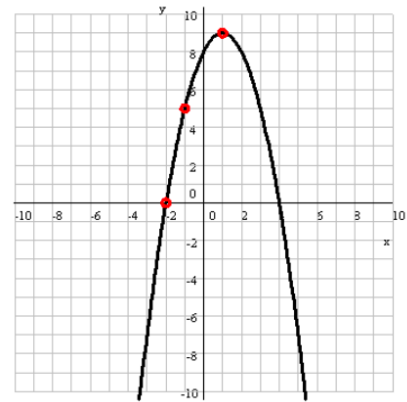
Example 1. Consider the graph shown. Three points on the graph are marked. These are $A(-2, 0)$, $B(-1, 5)$, and $C(1, 9)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.

The possible equations offered are:

$$y = 3x + 6$$

$$(x - 4)^2 + (y - 5)^2 = 25$$

$$y = -x^2 + 2x + 8$$



Solution: Let us consider first the equation $y = 3x + 6$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A , B , and C . Let's check.

Point $A(-2, 0)$ is on the graph if and only if its coordinates are a solution of $y = 3x + 6$.

Check $y = 3x + 6$ with $x = -2$ and $y = 0$.

The left-hand side is: $\text{LHS} = 0$

and the right-hand side is: $\text{RHS} = 3(-2) + 6 = 0$. $\text{RHS} = \text{LHS} \checkmark$

Point A is on the graph of $y = 3x + 6$. This does not mean that $y = 3x + 6$ is the right equation. It only means that we didn't rule it out based on point A alone.

Let's see about point $B(-1, 5)$. Is this point on the graph of $y = 3x + 6$?

Check $y = 3x + 6$ with $x = -1$ and $y = 5$.

$\text{LHS} = 5$ and $\text{RHS} = 3(-1) + 6 = -3 + 6 = 3$ $\text{RHS} \neq \text{LHS}$

At this point, we can conclude that the graph shown is not of the equation of $y = 3x + 6$, because point B is on the graph but its coordinates are not a solution of this equation. So, we can move on to the

next equation.

Consider now the equation $(x-4)^2 + (y-5)^2 = 25$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of *A*, *B*, and *C*. Let's check.

Point *A* $(-2, 0)$ is on the graph if and only if its coordinates are a solution of $(x-4)^2 + (y-5)^2 = 25$.

Check $(x-4)^2 + (y-5)^2 = 25$ with $x = -2$ and $y = 0$.

$$\text{LHS} = (-2-4)^2 + (0-5)^2 = (-6)^2 + (-5)^2 = 36 + 25 = 61$$

$$\text{RHS} = 25. \quad \text{RHS} \neq \text{LHS}$$

We can conclude that the graph shown is not of the equation of $(x-4)^2 + (y-5)^2 = 25$, because point *A* is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $y = -x^2 + 2x + 8$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of *A*, *B*, and *C*. Let's check.

Point *A* $(-2, 0)$ is on the graph if and only if its coordinates are a solution of $y = -x^2 + 2x + 8$.

Check $y = -x^2 + 2x + 8$ with $x = -2$ and $y = 0$.

$$\text{LHS} = 0 \quad \text{and} \quad \text{RHS} = -(-2)^2 + 2(-2) + 8 = -4 - 4 + 8 = 0. \quad \text{RHS} = \text{LHS} \quad \checkmark$$

This does not mean that $y = -x^2 + 2x + 8$ is the right equation. It only means that we didn't rule it out based on point *A* alone. Let's see point *B*.

Point *B* $(-1, 5)$ is on the graph if and only if its coordinates are a solution of $y = -x^2 + 2x + 8$.

Check $y = -x^2 + 2x + 8$ with $x = -1$ and $y = 5$.

$$\text{LHS} = 5 \quad \text{and} \quad \text{RHS} = -(-1)^2 + 2(-1) + 8 = -1 - 2 + 8 = 5. \quad \text{RHS} = \text{LHS} \quad \checkmark$$

This does not mean that $y = -x^2 + 2x + 8$ is the right equation. It only means that we didn't rule it out based on points *A* and *B*. Let's see point *C*.

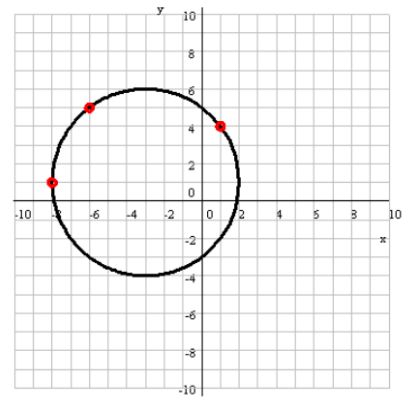
Point *C* $(1, 9)$ is on the graph if and only if its coordinates are a solution of $y = -x^2 + 2x + 8$.

Check $y = -x^2 + 2x + 8$ with $x = 1$ and $y = 9$.

$$\text{LHS} = 9 \quad \text{and} \quad \text{RHS} = -1^2 + 2 \cdot 1 + 8 = -1 + 2 + 8 = 9. \quad \text{RHS} = \text{LHS} \quad \checkmark$$

We found that all three points are on the graph of this equation. This still does not mean that $y = -x^2 + 2x + 8$ is the right equation. Given that we were given three equations with the assumption that the correct equation is among them, it can only be this one. So, our answer is that the graph shown is of the equation $y = -x^2 + 2x + 8$. We can find additional nice points on the graph (for example, $(4, 0)$ or $(2, 8)$) and test them against the equation. Soon we will learn how to graph such shapes.

Example 2. Consider the graph shown. Three points on the graph are marked. These are $A(-8, 1)$, $B(-6, 5)$, and $C(1, 4)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.



The possible equations offered are:

$$3y = x + 11$$

$$3y + x^2 = -8x + 3$$

$$(x + 3)^2 + (y - 1)^2 = 25$$

Solution: Let us consider first the equation $3y = x + 11$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A , B , and C . Let's check.

Point $A(-8, 1)$ is on the graph if and only if its coordinates are a solution of $3y = x + 11$.

Check $3y = x + 11$ with $x = -8$ and $y = 1$.

The left-hand side is: $\text{LHS} = 3 \cdot 1 = 3$

and the right-hand side is: $\text{RHS} = -8 + 11 = 3$. $\text{RHS} = \text{LHS} \checkmark$

Point A is on the graph of $3y = x + 11$. This does not mean that $3y = x + 11$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B .

Point $B(-6, 5)$ is on the graph if and only if its coordinates are a solution of $3y = x + 11$.

Check $3y = x + 11$ with $x = -6$ and $y = 5$.

$\text{LHS} = 3(-6) = -18$ and $\text{RHS} = -6 + 11 = 5$. $\text{RHS} \neq \text{LHS}$

We can conclude that the graph shown is not of the equation of $3y = x + 11$, because point B is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $3y + x^2 = -8x + 3$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A , B , and C . Let's check.

Point $A(-8, 1)$ is on the graph if and only if its coordinates are a solution of $3y + x^2 = -8x + 3$.

Check $3y + x^2 = -8x + 3$ with $x = -8$ and $y = 1$.

$\text{LHS} = 3 \cdot 1 + (-8)^2 = 3 + 64 = 67$ and $\text{RHS} = -8(-8) + 3 = 64 + 3 = 67$. $\text{RHS} = \text{LHS} \checkmark$

This does not mean that $3y + x^2 = -8x + 3$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B .

Point $B(-6, 5)$ is on the graph if and only if its coordinates are a solution of $3y + x^2 = -8x + 3$.

Check $3y + x^2 = -8x + 3$ with $x = -6$ and $y = 5$.

$\text{LHS} = 3 \cdot 5 + (-6)^2 = 15 + 36 = 51$ and $\text{RHS} = -8(-6) + 3 = 48 + 3 = 51$. $\text{RHS} = \text{LHS} \checkmark$

This does not mean that $3y + x^2 = -8x + 3$ is the right equation. It only means that we didn't rule it out based on points A and B . Let's see point C .

Point $C(1, 4)$ is on the graph if and only if its coordinates are a solution of $3y + x^2 = -8x + 3$.

Check $3y + x^2 = -8x + 3$ with $x = 1$ and $y = 4$.

$\text{LHS} = 3 \cdot 4 + 1^2 = 12 + 1 = 13$ and $\text{RHS} = -8 \cdot 1 + 3 = -5$. $\text{RHS} \neq \text{LHS}$

We can conclude that the graph shown is not of the equation of $3y + x^2 = -8x + 3$, because point C is on the graph but its coordinates are not a solution of this equation. So, we can move on to the next equation.

Consider now the equation $(x+3)^2 + (y-1)^2 = 25$. If the graph belongs to this equation, then the coordinates of *all* points on the graph are solutions of the equation, including those of A , B , and C . Let's check.

Point $A(-8, 1)$ is on the graph if and only if its coordinates are a solution of $(x+3)^2 + (y-1)^2 = 25$.

Check $(x+3)^2 + (y-1)^2 = 25$ with $x = -8$ and $y = 1$.

$$\text{LHS} = (-8+3)^2 + (1-1)^2 = (-5)^2 + 0^2 = 25 \text{ and } \text{RHS} = 25. \quad \text{RHS} = \text{LHS} \checkmark$$

This does not mean that $(x+3)^2 + (y-1)^2 = 25$ is the right equation. It only means that we didn't rule it out based on point A alone. Let's see point B .

Point $B(-6, 5)$ is on the graph if and only if its coordinates are a solution of $(x+3)^2 + (y-1)^2 = 25$.

Check $(x+3)^2 + (y-1)^2 = 25$ with $x = -6$ and $y = 5$.

$$\text{LHS} = (-6+3)^2 + (5-1)^2 = (-3)^2 + 4^2 = 9 + 16 = 25 \text{ and } \text{RHS} = 25 \quad \text{RHS} = \text{LHS} \checkmark$$

This does not mean that $(x+3)^2 + (y-1)^2 = 25$ is the right equation. It only means that we didn't rule it out based on points A and B . Let's see point C .

Point $C(1, 4)$ is on the graph if and only if its coordinates are a solution of $(x+3)^2 + (y-1)^2 = 25$.

Check $(x+3)^2 + (y-1)^2 = 25$ with $x = 1$ and $y = 4$.

$$\text{LHS} = (1+3)^2 + (4-1)^2 = 4^2 + 3^2 = 16 + 9 = 25 \text{ and } \text{RHS} = 25 \quad \text{RHS} = \text{LHS} \checkmark$$

We found that all three points are on the graph of this equation. This still does not mean that $(x+3)^2 + (y-1)^2 = 25$ is the right equation. Given that we were given three equations with the assumption that the correct equation is among them, it can only be this one. So, our answer is that the graph shown is of the equation $(x+3)^2 + (y-1)^2 = 25$. We can find additional nice points on the graph (for example, $(2, 1)$ or $(-3, -4)$) and test them against the equation.



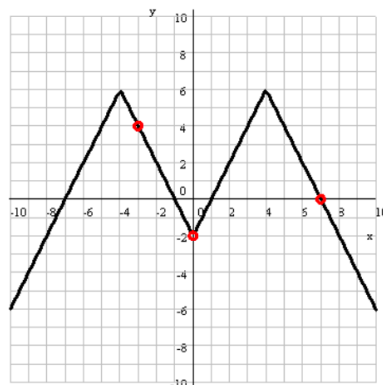
Practice Problems

1. Consider the graph shown. Three points on the graph are marked. These are $A(-3, 4)$, $B(0, -2)$, and $C(7, 0)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see.

The possible equations offered are:

$$y + 2 = |2x| \quad 6 - y = |8 - |2x||$$

$$x^2 + y^2 = 1 + 4(x + y + 5)$$

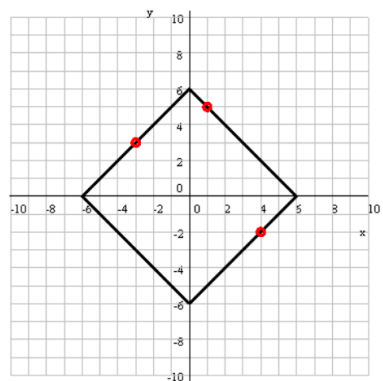


2. Consider the graph shown. Three points on the graph are marked. These are $A(-3, 3)$, $B(1, 5)$, and $C(4, -2)$. Use these points to determine, which of the given equations is the one whose graph is the shape we see. The possible equations offered are:

$$2y = x + 9$$

$$|x| + |y| = 6$$

$$y + 3 = 9 - |x|$$



9.3 Graphing a Line

Equations that are in x , or in y , or in x and y can be graphed. The graph of an equation is our way of representing a large solution set.

Definition: The **graph of an equation** in x and y is the set of all points $P(x,y)$ for which the coordinates x and y form a solution of the equation.

The word linear means: "of degree one". In case of linear equations, the graph is a straight line. There are several forms of a line's equation. Two of them are as follows. There are more, and we will study them later.

$$\begin{array}{ll} y = mx + b & \text{slope-intercept form} \\ Ax + By = C & \text{general form} \end{array}$$

There are several methods of graphing a line. We will start with the simplest one, by finding a few points and connecting the dots.

Example 1. Graph the line $y = -2x + 3$

Solution: We will find points on this line and connect the dots. Since the graph is a straight line, theoretically it doesn't matter which of its many points we will find. To safeguard against computational errors and to guarantee precision, at least four or five points should be plotted. Here is how we can find a point.

Step 1. Let us freely choose any value for x . We will go with $x = 4$. We will look for a point on this line with x -coordinate 4.

Step 2. To find the y -coordinate of this point, we will use the equation of the line. Once we have a value for x , we can solve for y using the equation of the line.

$$\begin{array}{l} y = ? \text{ if } x = 4 \\ x = 4 \text{ and } y = -2x + 3 \implies y = -2 \cdot 4 + 3 = -8 + 3 = -5 \end{array}$$

So if $x = 4$ and the point to be on the line, then y must be -5 . Thus we found the point $(4, -5)$ that is on this line. We repeat the process with other values for x to find other points on the line.

Let $x = 0$. We will compute the value of y .

$$\begin{array}{l} y = ? \text{ if } x = 0 \\ x = 0 \text{ and } y = -2x + 3 \implies y = -2 \cdot 0 + 3 = 0 + 3 = 3 \implies (0, 3) \end{array}$$

If $x = 0$, then $y = 3$. Thus we found the point $(0, 3)$ that is on this line.

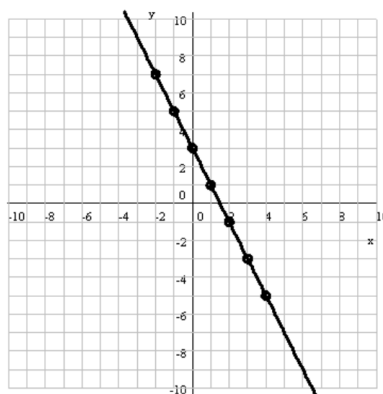
Let $x = -2$. We will compute the value of y .

$$\begin{array}{l} y = ? \text{ if } x = -2 \\ x = -2 \text{ and } y = -2x + 3 \implies y = -2(-2) + 3 = 4 + 3 = 7 \implies (-2, 7) \end{array}$$

If $x = -2$, then $y = 7$. Thus we found the point $(-2, 7)$ that is on this line.

We continue to find additional points in this manner. We plot these points and connect the dots. We organize the results in a table:

x	y	$\implies$	$P(x, y)$
-2	7		$(-2, 7)$
-1	5		$(-1, 5)$
0	3		$(0, 3)$
1	1		$(1, 1)$
2	-1		$(2, -1)$
3	-3		$(3, -3)$
4	-5		$(4, -5)$



Definition: The point where the graph intersects the x -axis is called **the x -intercept**. The point where the graph intersects the y -axis is called the **y -intercept**.

This line's y -intercept is $(0, 3)$.

Example 2. Graph the line $y = \frac{1}{2}x - 1$

Solution: We freely select any value for x . We find the y -coordinate of the point using the equation of the line.

Let $x = -4$. We will compute the value of y .

$$y = ? \text{ if } x = -4$$

$$x = -4 \text{ and } y = \frac{1}{2}x - 1 \implies y = \frac{1}{2}(-4) - 1 = -2 - 1 = -3$$

If $x = -4$, then $y = -3$. Thus we found the point $(-4, -3)$ on this line. We repeat the process with other values for x to find other points on the line.

Let $x = -3$. We will compute the value of y .

$$y = ? \text{ if } x = -3$$

$$x = -3 \text{ and } y = \frac{1}{2}x - 1 \implies y = \frac{1}{2}(-3) - 1 = -\frac{3}{2} - 1 = -\frac{5}{2}$$

If $x = -3$, then $y = -\frac{5}{2}$. Thus we found the point $(-3, -\frac{5}{2})$ on this line. Although the point we found is correct, its y -coordinate is not an integer. This makes the plotting of this point more difficult and also inaccurate. Whenever possible, we should rely on graphing lattice points. **A lattice point is a point of whose both coordinates are integers.**

Let $x = -2$. We will compute the value of y .

$$y = ? \text{ if } x = -2$$

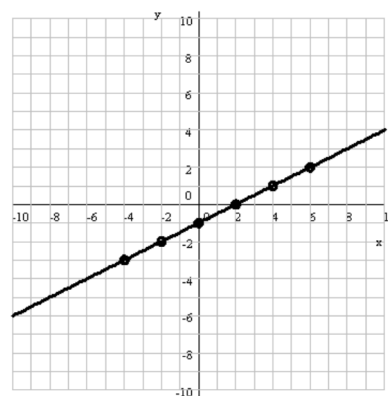
$$x = -2 \text{ and } y = \frac{1}{2}x - 1 \implies y = \frac{1}{2}(-2) - 1 = -1 - 1 = -2$$

If $x = -2$, then $y = -2$. Thus we found the point $(-2, -2)$ on this line.

We continue to find additional points in this manner. Notice that we get lattice points if we use even

numbers for x . We organize the results in a table, and then plot these points and connect the dots.

x	y	$\implies$	$P(x, y)$
-4	-3		$(-4, -3)$
-3	$-\frac{5}{2}$		$(-3, -\frac{5}{2})$
-2	-2		$(-2, -2)$
-1	$-\frac{3}{2}$		$(-1, -\frac{3}{2})$
0	-1		$(0, -1)$
1	$-\frac{1}{2}$		$(1, -\frac{1}{2})$
2	0		$(2, 0)$
4	1		$(4, 1)$
6	2		$(6, 2)$



We can see from our table that this line's x -intercept is $(2, 0)$ and y -intercept is $(0, -1)$.

Example 3. Graph the line $2x + 3y = -12$.

Solution: We freely select any value for x . We find the y -coordinate of the point using the equation of the line. We substitute the value for x and solve the equation for y .

Let $x = 0$. We will compute the value of y .

$$y = ? \text{ if } x = 0$$

$$x = 0 \text{ and } 2x + 3y = -12 \implies 2 \cdot 0 + 3y = -12 \quad \text{solve for } y$$

$$3y = -12 \quad \text{divide by 3}$$

$$y = -4$$

If $x = 0$, then $y = -4$. Thus we found the point $(0, -4)$ on this line. We repeat the process with other values for x to find other points on the line.

Let $x = 2$. We will compute the value of y .

$$y = ? \text{ if } x = 2$$

$$x = 2 \text{ and } 2x + 3y = -12 \implies 2(2) + 3y = -12 \quad \text{Solve for } y.$$

$$3y + 4 = -12 \quad \text{subtract 4}$$

$$y = -\frac{16}{3}$$

If $x = 2$, then $y = -\frac{16}{3}$. Thus we found the point $(2, -\frac{16}{3})$ on this line. Although the point we found is correct, its y -coordinate is not an integer. This makes the plotting of this point more difficult and also inaccurate. Whenever possible, we should rely on graphing lattice points. **A lattice point is a point of whose both coordinates are integers.**

Let $x = 3$. We will compute the value of y .

$$y = ? \text{ if } x = 3$$

$$x = 3 \text{ and } 2x + 3y = -12 \implies 2(3) + 3y = -12$$

$$2(3) + 3y = -12$$
$$3y + 6 = -12$$
$$3y = -18$$
$$y = -6$$

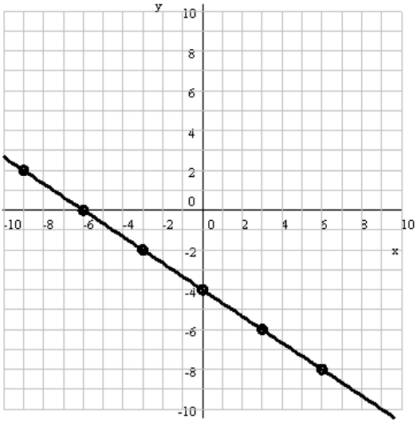
solve for y

subtract 6

divide by 3

If $x = 3$, then $y = -6$. Thus we found the point $(3, -6)$ on this line. We continue to find additional points in this manner. Notice that we get lattice points if we use numbers for x that are divisible by 3. We organize the results in a table. We plot the lattice points and connect the dots.

x	y	$\implies$	$P(x,y)$
-6	0		$(-6,0)$
-3	-2		$(-3,-2)$
0	-4		$(0,-4)$
3	-6		$(3,-6)$
-9	2		$(-9,2)$
-12	4		$(-12,4)$



This line’s x -intercept is $(-6,0)$ and y -intercept is $(0,-4)$.

Example 4. Graph the line $y = -2$

Solution: We freely select any value for x . We find the y -coordinate of the point using the equation of the line. We substitute the value for x and solve the equation for y .

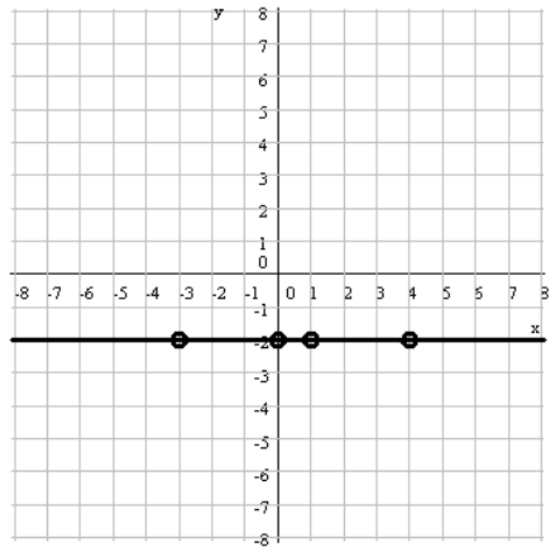
$$y = ? \text{ if } x = 0$$
$$x = 0 \text{ and } y = -2 \implies P(0,-2)$$

If $x = 0$, then $y = -2$. Thus we found the point $(0, -2)$ on this line. We repeat the process with other values for x to find other points on the line.

$$y = ? \text{ if } x = -3$$
$$x = -3 \text{ and } y = -2 \implies P(-3,-2)$$

If $x = -3$, then $y = -2$. Thus we found the point $(-3, -2)$ on this line. We continue to find additional points in this manner. It is clear that no matter what the value of x is, y will always be -2 . We organize the results in a table. We plot the lattice points and connect the dots.

x	y	$\implies$	$P(x, y)$
-3	-2		$(-3, -2)$
0	-2		$(0, -2)$
1	-2		$(1, -2)$
4	-2		$(4, -2)$



This line's y -intercept is $(0, -2)$ and it does not have an x -intercept.



Practice Problems

Graph each of the following lines.

1. $3x + 2y = 6$

4. $2x - 3y = 10$

7. $3x + 5y = -30$

2. $x = -4$

5. $y = 1$

8. $2x - y = 7$

3. $y = \frac{2}{5}x - 3$

6. $y = 3x + 6$

9. $y = \frac{1}{3}x$

9.4 Linear Inequalities

We will study solving linear inequalities. Let us first recall a few definitions.

Definition: An **inequality** is a statement in which two expressions (algebraic or numeric) are connected with one of $<$, $\leq$, $>$, or $\geq$. A **solution** of an inequality is a number that, when substituted into the variable in the inequality, makes the statement of inequality true. To **solve an inequality** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Example 1. Consider the inequality $-3x + 8 < -2(x - 1) + 5$. In case of each of the numbers given, determine whether it is a solution of the inequality or not.

a) -3 b) 4 c) 1 d) 8

Solution: a) We substitute the given number into the variable and check whether the inequality statement is true.

$$\text{LHS} = -3(-3) + 8 = 17 \text{ and } \text{RHS} = -2(-3 - 1) + 5 = -2(-4) + 5 = 8 + 5 = 13$$

The statement $17 < 13$ is false. Thus -3 is not a solution of the inequality.

b) We substitute 4 into the variable and check whether the inequality statement is true.

$$\text{LHS} = -3 \cdot 4 + 8 = -12 + 8 = -4 \text{ and } \text{RHS} = -2(4 - 1) + 5 = -2 \cdot 3 + 5 = -6 + 5 = -1$$

The statement $-4 < -1$ is true. Thus 4 is a solution of the inequality.

c) We substitute 1 into the variable and check whether the inequality statement is true.

$$\text{LHS} = -3 \cdot 1 + 8 = -3 + 8 = 5 \text{ and } \text{RHS} = -2(1 - 1) + 5 = -2 \cdot 0 + 5 = 0 + 5 = 5$$

The statement $5 < 5$ is false. Thus 1 is not a solution of the inequality.

d) We substitute 8 into the variable and check whether the inequality statement is true.

$$\text{LHS} = -3 \cdot 8 + 8 = -24 + 8 = -16 \text{ and } \text{RHS} = -2(8 - 1) + 5 = -2 \cdot 7 + 5 = -14 + 5 = -9$$

The statement $-16 < -9$ is true. Thus 8 is a solution of the inequality.

Inequalities have many, many solutions. To express these much larger solution sets, we developed interval notation. The steps of solving a linear inequality are almost identical to those of solving linear equations. However, there is a very important difference. Consider the true inequality $3 \leq 7$. If we add or subtract the same number from both sides, the inequality will remain true. The result is the same if we multiply both sides by a positive number. But if we multiply both sides of $3 \leq 7$ by -2 , we get $-6 \leq -14$, which is false.

When multiplying or dividing both sides of an inequality by a negative number, we must reverse the inequality sign.

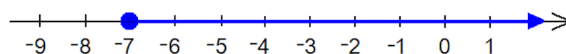
Example 2. Solve the inequality $\frac{-3x + 1}{2} \leq 11$

Solution: The steps are identical to those of solving equations, except for when multiplying or dividing by a negative number.

$$\frac{-3x + 1}{2} \leq 11 \quad \text{multiply by 2}$$

$$\begin{array}{rcl}
 -3x + 1 & \leq & 22 \quad \text{subtract 1} \\
 -3x & \leq & 21 \quad \text{divide by } -3 \implies \text{MUST reverse inequality sign} \\
 x & \geq & -7
 \end{array}$$

Notice that we reversed the inequality sign when we divided by -3 . Our solution set is the set of all numbers greater than or equal to -7 . We can present this set as an interval: $[-7, \infty)$. We can also depict the solution set on the number line:



Although we will not be asked to do so, we can check inequalities too. If we randomly pick a number inside our solution set and substitute it into the inequality, the inequality statement must be true. If we randomly pick a number outside our solution set and substitute it into the inequality, the inequality statement must be false. Finally, if substitute the boundary point (the one that separates the solutions from the non-solutions) the two sides should be equal. For an easy to substitute number inside our solution set, we choose $x = 0$. Substituting it into the inequality, we get $\frac{1}{2} \leq 11$, which is true. For a number outside of our solution set, we choose $x = -10$. This value results in $\frac{31}{2} \leq 11$, which is false. Finally, setting $x = -7$, we get the two sides equal.

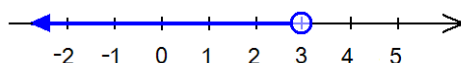
Example 3. Solve the given inequality. Present the solution set using interval notation and plot it on a number line.

$$\frac{5x+1}{4} - \frac{2x-1}{5} > 2x-3$$

Solution: We will bring both sides to the common denominator, and then clear all denominators by multiplying by that number. In fact, we can speed up the process by just multiplying both sides by the common denominator. In this case, that common denominator is 20.

$$\begin{array}{rcl}
 \frac{5x+1}{4} - \frac{2x-1}{5} & > & 2x-3 \quad \text{multiply by 20} \\
 5(5x+1) - 4(2x-1) & > & 20(2x-3) \quad \text{expand products} \\
 25x+5-8x+4 & > & 40x-60 \quad \text{combine like terms} \\
 17x+9 & > & 40x-60 \quad \text{subtract } 17x \\
 9 & > & 23x-60 \quad \text{add 60} \\
 69 & > & 23x \quad \text{divide by 23} \\
 3 & > & x
 \end{array}$$

Our solution is $(-\infty, 3)$. We depict the solution set on a number line:



In case it is not clear how we got from the first line to the second line, here is the detailed computation:

$$20 \left(\frac{5x+1}{4} - \frac{2x-1}{5} \right) = 20 \cdot \frac{5x+1}{4} - 20 \cdot \frac{2x-1}{5} = \frac{20(5x+1)}{4} - \frac{20(2x-1)}{5} = 5(5x+1) - 4(2x-1)$$

We recommend however to perform these steps on the margin or mentally.



Sample Problems

Solve each of the following inequalities. Graph the solution set.

1. $-7 > -5x + 3$

3. $5(4x - 1) - (x - 3) \geq -x - 2$

2. $3(x - 2) \leq 2x + 1$

4. $\frac{m+4}{2} - \frac{4m+3}{5} > 2$



Practice Problems

Solve each of the following inequalities. Graph the solution set.

1. $x - 17 > -4x + 3$

11. $2x + 5 > \frac{3x - 1}{2} - \frac{2x + 1}{3}$

2. $-3x + 5 \leq 12$

12. $5(x - 1) - 3(x + 1) \geq 3x - 8$

3. $5y + 3 < y - 7$

13. $3(x - 4) + 5(x + 8) \leq 2(x - 1)$

4. $-2x - (3x - 1) \geq 2(5 - 3x)$

14. $2x + 6 > \frac{3x - 1}{5} - \frac{7 - x}{3}$

5. $\frac{2}{3}x - 1 \geq x$

15. $-\frac{2}{5}(x + 1) + \frac{1}{2}(x - 4) \geq \frac{3}{10}x$

6. $5 - (3a - 2) < -2$

16. $\frac{3x - 1}{4} + \frac{8 - 4x}{3} \leq -3 - x$

7. $5x - 2 > 3(x - 1) - 4x + 1$

17. $\frac{x - 2}{5} - \frac{x}{2} < x - 16$

8. $-3(x - 2) \leq -2x + 5$

9. $3x - 2(x - 1) < -2x - 1$

18. $\frac{2x + 1}{3} + 2 \geq x + \frac{3 - x}{2}$

10. $-w + 13 \geq 2w + 1$

Problem Set 9

- Label each of the following statements as true or false.
 - $-x$ represents a negative number.
 - x^{-1} represents a negative number.
 - Every equation linear in x , or in y , or in x and y , when graphed, is a straight line.
- Express each of the given sets of numbers using interval notation.
 - $\{x : x < 6\}$
 - $\{x : x \leq 7 \text{ and } x > 3\}$
 - $\{x : x \leq 7 \text{ or } x > 3\}$
- Perform the given set operations. Present your answer in interval notation.
 - $(3, 8) \cup [5, 11]$
 - $(-\infty, 9) \cap (5, \infty)$
 - $(2, \infty) \cup [5, \infty)$
 - $(3, 8) \cap [5, 11]$
 - $(-\infty, 9) \cup (5, \infty)$
 - $(2, \infty) \cap [5, \infty)$
- Compute the greatest common factor and least common multiple of 48 and 64.
- Find the average of the first five prime numbers.
- Write each of the following percents as a reduced fraction.
 - 50%
 - 60%
 - 124%
 - 100%
- $\frac{2}{5}$ of what number is 28?
- Use conversion factors (also called unit multipliers) to convert. (100cm = 1m and 12in = 1ft)
 - 38 meters to centimeters
 - $3\frac{7}{12}$ feet to inches
- Simplify each of the following. Present your answer using only positive integer exponents.
 - $2x^{-3}$
 - $(2x)^{-3}$
 - $\frac{1}{x^{-3}}$
 - $\frac{x^{-2}}{y^{-3}}$
 - $\frac{x^{-2}}{y^3}$
 - $y^{-5}(xy^{-2})^{-3}$
- Simplify each of the following expressions.
 - $2x^3(x^5)$
 - $(2x)^{-3}(x^5)$
 - $(-a^2b)^3(a^4b)^2$
 - $(-2xy^2)^2$
 - $(2x)^3(x^5)$
 - $(-2xy^{-2})^{-2}$
 - $(-a^{-2}b)^{-3}(a^4b^{-1})^{-2}$
 - $\left(\frac{-2a^3b^{-2}}{a^5b^{-3}}\right)^0$
 - $\left(\frac{2x^2}{3y^{-3}}\right)^2\left(\frac{-2x}{y^2}\right)^{-3}$
- Simplify each of the following.
 - $3(2x - 1) - 5(2x + 1)$
 - $3((2x - 1) - 5(2x + 1))$
 - $(2x - 5)(4x^2 + 10x + 25)$
 - $\frac{2}{3} - \frac{2}{7}\left(\frac{3}{4} - \frac{2}{5}\right)$
- Solve each of the given equations. Make sure to check your solutions.
 - $\frac{5}{12}x - \frac{1}{6} = -\frac{2}{9}$
 - $\frac{x - \frac{3}{5}}{2} = -2\frac{1}{4}$
 - $\frac{2}{3}\left(x - \frac{3}{4}\right) - \frac{1}{2}\left(x + \frac{2}{3}\right) = \frac{7}{6}$
 - $4x - 2 = -\frac{1}{2}$
 - $7 - 2x = 18$

13. Graph each of the following.

a) $y = 2x - 3$ b) $2x + 3y = 6$ c) $x = -4$ d) $y = 2$

14. Solve each of the following inequalities. Graph the solution set.

a) $\frac{3x+10}{2} - \frac{2x-6}{3} < 7$ b) $x(4-x) - (2x-1)^2 \geq (1-x)(5x-2)$

15. Three vertices of rectangle $ABCD$ are given as $A(-5, -1)$, $B(3, -1)$, and $C(3, 3)$.

- a) Find the coordinates of vertex D .
- b) Find the perimeter and area of the rectangle.
- c) Find the coordinates of the midpoint of line segment AC .

16. a) Express a 20% increase and then later a 30% increase as a single change. What percentage of a change is this?

b) Express a 20% increase and then later a 30% decrease as a single change. What percentage of a change is this?

17. Three fifth of one sixth of a number is 12. Find this number.

18. Ann got a 5% raise at work, so now she is making 3360 dollars a month. What was her pay before the increase?

19. Julia is 5 years younger than her brother, Tom. How old are they if the sum of their ages is 43?

20. When we dissolve a solid or liquid in water, the resulting liquid is called a solution. Solutions have concentration: A 7% sugar solution means that the solution contains 7% sugar, and the remaining 93% is water.

- a) What is the concentration of a sugar solution if 50 gallons of the solution contains 8 gallons of sugar?
- b) How much sugar is in 20 gallons of a 15% sugar solution?
- c) We poured together 20 gallons of a 15% sugar solution with 5 gallons of a 40% sugar solution. How much sugar is in this solution?
- d*) We have 20 gallons of a 15% sugar solution. How much water should we add to this solution if we wanted to change the concentration to 10%?

21. One side of a rectangle is 6in shorter than twice the other side. Find the sides of the rectangle if its perimeter is 120in.

22. One side of a rectangle is 5 units longer than another side. If we increased both sides by 1 unit, the area of the rectangle would increase by 20 unit². Find the sides of the original rectangle.

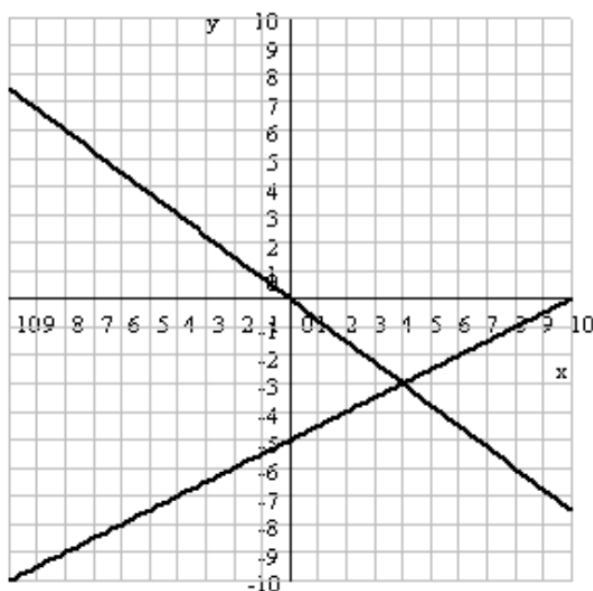
23. Banks X and Y offer slightly different business checking accounts. Bank X charges \$ 10 per month for the account and then 12 cents for every check cashed. Bank Y charges \$ 14 per month for the account and 10 cents for every check cashed.

- a) Which deal is better if we cash 85 checks per month? Explain your answer.
- b) Which deal is better if we cash 300 checks per month? Explain your answer.
- c) If we cash n checks in a month, the two offers happen to be identical. Find the value of n .

Chapter 10

10.1 Solving Systems of Linear Equations by Elimination

Recall that the graph of an equation is the set of all points whose coordinates form a solution to the equation. Consider the lines $y = -\frac{3}{4}x$ and $x - 2y = 10$. We graph these lines in the same coordinate system. What can we say about the point where the two lines intersect each other? In this particular case, we can read the coordinates of this point: $(4, -3)$.



This point is on the line $y = -\frac{3}{4}x$ if and only if its coordinates form a solution of the equation. We check:

$$\text{LHS} = -3 \quad \text{and} \quad \text{RHS} = -\frac{3}{4}(4) = -3 \quad \checkmark$$

Therefore, this point is on the line $y = -\frac{3}{4}x$.

Similarly, this point is on the line $x - 2y = 10$ if and only if its coordinates form a solution of the equation.

$$\text{LHS} = 4 - 2(-3) = 4 + 6 = 10 \quad \text{and} \quad \text{RHS} = 10 \quad \checkmark$$

Therefore, this point is also on the line $x - 2y = 10$.

Two different lines cannot have more than one point in common. Two points uniquely determine a line. Therefore, if two lines have two points in common, they are really the same line. In this sense, the intersection point is special. Algebraically, the intersection point is the unique point whose coordinates are a solution for the equations of both lines.

Definition: Two equations in x and y form a **system of equations**. The **solution(s)** of the system are the point(s) whose coordinates form a solution of both equations. To **solve** a system means to find all solutions of it.

In our example above, $(-4, 3)$ is the only solution of the system
$$\begin{cases} y = -\frac{3}{4}x \\ x - 2y = 10 \end{cases}.$$

In this case, the intersection was a point whose both coordinates happen to be integers. (Such points are called lattice points.) In case we are less lucky, we will need more precise tools for solving than graphing the two equations in the same coordinate system.

There are several algebraic methods to solve a system of linear equations.

Solving Linear Systems Using Elimination

This method is sometimes called the addition method. We will refer to it as elimination. The basic principle behind this method is the following. We can multiply both sides of an equation by the same non-zero number and this step will not change the solution set. We can also add equations: If $x = y$ and $A = B$, then $x + A = y + B$.

Example 1. Solve the given system of linear equations using elimination.
$$\begin{cases} 3x - 5y = 11 \\ 2x + 3y = 20 \end{cases}$$

Solution: We will multiply both sides of one or both equations with the goal to end up with one unknown with opposite coefficients. This is possible for both x and y . To eliminate y , we will multiply the first equation by 3 and the second equation by 5. That way the coefficients of y will be 15 and -15 . Once we add the two equations, y will be eliminated.

$$\begin{cases} 3x - 5y = 11 & \text{multiply by 3} \\ 2x + 3y = 20 & \text{multiply by 5} \end{cases} \implies \begin{cases} 9x - 15y = 33 \\ 10x + 15y = 100 \end{cases}$$

We will eliminate y by adding the two equations (i.e adding the left-hand side to left-hand side and right-hand side to right-hand side.) Then the equation will become an equation in only x , so we can solve for it.

$$\begin{array}{r} \begin{cases} 9x - 15y = 33 \\ + \quad 10x + 15y = 100 \end{cases} \\ \hline 19x \qquad = 133 \end{array} \quad \text{divide by 19} \\ x = 7$$

Now that we know the value of x , we can use either one of the two equations to find the value of y . The second equation, in its original form, will be transformed from $2x + 3y = 20$ to $2 \cdot 7 + 3y = 20$. Now we can easily solve for y .

$$\begin{array}{r} 2 \cdot 7 + 3y = 20 \\ 14 + 3y = 20 \quad \text{subtract} \\ 3y = 6 \\ y = 2 \end{array}$$

Thus, the solution of this system is $x = 7$ and $y = 2$, or, in short, $(7, 2)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $3x - 5y = 11$

$$\text{LHS} = 3 \cdot 7 - 5 \cdot 2 = 21 - 10 = 11$$

$$\text{RHS} = 11 \checkmark$$

Checking $2x + 3y = 20$

$$\text{LHS} = 2 \cdot 7 + 3 \cdot 2 = 14 + 6 = 20$$

$$\text{RHS} = 20 \checkmark$$

Therefore, our solution, $(7, 2)$ is correct.

Example 2. Solve the given system of linear equations using elimination.
$$\begin{cases} 2x - y = -19 \\ -x + 3y = 12 \end{cases}$$

Solution: We will multiply both sides of one or both equations with the goal to end up with one unknown with opposite coefficients. This is possible for both x and y . To eliminate x , we will leave the first equation as it is, and multiply the second equation by 2. This way the coefficients of x will be 2 and 2. Then, when we add the two equations, x will be eliminated, and we can solve the equation for y .

$$\begin{cases} 2x - y = -19 \\ -x + 3y = 12 \end{cases} \text{ multiply by 2} \implies \begin{cases} 2x - y = -19 \\ -2x + 6y = 24 \end{cases}$$

We now add the two equations:

$$\begin{cases} 2x - y = -19 \\ + \quad -2x + 6y = 24 \end{cases}$$

$$5y = 5 \quad \text{divide by 5}$$

$$y = 1$$

Now that we know the value of y , we can use either one of the two equations to find the value of x . The first equation, in its original form, will be transformed from $2x - y = -19$ to $2x - 1 = -19$. Now we can easily solve for x .

$$2x - 1 = -19 \quad \text{add 1}$$

$$2x = -18 \quad \text{divide by 2}$$

$$x = -9$$

Thus, the solution of this system is $x = -9$ and $y = 1$, or, in short, $(-9, 1)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $2x - y = -19$

$$\text{LHS} = 2(-9) - 1 = -18 - 1 = -19$$

$$\text{RHS} = -19 \checkmark$$

Checking $-x + 3y = 12$

$$\text{LHS} = -(-9) + 3 \cdot 1 = 9 + 3 = 12$$

$$\text{RHS} = 12 \checkmark$$

Therefore, our solution, $(-9, 1)$ is correct.

Most real-world problems boil down to systems of equations. In this sense, solving systems of equations is one of the most important tasks in problem solving.

Example 3. There is an animal farm where chickens and cows live. All together, there are 53 heads and 174 legs. How many chickens and how many cows are there on the farm?

Solution: We will denote the number of chickens by x and the number of cows by y . The first equation will express the number of heads. x many chickens come with x many heads, and y many cows come with y many heads. The second equation will express the number of legs. x many chickens come with $2x$ many legs, and y many cows come with $4y$ many heads.

$$\begin{cases} x + y = 53 \\ 2x + 4y = 174 \end{cases}$$

Before we start solving the system, let us notice that we can simplify the second equation by dividing both sides by 2. We can often make our life easier with simplifications such as this one.

$$\begin{cases} x + y = 53 \\ x + 2y = 87 \end{cases}$$

To eliminate x , we will multiply the first equation by -1 leave the second equation as is. Then we add the two equations.

$$\begin{cases} -x - y = -53 \\ + \quad x + 2y = 87 \\ \hline y = 34 \end{cases}$$

Now that we know the value of y , we use the first equation to find x .

$$\begin{aligned} x + 34 &= 53 && \text{subtract 34} \\ x &= 19 && \implies x = 19, y = 34 \end{aligned}$$

Thus we have 19 chickens and 34 cows. We check: the number of heads is $19 + 34 = 53$, and the number of legs is $2 \cdot 19 + 4 \cdot 34 = 38 + 136 = 174$. So our solution is correct.

Example 4. We invested \$10000 into two bank accounts. One account earns 14% per year, the other account earns 8% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$1238?

Solution: Let us denote the amount invested at 14% by x and the amount invested at 8% by y . The two equations will express the total amount invested, and the total interest earned.

$$\begin{aligned} x + y &= 10000 && \text{the amounts invested add up to \$10000} \\ 0.14x + 0.08y &= 1238 && \text{the interests earned add up to \$1238} \end{aligned}$$

We solve the system of equation by elimination. But let us first make the second equation simpler:

$$\begin{aligned} 0.14x + 0.08y &= 1238 && \text{multiply by 100} \\ 14x + 8y &= 123800 && \text{divide by 2} \\ 7x + 4y &= 61900 \end{aligned}$$

We now have

$$\begin{aligned}x + y &= 10000 \\7x + 4y &= 61900\end{aligned}$$

We will multiply the first equation by -4 to eliminate y . Then we add the two equations.

$$\begin{array}{r} -4x - 4y = -40000 \\ 7x + 4y = 61900 \\ \hline 3x = 21900 \quad \text{divide by 3} \\ x = 7300 \end{array}$$

Thus we invested \$7300 at 14%. The other amount can be found using the first equation:

$$\begin{aligned}7300 + y &= 10000 \\ y &= 2700\end{aligned}$$

We invested \$7300 at 14% and \$2700 at 8%. We check: the amounts add up to $\$7300 + \$2700 = \$10000$. The interest from the accounts are:

$$14\% \text{ of } 7300 \text{ is } 0.14(7300) = 1022 \text{ and } 8\% \text{ of } 2700 \text{ is } 0.08(2700) = 216$$

Since $1022 + 216 = 1238$, our solution is correct.

Example 5. We have a jar of coins, all pennies and dimes. All together, we have 372 coins, and the total value of all coins in the jar is \$20.91. How many pennies are there in the jar?

Solution: Let us denote the number of pennies by x and the number of dimes by y . The first equation will express the number of the coins. This equation is therefore $x + y = 372$. To express the value of all coins, x many pennies are worth $0.01x$ and y many dimes are worth $0.1y$. The total value of all coins is then

$$0.01x + 0.1y = 20.91$$

In order to clear the decimals, we may multiply both sides by 100. Then we have

$$x + 10y = 2091$$

Let us notice that this is the same equation that we would obtain if we expressed the value of all coins in cents and not in dollars. So our system is now

$$\begin{aligned}x + y &= 372 \\ x + 10y &= 2091\end{aligned}$$

We will eliminate x by multiplying the first equation by -1 and leaving the second equation as is. Then we add the two equations and solve for y .

$$\begin{array}{r} -x - y = -372 \\ x + 10y = 2091 \\ \hline 9y = 1719 \quad \text{divide by 9} \\ y = 191 \end{array}$$

Thus we have 191 dimes. The number of pennies can be found using any of the two equations. We will use the simplest one, the original first equation.

$$\begin{aligned}x + 191 &= 372 && \text{subtract } 191 \\x &= 181\end{aligned}$$

Thus we have 181 pennies and 191 dimes. We check: the number of all coins is $181 + 191 = 372$, and the value of the coins is $0.01 \cdot 181 + 0.1 \cdot 191 = 1.81 + 19.1 = 20.91$. Thus our solution is correct.



Practice Problems

1. Solve each of the following system of linear equations.

a) $\begin{cases} 2x - y = -8 \\ x + 2y = -9 \end{cases}$

d) $\begin{cases} \frac{1}{2}x + \frac{1}{4}y = -1 \\ \frac{1}{2}y - \frac{1}{3}x = 6 \end{cases}$

g) $\begin{cases} 3x - 2y = 2 \\ 2x + 3y = 5 \end{cases}$

b) $\begin{cases} 2(p-1) - 3(q-1) = 24 \\ p + q = -6 \end{cases}$

e) $\begin{cases} 3x - y = 4 \\ 2(y-3) = -2(x+1) \end{cases}$

h) $\begin{cases} x - 2y = 5 \\ 2x + 3y = 10 \end{cases}$

c) $\begin{cases} 3x - y = 10 \\ \frac{1}{3}x - y = 2 \end{cases}$

f) $\begin{cases} 2a + b = 17 \\ a + b = 5 \end{cases}$

i) $\begin{cases} 0.5x - 1.2y = -1.21 \\ x + 3.2y = 2.06 \end{cases}$

2. Given the equations of two straight lines, find both coordinates of all intersection points.

a) $2x - 5y = -41$ and $x + y = 4$

d) $3x - y = 9$ and $-\frac{2}{3}x + \frac{1}{2}y = -2$

3. There is an animal farm where chickens and cows live. All together, there are 60 heads and 164 legs. How many chickens and how many cows are there on the farm?

4. We invested \$6000 into two bank accounts. One account earns 7% per year, the other account earns 11% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$520?

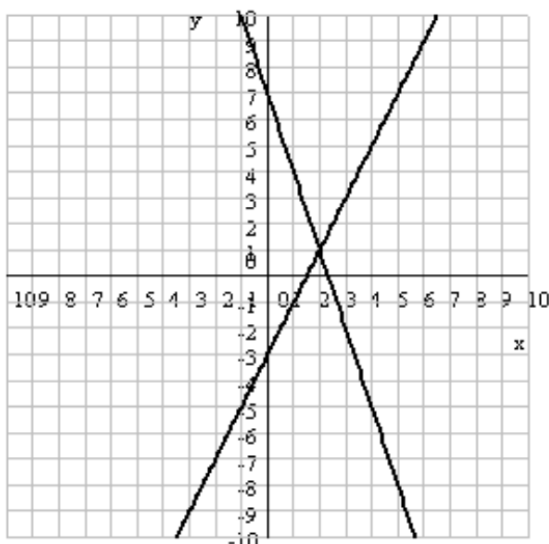
5. We have 51 coins, all dimes and quarters, in the total value of \$7.05. How many quarters and how many dimes are there?

6. We invested \$7600 in two bank accounts. One account earns 9% per year, the other account earns 13% per year. How much did we invest into each account if after the first year we have a total of \$8508 in the accounts?

10.2 Solving Systems of Linear Equations by Substitution

As we are progressing in algebra, we have learned how to solve linear equations and inequalities. The solution set of a linear equation was usually very simple: a single number. The set of all solutions of an inequality is much more complicated. We can no longer just list all elements in the solution set, and so we needed to develop new notation: interval notation.

Straight lines are even more complicated solution sets. They are solution set of a linear equation in two variables. Consider, for example, the graph of the equation $y = 2x - 3$. Every point on the graph of this line have coordinates that form a solution to the equation $y = 2x - 3$. For example, points such as $(7, 11)$ and $(10, 17)$, and $(0, -3)$ are all in this solution set, because $11 = 2 \cdot 7 - 3$, $17 = 2 \cdot 10 - 3$, and $-3 = 2 \cdot 0 - 3$. As a matter of fact, x can be any real number and then there is a unique real number that will work for y . This set is the set of points $P(x, 2x - 3)$. The most meaningful representation of this set might just be its graph.



Consider now two equations in x and y . Our example will be $y = 2x - 3$ and $y = -3x + 7$. If we graphed the two equations in the same coordinate system, we would see two straight lines. If the lines are not parallel, they intersect each other in a point. In this case, this point appears to be $(2, 1)$.

Can we use algebraic methods to see if the point $(2, 1)$ is the intersection point? The intersection point is the only point that is contained in both lines. Indeed, $1 = 2 \cdot 2 - 3$, and so $x = 2, y = 1$ is a solution of $y = 2x - 3$. Also, $1 = -3 \cdot 2 + 7$ and so $x = 2, y = 1$ is also a solution of $y = -3x + 7$. Therefore, the point $(2, 1)$ is on both lines and must be the intersection point.

We also say that the ordered pair $(2, 1)$ is a simultaneous solution of both equations.

Definition: Two equations in x and y form a **system of equations**. The **solution(s)** of the system are the point(s) whose coordinates form a solution of both equations. To **solve** a system means to find all solutions of it.

In our example above, $(2, 1)$ is the only solution of the system
$$\begin{cases} y = 2x - 3 \\ y = -3x + 7 \end{cases}.$$

In this case, the intersection was a point whose both coordinates happen to be integers. Such points are called lattice points. In case we are less lucky, we will need more precise tools for solving than graphing the two equations in the same coordinate system. There are several algebraic methods to solve a system of linear equations. Here we will explore a technique called substitution.

Solving Linear Systems using Substitution

The basic idea of this method is to absorb the information of one equation and to substitute that into the other equation, thereby reducing the number of unknowns to one.

Example 1. Solve the given system of linear equations using substitution.
$$\begin{cases} 2x - y = -19 \\ -x + 3y = 12 \end{cases}$$

Solution: We first inspect the two equations and look for coefficients such as 1 or -1 . In this case, the coefficient of y is -1 in the first equation. We solve for y in this equation. We can't solve for y and obtain a number, we can only solve for it in terms of x .

$$\begin{aligned} 2x - y &= -19 && \text{add } y \\ 2x &= y - 19 && \text{add } 19 \\ 2x + 19 &= y \end{aligned}$$

The information from the first equation can be expressed as $y = 2x + 19$. This is going to be what we substitute into the other equation by substituting $2x + 19$ into y . This way, the equation $x + 3y = 12$ will become $-x + 3(2x + 19) = 12$. This is now an equation in only one variable for which we can solve.

$$\begin{aligned} -x + 3(2x + 19) &= 12 \\ -x + 6x + 57 &= 12 \\ 5x + 57 &= 12 && \text{subtract } 57 \\ 5x &= -45 && \text{divide by } 5 \\ x &= -9 \end{aligned}$$

Now that we know the value of x , we return to what we used for substitution and get the value of the other unknown.

$$y = 2x + 19 = 2(-9) + 19 = -18 + 19 = 1$$

Therefore, the solution of this system is $x = -9$ and $y = 1$, or, in short, $(-9, 1)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $2x - y = -19$	Checking $-x + 3y = 12$
LHS = $2(-9) - 1 = -18 - 1 = -19$	LHS = $-(-9) + 3 \cdot 1 = 9 + 3 = 12$
RHS = $-19 \checkmark$	RHS = $12 \checkmark$

Therefore, our solution, $(-9, 1)$ is correct.

Of course, not all linear systems contain easy coefficients such as 1 or -1 .

Example 2. Solve the given system of linear equations.
$$\begin{cases} 3x - 5y = 11 \\ 2x + 3y = 20 \end{cases}$$

Solution: We will solve for x in the second equation and substitute the information into the first equation. First, we solve for x in $2x + 3y = 20$.

$$\begin{aligned} 2x + 3y &= 20 && \text{subtract } 3y \\ 2x &= -3y + 20 && \text{divide by } 2 \\ x &= \frac{-3y + 20}{2} \end{aligned}$$

This is the information we will substitute into the first equation. $3x - 5y = 11$ will become $3\left(\frac{-3y+20}{2}\right) - 5y = 11$. We solve this equation for y .

$$\begin{array}{rcll}
 3\left(\frac{-3y+20}{2}\right) - 5y & = & 11 & \text{multiply by 2} \\
 3(-3y+20) - 10y & = & 22 & \text{distribute 3} \\
 -9y + 60 - 10y & = & 22 & \text{combine like terms} \\
 -19y + 60 & = & 22 & \text{subtract 60} \\
 -19y & = & -38 & \text{divide by } -19 \\
 y & = & 2 &
 \end{array}$$

Now that we know that y is 2, we find x using the expression we used for the substitution.

$$x = \frac{-3y+20}{2} = \frac{-3 \cdot 2 + 20}{2} = \frac{-6 + 20}{2} = \frac{14}{2} = 7$$

Thus, the solution of this system is $x = 7$ and $y = 2$, or, in short, $(7, 2)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $3x - 5y = 11$

$$\text{LHS} = 3 \cdot 7 - 5 \cdot 2 = 21 - 10 = 11$$

$$\text{RHS} = 11 \checkmark$$

Checking $2x + 3y = 20$

$$\text{LHS} = 2 \cdot 7 + 3 \cdot 2 = 14 + 6 = 20$$

$$\text{RHS} = 20 \checkmark$$

Therefore, our solution, $(7, 2)$ is correct.

Most real-world problems boil down to systems of equations. In this sense, solving systems of equations is one of the most important tasks in problem solving.

Example 3. There is an animal farm where chickens and cows live. All together, there are 53 heads and 174 legs. How many chickens and how many cows are there on the farm?

Solution: We will denote the number of chickens by x and the number of cows by y . The first equation will express the number of heads. x many chickens come with x many heads, and y many cows come with y many heads. The second equation will express the number of legs. x many chickens come with $2x$ many legs, and y many cows come with $4y$ many heads.

$$\begin{cases} x + y = 53 \\ 2x + 4y = 174 \end{cases}$$

Before we start solving the system, let us notice that we can simplify the second equation by dividing both sides by 2. We can often make our life easier with simplifications such as this one.

$$\begin{cases} x + y = 53 \\ x + 2y = 87 \end{cases}$$

We will solve for x in the first equation and substitute that expression into the second equation.

$$x = 53 - y \implies (53 - y) + 2y = 87$$

We solve this equation for y .

$$\begin{aligned} 53 - y + 2y &= 87 \\ 53 + y &= 87 && \text{subtract 53} \\ y &= 34 \end{aligned}$$

Now that we know the value of y , we can easily find x .

$$x = 53 - 34 = 19 \implies x = 19, y = 34$$

Thus we have 19 chickens and 34 cows. We check: the number of heads is $19 + 34 = 53$, and the number of legs is $2 \cdot 19 + 4 \cdot 34 = 38 + 136 = 174$. So our solution is correct.

Example 4. We invested \$10000 into two bank accounts. One account earns 14% per year, the other account earns 8% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$1238?

Solution: Let us denote the amount invested at 14% by x and the amount invested at 8% by y . The two equations will express the total amount invested, and the total interest earned. Then the interest earned from the first account is 14% of x , and that of the second account is 8% of y . Recall that 14% of x can be written as $0.14x$ and 8% of y as $0.08y$.

$$\begin{cases} x + y = 10000 & \text{the amounts invested add up to \$10000} \\ 0.14x + 0.08y = 1238 & \text{the interests earned add up to \$1238} \end{cases}$$

We solve the system of equation by substitution, but let us first make the second equation simpler:

$$\begin{aligned} 0.14x + 0.08y &= 1238 && \text{multiply by 100} \\ 14x + 8y &= 123800 && \text{divide by 2} \\ 7x + 4y &= 61900 \end{aligned}$$

We now have

$$\begin{cases} x + y = 10000 \\ 7x + 4y = 61900 \end{cases}$$

We will solve for y in the first equation and substitute the result into the second equation.

$$x + y = 10000 \implies y = 10000 - x$$

Now the equation $7x + 4y = 61900$ becomes $7x + 4(10000 - x) = 61900$. We can solve this equation for x .

$$\begin{aligned} 7x + 4(10000 - x) &= 61900 && \text{distribute 4} \\ 7x + 40000 - 4x &= 61900 && \text{combine like terms} \\ 3x + 40000 &= 61900 && \text{subtract 40000} \\ 3x &= 21900 && \text{divide by 3} \\ x &= 7300 \end{aligned}$$

We can now easily find y using $y = 10000 - x$.

$$y = 10000 - x = y = 10000 - 7300 = 2700$$

Our solution, $x = 7300$ and $y = 2700$ means that we invested \$7300 at 14% and \$2700 at 8%. We check: the amounts add up to $\$7300 + \$2700 = \$10000$. The interest from the accounts are:

$$14\% \text{ of } 7300 \text{ is } 0.14(7300) = 1022 \text{ and } 8\% \text{ of } 2700 \text{ is } 0.08(2700) = 216$$

Since $1022 + 216 = 1238$, our solution is correct.

Example 5. We have a jar of coins, all pennies and dimes. All together, we have 372 coins, and the total value of all coins in the jar is \$20.91. How many pennies are there in the jar?

Solution: Let us denote the number of pennies by x and the number of dimes by y . The first equation will express the number of the coins. This equation is therefore $x + y = 372$. To express the value of all coins, x many pennies are worth $0.01x$ and y many dimes are worth $0.1y$. The total value of all coins is then

$$0.01x + 0.1y = 20.91$$

In order to clear the decimals, we may multiply both sides by 100. Then we have

$$x + 10y = 2091$$

Let us notice that this is the same equation that we would obtain if we expressed the value of all coins in cents and not in dollars. So our system is now

$$\begin{cases} x + y = 372 \\ x + 10y = 2091 \end{cases}$$

We solve for x in the first equation and substitute that into the second equation.

$$x + y = 372 \implies x = 372 - y$$

Now the other equation, $x + 10y = 2091$ becomes

$$\begin{array}{rcll} 372 - y + 10y & = & 2091 & \text{combine like terms} \\ 9y + 372 & = & 2091 & \text{subtract 372} \\ 9y & = & 1719 & \text{divide by 9} \\ y & = & 191 & \implies x = 372 - y = 372 - 191 = 181 \end{array}$$

The solution $x = 181$, $y = 191$ means that we have 181 pennies and 191 dimes. We check: the number of all coins is $181 + 191 = 372$, and the value of the coins is $0.01 \cdot 181 + 0.1 \cdot 191 = 1.81 + 19.1 = 20.91$. Thus our solution is correct.



Practice Problems

1. Solve each of the following system of linear equations.

$$\text{a) } \begin{cases} 3x + y = -4 \\ x - 3y = -8 \end{cases}$$

$$\text{b) } \begin{cases} 5(p-1) - 2(q-1) = 22 \\ p - q = 8 \end{cases}$$

$$\text{c) } \begin{cases} a + 3b = 10 \\ 3b - 5a = 22 \end{cases}$$

$$\text{d) } \begin{cases} \frac{1}{2}x + \frac{1}{4}y = 5 \\ \frac{1}{2}y - \frac{1}{3}x = -6 \end{cases}$$

$$\text{e) } \begin{cases} 2x - y = 1 \\ 2(y-3) = 6(x-1) \end{cases}$$

$$\text{f) } \begin{cases} 2a + 3b = -16 \\ (a+3)^2 = a^2 + 2b + 27 \end{cases}$$

$$\text{g) } \begin{cases} 2x + 3y = 3 \\ 5x - 2y = 4 \end{cases}$$

$$\text{h) } \begin{cases} 3x - 2y = -8 \\ -2x + 3y = 12 \end{cases}$$

$$\text{i) } \begin{cases} 2r - 0.5s = -1.7 \\ 1.5r + s = 0.65 \end{cases}$$

2. Given the equations of two straight lines, find both coordinates of all intersection points.

$$\text{a) } 2x - 5y = -41 \text{ and } x + y = 4$$

$$\text{d) } 5x - y = -35 \text{ and } y = -\frac{3}{4}x + \frac{1}{2}$$

$$\text{b) } x + y = -5 \text{ and } 2x - y = -7$$

$$\text{e) } y = -\frac{2}{3}x + 7 \text{ and } x + 2y = 6$$

$$\text{c) } y = \frac{3}{4}x - 2 \text{ and } 2y = x$$

3. There is an animal farm where chickens and cows live. All together, there are 52 heads and 134 legs. How many chickens and how many cows are there on the farm?

4. We invested \$9700 into two bank accounts. One account earns 7% per year, the other account earns 12% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$1004?

5. We have 54 coins, all dimes and quarters, in the total value of \$10.05. How many quarters and how many dimes are there?

6. We invested \$7800 into two bank accounts. One account earns 9% per year, the other account earns 10% per year. How much did we invest into each account if after the first year we have a total of \$8549 in the accounts?

10.3 The Zero Product Rule

What does factoring mean and why do we do it?

Definition: To factor something means to re-write it as a product.

We factor things for several reasons. For example, reducing a fraction to lowest terms involves factoring both numerator and denominator and then canceling out all common factors. Another, very important reason for factoring is the Zero Product Rule. It is our only method to solve equations of degree 2, 3, 4, and so on.

Theorem: (The Zero Product Rule) Suppose that we multiply some numbers and the result is zero. Then:

- 1.) One of the factors must be zero, and
- 2.) the values of all other factors are irrelevant.

Example 1. Solve the equation $(x - 3)(x - 7) = 0$.

Solution: If we were to expand the left-hand side, we would get a quadratic expression, and so this equation is quadratic. We will solve this equation by applying the Zero Product Rule.

We are multiplying only two factors, $x - 3$ and $x - 7$, and the result is zero. The only way this is possible if one of the two factors is zero.

Either $x - 3$ is zero (and then we can comfortably ignore the other factor, $x - 7$) and solve the linear equation $x - 3 = 0$ for x . Or, the other factor, $x - 7$ is zero (and now we don't need to worry about $x - 3$). Again, we solve the linear equation for x .

$$(x - 3)(x - 7) = 0$$

$$\begin{array}{llll} \text{Either } x - 3 & = & 0 & \text{or } x - 7 = 0 \\ x & = & 3 & \text{or } x = 7 \end{array}$$

Thus this equation has two solutions, $x_1 = 3$ and $x_2 = 7$.

Example 2. Solve the equation $(x - 1)(3x + 1)(2x - 5) = 0$

Solution: We multiplied three quantities, and the result was zero. There are only three ways that can happen:

$$\text{Either } x - 1 = 0 \quad \text{or} \quad 3x + 1 = 0 \quad \text{or} \quad 2x - 5 = 0$$

We solve each of the linear equations for x and obtain:

$$\begin{array}{llll} x = 1 & \text{or} & 3x + 1 = 0 & \text{or} & 2x - 5 = 0 \\ & & 3x = -1 & & 2x = 5 \\ & & x = -\frac{1}{3} & & x = \frac{5}{2} \end{array}$$

So this equation has three solutions: $x_1 = 1, x_2 = -\frac{1}{3},$ and $x_3 = \frac{5}{2}$.

We will leave checking to the reader. It is clear that when we substitute each solution into the original equation, a different factor will be zero, making the product zero.

Example 3. Solve the equation $5(x-2)(x+8) = 0$.

Solution: When we apply the Zero Product Rule for the three factors, the first factor, 5 will never be zero, no matter what the value of x . So, even though we have three factors, there are only two solutions of this equation, $x = 2$ and $x = -8$.

Example 4. Solve the equation $x^2(x+1)(x-3) = 0$.

Solution: We can re-write the product on the left-hand side without exponents:

$$x \cdot x \cdot (x+1)(x-3) = 0$$

When we apply the Zero Product Rule, the four factors will give us four solutions: 0, 0, -1 , and 3. It is clear that these four factors will produce only three solutions since the first and second solutions are identical. The solutions of this equation are -1 , 0, and 3.



Practice Problems

Solve each of the following equations.

1. $(x+2)(x-5) = 0$

3. $x(x+7)^2(x-10)^3 = 0$

5. $x(x+1)(x-1)(x+6) = 0$

2. $x(x-3)(x+1) = 0$

4. $5(x+2)(x-4) = 0$

6. $4x^2(2x-1)(3x+7)^2(x+8) = 0$

7. Write an equation (it can be in a factored form) with solutions 3 and -6 .

8. Write an equation (it can be in a factored form) with solutions 0, 8 and -4 .

9. Is it possible to have a seven degree equation with just one solution? Find an example.

10.4 Factoring – Part 1 (The GCF and the Difference of Squares Theorem)

Part 1 – Factoring out the GCF

Definition: To **factor** something means to re-write it as a product.

Factoring will be a very important step in solving many types of problems. Most importantly, factoring is key in solving equations of degree 2 (also called *quadratic*), degree 3 (also called *cubic*), degree 4, and so on. This is because of the zero product rule. Let us recall this rule first.

Theorem: Suppose that we multiply some numbers and the result is zero.

Then:

- 1.) One of the factors must be zero, and
- 2.) the values of all other factors are irrelevant.

This property is only true for zero. Suppose that the product of two numbers is 100. The value of the two factors depend on each other. Let's say we start with $1 \cdot 100$. If we increase the first factor, the second factor must decrease, as in $2 \cdot 50$ or $5 \cdot 20$. It is a balancing act. Only zero has the very special property that allows us to focus on only one factor while ignoring all other factors.

For example, the zero product rule can be used to solve the equation $(x + 3)(x - 1) = 0$. If two factors multiply to zero, one of the factors must be zero. So, there are only two possibilities: either $x + 3 = 0$ (and we don't need to worry about the second factor), or $x - 1 = 0$ (and we don't need to worry about the value of the first factor.) The zero product rule allowed us to trade in one quadratic (of degree 2) equation for two linear equations: $x + 3 = 0$ and $x - 1 = 0$. We solve these equations and obtain -3 and 1 as solution.

Equations with degree 2, 3, 4, 5, and beyond can be solved by the zero product rule. So, if an equation is of a degree higher than 1, we will reduce one side to zero, factor the other side and apply the zero product rule. For this reason, factoring algebraic expressions is a very important task.

There are many factoring techniques, and we will learn many of them. Different techniques work on different expressions. The process of factoring starts with inspecting the expression to decide which techniques would work. There is one exception to this: in all cases, our first step must be **factoring out the greatest common factor**. We will see later examples in which the additional techniques can not even be applied unless we factor out the greatest common factor or GCF first.

Recall the distributive law:

Axiom (The Distributive Law): For all real numbers a , b , and c ,

$$a(b + c) = ab + ac$$

Consider the expression $2(5x - 9)$. We can apply the distributive law to expand this expression:

$$2(5x - 9) = 10x - 18$$

Factoring out the greatest common factor is the reversal of this process.

Example 1. Factor out the greatest common factor in $12x - 18$.

Solution: The first step is to identify the greatest common factor or GCF. Both $12x$ and -18 are divisible by 6.

We write $6(\quad)$ and the rest is a few division problems.

We ask: 6 times what will give us $12x$? The answer is $2x$ because $6 \cdot 2x = 12x$. Similarly, 6 times what will give us -18 ? The answer is -3 . We can now write:

$$12x - 18 = \boxed{6(2x - 3)}$$

After we wrote down what we think the answer is, we need to ask two questions. Does the multiplication backward work? Did we get all common divisors out? We distribute 6 in $6(2x - 3)$ and see that we get the correct product. If we inspect $2x - 3$, we see that the two terms do not share any divisors, and so we did factor out the greatest common factor.

Example 2. Factor out the greatest common factor in $10a^3b^2 - 5ab + 30ab^3$.

Solution: We first identify the greatest common factor between the three terms in $10a^3b^2 - 5ab + 30ab^3$. The numbers multiplying the variables, also called coefficients are 10, -5 , and 30. Their greatest common factor is 5. Then we look for a -powers. The first term is divisible by a^3 , the second term by a , and the third term by a . The greatest common factor between them is a . Similarly, the greatest common factor of b^2 , b , and b^3 is b . Therefore, the greatest common factor is $5ab$. So we write $5ab(\quad)$ and the rest is three division problems.

$$10a^3b^2 - 5ab + 30ab^3 = 5ab(\quad)$$

We will need to write three terms into the parentheses. In case of all factoring, we usually ask: does the multiplication backward work? $5ab$ must be multiplied by what, so that the product is $10a^3b^2$. The answer is $2a^2b$. So now we have:

$$10a^3b^2 - 5ab + 30ab^3 = 5ab(2a^2b \quad)$$

Once we wrote down the first term, we can check whether the multiplication backwards work. For the second term, $-5ab$, nearly everything was factored out. If this happens, we are left with 1. In this case, we are left with -1 .

$$10a^3b^2 - 5ab + 30ab^3 = 5ab(2a^2b - 1 \quad)$$

For the third term, we ask: $5ab$ times what is $30ab^3$? The answer is $6b^2$, and so we have

$$10a^3b^2 - 5ab + 30ab^3 = \boxed{5ab(2a^2b - 1 + 6b^2)}$$

We ask the two questions. *Does the multiplication backward work?* and *Did we get all the common factors out?* Applying the distributive law, we see that the multiplication backward does work. Inspecting the three terms inside the parentheses, we see that they do not share any divisors. This is especially easy, given that the second term is -1 . Thus our solution is correct.

Sometimes we will need to factor out -1 from an expression. This step is usually needed when the coefficient of the highest degree term is -1 .

Example 3. Factor out -1 from $8x^5 - x^6 + 3x - 2$.

Solution: It is always a good idea to rearrange the terms by degree. Then we write $-1(\quad)$. Inside the parentheses, we write the opposite of our expression, i.e. change all signs.

$$8x^5 - x^6 + 3x - 2 = -x^6 + 8x^5 + 3x - 2 = \boxed{-1(x^6 - 8x^5 - 3x + 2)}$$

We often omit the 1 and write only $-(x^6 - 8x^5 - 3x + 2)$.

Sometimes the greatest common factor is more complicated.

Example 4. Factor out the GCF from $12a^3(a-2) - 6a^2(a-2) + 24(a-2)$.

Solution: In this case, $a-2$ is part of the GCF. We factor it out:

$$12a^3(a-2) - 6a^2(a-2) + 24(a-2) = (a-2)(12a^3 - 6a^2 + 24)$$

If we look at the expression in the second pair of parentheses, we see that there is a common factor of 6. Thus the final answer is

$$(a-2)6(2a^3 - a^2 + 4) = \boxed{6(a-2)(2a^3 - a^2 + 4)}$$

Factoring out the GCF must always be the first step in factoring. In case of the next example, this is all we need.

Example 5. Solve the equation $x^2 = 6x$

Solution: We realize that this is a quadratic equation. Therefore, we need to reduce one side to zero, factor, and apply the zero product rule. The number multiplying the variables in the highest degree term is called **the leading coefficient**. When reducing one side to zero, we should try to avoid creating negative leading coefficients. In this case, we should subtract $6x$ from both sides.

$$\begin{aligned} x^2 &= 6x && \text{subtract } 6x \\ x^2 - 6x &= 0 && \text{factor out the GCF} \\ x(x-6) &= 0 \end{aligned}$$

We apply the zero product rule to the two factors:

$$\begin{aligned} x = 0 \quad \text{or} \quad x - 6 &= 0 \\ x &= 6 \end{aligned}$$

Therefore, there are two solutions, $\boxed{0 \text{ and } 6}$. We check: if $x = 0$, then both sides are zero. If $x = 6$, then both sides are 36. Thus our solution is correct.

Example 6. Find all numbers with the following property. The number raised to the third power is five times the number we get if we double the number and then square the result.

Solution: We label this number by x . Then the number raised to the third power is x^3 . If we double the number, we get $2x$. We write the equation comparing the square of $2x$ and x^3 .

$$\begin{aligned} 5((2x)^2) &= x^3 \\ 5(4x^2) &= x^3 \\ 20x^2 &= x^3 && \text{subtract } 20x^2 \\ 0 &= x^3 - 20x^2 && \text{factor out the GCF} \\ 0 &= x^2(x-20) && \text{apply the zero product rule} \\ x = 0 \quad \text{or} \quad x &= 20 \end{aligned}$$

So there are two such numbers: $\boxed{0 \text{ and } 20}$. We check: 0 clearly works. If the number is 20, it raised to the third power is $20^3 = 8000$. If we double 20, we get 40. The square of 40 is $40^2 = 1600$, and indeed 8000 is five times 1600, thus our solution is correct.

Part 2 – The Difference of Squares Theorem

Consider a sum or a difference such as $x - 5$ or $2a + 1$. If we change both signs in such an expression, we obtain its opposite. When we change only one of the two signs, we obtain its conjugate.

Definition: Two algebraic expressions are **conjugates** if they both have two terms and are identical except for the sign of one of the terms. For example, $x - 5$ and $x + 5$ are conjugates of each other. So are $2a - 1$ and $2a + 1$.

Conjugates are very useful in algebra for all kinds of reasons. Perhaps their most important advantage is their behavior when multiplied. Consider a few examples.

$$\begin{aligned}(x - 5)(x + 5) &= x^2 - 5x + 5x - 25 = x^2 - 25 \\ (2a + 1)(2a - 1) &= 4a^2 - 2a + 2a - 1 = 4a^2 - 1\end{aligned}$$

Because of the identical terms and alternating signs, O and I from FOIL completely cancel out each other, and we are left with only two terms. In general, when we multiply conjugates $A + B$ and $A - B$, where A could be any number or expression, $(A + B)(A - B) = A^2 - AB + AB - B^2 = A^2 - B^2$ and therefore

$$(A + B)(A - B) = A^2 - B^2$$

This statement is very clear, easy to understand, and completely mechanical. But it becomes much less clear, almost mysterious when we apply the equality backwards.

Theorem: (The Difference of Squares Theorem) If A and B are any number or expression, the difference of their squares can always be factored into a pair of conjugates:

$$A^2 - B^2 = (A + B)(A - B)$$

Example 7. Completely factor each of the following.

a) $x^2 - 25$ b) $18a^2x - 8b^2x$ c) $(5m + 3n - 1)^2 - (-2m - n + 5)^2$

Solution: a) We realize that we are looking at a difference between two squares. By the difference of squares theorem, such an expression can always be factored into a pair of conjugates.

$$\begin{aligned}x^2 - 25 &= x^2 - 5^2 \\ &= (x + 5)(x - 5)\end{aligned}$$

So our answer is $\boxed{(x + 5)(x - 5)}$.

- b) The terms in $18a^2x - 8b^2x$ are not all squares. This is because there is a GCF that needs to be factored out before we could apply the difference of squares theorem.

$$\begin{aligned} 18a^2x - 8b^2x &= 2x(9a^2 - 4b^2) && \text{realize the setup for the difference of squares theorem} \\ &= 2x((3a)^2 - (2b)^2) && \text{factor via the theorem} \\ &= 2x(3a + 2b)(3a - 2b) \end{aligned}$$

To completely factor an expression, we often use several techniques. **The GCF must always be the first one** because, as this example shows, sometimes the GCF is an obstacle to applying other factoring techniques.

- c) This example is here to remind students how mechanical this theorem really is. In the statement $A^2 - B^2 = (A + B)(A - B)$, A and B could be any algebraic expressions, not just a number. For example, $A = 5m + 3n - 1$ and $B = -2m - n + 5$. If we state the difference of squares theorem with these expressions, then $A^2 - B^2 = (A + B)(A - B)$ becomes

$$\begin{aligned} (5m + 3n - 1)^2 - (-2m - n + 5)^2 &= \\ &= [(5m + 3n - 1) + (-2m - n + 5)][(5m + 3n - 1) - (-2m - n + 5)] \\ &= (5m + 3n - 1 - 2m - n + 5)(5m + 3n - 1 + 2m + n - 5) \quad \text{combine like terms} \\ &= (3m + 2n + 4)(7m + 4n - 6) \end{aligned}$$

This problem would be quite difficult to solve using other methods.

Example 8. Completely factor each of the following.

a) $x^2 + 9$ b) $x^{18}y - 25x^2y^5$ c) $80x^4 - 5$

Solution: a) The expression $x^2 + 9$ is not the difference of two squares, rather, it is their sum. **The sum of two squares can not be factored.** Therefore, the final answer is $\boxed{x^2 + 9}$.

- b) We factor out the GCF first. The GCF in $x^{18}y - 25x^2y^5$ is x^2y .

$$x^{18}y - 25x^2y^5 = x^2y(x^{16} - 25y^4)$$

We might be tempted to think that the square root of x^{16} is x^4 . This is not true, however. Recall that $(a^n)^m = a^{nm}$ and so $(x^8)^2 = x^{16}$. The square root of x^{16} is x^8 . We realize the difference of squares and then factor it into a pair of conjugates.

$$x^2y(x^{16} - 25y^4) = x^2y((x^8)^2 - (5y)^2) = \boxed{x^2y(x^8 + 5y)(x^8 - 5y)}$$

- c) We factor out the GCF first. Then, if we see that difference of squares theorem, we apply it.

$$80x^4 - 5 = 5(16x^4 - 1) = 5((4x^2)^2 - 1^2) = 5(4x^2 + 1)(4x^2 - 1)$$

We are not done yet. The expression $4x^2 + 1$ is a sum of two squares, therefore it can not be factored further. But $4x^2 - 1$ is a difference of two squares, and can be therefore factored into a pair of conjugates.

$$5(4x^2 + 1)(4x^2 - 1) = 5(4x^2 + 1)((2x)^2 - 1^2) = \boxed{5(4x^2 + 1)(2x + 1)(2x - 1)}$$

This happens when we have the difference of two quantities raised to the fourth power. The difference of squares theorem can be applied twice.

Example 9. Solve the equation $3x^3 = 12x$.

Solution: If the equation is of a degree higher than one, we need to apply the zero product rule. We reduce one side to zero, and factor the other side.

$$\begin{aligned}
 3x^3 &= 12x \\
 3x^3 - 12x &= 0 && \text{factor out the GCF} \\
 3x(x^2 - 4) &= 0 && \text{realize the difference of squares} \\
 3x(x^2 - 2^2) &= 0 && \text{apply it} \\
 3x(x+2)(x-2) &= 0
 \end{aligned}$$

We apply the zero product rule. We can treat $3x$ as two different factors or just one factor.

$$\begin{aligned}
 3x = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 2 = 0 \\
 x = 0 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 2
 \end{aligned}$$

We check our solutions.

$$\text{If } x = 0, \text{ then } \text{LHS} = 3 \cdot 0^3 = 0 \text{ and } \text{RHS} = 12 \cdot 0 = 0 \quad \checkmark.$$

$$\text{If } x = -2, \text{ then } \text{LHS} = 3 \cdot (-2)^3 = 3(-8) = -24 \text{ and } \text{RHS} = 12(-2) = -24 \quad \checkmark.$$

$$\text{If } x = 2, \text{ then } \text{LHS} = 3 \cdot 2^3 = 3 \cdot 8 = 24 \text{ and } \text{RHS} = 12 \cdot 2 = 24 \quad \checkmark.$$

Thus, our solution, $\boxed{0, -2, \text{ and } 2}$ is correct.

Example 10. If we raise a number to the third power, we get nine times the number. Find all numbers with this property.

Solution: We label the unknown number by x . The equation is then $x^3 = 9x$. We solve this equation.

$$\begin{aligned}
 x^3 &= 9x && \text{subtract } 9x \text{ to reduce one side to zero} \\
 x^3 - 9x &= 0 && \text{factor out the GCF} \\
 x(x^2 - 9) &= 0 && \text{realize the difference of two squares} \\
 x(x^2 - 3^2) &= 0 && \text{and then apply it} \\
 x(x+3)(x-3) &= 0 && \text{apply the zero product rule} \\
 x = 0 \quad \text{or} \quad x + 3 = 0 \quad \text{or} \quad x - 3 = 0 \\
 x = 0 \quad \text{or} \quad x = -3 \quad \text{or} \quad x = 3
 \end{aligned}$$

We check against the conditions stated in the problem. Clearly, 0^3 is nine times 0. Similarly, $3^3 = 27$ is nine times 3, and $(-3)^3 = -27$ is nine times -3 . Therefore, our solution, $\boxed{0, 3, \text{ and } -3}$ is correct.

The difference of squares theorem also has some practical applications to arithmetic. If we have to compute the difference of two large squares that have an easily computable sum or difference, we can apply the theorem to cut down on computation.

Example 11. Compute each of the following without using a calculator.

$$\text{a) } 52^2 - 48^2 \quad \text{b) } 100^2 - 99^2$$

Solution: a) Notice that the sum of 52 and 48 is 100. (Before the addition, take away 2 from 52 and add it to 48) and their difference is 4. Let us apply the difference of squares theorem.

$$52^2 - 48^2 = (52 + 48)(52 - 48) = 100 \cdot 4 = \boxed{400}.$$

b) The difference between 100 and 99 is 1, therefore $100^2 - 99^2$ will be the same as the sum of 100 and 99.

$$100^2 - 99^2 = (100 + 99)(100 - 99) = 199 \cdot 1 = \boxed{199}.$$



Discussion: While $x^2 - 9$ can be factored via the difference of squares theorem, $x^2 + 9$ can not be factored. How are these two facts related to the equations $x^2 = 9$ and $x^2 = -9$?



Sample Problems

1. Completely factor each of the following.

- | | | | | |
|------------------|-----------------|--------------|--------------------|-----------------|
| a) $3x - 12$ | c) $3a^2 - 12$ | e) $x^2 - 1$ | g) $-49 + x^6$ | i) $2p^4 - 162$ |
| b) $x^2 - 25y^2$ | d) $3a^2 - 12a$ | f) $x^2 + 1$ | h) $3a^3 - 27ab^2$ | j) $20x + 5x^3$ |

2. Solve each of the following equations. Make sure to check your solution.

- | | | |
|---------------------------------|---------------|-------------------|
| a) $(x - 2)(x + 3)(2x + 1) = 0$ | c) $x^2 = 9$ | e) $8x^3 = 50x^2$ |
| b) $m(m + 7) = 0$ | d) $x^2 = 9x$ | f) $8p^3 = 50p$ |

3. Word Problems

- Find all numbers that satisfy the following condition: if we square the number, we get back the same number.
- Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.



Practice Problems

1. Factor out the greatest common factor from each of the following.

- | | | |
|------------------------------------|--------------------------------|---|
| a) $10a^2b^2 - 15ab^3 + 25a^2b^3c$ | c) $a^2 - a^3 + a^4$ | e) $x^5 - 2x^4 + 4x^3$ |
| b) $6x^3 - 3x^2 - 15x^4$ | d) $6a^2b + 12a^3b - 30a^3b^2$ | f) $3xy(a - 3) + 8t(a - 3) - 200x^5(a - 3)$ |

2. Factor out -1 from each of the following.

- | | | |
|--------------------|--------------------|--------------------|
| a) $x^3 - x^5 + 2$ | b) $-x^2 + 3x - 1$ | c) $-x^2 + 3x - 5$ |
|--------------------|--------------------|--------------------|

3. Factor each of the following via the difference of squares theorem.

- | | | | |
|---------------|----------------|--------------|----------------|
| a) $x^2 - 49$ | b) $9a^2 - 25$ | c) $x^2 - 1$ | d) $y^6 - 100$ |
|---------------|----------------|--------------|----------------|

4. Completely factor each of the following.

- | | | | |
|------------------|----------------------------|-------------------------|---------------------|
| a) $5a^2 - 45$ | e) $x^3 - x$ | i) $a^2 - (x - 1)^2$ | m) $-2x^4 + 162$ |
| b) $2m^4 - 2n^4$ | f) $5x^3y^4 - 80x^3$ | j) $-16 + a^4$ | n) $5a^3b^2 - 15ab$ |
| c) $2x^4 - 8x^2$ | g) $a^2(x - 1) - 9(x - 1)$ | k) $600ab^2 - 6ab^4$ | |
| d) $3a - 12ab^2$ | h) $18a^2x^2 - 50x^2$ | l) $36x^2y^3 + 4x^4y^3$ | |

5. Solve each of the following equations. Make sure to check your solutions.

$$\begin{array}{llll} \text{a) } (w+5)(w-1) = 0 & \text{c) } 2(x-2)(x+3) = 0 & \text{e) } x^2 + 6x = 0 & \text{g) } 3x^3 = 75x \\ \text{b) } x(x-2)(x+3) = 0 & \text{d) } x^2 = 4 & \text{f) } 3x^3 = 75x^2 & \text{h) } 45a^4 = 20a^2 \end{array}$$

6. Find all numbers satisfying the given conditions.

- a) The cube of the number is three times as large as the square of twice the number.
- b) The cube of the number is five times as large as the opposite of the square of the number.
- c) The cube of a number is the same as the four times the number.

7. Use the difference of squares theorem to compute the following without a calculator.

$$\text{a) } 51^2 - 49^2 \quad \text{b) } 2001^2 - 2000^2 \quad \text{c) } 120^2 - 20^2 \quad \text{d) } 28^2 - 22^2$$

Problem Set 10

1. Label each of the following statements as true or false.

- a) For any sets, A and B , $A \cap B \subseteq A$.
- b) For any sets A and B , $A \subseteq A \cup B$.
- c) For any sets A and B , $A \cup B \subseteq A \cap B$.
- d) For any sets A and B , if $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- e) For any sets A , $\emptyset \subseteq A$.
- f) For any sets A , B , and C , if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- g) For any sets A , B , and C , $(A \cap B) \cup C = A \cap (B \cup C)$.
- h) For any sets A , B , and C , $(A \cap B) \cap C = A \cap (B \cap C)$.
- i) For any sets A , B , and C , $(A \cup B) \cup C = A \cup (B \cup C)$.

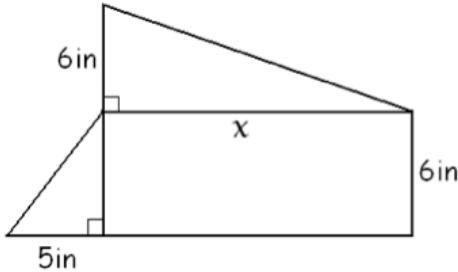
2. Suppose that $A = \{1, 2, 3, 4, 5\}$. List all two-element subsets of A .

3. Suppose that $P = \{1, 3, 6, 10\}$, and $S = \{1, 2, 5, 6, 9, 10\}$, and $T = \{3, 5, 6, 7, 8, 9, 10\}$

- a) Draw a Venn-diagram depicting P , S , and T .
- b) Find each of the following.
 - i) $S \cap P \cap T$ ii) $S \cup (P \cap T)$ iii) $(S \cup P) \cap T$ iv) $P \cup (S \cap T)$
- c) Find an operation or (operations) on P , S , and/or T so that the result is the set $\{1, 6, 10\}$.
- d) True or false? $P \subseteq S \cup T$
- e) True or false? $S \cap T \subseteq P$

4. Suppose that $U = \{1, 2, 3, \dots, 14, 15\}$. Find each of the following sets.

- a) $A = \{x \in U : x \text{ is divisible by } 3 \text{ or } x \geq 11\}$
- b) $A = \{x \in U : x \text{ is divisible by } 3 \text{ and } x \geq 11\}$

5. Suppose that $T = \{n \in \mathbb{Z} : n \text{ is divisible by } 3\}$, $S = \{n \in \mathbb{Z} : n \text{ is divisible by } 6\}$, and $E = \{n \in \mathbb{Z} : n \text{ is divisible by } 2\}$. Label each of the statements as true or false.
- a) $T \cap E = S$ b) $E \subseteq S$ c) $S \subseteq T$ d) $T \cup E = S$
6. Find the prime factorization for each of the following numbers.
- a) 1200 b) 2016 c) 1820
7. Find the prime factorization for x if
- a) $x = 30^{20}$ c) $x = 15!$ ($15! = 15 \cdot 14 \cdot 13 \cdot \dots \cdot 2 \cdot 1$)
- b) $x = 72^{10}$
8. Find the greatest prime factor of $20!$.
9. Find the value of x if we know that the object shown on the picture has area 195in^2 . (Hint: Set up and solve an equation expressing the area!)
- 
10. Label each of the following statements as true or false.
- a) Every integer greater than 1 is either a prime or a composite.
- b) If n^2 is divisible by 12, then n^2 is divisible by 36.
- c) If n^3 is divisible by 2, then it is also divisible by 8.
- d) The sum of two prime numbers is never a prime.
- e) If a number n is divisible by 10 and by 12, then it is also divisible by 120.
- f) There is a number less than 100 that is divisible by all of 1, 2, 3, 4, 5, and 6.
- g) If n is a natural number, the all exponents in the prime factorization of n^3 are divisible by 3.
11. Perform the indicated operations.
- a) $\frac{2}{3} - \frac{1}{4} \div \frac{5}{8}$ b) $\frac{-\frac{4}{5} - \frac{3}{4}}{\frac{1}{2} - \frac{1}{5}}$ c) $2 - 3\frac{1}{3} \div 2\frac{1}{7}$ d) $-\left(-\frac{2}{3}\right)^2 - \left(-\frac{1}{2}\right)^2$
12. Evaluate each of the given numerical expressions.
- a) $5 - 2(20 - 12 \div 2(-3^2 + 15))$ b) $\sqrt{2\sqrt{3 \cdot 7 - 5} - \sqrt{2^3 - (-1)^3} - 1}$ c) $\frac{-1^2 - 2(-3^2 - 2(3^2 - 7))}{-2^2 - 1}$
13. Suppose that $x = 2.5 \cdot 10^9$ and $y = 1.6 \cdot 10^3$. Find each of the following. Express your answer using scientific notation.
- a) x^2 b) xy c) $\frac{x}{y}$ d) $\sqrt{x}$ e) $\frac{1}{x}$ d) $\frac{y}{x}$
14. a) Sally is making \$2400 a month. How much would she make after a promotion that comes with a 10% raise?
- b) The price of a book was raised from 120 dollars to 150 dollars. What percent of a change was this?
- c) For the next sale, the book's price was lowered from 150 dollars back to 120 dollars. What percent of a change was this?
- d) The population of a town has increased by 18%. If the town's population is 76 700 after this increase, what was it before the increase?

15. Evaluate the algebraic expression $\frac{-2x^2 + 9x - 10}{2x - 5}$ if

- a) $x = 3$ b) $x = -2$ c) $x = \frac{1}{2}$ d) $x = -\frac{3}{2}$ e) $\frac{5}{2}$

16. Simplify each of the following. Assume that all variables represent non-zero numbers.

- a) $(-x)^3(-x)^{-8}$ f) $2a - 3 - (5a - 3)$ l) $(-3x + 2)^2$
 b) $(x(-x)^3)^{-4}$ g) $(2x - 5)(x + 2)$ m) $5x - (3(4 - (x - 3(2x - 1)) + x) - x) - 1$
 c) $(-x(-x)^{-3})^{-4}$ h) $-3(2x - 5) - 4(-x + 2)$ n) $5(x - (3(4 - (x - 3(2x - 1)) + x) - x)) - 1$
 d) $\frac{-xy^{-4}}{(-xy)^{-5}}$ i) $(-x + 4)(5x + 1)$ o) $x^2 - (x - 3)^2$
 e) $2 \cdot 3^{-2} - 6^{-1}$ j) $(5x + 7)(5x - 7)$ p) $(x - 2)^3$
 k) $-3a - 2(5a - 3(4a - b))$

17. Factor out the GCF (greatest common factor) or -1 in each of the following expressions.

- a) $18x - 3x^2$ c) $4a^2b^3 - 20ab^4 + 12a^2b^2$
 b) $-x^2 + x - 3$ d) $4a^2(2a - 1) - 3a(2a - 1) + 5(2a - 1)$

18. Completely factor each of the following.

- a) $x^2 - 36$ c) $6x^2 - 15x + 8x - 20$ e) $2a^2x^5 - 32a^2x$
 b) $25p^2 - 49q^{10}$ d) $12x^3 + 3x$ f) $3x(x - 2) - 12(x - 2)$

19. Solve each of the given equations.

- a) $-3x + 7 = 16$ g) $3(2x - 1) - 2(5x - 7) = -5$ n) $\frac{\frac{3x - 1}{4} + 9}{\frac{5}{2}} - 7 = -3$
 b) $\frac{3}{4}x - \frac{1}{3} = -\frac{1}{30}$ h) $\frac{2}{3}\left(x - \frac{3}{4}\right) - \frac{1}{6}\left(2x + \frac{3}{5}\right) = -\frac{13}{30}$ o) $5(3x - 1) - 4(4x - 3) = 7$
 c) $\frac{a}{-3} + 7 = -1$ i) $2x - (3x - 4(2x - 1)) = 17$ p) $2(x + 3)(x - 1) = 0$
 d) $\frac{2}{3}\left(x - \frac{1}{2}\right) = -\frac{5}{6}$ j) $-\frac{3}{4}x - \frac{5}{6} = -x + \frac{13}{6}$ q) $x(x + 3)(x - 1) = 0$
 e) $\frac{a + 7}{-3} = -1$ k) $3x - 8 - 5(2x - 8) = -3$ r) $3x^5 = 12x^4$
 f) $5x - 8 = -8$ l) $(2x - 3)(x + 5) = 2x(x + 3)$ s) $3x^5 = 12x^3$
 m) $(2x - 5)^2 = 5x^2 - (x + 1)(x - 3)$ t) $x^2 + 2x = 0$

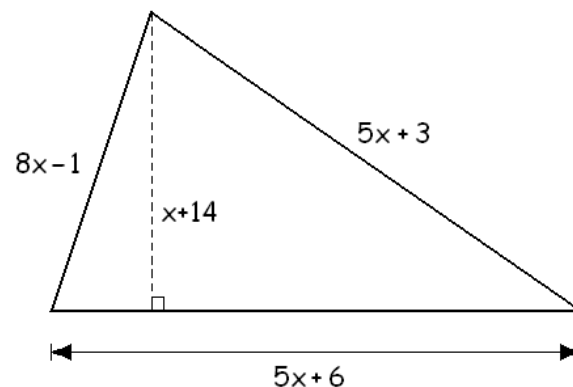
20. Solve each of the following system of equations.

- a) $\begin{cases} 2x - y = -1 \\ 5x - 2y = 2 \end{cases}$ c) $\begin{cases} 3x + 5y = -20 \\ \frac{1}{3}x - \frac{1}{2}y = 2 \end{cases}$ e) $\begin{cases} 3x + 5y = -4 \\ 4x - 3y = -15 \end{cases}$
 b) $\begin{cases} 2x - 5y = -9 \\ x - y = -3 \end{cases}$ d) $\begin{cases} (x - 2)^2 + (y - 5)^2 = (x + 1)^2 + (y - 8)^2 \\ \frac{x - 1}{2} = \frac{y - 1}{5} \end{cases}$ f) $\begin{cases} 2x - y = 15 \\ x + 3y = -17 \end{cases}$

21. Solve each of the following application problems. In each case, set up and solve an equation or a system of equations.

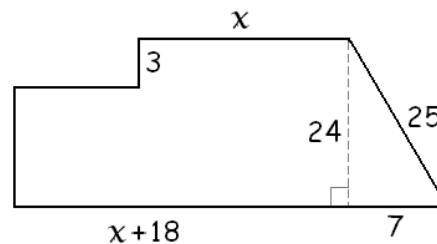
- The sum of three consecutive multiples of 4 is -24 . Find these numbers.
- One side of a rectangle is five inches shorter than four times another side. Find the sides of the rectangle if its perimeter is 110in.
- There is a farm where chickens and cows live. There are 93 heads and 300 legs. How many chickens, how many cows?

- Consider the triangle shown on the picture. Find the value of x , given that the perimeter of the triangle is 98 unit.
- Samantha is asked about her age. She answers as follows. *"My age is ten less than twice the age of my younger brother. The sum of our ages is 32"*. How old is Samantha?



- The local library held a sale of their old books. Each softcover book was priced at 3 dollars, and hardcover books costed 5 dollars each. We purchased a whole lot of books! The number of softcover books we got was eight less than three times the number of hardcover books. How many softcover books did we buy if we paid a total of 74 dollars?
- Find all numbers with the following property. If we double the number and subtract that from the square of the number, the result is exactly seven times the original number.
- Find all numbers with the following property. If we double the number and subtract that from the cube of the number, the result is exactly seven times the original number.
- The first night in a hotel costs 38 dollars, and each additional night cost 25 dollars. How long did we stay in the hotel if the bill was 163 dollars?
- If we increase the side of a square by 1 meter, its area will increase by 23 square-meters. How long are the sides of the square before the increase?
- We have a jar of coins, all quarters and dimes. All together, we have 289 coins, and the total value of all coins in the jar is \$54.55. How many quarters are there in the jar?

- Consider the object shown on the picture. Angles that look like right angles are right angles. Find the perimeter of the object, given that its area is 1182 unit^2 .



Chapter 11

11.1 Completing the Square - Part 1

In this section, we will be factoring quadratic expressions whose leading coefficient is 1 and linear coefficient is even. Factoring by completing the square is an extremely powerful factoring technique. We will see later that this is the only method that does not break down once numbers stop being "nice". Recall that a **complete square** is the square of a sum or a difference. For example, the expressions $(2a - 3)^2$ and $(x + 7)^2$ are complete squares.

Example 1. Factor $-4x + x^2 - 21$ by completing the square.

Solution: Step 1. We re-arrange the terms by decreasing order of degree.

$$-4x + x^2 - 21 = x^2 - 4x - 21$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. The linear coefficient is the number multiplying x , **sign included**. In our example, the magic number is $\frac{-4}{2} = -2$. We do not write this line in the main computation.

Step 2B. We place an x in front of the magic number, and square the expression we obtained. Work out this computation on the margin, not in the main computation.

$$(x - 2)^2 = (x - 2)(x - 2) = x^2 - 2x - 2x + 4 = x^2 - 4x + 4$$

Step 2C. We write the "helper line" $(x - 2)^2 = x^2 - 4x + 4$ in the upper right hand side of the paper. We will use it twice. Our computation so far looks like this:

$$-4x + x^2 - 21 = x^2 - 4x - 21 \qquad (x - 2)^2 = x^2 - 4x \boxed{+4}$$

Step 3. The smuggling step. What we have achieved in Step 2, is to have found the only perfect square that begins with the same two terms, $x^2 - 4x$ as our expression to be factored. We can see that the last term, $+4$ is missing. We complete the square as follows.

Step 3A. Write down our expression with one modification: we leave a gap between the second and third terms.

$$x^2 - 4x \qquad - 21$$

Step 3B. We add zero to the expression by adding and then immediately subtracting 4 into the gap.

$$x^2 - 4x \quad +4 - 4 - 21$$

Step 4. We have obtained five terms. We re-write the first three terms as a perfect square (the second time we used the helper line) and combine the last two terms.

$$x^2 - 4x - 21 = \underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4 - 21 = (x - 2)^2 - 25$$

Step 5. We re-write the last number as a square.

$$(x-2)^2 - 25 = (x-2)^2 - 5^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-2)^2 - 5^2 = (x-2+5)(x-2-5)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x-2+5)(x-2-5) = \boxed{(x+3)(x-7)}$$

Step 8. We check our result by multiplication.

$$(x+3)(x-7) = x^2 - 7x + 3x + 21 = x^2 - 4x + 21$$

Thus our result, $(x+3)(x-7)$ is correct.

The entire computation should look like this:

$$-4x + x^2 - 21 =$$

$$\begin{aligned} &= x^2 - 4x - 21 \\ &= \underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4 - 21 \\ &= (x-2)^2 - 25 \\ &= (x-2)^2 - 5^2 \\ &= (x-2+5)(x-2-5) \\ &= \boxed{(x+3)(x-7)} \end{aligned}$$

$$(x-2)^2 = x^2 - 4x + \boxed{4}$$

$$\text{We check: } (x+3)(x-7) = x^2 - 7x + 3x - 21 = x^2 - 4x - 21$$

Example 2. Factor $32 - 18x + x^2$ by completing the square.

Solution: Step 1. We re-arrange the terms by decreasing order of degree.

$$32 - 18x + x^2 = x^2 - 18x + 32$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. (The linear coefficient is the number multiplying x , sign included.) In our example, the magic number is $\frac{-18}{2} = -9$.

Step 2B. We place an x in front of the magic number, and square the expression we obtained. Do not write this computation in the main computation.

$$(x-9)^2 = (x-9)(x-9) = x^2 - 9x - 9x + 81 = x^2 - 18x + 81$$

Step 2C. We write the "helper line" $(x-9)^2 = x^2 - 18x + 81$ in the upper right hand side of the paper.

$$x^2 - 18x + 32 \qquad (x-9)^2 = x^2 - 18x + \boxed{81}$$

Step 3. (The smuggling step.) What we have achieved in Step 3, is to have found the only perfect square that begins with the same two terms, $x^2 - 18x$ as our expression to be factored. We can see that the last term, $+81$ is missing. We complete the square by writing down our expression with a gap between the second and third terms, and then adding zero to the expression by adding and then immediately subtracting 81 in the gap.

$$x^2 - 18x + 32 = x^2 - 18x + 81 - 81 + 32$$

Step 4. We obtained five terms. We re-write the first three terms as a perfect square and combine the last two terms.

$$x^2 - 18x + 32 = \underbrace{x^2 - 18x + 81}_{(x-9)^2} - 81 + 32 = (x-9)^2 - 49$$

Step 5. We re-write the last number as a square.

$$(x-9)^2 - 49 = (x-9)^2 - 7^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-9)^2 - 7^2 = (x-9+7)(x-9-7)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x-9+7)(x-9-7) = (x-2)(x-16)$$

Step 8. We check back by multiplication. (See below.)

The entire computation should look like this:

$$32 - 18x + x^2 =$$

$$= x^2 - 18x + 32$$

$$(x-9)^2 = x^2 - 18x + \boxed{+ 81}$$

$$= \underbrace{x^2 - 18x + 81}_{(x-9)^2} - 81 + 32$$

$$= (x-9)^2 - 49$$

$$= (x-9)^2 - 7^2$$

$$= (x-9+7)(x-9-7)$$

$$= \boxed{(x-2)(x-16)}$$

We check: $(x-2)(x-16) = x^2 - 2x - 16x + 32 = x^2 - 18x + 32$

Thus our result, $(x-2)(x-16)$ is correct.

Example 3. Factor $28x + x^2 - 1173$ by completing the square.

Solution: We first rearrange the terms by degrees.

$$28x + x^2 - 1173 =$$

$$= x^2 + 28x - 1173$$

$$\text{half of the linear coefficient is } \frac{28}{2} = 14$$

We work out $(x + 14)^2 = x^2 + 28x + 196$ on the margin. So we know to smuggle in 196.

$$= x^2 + 28x - 1173$$

$$(x + 14)^2 = x^2 + 28x + \boxed{+ 196}$$

$$= \underbrace{x^2 + 28x + 196}_{(x+14)^2} - 196 - 1173$$

realize complete square, combine like terms

$$= (x + 14)^2 - 1369$$

$$\text{from calculator, } \sqrt{1369} = 37$$

$$= (x + 14)^2 - 37^2$$

difference of squares theorem

$$= (x + 14 + 37)(x + 14 - 37)$$

combine like terms

$$= \boxed{(x + 51)(x - 23)}$$

We check by multiplication: $(x + 51)(x - 23) = x^2 - 23x + 51x - 1173 = x^2 + 28x - 1173$

Thus our result, $(x + 51)(x - 23)$ is correct.

Example 4. Factor $17 - 2a + a^2$ by completing the square.

Solution: We first rearrange the terms by degree.

$$17 - 2a + a^2 =$$

$$= 17 - 2a + a^2 \quad \text{rearrange terms}$$

$$= a^2 - 2a + 17 \quad \text{the "magic number" is } \frac{-2}{2} = -1$$

We work out $(a - 1)^2 = a^2 - 2a + 1$ on the margin.

$$= a^2 - 2a + 17$$

$$(a - 1)^2 = a^2 - 2a + \boxed{+ 1}, \text{ so we smuggle in 1}$$

$$= \underbrace{a^2 - 2a + 1}_{(a-1)^2} - 1 + 17$$

realize complete square, combine like terms

$$= (a - 1)^2 + 16$$

We can not apply the difference of squares theorem, since 16 is added, not subtracted. **The sum of squares can not be factored**, and so the expression $a^2 - 2a + 17$ can not be factored.



Practice Problems

Factor each of the following by completing the square.

1. $x^2 - 10x + 21$

6. $b^2 - 10b + 26$

11. $m^2 - 42m + 432$

16. $q^2 - 2q - 48$

2. $x^2 - 6x + 8$

7. $3 + x^2 - 4x$

12. $x^2 - 50x + 525$

17. $x^2 - 18x + 81$

3. $22y + y^2 + 105$

8. $d^2 + 2d + 2$

13. $10y + y^2 - 375$

18. $t^2 - 36t - 4437$

4. $b^2 - 4b - 45$

9. $6x + x^2 - 432$

14. $x^2 - 40x + 336$

19. $x^2 - 46x + 360$

5. $14a + a^2 - 51$

10. $x^2 - 14x + 58$

15. $x^2 - 6x + 25$

20. $14q + q^2 - 2352$

11.2 Completing the Square – Part 2

Factoring by completing the square is an extremely powerful factoring technique. We will see later that this is the only method that does not break down once numbers stop being "nice". In this section, we will see expressions in which the leading coefficient is not 1, but it can be easily factored out.

Example 1. Factor $18x - 3x^2 + 165$ by completing the square.

Solution: Step 1. We re-arrange the terms by decreasing order of degree.

$$18x - 3x^2 + 165 = -3x^2 + 18x + 165$$

Step 2. We factor out the greatest common factor.

$$-3x^2 + 18x + 165 = -3(x^2 - 6x - 55)$$

Step 3. We factor the expression within the parentheses by completing the square.

$$\begin{aligned} -3(x^2 - 6x - 55) &= (x - 3)^2 = x^2 - 6x + 9 \\ &= -3\left(\underbrace{x^2 - 6x + 9}_{(x-3)^2} - 9 - 55\right) \\ &= -3\left((x - 3)^2 - 64\right) \\ &= -3\left((x - 3)^2 - 8^2\right) \\ &= -3(x - 3 + 8)(x - 3 - 8) = \boxed{-3(x + 5)(x - 11)} \end{aligned}$$

Step 4. We check our result by multiplication.

$$-3(x + 5)(x - 11) = -3(x^2 - 11x + 5x - 55) = -3(x^2 - 6x - 55) = -3x^2 + 18x + 165$$

Thus our result, $-3(x + 5)(x - 11)$ is correct.

Example 2. Factor $267x^2 - 48x^3 + 3x^4$ by completing the square.

Solution: We first rearrange the terms by degree.

$$\begin{aligned} 267x^2 - 48x^3 + 3x^4 &= \\ &= 3x^4 - 48x^3 + 267x^2 && \text{factor out } 3x^2 \\ &= 3x^2(x^2 - 16x + 89) && (x - 8)^2 = x^2 - 16x + \boxed{64} \\ &= 3x^2\left(\underbrace{x^2 - 16x + 64}_{(x-8)^2} - 64 + 89\right) && \text{realize complete square, combine like terms} \\ &= 3x^2((x - 8)^2 + 25) \end{aligned}$$

We can not apply the difference of squares theorem, since 25 is added, not subtracted. The sum of squares cannot be factored. Therefore, the completely factored form of the expression

is $\boxed{3x^2(x^2 - 16x + 89)}$

Example 3. Factor $5x^2 - 240x + 2160$ by completing the square.

Solution: We first factor out 5.

$$\begin{aligned}
 5x^2 - 240x + 2160 &= 5(x^2 - 48x + 432) & (x - 24)^2 &= x^2 - 48x + \boxed{+ 576} \\
 &= 5\left(\underbrace{x^2 - 48x + 576}_{(x-24)^2} - 576 + 432\right) \\
 &= 5\left((x - 24)^2 - 144\right) \\
 &= 5\left((x - 24)^2 - 12^2\right) \\
 &= 5(x - 24 + 12)(x - 24 - 12) \\
 &= \boxed{5(x - 12)(x - 36)}
 \end{aligned}$$

We check: $5(x - 12)(x - 36) = 5(x^2 - 48x + 432) = 5x^2 - 240x + 2160$.

Thus our result, $5(x - 12)(x - 36)$ is correct.

Example 4. Factor $9 - y^2 - 8y$ by completing the square.

Solution: We first rearrange the terms by degree and then factor out the leading coefficient.

$$\begin{aligned}
 9 - y^2 - 8y &= -y^2 - 8y + 9 \\
 &= -1(y^2 + 8y - 9) & (y + 4)^2 &= y^2 + 8y + \boxed{+ 16} \\
 &= -\left(\underbrace{y^2 + 8y + 16}_{(y+4)^2} - 16 - 9\right) & \text{realize complete square, combine like terms} \\
 &= -\left((y + 4)^2 - 25\right) & \text{re-write 25 as a square} \\
 &= -\left((y + 4)^2 - 5^2\right) & \text{factor via the difference of squares theorem} \\
 &= -(y + 4 + 5)(y + 4 - 5) & \text{combine like terms, drop extra parentheses} \\
 &= \boxed{-(y + 9)(y - 1)}
 \end{aligned}$$

We check: $-(y + 9)(y - 1) = -(y^2 - y + 9y - 9) = -(y^2 + 8y - 9) = -y^2 - 8y + 9$

Thus our result, $-(y + 9)(y - 1)$ is correct.



Practice Problems

Completely factor each of the following by completing the square.

1. $4x + 2x^2 - 30$

5. $18c - 24c^2 + 6c^3$

9. $10abc - 600ac + 5ab^2c$

2. $70a^2 - 255a + 5a^3$

6. $-2d - 2d^2 - d^3$

10. $70y^3 + 24y^4 + 2y^5$

3. $78b^2 - 30b^3 + 3b^4$

7. $432 - x^2 - 6x$

11. $18x^2y^2 - 216x^2y + 3x^2y^3$

4. $32x + 2x^2 - 594$

8. $x^2 - 14x + 58$

12. $1000x - 50x^2 - 5x^3$

11.3 Summation

Carl Friedrich Gauss (1777-1855) is probably one of the greatest mathematicians of all time. He made major contributions to most areas within mathematics. The following story is from when Gauss was still at primary school. One day Gauss' teacher asked his class to add together all the numbers from 1 to 100, assuming that this task would occupy them for quite a while. He was shocked when young Gauss, after a few seconds thought, wrote down the answer 5050. The teacher couldn't understand how his pupil had calculated the sum so quickly in his head, but the eight year old Gauss pointed out that the problem was actually quite simple.

[Solution - YouTube Link](#)

Here is the question: $1 + 2 + 3 + \dots + 100 = x$

We repeat this line, but this time, backward: $100 + 99 + 98 + \dots + 1 = x$

And we will add the two lines - which will be twice the desired sum.

$$\begin{array}{ccccccccccc} 1 & + & 2 & + & 3 & + & . & . & . & + & 100 & = & x \\ 100 & + & 99 & + & 98 & + & . & . & . & + & 1 & = & x \end{array}$$

But instead of adding row 1 and then row 2, we add the numbers column by column. The sum in the first column is $1 + 100 = 101$. The sum of the second column is $2 + 99 = 101$. The sum of the third column is $3 + 98 = 101$. And so on, the sum of each column is 101 because as we step to the right, the number in the first row increase by 1 and the number in the second row decrease by 1. Thus the sum remains 101.

$$\begin{array}{ccccccccccc} 1 & + & 2 & + & 3 & + & . & . & . & + & 100 & = & x \\ + & 100 & + & 99 & + & 98 & + & . & . & . & + & 1 & = & x \\ \hline 101 & + & 101 & + & 101 & + & . & . & . & + & 101 & = & 2x \end{array}$$

When we add the same number to itself repeatedly, that can be re-written as multiplication. We have 100 columns, so we added 101 to itself 100 times.

$$\begin{aligned} 100 \cdot 101 &= 2x \\ 10100 &= 2x \\ 5050 &= x \end{aligned}$$

This method can be applied to adding long sums, as long as the numbers increase or decrease by the same amount.

Example 1. Find each of the given sums.

a) $4 + 8 + 12 + \dots + 600$ b) $3 + 6 + 9 + \dots + 2019$

Solution: a) We will apply Gauss's method.

$$\begin{array}{ccccccccccc} 4 & + & 8 & + & 12 & + & . & . & . & + & 600 & = & x \\ + & 600 & + & 596 & + & 592 & + & . & . & . & + & 4 & = & x \\ \hline 604 & + & 604 & + & 604 & + & . & . & . & + & 600 & = & 2x \end{array}$$

The easy question is: what is the sum in each column? Clearly 604. It takes a bit more work to figure out how many columns are there. We label the numbers in the first row as 1, 2, 3, etc

$$\begin{array}{ccccccc} 1 & & 2 & & 3 & & ? \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 4 & + & 8 & + & 12 & + & . & . & . & + & 600 & = & x \end{array}$$

and see that the last number will have to get the label 150.

So, we have 150 columns and each of them adds up to 604

$$\begin{aligned} 604 + 604 + 604 + \dots + 604 &= 2x \\ 150 \cdot 604 &= 2x \\ 90600 &= 2x \\ 45300 &= x \end{aligned}$$

So the sum $4 + 8 + 12 + \dots + 600 = \boxed{45300}$.

b) $3 + 6 + 9 + \dots + 2019$ [Solution - Youtube link](#)

Example 2. Find each of the given sum

a) $31 + 38 + 45 + \dots + 423$.

b) $30 + 37 + 44 + \dots + 2018$ [Solution - Youtube link](#)

Solution: a) We will apply Gauss's method.

$$\begin{array}{r} 31 + 38 + 45 + \dots + 423 = x \\ + 423 + 416 + 409 + \dots + 31 = x \\ \hline 454 + 454 + 454 + \dots + 454 = 2x \end{array}$$

The easy question is: what is the sum in each column? Clearly 454. It takes a bit more work to figure out how many columns are there. We label the numbers in the first row as 1, 2, 3, etc

$$\begin{array}{ccccccc} 1 & 2 & 3 & & & ? & = x \\ \downarrow & \downarrow & \downarrow & & & \downarrow & \\ 31 & + & 38 & + & 45 & + & \dots + 423 = x \end{array}$$

We can see that as the label increases by 1, the number in the first row increases by 7. Because of this, we might suspect that the number belonging to label n has to do with $7n$. Let us try: If $n = 1$, then $7 \cdot 1 + b = 31$ gives us $b = 24$. Thus the connection between the label and the number with that label is

$$M = 7n + 24$$

Let's try this for $n = 2$. If $n = 2$, then $M = 7 \cdot 2 + 24 = 38$.

Let's try this for $n = 3$. If $n = 3$, then $M = 7 \cdot 3 + 24 = 45$.

And so on, our formula works. So what label n belongs to $M = 423$? We just need to solve the equation

$$\begin{aligned} 423 &= 7n + 24 \\ 399 &= 7n \\ 57 &= n \end{aligned}$$

Thus, there are 57 columns. The rest is easy:

$$\begin{aligned} 454 + 454 + 454 + \dots + 454 &= 2x \\ 57 \cdot 454 &= 2x \\ 25878 &= 2x \\ 12939 &= x \end{aligned}$$

So the sum $31 + 38 + 45 + \dots + 423 = \boxed{12939}$

b) $30 + 37 + 44 + \dots + 2018$ [Solution - Youtube link](#)

Example 3. The first row in a theater has 31 seats in it. The second row has one more seats than the first row. The third row has one more seats than the second row. And so on, each row has one more seats than the row before. If the last row has 191 seats in it, how many seats are there in the entire theater?

[Solution - Youtube link](#)

Solution: To find the number of all seats in the theater, we need to add the number of seats in each row. The number of total seats is

$$31 + 35 + 39 + \dots + 191$$

We will apply Gauss's method.

$$\begin{array}{cccccccc} 31 & + & 35 & + & 39 & + & . & . & . & + & 191 & = & x \\ 191 & + & 187 & + & 183 & + & . & . & . & + & 31 & = & x \end{array}$$

The easy question is: what is the sum in each column? Clearly 222. It takes a bit more work to figure out how many columns are there. We label the numbers in the first row as 1, 2, 3, etc

$$\begin{array}{cccccccc} 1 & & 2 & & 3 & & & & ? & = & x \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\ 31 & + & 35 & + & 39 & + & . & . & . & + & 191 & = & x \end{array}$$

We can see that as the label increase by 1, the number in the first row increase by 4. Because of this, we might suspect that the number belonging to label n has to do with $4n$. Let us try: If $n = 1$, then $4 \cdot 1 + b = 31$ gives us $b = 27$. Thus the connection between the label and the number with that label is

$$M = 4n + 27$$

Let's try this for $n = 2$. If $n = 2$, then $M = 4 \cdot 2 + 27 = 35$.

Let's try this for $n = 3$. If $n = 3$, then $M = 4 \cdot 3 + 27 = 39$.

And so on, our formula works. So what label n belongs to $M = 191$? We just need to solve the equation

$$\begin{array}{rcl} 191 & = & 4n + 27 & \text{subtract 27} \\ 164 & = & 4n & \text{divide by 4} \\ 41 & = & n \end{array}$$

Thus, there are 41 columns. The rest is easy:

$$\begin{array}{rcl} 222 + 222 + 222 + \dots + 222 & = & 2x \\ 41 \cdot 222 & = & 2x \\ 9102 & = & 2x & \text{divide by 2} \\ 4551 & = & x \end{array}$$

So the theater has 4551 seats.

Example 4. Compute the sum $\frac{1}{1001} + \frac{2}{1001} + \frac{3}{1001} + \dots + \frac{1000}{1001}$ [Solution - Youtube link](#)



Practice Problems

Compute each of the following sums.

1. $19 + 30 + 41 + \dots + 4078$

2. $29 + 34 + 39 + \dots + 374$

3. $15 + 17 + 19 + \dots + 643$

4. $32 + 41 + 50 + \dots + 698$

5. $11 + 17 + 23 + \dots + 1325$

6. $16 + 23 + 30 + \dots + 1094$

7. $20 + 27 + 34 + \dots + 783$

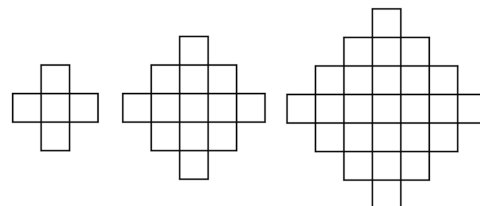
8. $31 + 37 + 43 + \dots + 1015$

9. $\frac{1}{2019} + \frac{2}{2019} + \frac{3}{2019} + \dots + \frac{2018}{2019}$

10. What is the sum of all numbers shown in the square?

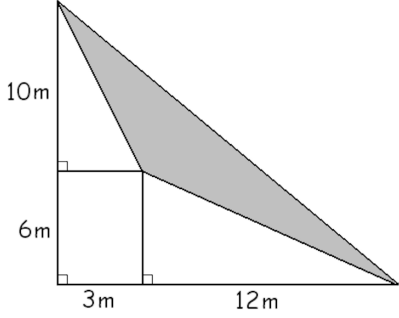
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18
10	11	12	13	14	15	16	17	18	19

11. Consider the objects shown on the picture. Each are made up of unit squares. (A unit square is a square with sides 1 unit long.) The first figure consists of 5 unit squares. The second consists of 13 unit squares, the third of 25 little squares. How many unit squares make up the 100th such figure?



Problem Set 11

- List all factors of 56.
 - Find the prime factorization of 240.
 - Use the prime factorization to compute the greatest common divisor and least common multiple of 240 and 540.
- Find the prime factorization for x if
 - $x = 48^{99}$
 - $x = (5!)^3$ Recall that $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
- Perform the division with remainder: $2019 \div 11$
- Label each of the following as true or false.

a) There is no prime number divisible by 3.	e) If a number x is divisible by 3, then its square x^2 is divisible by 9.
b) For all sets A , $A \cup \emptyset = A$.	f) If n^2 is divisible by 18, then n is divisible 36.
c) If the product xy is divisible by 6, then x is divisible by 6 or y is divisible by 6.	g) Every rectangle is a square.
d) If the product xy is divisible by 5, then x is divisible by 5 or y is divisible by 5.	h) No rectangle is a square.
- What is the last digit of 7^{2020} ?
- Compute the perimeter and area of a right triangle with sides 34cm, 16cm, and 30cm long. Include units in your computation and answer.
 - Compute the area of the right triangle determined by the points $A(-8, -3)$, $B(2, -3)$, and $C(-8, 4)$.
- Compute the area of the shaded region on the picture. Include units in your computation and answer.
 
- Suppose that $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 5, 7, 10\}$, and $C = \{1, 3, 5, 8, 10\}$.
 - Draw a Venn diagram depicting these sets.
 - Find each of the following.

i) $A \cap B$	ii) $A \cup (B \cap C)$	iii) $A \cap (B \cup C)$
---------------	-------------------------	--------------------------
- Compute each of the following sets.

a) $(0, 5) \cup [3, 7]$	d) $(-\infty, 2) \cap (-4, \infty)$	g) $(-\infty, -3) \cup (1, 8)$	j) $(4, \infty) \cap [1, 6]$
b) $(0, 5) \cap [3, 7]$	e) $(3, 5) \cup [1, 8]$	h) $(-\infty, -3) \cap (1, 8)$	
c) $(-\infty, 2) \cup (-4, \infty)$	f) $(3, 5) \cap [1, 8]$	i) $(4, \infty) \cup [1, 6]$	
- We place \$3000 into a bank account with an annual interest rate of 5%. How much money is in the account after a year?
 - Sally is currently making \$2000 a month. What will be her salary if she gets a pay increase of 2%?
 - A TV set is currently priced at \$810. What would be the sale price next week when the TV will go on a 20% off sale?

11. Simplify each of the following.

$$\begin{array}{llll} \text{a) } |11 - 3| - 5|| & \text{c) } |11| - 3 - 5|| & \text{e) } -3^2 - 12 \div 2 \cdot 3 & \text{g) } |3 - |-7 + 2|| \\ \text{b) } |11 - |3 - 5|| & \text{d) } 12 - 2(5 - 3(-2)) & \text{f) } \frac{18 - 5 + 3}{-2^2 - (-2)^2} & \text{h) } \frac{6 - 2(-3)}{-2^2 - (-1)} \end{array}$$

12. Simplify each of the following.

$$\begin{array}{llll} \text{a) } -3^2 - 4(-5) + 24 \div 3 \cdot 2 & \text{e) } \frac{13 - 5 + 6}{-2^3 + 8} & & \text{h) } \frac{3 - \frac{1}{5}}{2 + \frac{1}{3}} \\ \text{b) } \frac{3}{8} - \frac{1}{4} \cdot \frac{3}{5} & \text{f) } \frac{2^{-1} + 3^{-1}}{2^{-1} - 3^{-1}} & & \\ \text{c) } \left(-\frac{2}{5}\right)^2 - \frac{1}{3} + \frac{1}{5} & \text{g) } \frac{\frac{7}{5} - \left(-\frac{3}{4}\right)^2 + 1}{\frac{1}{10} - \frac{1}{20} + \frac{1}{40}} & \text{i) } (-2)^{-2} - (-2)^{-3} - (-2)^{-4} & \\ \text{d) } 2 - 5(8 - 3(2 - (-1)^3)) & & \text{j) } 2^{-3} - 3^{-1} & \end{array}$$

13. Evaluate $\frac{-x^2 + 10x - 21}{3 - x}$ if

$$\text{a) } x = 2 \quad \text{b) } x = 3 \quad \text{c) } x = \frac{5}{2} \quad \text{d) } x = -\frac{3}{8}$$

14. Suppose that we denote 3^{1000} by A . Express each of the following in terms of A .

$$\begin{array}{lll} \text{a) } 3^{1000} + 3^{1001} & \text{c) } 3^{6000} & \text{e) } 3^{500} \\ \text{b) } 3^{998} & \text{d) } 9^{500} & \end{array}$$

15. What number do we get if we increase 5000 by

$$\begin{array}{lll} \text{a) } 2\% & \text{c) } 20\% & \text{e) } 200\% \\ \text{b) } 12\% & \text{d) } 120\% & \end{array}$$

16. Express each of the following as a single change.

$$\begin{array}{l} \text{a) First a 20\% increase and then a 25\% increase.} \\ \text{b) First a 30\% decrease and then a 40\% decrease.} \\ \text{c) First a 20\% increase and then a 20\% decrease.} \\ \text{d) First a 20\% decrease and then a 20\% increase.} \end{array}$$

17. Suppose that $x = 125000000000$

$$\text{and } y = 0.00000000025.$$

a) Write x and y using scientific notation.

b) Compute each of the following. Present your answers using scientific notation.

$$\text{i) } xy \quad \text{ii) } x^2y \quad \text{iii) } \sqrt{10y} \quad \text{iv) } \frac{1}{y}$$

18. Compute each of the following sums.

$$\begin{array}{l} \text{a) } 8 + 16 + 24 + \dots + 2016 \\ \text{b) } 20 + 23 + 26 + \dots + 98 \\ \text{c) } 320 + 335 + 350 + \dots + 2120 \\ \text{d) } \frac{1}{99} + \frac{2}{99} + \dots + \frac{98}{99} \end{array}$$

19. Simplify each of the following. Express your answer using only positive exponents.

$$\begin{array}{llll} \text{a) } \frac{(-a^{-4}b^5)^{-2}}{a^5b^{-3}a} & \text{b) } \left(\left((-x)^2x\right)^3x\right)^4 & \text{c) } \frac{(-8ax^3)^0(-3ax^{-1}a^2)^3}{(-6a^4x)^2} & \text{d) } \frac{(2xy^{-2})^{-3}(-x^0y^2x^{-3})^{-1}}{x^{-2}(-2^{-3}x^0y^{-4})^2} \end{array}$$

20. Simplify each of the following.

$$\begin{array}{lll} \text{a) } (3x - 8) + (-2x + 1) & \text{e) } (3x^2 + 5) + (3x^2 - 5) & \text{i) } (2x - 3)^3 \\ \text{b) } (3x - 8) - (-2x + 1) & \text{f) } (3x^2 + 5) - (3x^2 - 5) & \text{j) } (x - 2)(x + 3)(x - 1) \\ \text{c) } 2(3x - 8) - 3(-2x + 1) & \text{g) } (3x^2 + 5)(3x^2 - 5) & \text{k) } (x + 2)^2(x - 2)^2 \\ \text{d) } (3x - 8)(-2x + 1) & \text{h) } (2x - 3)^2 & \end{array}$$

21. Completely factor each of the following.

a) $2x^2 - 18$ b) $12a^2x^2 - 75x^2$ c) $x^2 + 9$ d) $5a^7 - 5a^3$ e) $-x^3 + 16x$ f) $x^{16} - 25$

22. Completely factor each of the following.

a) $-2x^5 + 20x^4 + 150x^3$ b) $6x^2 - 60x + 96$ c) $87a - 30a^2 + 3a^3$

23. Solve each of the following equations.

a) $(3x - 1)(x + 1) - 2(x - 2)^2 = 14x - 9$ i) $x^2 - 2x = -5$
 b) $5m^6 = 80m^2$ j) $2 - (2x - 5) = (x - 4)^2$
 c) $(3x - 1)^2 - (2x + 5)^2 = 24 - 5x(4 - x)$ k) $(3a + 1)^2 - (3a + 4)(3a - 2) = 3(a + 3)$
 d) $(2x - 3(4x + 5(-x + 2) - 3)) = 2(3(x - 5) + 1)$ l) $(3 - 4(5 - (6 - x) + 1) - 1) + 1 = x^2 + 3$
 e) $\frac{1}{2}(3x - 5) - \frac{3}{4}(2x + 1) = x - \frac{1}{4}$ m) $3(x - 2)^2 - (2x - 1)^2 = 15 - (x + 6)^2$
 f) $3((9x - 1) - 5(2x + 1)) = -18$ n) $x(x + 1)^2(x - 5) = 0$
 g) $5(2x + 3) = (x + 4)^2 - (x - 1)^2$ o) $(2x - 5)^2 = (5x - 2)^2$
 h) $\frac{2}{5}x - \frac{3}{4} - \frac{1}{3}\left(x + \frac{3}{5}\right) = -\frac{13}{15}$ p) $2(x - 3(x - 2(x - (3x - 1)))) = 4(5 - 7(x - 2))$

24. Solve each of the following inequalities. Present your answer in interval notation.

a) $\frac{2x - 1}{3} - \frac{3x + 1}{4} \leq x - 6$ c) $\frac{2}{3}x - \frac{5}{6} \geq -\frac{1}{3}$
 b) $(x + 6)^2 < (x + 4)^2$ d) $2(x - 3) - x - 16 > 3x + 2 - 4(1 - 2x)$

25. Solve each of the following systems of linear equations.

a) $\begin{cases} 3x - 2y = 12 \\ 2x - y = 4 \end{cases}$ b) $\begin{cases} x + 5y = 3 \\ (x - 3)^2 + (y - 1)^2 = (x - 2)^2 + (y + 2)^2 \end{cases}$ c) $\begin{cases} 2x - y = 6 \\ \frac{2}{5}x + \frac{3}{7}y = x \end{cases}$

26. a) Graph the line $3x - 4y = -5$. Make sure to find two lattice points.

b) Graph the straight lines $3x + 5y = 5$ and $y = -x - 1$ in the same coordinate system. Use your graph to find the coordinates of the point where the lines intersect.

27. We throw a small object upward from the top of a 1200ft tall building. The vertical location of the object, (measured in feet) t seconds after we threw it is

$$L = -16t^2 + 160t + 1200$$

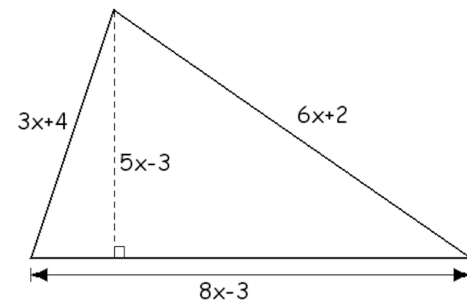
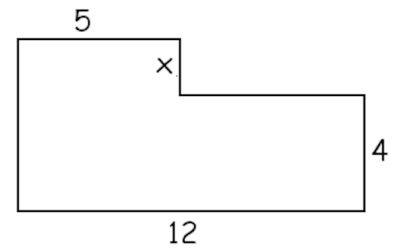
a) Where is the object 3 seconds after we threw it?

b) How long does it take for the object to hit the ground?

28. One side of a rectangle is 12ft shorter than three times another side. Find the sides if its area is 231ft^2 .

29. The freshman class had 60 students. 45 students took English, 38 students took Mathematics, and 28 took both English and Mathematics. How many students took neither of these subjects?

30. The population in our town has increased by 20%. If the town has now 72000 people, how many lived there before the increase?
31. A bank teller has 23 more five-dollar bills than ten-dollar bills. The total value of the money is \$610. How much of each denomination of bill does he have?
32. Three times a number is one less than twice the difference of the number and three. Find this number.
33. Amy's age is three less than five times her son's age. How old are they if the sum of their ages is 33?
34. Consider the figure shown on the picture. Angles that look like right angles are right angles. Find the value of x if we know that the area of this object is 58 unit^2 .
35. a) Find the value of x if the triangle shown on the picture has a perimeter 54 units.
b) Given the value of x you found, compute the area of the triangle.
36. There were a lot of coins in that jar, all quarters and dimes. The number of dimes was two less than five times the number of quarters. How many of each coins were there if all the coins in the jar were worth 8 dollars and 80 cents? (Hint: think in terms of cents)
37. If we increase each side of a square by 4 units, its area increases by 200 unit^2 . How long is each side of the square?
38. A living room set went to a 15% off sale. The sale price is \$1062.50. What was the price before the sale?
39. The first row in a theater has 25 seats in it. The second row has three more seats than the first row. The third row has three more seats than the second row. And so on, each row has three more seats than the row before. If the last row has 163 seats in it, how many seats are there in the entire theater?
40. The tickets for the field trip were purchased yesterday for both students and instructors. Children tickets cost \$12, adult tickets cost \$19. The number of children ticket purchased was three more than four times the number of adults tickets purchased. How many of each were purchased if all of the tickets cost a total of \$304 dollars?



Chapter 12

12.1 Square Root of 2 is Irrational

A very powerful proving technique is what we call **indirect proof**, or **proof by contradiction**.

The logic behind this proving technique is as follows. Suppose that we start with a true statement and arrive to other statements by making logically correct steps. Then these new statements must all be true.

Suppose we start with a statement and use logically correct steps to arrive to other statements, including one that is obviously false. Then we must have started with a false statement.

True statements only imply true statements. If our conclusion is false, we must have started with a false statement.

Suppose we want to prove a statement to be true. In case of a proof by contradiction, we formulate the exact opposite of our statement, and, using logically correct steps, we derive an obviously false statement. This proves that we started with a false statement. Therefore, the opposite of our statement is false, which means that our statement is true.

The fact that $\sqrt{2}$ is irrational can be proven by contradiction.

Definition: A number is **rational** if it can be written as a fraction of two integers.

Definition: A number is **irrational** if it is not rational, i.e. it can not be written as a fraction of two integers.

Theorem: $\sqrt{2}$ is an irrational number.

Proof. Suppose, for a contradiction, that $\sqrt{2}$ is rational, i.e. there exist two integers, a and b ($b \neq 0$) such that

$$\sqrt{2} = \frac{a}{b}$$

We may also assume that the fraction $\frac{a}{b}$ is in lowest terms, otherwise we could reduce the fraction $\frac{a}{b}$ and replace it with the reduced equivalent. So, let us assume that $\frac{a}{b}$ is in lowest terms, which means that a and b do not share any divisor larger than 1. Now let us square both sides.

$$2 = \frac{a^2}{b^2}$$

Let us multiply both sides by b^2 .

$$2b^2 = a^2$$

Since a^2 is twice another integer, it is even. This means that a itself must be even. Let us re-write $a = 2k$ where k is some integer.

$$\begin{aligned} 2b^2 &= (2k)^2 \\ 2b^2 &= 4k^2 \end{aligned}$$

Let us divide both sides by 2. Then we have

$$b^2 = 2k^2$$

Since b^2 is twice another integer, it is even. This means that b itself must be even. We are now done, because the following statements cannot all be true.

1. a and b are two integers that do not share any divisors.
2. a is even.
3. b is even

This is a contradiction, guaranteeing that there is at least one false statement among the three. This means that the assumption that $\sqrt{2}$ is rational must be false, its opposite therefore true. This completes our proof. ■

Part 1: Converting a Fraction to a Decimal

Example 1. Convert $\frac{3}{8}$ to a decimal.

[illegible][illegible]

Part 2: Converting a Terminating Decimal to Fraction

A decimal is **terminating** if it has a last digit. It is quite easy to turn a terminating decimal to a fraction of integers.

Example 3. Convert 0.45 to a reduced fraction.

Solution: Step 1. Write the number as a fraction of any kind first.

$$0.45 = \frac{0.45}{1}$$

We can mentally check it as division: any number divided by one results in the same number.

Step 2. We ask ourselves: "How many digits do we need to move the decimal point to the right in 0.45 to obtain an integer"? The answer is: two digits. Moving the decimal point to the right by two digits is the same as multiplication by 100. Thus, to fix the numerator, we need to multiply it by 100. Because we also want to preserve the value, we multiply both upstairs and downstairs by 100.

$$\frac{0.45}{1} = \frac{0.45 \cdot 100}{1 \cdot 100} = \frac{45}{100}$$

Step 3. We simplify the fraction by dividing numerator and denominator by the greatest common factor.

$$\frac{45}{100} = \frac{-5 \cdot 9}{-5 \cdot 20} = \frac{9}{20}$$

Example 4. Convert 0.0005 to a reduced fraction.

Solution: We will write the number as a fraction and then multiply numerator and denominator by 10000.

$$0.0005 = \frac{0.0005}{1} = \frac{0.0005 \cdot 10000}{1 \cdot 10000} = \frac{5}{10000} = \frac{-5 \cdot 1}{-5 \cdot 2000} = \frac{1}{2000}$$

Example 5. Convert 23.044 to a reduced fraction.

Solution:

$$23.044 = 23 + 0.044 = 23 + \frac{0.044}{1} = 23 + \frac{0.044 \cdot 1000}{1 \cdot 1000} = 23 \frac{44}{1000} = 23 \frac{4 \cdot 11}{4 \cdot 250} = 23 \frac{11}{250}$$

Part 3: (The Fun Stuff)

Converting a Non-Terminating Decimal to Fraction

A decimal is non-terminating if it has infinitely many digits. If there is a repeating block, we denote it by a bar drawn over the repeating digit. For example, the number $2.\overline{35}$ denotes $2.35353535\dots$.

Example 6. Re-write each of the given repeating decimals without the bar notation.

$$\text{a) } 1.201\overline{7} \quad \text{b) } 1.20\overline{17} \quad \text{c) } 1.2\overline{017} \quad \text{d) } 1.\overline{2017}$$

Solution: Only the digit(s) under the bar are repeating. The rest is there as is.

$$\text{a) } 1.201\overline{7} = 1.20177777\dots$$

$$\text{c) } 1.2\overline{017} = 1.2017017017017\dots$$

$$\text{b) } 1.20\overline{17} = 1.20171717171717\dots$$

$$\text{d) } 1.\overline{2017} = 1.2017201720172017\dots$$

Turning these decimals into fractions of integers is an interesting and fun application of linear equations.

Example 7. Convert the repeating decimal $7.\overline{4}$ to a fraction.

Solution: Step 1. We label our number x and write it without the bar notation. The dots are important: they indicate that we have infinitely many 4's there and not just three.

$$7.444\dots = x$$

Step 2. We multiply both sides of this equation by 10.

$$74.444\dots = 10x$$

Step 3. We write these equations together, starting with the second one.

$$74.444\dots = 10x$$

$$7.444\dots = x$$

Step 3. (Chop, chop.) We subtract the second equation from the first one.

$$\begin{array}{r} 74.444\dots = 10x \\ - \quad 7.444\dots = x \\ \hline 67 \qquad = 9x \end{array}$$

Step 4. We solve the equation for x .

$$\begin{array}{l} 67 = 9x \quad \text{divide by 9} \\ \frac{67}{9} = x \end{array}$$

Thus the answer is $\frac{67}{9}$. We can check by long division. Indeed, $67 \div 9 = 7.4444444\dots$

Example 8. Convert the repeating decimal $0.\overline{405}$ to a fraction.

Solution: Step 1. We label our number x and write it without the bar notation. $0.405405405405405\dots = x$

Steps 2 and 3. This decimal has a three-digit long repeating block. To obtain proper alignment of the digits, we will move the decimal point by three digits, i.e. we will multiply by 1000. We multiply both sides of this equation by 1000. We write these equations together, starting with the second one.

$$\begin{aligned} 405.405405405405\dots &= 1000x \\ 0.405405405405\dots &= x \end{aligned}$$

Step 3. (Chop, chop.) We subtract the second equation from the first one.

$$\begin{array}{r} 405.405405405405\dots = 1000x \\ - 0.405405405405\dots = x \\ \hline 405 \qquad \qquad \qquad = 999x \end{array}$$

Step 4. We solve the equation for x .

$$\begin{aligned} 405 &= 999x \quad \text{divide by 999} \\ \frac{405}{999} &= x \end{aligned}$$

Please note that the fraction obtained is not reduced. However, the essential point in the problem is to find a fraction, not the reduced form of it. Thus the answer is $\frac{405}{999}$. We can check by long division. Indeed, $405 \div 999 = 0.405405405\dots$

Example 9. Convert the repeating decimal $18.29\overline{04}$ to a fraction.

Solution: Step 1. We label our number x and write it without the bar notation.

$$18.2904040404\dots = x$$

Steps 2 and 3. This decimal has a two-digit long repeating block. To obtain proper alignment of the digits, we will move the decimal point by two digits, i.e. we will multiply by 100. We multiply both sides of this equation by 100. We write the two equations together, starting with the second one.

$$\begin{aligned} 1829.0404040404\dots &= 100x \\ 18.2904040404\dots &= x \end{aligned}$$

Step 3. (Chop, chop.) We subtract the second equation from the first one.

$$\begin{array}{r} 1829.0404040404\dots = 100x \\ - 18.2904040404\dots = x \\ \hline 1810.75 \qquad \qquad = 99x \end{array}$$

It appears that we have a problem: the right-hand side is not an integer after the subtraction.
This is quite easy to fix: we just multiply both sides by 100.

$$\begin{aligned} 1810.75 &= 99x && \text{multiply by 100} \\ 181075 &= 9900x \end{aligned}$$

Step 4. We solve the equation for x .

$$\begin{aligned} 181075 &= 9900x && \text{divide by 9900} \\ \frac{181075}{9900} &= x \end{aligned}$$

Thus the answer is $\frac{181075}{9900}$. We can check by long division. Indeed, $181075 \div 9900 = 18.29040404\dots$



Practice Problems

1. Perform each of the following conversions.

a) Convert the given fraction to decimals.

i) $\frac{4}{5}$

ii) $\frac{26}{3}$

iii) $\frac{26}{25}$

iv) $\frac{26}{7}$

How many digits long is the repeating block?

b) Convert the given decimal to a fraction of integers. (You do not have to reduce them!)

i) 2.18

ii) $2.\overline{9}$

iii) $6.\overline{47}$

iv) $1.8\overline{705}$

2. Based on your answer for 1b ii), what is a surprising new fact about decimal presentation of numbers?

3. Consider the fraction $\frac{1}{n}$. What numbers n will result in a terminating decimal?

4. The decimal presentation of $\frac{2}{13}$ does not appear to be repeating. Is it?

5. Find a fraction formed of two integers that will result in a non-terminating, non-repeating decimal.

12.3 The Real Number System

Definition: The set of all natural numbers, denoted by $\mathbb{N}$, is the infinite set

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

If we add two natural numbers, the sum is also a natural number. In other words, if x and y are natural numbers, then the sum $x + y$ is also a natural number. When this is true, we say that the set of all natural numbers is **closed under addition**. On the other hand, the set of all natural numbers is not closed under subtraction: while $10 - 3$ is a natural number, $3 - 10$ is not.

Theorem: The set of all natural numbers is closed under addition and multiplication, but not under subtraction and division.

Definition: The set of all integers, denoted by $\mathbb{Z}$, is the infinite set

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$$

Notice that the set of all integers contains all natural numbers. When this happens, we say that the set of all natural numbers is a subset of the set of all integers. Notation: $\mathbb{N} \subseteq \mathbb{Z}$.

Theorem: The set of all integers is closed under addition, multiplication, and subtraction, but not under division.

Definition: A number is **rational** if it can be written as a quotient of two integers.

For example, $\frac{3}{8}$ is a rational number because both 3 and 8 are integers and so $\frac{3}{8}$ is a quotient of two integers.

Definition: The set of all rational numbers, denoted by $\mathbb{Q}$, is the infinite set

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \text{ and } b \text{ are integers, } b \neq 0 \right\}$$

Notice that the set of all rational numbers entirely contains the set of all integers, i.e. $\mathbb{Z} \subseteq \mathbb{Q}$. In fact, the three sets are such that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$$

For example, the number -5 is an integer and also a rational number, because we can write it as a quotient $\frac{-5}{1}$ where both -5 and 1 are integers. The number zero is also a rational number because it can be written as $\frac{0}{3}$.

Theorem: The set of all rational numbers is closed under addition, multiplication, subtraction, and division.

For some strange reason, mathematicians still needed additional types of numbers.

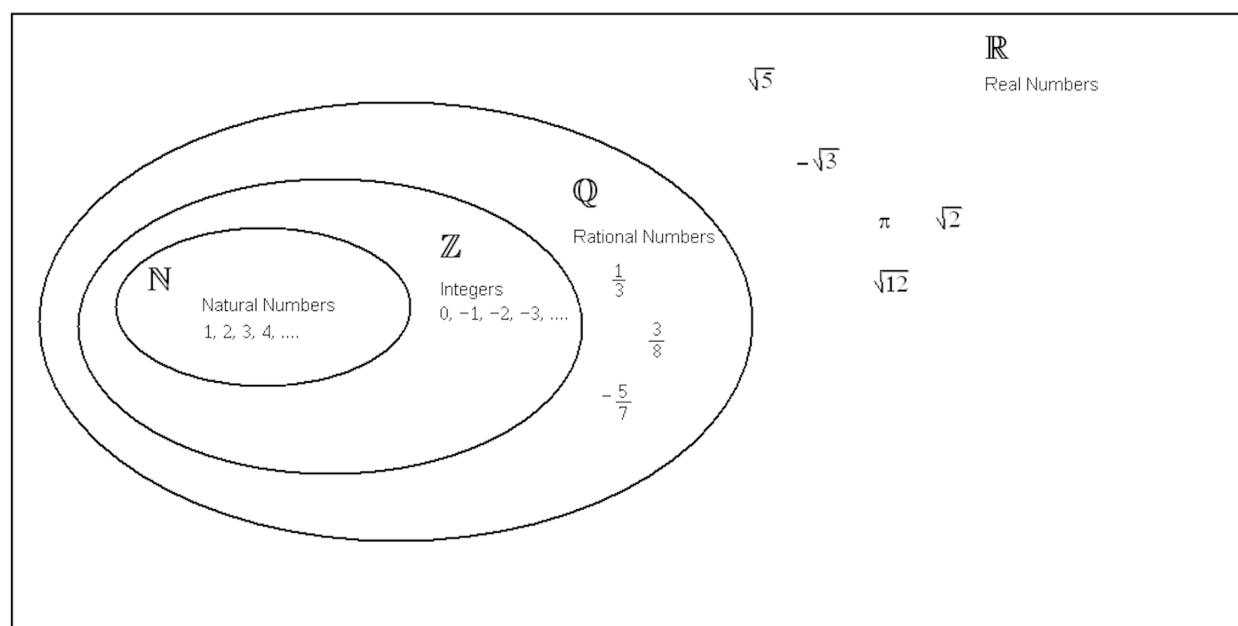
Definition: A number is **irrational** if it cannot be written as a quotient of two integers.

This is a very strange property because there are so many different integers from which to choose. However, irrational numbers exist. For example, π and $\sqrt{2}$ are irrational numbers. Surprisingly, in a sense, there are many more irrational numbers than rational numbers. (In a fascinating subject within mathematics called set theory, mathematicians have developed language to compare infinite sets. In that comparison, the set of irrational numbers proved to be much greater than the set of rational numbers.)

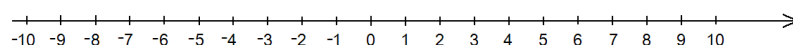
Definition: The set of all **real numbers**, denoted by $\mathbb{R}$, is the collection of all rational and irrational numbers.

The set of all real numbers contain all previous number sets as a subset. For example, every rational number is a real number.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$



Mathematicians proved that there are exactly as many real numbers as many points there are on a straight line. Given a line, for every point on it, we can uniquely assign a real number to that point. Vice versa, for every real number, there is exactly one point on the line. This correspondence is expressed by the concept of the **number line**.



Theorem: Every terminating decimal represents a rational number.

We have seen this before. To convert a terminating decimal to a fraction of integers, see the previous section.

Theorem: Every non-terminating, repeating decimal represents a rational number.

We have seen this before. To convert a terminating decimal to a fraction of integers, see the previous section.

There is an important conclusion that can be drawn from these two facts. Consider $\sqrt{2}$, for example. If we accept the fact that $\sqrt{2}$ is irrational (which can be proved at this level) then it follows that its decimal presentation can not be terminating. (Why not?) And also, the decimal presentation of $\sqrt{2}$ can not be repeating. What is left for poor irrational numbers?

Theorem: The decimal presentation of irrational numbers is non-terminating and non-repeating.

By definition of irrational numbers, $\sqrt{2}$ can not be expressed as a fraction of two integers. We have just seen that $\sqrt{2}$ can not really be written as a decimal. If we attempted to write $\sqrt{2}$ as a decimal, we can only write approximations of the number, and never the exact value.

When we are prompted to give a number's *exact value*, in case of $\sqrt{2}$, the symbol $\sqrt{2}$ is our only option. Fractions formed from integers such as $\frac{141}{100}$ or decimals such as 1.41 are only *approximations*.

Problem Set 12

1. Label each of the following statements as true or false.

- | | |
|---|---|
| a) If $A \subseteq B$, then $A \cap B = A$ | h) There is no prime number divisible by 5. |
| b) If $A \subseteq B$, then $A \cup B = B$ | i) Every integer has at least two positive divisors. |
| c) 1 is a prime number | j) The product of any two rational numbers is rational. |
| d) If a number m is divisible by 4 and by 6, then it is also divisible by 24. | k) The product of any two irrational numbers is irrational. |
| e) If a number m is divisible by 3 and by 8, then it is also divisible by 24. | l) $ x - 3 $ can be simplified as $x + 3$.
(Hint: try a few values for x !) |
| f) If a number n is divisible by 3, then its square, n^2 is divisible by 9. | m) $\mathbb{Z} \cap \mathbb{N} = \mathbb{Z}$ |
| g) For all sets A and B , $A \cap B \subseteq A \cup B$. | |

2. Suppose that S is the set of all squares and R is the set of all rectangles. Label each of the following statements as true or false.

- | | | | |
|--------------------|----------------------------|-------------------|-------------------|
| a) $S \subseteq R$ | c) $R \subseteq S$ | e) $R \cup S = S$ | g) $R \cap S = S$ |
| b) $R \subseteq S$ | d) $\emptyset \subseteq S$ | f) $R \cup S = R$ | h) $R \cap S = R$ |

3. Suppose that F is the set of all integers divisible by four, S is the set of all integers divisible by six, and T is the set of all integers divisible by three. Label each of the following statements as true or false.

- a) $S \subseteq T$ b) $F \subseteq S$ c) $F \cap T = S$ d) $F \cap T \subseteq S$ e) $F \subseteq S \cup T$

4. We know that addition is associative, i.e, for every real numbers x , y , and z , $(x + y) + z = x + (y + z)$. This is the reason we are allowed to write something such as $x + y + z$. Are the two set operations we know also associative?

Suppose that $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 7, 9, 10\}$, and $C = \{1, 3, 4, 7, 8, 10\}$. Find each of the following.

- | | | | |
|------------------------|------------------------|------------------------|------------------------|
| a) $A \cap B$ | c) $B \cap C$ | e) $A \cup B$ | g) $B \cup C$ |
| b) $(A \cap B) \cap C$ | d) $A \cap (B \cap C)$ | f) $(A \cup B) \cup C$ | h) $A \cup (B \cup C)$ |

i) Based on your results, what is your suspicion about union and intersection being associative? (If we suspect something to be true in mathematics, but we haven't proved it yet, the statement is called a **conjecture**.)

5. Find the least common multiple and greatest common factor of 120 and 80.

6. *Recall the definition of factorial: $100!$ is short for $100 \cdot 99 \cdot 98 \cdot \dots \cdot 2 \cdot 1$. We read it as '100 factorial'. Find the exponent of 7 in the prime factorization of $100!$.

7. Simplify each of the following.

- a) $(5^{20})^3$ b) $(2^{50})^2$ c) $(7^{-3})^{-20}$ d) Given your result for b), how can we simplify $\sqrt{2^{100}}$?

8. Suppose that x is a natural number such that the least common multiple of 60 and x is 300. What numbers are possible for x ?

9. Compute each of the following sums:

- | | | |
|----------------------------------|---|-----------------------------------|
| a) $17 + 22 + 27 + \dots + 2022$ | b) $\frac{1}{99} + \frac{2}{99} + \frac{3}{99} + \dots + \frac{98}{99}$ | c) $-10 + (-9) + \dots + 19 + 20$ |
|----------------------------------|---|-----------------------------------|

10. For each of the given decimals, prove that they are rational by re-writing them as fractions. You do NOT have to simplify the fraction.

a) 2.04 b) $0.3\overline{7} = 0.3777\dots$ c) $0.2\overline{05} = 0.2050505\dots$ d) $5.\overline{425} = 5.425425425\dots$

11. a) Our stocks first increased 15% in value, but then later it lost 20%. Express the two changes as a single change. What percent of a change is this?

b) The population of our town has increased by 2%. Now we have 24480 residents. How many people lived in the town before the increase?

12. Perform the indicated operations and simplify.

a) $\frac{2^{-3} - 3^{-2}}{2^{-3} + 3^{-2}}$

e) $|2 - 3| - 8||$

j) $\sqrt{4} \cdot \sqrt{9}$

b) $\frac{3 - 4 + 5}{2^4 - (-2)^4}$

f) $|2 - |3 - 8||$

k) $\sqrt{4 \cdot 9}$

c) $(-1)^{100} + (-1)^{101} + (-1)^{102}$

g) $||2 - 3| - 8|$

l) $\sqrt{7 + \sqrt{3(-2)^2 - \sqrt{100} + \sqrt{4}}}$

d) $1 - 2(3 - 4(5 - 6))$

i) $\sqrt{25} - \sqrt{16}$

m) $3(-2^{-2})$

13. Simplify each of the following.

a) $3(4a - b) + (-3a + 2b)$

c) $(5a - 2)^3$

e) $\frac{2a^2b^{-5}(aba^{-2})}{(-2a)^2(ab^{-3})^2}$

b) $(2x^5 - 1)^2(2x^5 + 1)^2$

d) $\frac{2}{3}\left(6x - \frac{3}{5}\right) - \frac{3}{4}(12x - 1)$

14. Completely factor each of the following.

a) $3x^4y - 75x^2y$

d) $-5x^2 - 30x + 80$

g) $(a + b + c)^2 - (a + b - c)^2$

b) $2x^5 - 162x$

e) $26xy - 8x^2y + 2x^3y$

h) $5b^2(2a - 3) - 10b(2a - 3) + 10a - 6$

c) $20ab + 5a^3b$

f) $(3m - 1)^2 - (m + 5)^2$

i) $x^2 - 10x + 25 - y^{10}$

15. Solve each of the given equations. Make sure to check your solutions.

a) $\frac{2}{3}x - \frac{1}{6} = x - \frac{5}{2}$

f) $4\left(\frac{3x - 1}{2} + 5\right) - 2$

k) $2x^4 - 32x^3 = 0$

b) $\frac{x + 7}{2} = -3$

f) $\frac{\frac{5}{3}}{3} - 6 = -3$

l) $2x^5 - 32x^3 = 0$

c) $\frac{x}{2} + 7 = -3$

g) $4(8a + 5) - 3(5a + 1) - 7(2a - 1) = 6$

m) $2x^6 - 32x^2 = 0$

d) $5a - 8 = -8$

h) $(2x - 1)^2 - (x - 3)^2 = (x + 2)(3x - 4)$

e) $\frac{\frac{x - 1}{3} + 2}{5} - 7$

i) $4(2(3m + 5) - 2(3(5m + 1) - 7(2m - 1))) = 40$

e) $\frac{5}{2} + 7 = 4$

j) $(2x - 1)^2 - (x + 2)(3x - 4) = -8x + 17$

16. Solve each of the given inequalities. Present your answer using interval notation.

a) $(3y - 1)(y + 2) - (2y + 5)^2 \geq (y + 1)(2 - y) + 3$ b) $-\frac{1}{2}\left(\frac{4}{5}x - \frac{1}{3}\right) < \frac{4}{15} - \frac{8}{15}\left(\frac{3}{4}\right)$

17. Graph each of the given equations.

a) $y = -\frac{2}{3}x + 5$ b) $3x - 5y = 15$

18. *a) Prove that if x and y are rational numbers, then so is $x + y$. (i.e. The set of all rational numbers is closed under addition.)

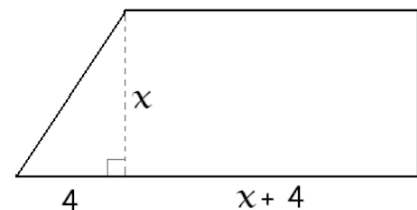
*b) Prove that if x and y are both irrational numbers, that does not mean that $x + y$ is also irrational. (i.e. The set of all irrational numbers is not closed under addition.)

*c) Prove that $\sqrt{2} - 1$ is irrational.

19. Solve each of the following application problems.

- The difference between two numbers is 22. Find the numbers if their product is 1035. Find these numbers.
- We throw a small object upward from the top of a 256ft tall building. The vertical location of the object, (measured in feet) t seconds after we threw it is $L = -16t^2 + 96t + 256$. How long does it take for the object to hit the ground?
- The sum of four consecutive even numbers is 84. Find these numbers.
- We have some five-dollar bills and ten-dollar bills in the register. All together, we have 32 bills, totaling 265 dollars. How many ten- and five-dollar bills do we have?
- Kendall's age is eight years less than four times the age of his sister, Meloney. How old are they if the product of their ages is 60?
- Ann and Bonnie are discussing their financial situation. Ann said: *'If you gave me five dollars, we would end up with the same amount of money.'* Bonnie answers: *'But if you gave me just one dollar then I would end up with four times as much money as you.'* How much do they each have?
- Dawn was thinking about the number. If we add three to it and then square the sum, we get the same result if we add 27 to the number and then double the sum. Of what number is Dawn thinking?
- The greatest angle in a triangle 20° less than three times the smallest angle. The third angle is 20° more than the smallest angle. Find the three angles.
- Stephen's job offered him a temporary position. He would be paid every week. His first paycheck would be \$500. A week later, his second check would be for \$520, the next week \$540, and so on. His last paycheck will be for \$1280. What would be the total amount paid to Stephen?

20. Consider the figure shown on the picture. Angles that look like right angles are right angles. Find the value of x if the figure shown on the picture has area 72 unit^2 .



Appendix A – Answers

1.2 – The Words And and Or

Practice Problems

1. true 2. false 3. true 4. true 5. false 6. true 7. false 8. true 9. false 10. false 11. true
 12. false 13. true 14. false 15. true 16. false 17. true 18. true 19. true 20. false
 21. a) 1, 3, and 5 b) 1, 2, 3, 4, 5, 7

Enrichment

1. there are 8 cases

A	B	C	A or B or C
true	true	true	true
true	true	false	true
true	false	true	true
false	true	true	true
true	false	false	true
false	true	false	true
false	false	true	true
false	false	false	false

3. there are 8 cases

A	B	C	(A and B) or C
true	true	true	true
true	true	false	true
true	false	true	true
false	true	true	true
true	false	false	false
false	true	false	false
false	false	true	true
false	false	false	false

2. there are 8 cases

A	B	C	A and B and C
true	true	true	true
true	true	false	false
true	false	true	false
false	true	true	false
true	false	false	false
false	true	false	false
false	false	true	false
false	false	false	false

4. there are 8 cases

A	B	C	A and (B or C)
true	true	true	true
true	true	false	true
true	false	true	true
false	true	true	false
true	false	false	false
false	true	false	false
false	false	true	false
false	false	false	false

Problem Set 1

1. a) true b) false c) true d) false e) true f) false g) true h) true
 2. a) true b) false c) true d) false e) true f) true
 3. a) true b) false c) true d) true e) false f) true
 4. a) 1,3 b) 1,2,3,4,5,7 c) 4,6,8 d) 2,4,5,6,7,8 e) 1,2,3,4,5,6,7,8 f) none

2.2 – The Order of Operations Agreement

Sample Problems

1. 5 2. 8 3. 13 4. 55 5. 25 6. 81 7. 7

Practice Problems

1. 43 2. 51 3. 9 4. 12 5. 4 6. 4 7. 1 8. 40 9. 15 10. 5 11. 3
12. 20 13. 85 14. 2 15. 1

2.3 – Perimeter and Area of a Rectangle

Discussion

1. $1\text{ yd}^2 = 9\text{ft}^2$
Algebraically: if $1\text{ yd} = 3\text{ft}$, then $1\text{ yd}^2 = 1\text{ yd} \cdot 1\text{ yd} = 3\text{ft} \cdot 3\text{ft} = 9\text{ft}^2$

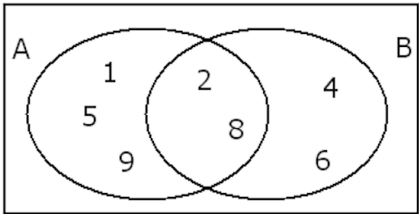
Practice Problems

1. $P = 40\text{cm}$, $A = 96\text{cm}^2$ 2. $P = 56\text{ft}$, $A = 159\text{ft}^2$ 3. $P = 36\text{ft}$, $A = 54\text{ft}^2$

2.4 – Introduction to Set Theory

Practice Problems

1. a) true b) false c) true d) false e) true
2.



3. a) true b) false c) true d) false e) true
4. a) $\{1, 2\}$ b) $\{1, 2, 3, 4, 5, 6\}$ c) $\{4, 5, 6, 7\}$
d) $\mathbb{Z}$ (the set of all integers)
5. Y is not an element of set X . Instead, Y is a subset of X , denoted by $Y \subseteq X$.

6. This is naturally true if $A = B$ as every set is a subset of itself. However, if A and B are different sets, at least one of $A \subseteq B$ and $B \subseteq A$ will be false.
7. We will list the subsets of A by organizing by the number of their elements.

0-element subsets:	$\emptyset$	1 subset
1-element subsets:	$\{1\}, \{2\}, \{3\}, \{4\}$	4 subsets
2-element subsets:	$\{1, 2\}$ $\{1, 3\}$ $\{2, 3\}$ $\{1, 4\}$ $\{2, 4\}$ $\{3, 4\}$	6 subsets
3-element subsets:	$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$	4 subsets
4-element subsets:	$\{1, 2, 3, 4\}$	1 subset

So there are 16 subsets.

Problem Set 2

1. a) true b) true c) false d) true e) false f) false g) true h) true
2. a) true b) false c) false d) true
3. a) 1, 2, 3, 4, 5 b) 4, 5, 6 c) all natural numbers d) 2, 4, 6, 8, 10
4. a) 9 b) 10 c) 12 d) 5 e) 4 f) 2 g) 8 h) 49 i) 2 j) 12 k) 5 l) 27 m) 1
5. $36 - 2 \cdot ((5 - 2)^2 + 4) = 10$ 5. a) $P = 84\text{m}, A = 360\text{m}^2$ b) $P = 32\text{in}, A = 48\text{in}^2$

3.1 – The Set of All Integers

Practice Problems

1. a) true b) true c) false d) true e) true f) true
2. a) false b) true c) true d) false e) true f) true g) false
3. a) $5 > -7$ or $5 \geq -7$ b) $-12 < -4$ or $-12 \leq -4$ c) $0 > -8$ or $0 \geq -8$
d) $-1 > -4$ or $-1 \geq -4$ e) $-7 \geq -7$ or $-7 \leq -7$
4. a) 5 b) 5 c) -5 d) -5 e) 0 f) 3 g) 21
5. a) 5 b) -3 c) -6 d) -15 e) 0 f) 7 g) undefined h) -4 i) -28 j) -13
k) 1 l) 8 m) -3 n) 0 o) undefined p) 0 q) 10 r) 8

3.2 – Order of Operations on Integers

Practice Problems

1. 43 2. -29 3. 49 4. -45 5. -162 6. 0 7. 9 8. 12 9. 22 10. 28 11. -2
12. undefined 13. -16 14. -21 15. 24 16. -4 17. 19 18. -33 19. -17 20. 2
21. -2 22. -6 23. -8 24. -9 25. 20 26. 3 27. 15 28. -14 29. 0 30. 2
31. -2 32. -4

Enrichment

1. a) 69 and 11 b) 0 and 40
The pairs are colored differently. If your printout is black and white, look at the document at your computer.
a) $|5 - 2| - 4 + 7| - 10| = 69$ and $|5 - 2| - 4 + 7| - 10| = 11$
b) $2|-1 - 5| - 3|-4| = 0$ and $2|-1 - 5| - 3|-4| = 40$
2. a) $5 - 2 \left((-3)^2 - ((4 - 7)^2 + 2) \right) = 9$ b) $5 - \left((2 - 3)^2 - (4 - 7) \right)^2 + 2 = -9$

3.3 – Perimeter and Area of a Right Triangle

Practice Problems

1. $P = 56\text{ft}, A = 84\text{ft}^2$ 2. $P = 26\text{m}, A = 31\text{m}^2$ 3. $A = 9\text{ft}^2$ 4. $P = 84\text{cm}, A = 200\text{cm}^2$

Enrichment

The two areas are equal. Both the triangle and the rectangle have area 30cm^2 , and then they both lose the same region. So they remain equal.

3.4 – Square Root of an Integer

Practice Problems

1. a) 10 b) undefined c) -10 d) -7 e) undefined d) 1 e) 0
 2. a) 4 b) 3 c) 2 d) 2 3. a) -7 b) 4 c) 2 4. a) 2 b) 4

3.5 – Factors of a Number

Practice Problems

1. a) 80, 75, 270 b) 75, 270 c) 128, 80, 64
 2. a) 1296, 420, 55050 b) 9800, 420, 55050 c) 1296, 420, 55050
 3. a) 28072, 67808, 1296, 7620 b) 60610, 7620 c) 1296 d) 28072, 60610, 7620
 4. 1, 2, 3, 4, 6, 8, 12, 16, 24, 48 5. 91

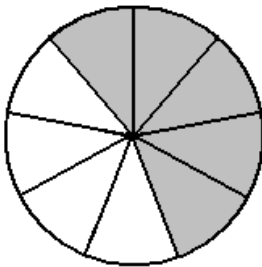
Problem Set 3

1. a) true b) true c) false d) false e) false g) true h) true 2. a) true b) true c) true
 3. a) 678678, 5019, 49740 b) 46228, 49740 c) 983305, 49740 d) 678678
 4. 1, 2, 4, 5, 8, 10, 20, 40 5. a) true b) false c) false d) true e) true f) false g) true
 6. a) -3 b) -70 c) -9 d) 9 e) -1 f) 16 g) 6 h) 1 i) 3 j) -40 k) 16 l) 4
 7. a) -8 b) 15 c) 2 d) -8
 8. a) -1 b) 13 c) -1 d) -72 e) 22 f) -42 g) 112 h) 40 i) -23
 9. a) 22 b) 9 c) 4 d) 4 e) -3 f) 2 g) 10 h) undefined
 10. $P = 12\text{ft}$, $A = 6\text{ft}^2$ 11. $P = 68\text{cm}$ $A = 164\text{cm}^2$ 12. 60

4.1 – Fractions 1: The Definition

Practice Problems

1. 16 2. 20
 3.



4. \$420 5. \$595 6. 42 7. \$2120
 8. a) 180 b) 168 c) $\frac{3}{7}$ is larger
 9. \$17280 in taxes and will keep \$36720.
 10. \$2472 11. \$30000

4.2 – Division With Remainder

Practice Problems

1. a) 18 R 6 b) 95 R 5 c) 114 R 6 d) 16 R 9 2. a) 7 b) 1 c) 2 d) 9

4.3 – Algebraic Expressions and Statements

Sample Problems

Practice Problems

1. a) 22 b) 16 c) -16 d) 16 e) -5 f) 5
2. a) 22 b) -30 c) 1 d) undefined e) 40 f) 100 g) 18 h) 36 i) 17 j) 15 k) -34
3. a) 5 b) 9 4. a) 784ft b) 208ft 5. a) 12 b) 36 c) 25 d) 2 e) undefined
6. a) $37 \neq 13$ no b) $55 = 55$ yes c) $70 \neq 76$ no d) $139 = 139$ yes
7. a) $-4 \neq 20$ no b) $12 = 12$ yes c) $8 = 8$ yes d) $-12 \neq 28$ no
8. a) $13 = 13$ yes b) $15 \neq 13$ no
9. a) $-62 \leq -57$ yes b) $16 \not\leq 8$ no c) $-32 \leq -32$ yes d) $-2 \not\leq -7$ no
10. a) $6 \not< 1$ no b) $14 < 19$ yes c) $10 < 10$ no d) $2 < -8$ no
11. a) $y = 4x - 3$ b) $-A = (B - 3C) + 1$ c) $2M = N(-M) - 5$ 12. $5x - 3$
13. a) $x, x + 1$, and $x + 2$ b) $y - 1, y$, and $y + 1$ c) $L - 2, L - 1$, and L
1. a) 20 b) 16 c) -1 d) 1 e) -11 f) -3 g) 20 h) 4 i) 13 j) 17
2. a) 17 b) 5 c) 35 d) 100 e) 196 f) 18 g) 6 h) 11 i) 26 j) 2
3. a) -1 b) undefined c) 3 d) -7 4. a) -6 b) -2 c) 0 d) -30 e) -20
5. a) -3 b) 4 c) -7 d) undefined e) -4
6. a)

	$x = 2$	$x = 5$	$x = 6$	$x = 10$	$x = -1$	$x = -5$	$x = -8$
$2x - 3$	1	7	9	17	-5	-13	13
$2x + 3$	7	13	15	23	1	-7	-19

b)

	$x = 2$	$x = 5$	$x = 6$	$x = 10$	$x = -1$	$x = -5$	$x = -8$
$2x - 3$	1	7	9	17	-5	-13	13
$-2x + 3$	-1	-7	-9	-17	5	13	-13

7. a) -1 b) -1 c) undefined d) -1 8. a) 11 b) -13 c) 3
9. a) 5 b) 6 c) 7 d) 3 $|a - b|$ always gives us the distance between a and b on the number line
10. a) $1 = 1$ yes b) $6 = 6$ yes c) $3 = 3$ yes
11. a) $0 \neq 2$ no b) $-5 \neq 1$ no c) $3 = 3$ yes d) $0 = 0$ yes e) $-8 \neq 4$ no
12. a) $1 = 1$ yes b) $4 \neq 21$ no c) $6 = 6$ yes d) $41 = 41$ no
13. a) $32 \neq 36$ no b) $40 \neq 32$ no
14. a) $7 \not< -23$ no b) $4 \not< 4$ no c) $2 < 12$ yes d) $-1 < 9$ yes e) $-5 \not< -23$ no
15. a) $-2 \geq -5$ yes b) $0 \geq -2$ yes c) $10 \not\geq 13$ no d) $6 \not\geq 7$ no e) $-4 \geq -8$ yes
16. a) $A - B = A(-B) - 4$ b) $x^2 = 5(-x) - 8$ c) $P - 10 = (Q + 2R) - 5$ d) $4y = 2(y + 7) + 1$
e) $x = -x + 10$ f) $(x + y) = x^2 - y^2 + 10$ 17. $3x + 7$
18. a) $x, x + 2, x + 4$, and $x + 6$ b) $x - 6, x - 4, x - 2$, and x

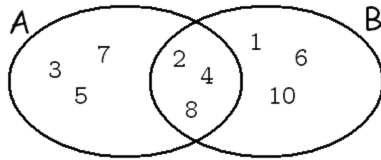
Problem Set 4

1. a) true b) false c) true d) false e) true f) true g) false h) true i) false j) true
k) false l) true m) false 2. 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90

3. a) 1189188 b) 101010, 1189188 c) 1189188 d) 1189188, 35530, 1234321

4. 101 and 2017 5. 2, 3, 5, 7, 11 6. 288 R 2

7.



8. a) $\{0, 3, 6, 9, 12, 15, 18\}$
b) $\{0, 1, 2, 3, 4, 5, 6, 7, 10, 15, 20\}$
c) $\{0, 5\}$ d) U or $\{0, 1, 2, 3, \dots, 19, 20\}$
e) $\{7, 8, 9, 10, 11\}$ f) $\{0, 1, 2, 3, 4, 5, 6, 7\}$
g) $\{0, 1, 2, 3\}$ h) $\{0, 4, 8, 12, 16, 20\}$
i) $\{0, 3, 4, 6, 8, 9, 12, 15, 16, 18, 20\}$ j) $\{0, 12\}$

9. a) i) true ii) false iii) false iv) true

b) x is a rectangle that is NOT a square, i.e. a rectangle that has two sides with different lengths.

10. a) $\{1, 2, 3, 4, 5\}$ b) $\{4, 5, 6\}$ c) $\mathbb{N}$ (all natural numbers) d) $\{2, 4, 6, 8, 10\}$ 11. $B \subseteq A$

12. a) 16 b) 15 c) 105

13. a) -5 b) 6 c) 1 d) -5 e) 4 f) -8 g) 16 h) -32 i) -4 j) -8 k) -16
l) -32 m) -40 n) 16 o) 58 p) 100 q) 34 r) 15 s) -110 t) -56 u) 50

14. a) 4 b) -4 c) -4 d) -4 e) 20 f) 1 g) -20 h) 1 i) 29 j) 49 k) 27 l) -21 m) 9
n) -23 o) 100 p) -100 q) 1 r) -21 s) 29 t) 49 u) -23 v) 100 w) -100 x) 100

15. a) 72 b) undefined c) 5 d) 10 e) -24 f) 3 g) undefined h) 45 i) 61 j) -1 k) -6
l) -27 m) 14 n) 20 o) 31 p) 12 q) 120 16. a) 15 b) 24 c) 4

17. a) 3 b) undefined c) 14 d) 12 e) 2 f) -2 18. a) 12 b) 135 c) 5 d) 31 e) 1

19. -2 and 1 20. 0, 3, and 4 21. $P = 42m$, $A = 96m^2$

22. a) $X = 2Y - 3$ b) $-x = \left(y + \frac{z}{2}\right) + 5$ c) $ab = 3(a + b) - 14$ d) $m + 2n = \frac{m}{n} + 1$
e) $3(x - y) = x(-y) - 1$ f) $P = 2D - 10$ g) $2.5 + (x - 1)1.5$ h) $x, x + 1$, and $x + 2$ i) $x(4x - 3)$
j) Ann has $A - 30$ dollars and Beatrix has $B + 30$. 23. 3 24. a) 203 b) 0 25. 5

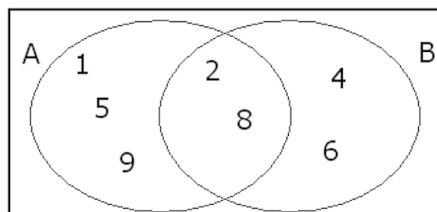
26. Hint: write down all the possible number triples that multiply to 36. Look at the sum of each triple.

5.1 – Set Operations

Practice Problems

1. a) $\{1, 4\}$ b) $\{1, 2, 3, 4, 5, 6, 9\}$ c) $\emptyset$ d) $\{1, 2, 3, 4, 5\}$ or Q

2. a)



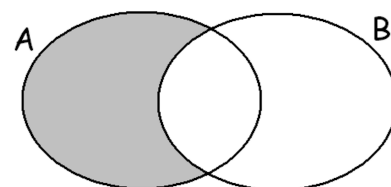
- b) i) $\{2, 8\}$ ii) $\{1, 2, 4, 5, 6, 8, 9\}$ iii) $\{2, 4, 6, 8\}$

- c) i) false ii) true iii) true

3. a) $P \cap M$ - the set of all students taking mathematics and physics at Truman College.
b) $P \cup M$ - the set of all students taking mathematics or physics or both at Truman College.

4. a) true b) false c) false d) true 5. a) true b) true c) true d) true

6. If A is a subset of B , then there is no element that belongs to A but not to B . This will show up on the Venn diagram by the shaded region shown containing no elements.



5.2 – Introduction to Number Theory

1. 1, 2, 3, 4, 6, 8, 12, 16, 24, 48 2. 91 3. a) 80, 75, 270 b) 75, 270 c) 128, 80, 64
4. a) $600 = 2^3 \cdot 3 \cdot 5^2$ b) $5500 = 2^2 \cdot 5^3 \cdot 11$ c) $2016 = 2^5 \cdot 3^2 \cdot 7$ d) $2015 = 5 \cdot 13 \cdot 31$
5. a) yes b) no, $2019 = 3 \cdot 673$ c) no, $2021 = 43 \cdot 47$ d) no, $2023 = 7 \cdot 289$ e) yes f) yes g) yes h) no, $4009 = 19 \cdot 211$
6. a) 101 b) 1009, because $1001 = 7 \cdot 143$, $1003 = 17 \cdot 59$ and $1007 = 19 \cdot 53$ (1002, 1004, 1006, and 1008 are divisible by 2 and 1005 by 5)

5.3 - One- and Two-Step Equations

Discussion

- a) 1 b) 0 c) 0 d) 0

One thing that is unusual in this problem is the idea of cancellation. Cancellation results in 0 or 1, depending on the operation.

Sample Problems

1. 11 2. -5 3. 24 4. 53 5. 2 6. -14 7. -30 8. 0 9. 5 10. 0 11. -4 12. -18
13. \$340 14. 9 nights 15. a) $x = \frac{z+4y}{3}$ b) $y = \frac{3x-z}{4}$

Practice Problems

1. -4 2. -5 3. 4 4. -11 5. 14 6. -4 7. -37 8. 0 9. -14 10. 12
11. -10 12. -2 13. -6 14. -6 15. 42 16. 11 17. -6 18. 2
19. Ann has \$200 and Bonnie has \$175 20. 9 years
21. a) $a = 2c - b$ b) $y = \frac{-5x+15}{3}$ c) $F = \frac{C}{M} + S$

5.4 – Fractions – Part 2: Equivalent Fractions

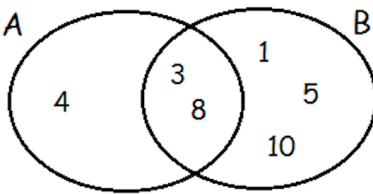
Practice Problems

1. a) $\frac{2}{9} = \frac{10}{45}$ b) $\frac{2}{9}$ of 90 is 20, and so is $\frac{10}{45}$ of 90. 2. a) $\frac{30}{36}$ b) $\frac{25}{30}$
3. a) $\frac{3}{5}$ b) $\frac{1}{4}$ c) 1 d) $\frac{1}{2}$ e) $\frac{3}{2}$ f) $\frac{11}{5}$
4. a) 30% b) 85% c) 240% d) 6% e) 500% 5. a) 24 b) 41 c) 48
6. a) $\frac{7}{10} = \frac{21}{30}$ and $\frac{2}{3} = \frac{20}{30}$. Clearly, $\frac{21}{30} > \frac{20}{30}$ and so $\frac{7}{10} > \frac{2}{3}$.
 b) $\frac{7}{10}$ of 60 is 42, and $\frac{2}{3}$ of 60 is 40. Since $42 > 40$, we get again that $\frac{7}{10}$ is greater.

Problem Set 5

1. a) {1, 3, 4, 5, 8, 10} b) {3, 8}

c)



2. a) {1, 2, 3, 4, 6, 7, 8, 9} b) {1, 9} c) = {1, 3, 6, 9}
 d) {1, 2, 3, 6, 7, 9} e) {1, 9}
3. a) true b) false c) false d) true e) true
 f) true g) true

4. a) $120 = 2^3 \cdot 3 \cdot 5$ b) $480 = 2^5 \cdot 3 \cdot 5$ c) $2016 = 2^5 \cdot 3^2 \cdot 7$
5. 1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75, 150 6. 201 7. a) 25 b) 45 c) 40
8. a) $\frac{4}{5}$ b) i) $\frac{4}{5}$ ii) $\frac{20}{25}$ c) i) $\frac{16}{20}$ ii) $\frac{28}{35}$ d) 80% 9. a) 55 b) 2 c) 3
10. a) 5 b) -2 c) 7 11. 4, -1 12. 5, -2, 0 13. a) -2 b) 40 c) 32 d) 0
14. a) 3 b) 25 15. a) $\frac{A+M}{3}$ b) $R(P-T)$ 16. 7 17. 12 18. 9
19. 4ft 20. Ann has \$140, Beatrix has \$130 21. 10 nights 22. 21 units

6.1 – Simplifying Algebraic Expressions

Sample Problems

1. a) 8 b) 192 2. a) $5x - y$ b) $a - 13b + 6$
3. a) $6x + 12y - 15$ b) $-10a + 5b - 40$ c) $p - 3q + 8m - 6$ d) $a - 3b + 7$
4. a) $a + 2b$ b) $x - 9y$ c) $3m + 3n$ d) $3a - 3b$ e) $7a - 3$
5. a) $-2x - 10$ b) $-7a + 2b$ c) $3b$ d) $a + 1$ 6. a) -5 b) 10 c) 2 7. $6x + 12$
8. a) Betty paid \$18 and Ann paid \$20. b) There were 26 women and 29 men on the party.
 c) 11cm by 48cm d) 21cm by 87cm e) -86 and -84 f) 17, 19, and 21
 g) 7 big and 17 small h) 41 dimes and 54 quarters i) 10 blue and 43 red pens

Practice Problems

1. a) 11 b) -13 c) 3 2. a) $4a - 14y$ b) a c) $9p - 9q - 1$ d) $-6y$ e) 0
3. a) $5b - 15a - 25$ b) 0 c) $-6n + 18m + 48$ d) $-p + 5q - 3r + 1$
4. a) 0 b) $-5a - 10b$ c) $2x - 4y - 2$ d) $4m + 16$
5. a) 2 b) $-x + 19$ c) 0 d) $-5a + 9b - 14c$ 6. a) $-22x + 12$ b) $60x - 88$ c) $-50x + 100$
7. a) 3 b) 8 c) -1 d) -6 e) 0 f) 20 8. $P = (6x + 22)$ unit $A = (36x - 12)$ unit²
9. a) 9ft by 23ft b) 33 ten-dollar bills and 56 five-dollar bills c) \$12 d) 4 and 29
e) 6 years and 27 years f) -2 g) $p = 7, A = 126$ unit² h) -7
i) 12 quarters and 58 dimes j) 3 k) 4 adult and 19 children l) 245, 250, 255
m) 44 fiction and 49 textbooks

6.2 – The Greatest Common Factor and Least Common Multiple**Practice Problems**

1. a) 12 and 1800 b) 240 and 3600 c) 20 and 2100
2. Yes, it is possible. For example, $\gcd(4, 4) = 4$ and $\text{lcm}(4, 4) = 4$.
But if $m \neq n$, then the two will never be equal. 3. 6, 12, 24, 30, 60, 120

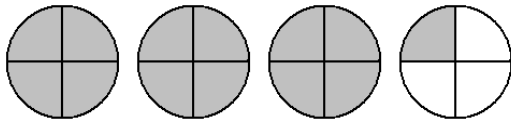
6.3 – Fractions – Part 3: Improper Fractions and Mixed Numbers

Practice Problems

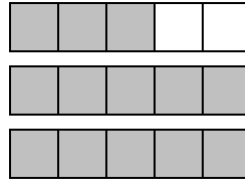
1. a) $3\frac{1}{3}$ b) $1\frac{1}{2}$ c) $3\frac{1}{2}$ d) $17\frac{1}{7}$ e) $5\frac{1}{4}$ f) $7\frac{3}{20}$ g) $3\frac{1}{5}$

2. a) $\frac{19}{5}$ b) $\frac{11}{8}$ c) $\frac{41}{7}$ d) $\frac{17}{5}$ e) $\frac{75}{7}$ f) $\frac{21}{4}$

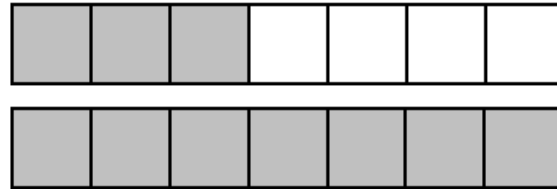
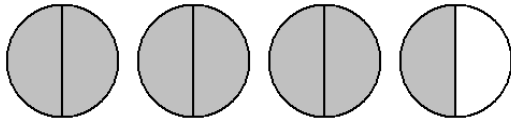
3. a) $\frac{13}{4} = 3\frac{1}{4}$ b) $2\frac{3}{5} = \frac{13}{5}$



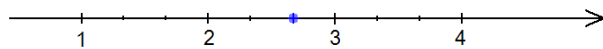
c) $\frac{7}{2} = 3\frac{1}{2}$



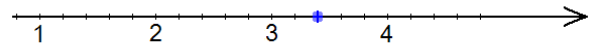
d) $\frac{10}{7} = 1\frac{3}{7}$



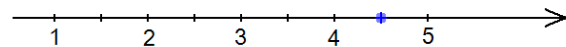
4. a) $\frac{8}{3} = 2\frac{2}{3}$



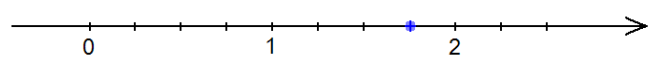
b) $3\frac{2}{5}$



c) $\frac{9}{2} = 4\frac{1}{2}$



d) $\frac{7}{4} = 1\frac{3}{4}$



6.4 – Fractions – Part 4: Adding and Subtracting

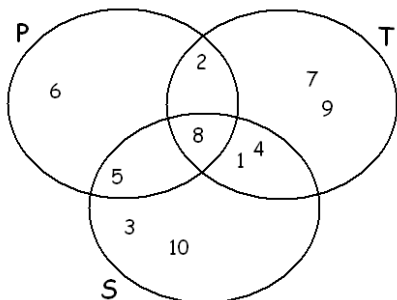
Practice Problems

1. $\frac{3}{2}$ 2. $-\frac{1}{6}$ 3. $-\frac{8}{3}$ 4. $\frac{21}{10}$ 5. $\frac{11}{8}$ 6. $\frac{13}{12}$ 7. $\frac{4}{3}$ 8. $-\frac{11}{8}$ 9. $-\frac{7}{10}$ 10. $-\frac{73}{12}$
 11. 0 12. $-\frac{19}{12}$ 13. $-\frac{1}{3}$ 14. $\frac{7}{2}$ 15. 3

Problem Set 6

1. a) true b) true c) false d) false e) true f) true g) true h) false (5 is the only one though) i) false
 (1 only has one) j) true k) false l) false (try $x = -2$ or -10) m) false
2. a) {4} b) {1, 2, 4, 6, 7, 9} c) {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} d) {5, 6, 7}
3. a) true b) false c) true d) true e) false f) true g) true h) false
4. a) true b) false c) false d) true e) false

5. a) $\{2, 4\}$ b) $\{4\}$ c) $\{4, 7, 10\}$ d) $\{4\}$ e) $\{1, 2, 3, 4, 5, 6, 7, 9, 10\}$ f) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
 g) $\{1, 2, 3, 4, 6, 7, 8, 9, 10\}$ h) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ i) Both set operations are associative.

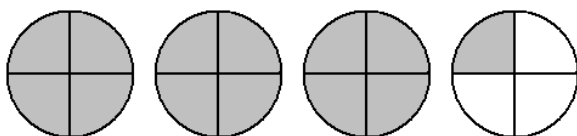


7. a) $\{2, 8\}$ b) $\{1, 2, 3, 4, 5, 7, 8, 9, 10\}$
 c) $\{1, 2, 3, 4, 5, 8, 10\}$ d) $\{2, 5, 8\}$
 e) $\{1, 2, 4, 8\}$ f) $\{1, 2, 4, 5, 6, 8\}$
 8. a) 65 R 2 b) 14 R 1 c) 56 R 11
 9. 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 10. 71

6.

11. a) 72, 40, 150, 190, 360 b) 72, 99, 150, 135, 360 c) 40, 150, 135, 190, 360
 d) 72, 99, 135, 190, 360 12. $\gcd = 40$ $\text{lcm} = 240$ 13. a) 30 b) 40

14. a) $\frac{4}{7}$ b) i) $\frac{8}{14}$ ii) $\frac{20}{35}$ c) i) $\frac{12}{21}$ ii) $\frac{28}{49}$



16. a) $5\frac{3}{5}$ b) 44% c) $\frac{43}{8}$

17. a) $-\frac{1}{12}$ b) $\frac{1}{8}$

15.

18. a) -3 b) -4 c) 3 d) 1 e) 22 f) 3 g) 7 h) 3 i) 1 j) 2 k) 1 l) 3
 m) -12 n) undefined

19. a) $a + b$ b) $7a - 3b$ c) $26a - 9b$ d) $14a - b$ e) $15b - 10a$ f) $5x - 2$
 g) x h) $9x - 6$ i) $x + 2$ j) $30 - 55x$ k) $x + 6$ l) $4m$ m) 10

20. a) 6 b) -13 c) -20 d) 0 e) -5 f) 5 g) 0 h) -6 i) 5 j) 0 k) 16
 l) 2 m) -2 n) 10 o) -1

21. a) 28 and 45 b) 18, 20, 22, and 24 c) Meloney is 28, Kendall is 41 d) 12m by 41m e) 2
 f) Ann has \$200, Bonnie has \$175 g) -9 h) 8 ten-dollar bills and 37 five-dollar bills
 i) 36° , 56° , and 88° j) 11 five-dollar bills and 21 ten-dollar bills

22. a) 24 unit b) 50 unit 23. 9

7.1 – Fractions – Part 5: Multiplying and Dividing

Practice Problems

1. a) $-\frac{2}{5}$ b) $\frac{1}{4}$ c) $\frac{4}{7}$ d) $-\frac{4}{9}$ e) $\frac{7}{10}$ f) $-\frac{4}{21}$ g) $\frac{7}{15}$ h) $\frac{4}{3}$ i) $-\frac{2}{3}$ j) $\frac{1}{5}$
2. $P = \frac{62}{3}\text{in}$ $A = \frac{148}{9}\text{in}^2$ 3. a) $\frac{9}{4}$ b) $-\frac{39}{4}$ c) $\frac{31}{9}$
4. a) $-\frac{3}{2}$ b) undefined d) $-\frac{1}{3}$ d) $-\frac{1}{6}$ 5. a) 500ft b) 6250ft² c) $36\frac{\text{m}}{\text{s}}$

7.2 – Rules of Exponents

Discussion

1. Two minus signs still cancel out each other, it is just that there are really three negative signs in the expression $-(-5)^2 = -(-5)(-5)$.
2. a and b appear in the denominator and division by zero is not allowed.
3. $2 \cdot 5^x = 2 \cdot \underbrace{5 \cdot 5 \cdot \dots \cdot 5}_{x \text{ times}}$ Imagine that x is very large. Then we have lots of 5-factors but just one 2-factor.

In order to be able to simplify to 10^x , we would need to see $2^x \cdot 5^x$.

Sample Problems

1. a) $2x^9$ b) $32x^9$ c) $16x^{20}$ d) $-x^5y^8$ e) $-8a^{11}$ f) $-16a^7b^8$ g) $2x^7y$ h) $-2a^4b$
2. a) 27 b) 2 c) 3125 3. a) $3M$ b) $4M$ c) $\frac{M}{3}$ d) M^2
4. a) $2^{2018} \cdot 5^{2018}$ b) $2^{1000} \cdot 3^{2000}$ c) $2^{150} \cdot 3^{100} \cdot 5^{50}$ d) $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$
5. a) $3.8 \cdot 10^9$ b) $6.25 \cdot 10^{12}$ 6. a) $4 \cdot 10^{26}$ b) $2.5 \cdot 10^{37}$ c) $1 \cdot 10^{19}$ d) $2.5 \cdot 10^{11}$
6. a) $9 \cdot 10^{20}$ b) $1.8 \cdot 10^{15}$ c) $5 \cdot 10^5$ d) $2.5 \cdot 10$
1. a) -9 b) 9 c) $16a^{12}$ d) $-8a^{12}$ e) $8a^6b^3$ f) $64a^6b^3$ g) $4a^2b^3$ h) m^6 i) $6p^3q^8$
j) a^9 k) $-20s^{11}t^2$ l) $4x^5y^{11}$ m) $-18a^{11}b^5$ n) x^3y^4 o) $-32a^7b^9$
2. a) 8 b) $250 \cdot 5^x$ c) 96
3. a) $5P$ b) $25P$ c) $\frac{P}{5}$ d) P^2 e) $11P$
4. a) $2^{200}5^{100}$ b) $2^{2000} \cdot 3^{4000}$ c) $2^{360} \cdot 3^{120} \cdot 5^{120}$ d) $2^9 \cdot 3^3 \cdot 5^3$
5. a) $2.1 \cdot 10^{10}$ b) $3 \cdot 10^{14}$ c) $3.25 \cdot 10^8$
6. a) $9 \cdot 10^{20}$ b) $1.8 \cdot 10^{15}$ c) $5 \cdot 10^5$ d) $2.5 \cdot 10$

Practice Problems

7.3 – The n th Root of a Number

Practice Problems

1. -2 2. 2 3. 1 4. 0 5. undefined 6. 3 7. -4 8. 0 9. 4 10. 2
11. -1 12. -3 13. undefined 14. undefined 15. 0 16. undefined 17. -1
18. -1 19. undefined 20. 2 21. undefined 22. -12 23. 144 24. 32 25. 16
26. undefined 27. 16 28. x^4 29. -3 30. 3 31. 1 32. -1 33. 0 34. 2

7.4 – Linear Equations 2

Discussion

a) contradiction b) conditional c) conditional d) identity

Sample Problems

1. -3 2. -5 3. 0 4. $\frac{13}{5}$ 5. There is no solution. 6. -10 7. $\frac{1}{2}$ 8. $\frac{25}{4}$ 9. 6
 10. There is no solution 11. -41 12. -4

Practice Problems

1. 3 2. 4 3. 2 4. $-\frac{1}{4}$ 5. -15 6. 4 7. 2 8. $\frac{12}{5}$ 9. 22 10. 0 11. 1
 12. $\frac{1}{2}$ 13. 4 14. 3 15. 15 16. $\frac{1}{2}$ 17. 9 18. -5 19. no solution
 20. $\frac{4}{13}$ 21. -4 22. all numbers are solution 23. 6 24. $\frac{1}{4}$ 25. 2 26. -5
 27. 0 28. -4 29. 7 30. 18

7.5 – Intervals

Practice Problems

1. a) $[3, \infty)$ b) $(-2, 9]$ c) $(-\infty, \infty)$ d) $(-\infty, -5) \cup (4, \infty)$ e) $(-\infty, 12)$ f) $[6, 10]$
 g) $(-\infty, -2) \cup (2, \infty)$ h) $(0, 7)$ i) $[3, 4]$
 2. a) $(1, 7)$ b) $(2, 5)$ c) $[1, 7]$ d) $[2, 5]$ e) $[1, 7)$ f) $(2, 5]$ g) $(-1, 10)$ h) $(3, 8)$
 i) $[-1, 10]$ j) $[3, 8]$ k) $(-1, 10)$ l) $[3, 8]$ m) $[-2, 2] \cup (4, 7)$ n) $\emptyset$ o) $[-2, 8]$
 p) $(0, 1)$ q) $[5, 11)$ r) $[7, 10)$
 3. a) $(-\infty, 8)$ b) $(-\infty, 4)$ c) $(-\infty, 8]$ d) $(-\infty, 4]$ e) $(-\infty, 8)$ f) $(-\infty, 4]$ g) $\mathbb{R} = (-\infty, \infty)$
 h) $(3, 5)$ i) $\mathbb{R} = (-\infty, \infty)$ j) $[3, 5]$ k) $\mathbb{R} = (-\infty, \infty)$ l) $(3, 5]$
 m) $(-\infty, -2) \cup (1, \infty)$ n) $\emptyset$ o) $(-\infty, -2] \cup [1, \infty)$ p) $\emptyset$ q) $(-\infty, -2] \cup (1, \infty)$ r) $\emptyset$

Problem Set 7

1. a) $\{1, 3, 5, 6, 7, 9, 11, 12, 13, 15\}$ b) $\{3, 9, 15\}$ c) $\{4, 8, 12\}$ d) $\{12\}$ e) $\{3, 4, 8, 9, 12, 15\}$
 2. a) $(-\infty, 5)$ b) $(-\infty, 10]$ c) $[4, 8)$ d) $(3, 11]$ e) $(-\infty, \infty)$ f) $[2, 9)$ g) $\emptyset$ h) $(-\infty, 1] \cup (3, \infty)$
 3. a) false b) true c) false d) true e) false f) true g) false
 4. a) $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ b) $2^{500} \cdot 3^{200} \cdot 7^{100}$ c) $7^2 \cdot 41$ d) $3 \cdot 5 \cdot 11^4$ 5. a) 29 b) 7
 6. $1, 2, 4, 5, 8, 10, 16, 20, 40, 80$ 7. $80 = 2^4 \cdot 5$ 8. $\gcd = 16$ $\text{lcm} = 480$ 9. a) 90 b) 160
 10. a) $\frac{9}{10}$ b) 90% c) $\frac{63}{70}$ d) $\frac{18}{20}$ 11. a) $7\frac{3}{11}$ b) $\frac{26}{7}$ c) 15%
 12. a) $-\frac{2}{15}$ b) $\frac{17}{45}$ c) 1 d) 14 e) $-\frac{11}{14}$
 13. a) $x = 1.25 \cdot 10^{-10}$ and $y = 4.5 \cdot 10^{13}$
 b) $i) 5.625 \cdot 10^3$ ii) $1.5625 \cdot 10^{-20}$ iii) $4.5 \cdot 10^{13}$ iv) $3.6 \cdot 10^{23}$ v) $6 \cdot 10^{11}$
 14. 15% 15. a) 56% increase b) 2% decrease c) 28% decrease
 16. a) x^{11} b) 6^{2x+1} c) $-x^5$ d) x^5 e) $2x^5y$
 17. a) -3 b) undefined c) 3 d) $\frac{1}{2}$ e) -1 f) 2 g) undefined h) 16
 18. a) 4 b) -3 c) $-\frac{2}{3}$ d) $-\frac{3}{2}$ e) undefined f) 0 19. x^{10}

20. a) x^{18} b) x^{12} c) x^6 d) x^4 21. a) 8 b) 4 c) x^8 d) x^3
 22. a) $6N$ b) $36N$ c) $30N$ d) $\frac{N}{36}$ e) $\sqrt{N}$ f) $\sqrt[10]{N}$
 23. a) -7 b) $\frac{1}{2}$ c) -11 d) $\frac{2}{3}$ e) -13 f) $-\frac{18}{5}$ g) -3 h) 1 i) $-\frac{3}{5}$
 j) -1 k) no solution l) -17 j) all numbers are solution
 24. 45 25. 30, 35, 40 26. 19 children tickets and 8 adult tickets 27. 18m by 121m
 28. 0 29. 2 30. after 99 minutes

8.1 – Multiplying Algebraic Expressions

Sample Problems

1. a) $6x^3y + 12xy^2 - 15xy$ b) $-10x^5 + 5x^4 - 40x^3$ c) $x - 3y + 8z^2 - 6$ d) $2a^2 - 6ab^2 + 4a^2b - 14a$
 2. a) $5x^2 + 12x - 9$ b) $4x^2 - 20x + 25$ c) $-2x^2 - 7x + 4$ d) $-2x^2 + 12x - 18$
 3. a) $-x^2 - 15x + 28$ b) $-m^2 + 6m - 9$ c) $-4x^2 - 2x + 9$
 4. a) 1 b) 0 c) 3 d) -9 5. 7cm 6. -4 7. 4 units by 5 units

Practice Problems

1. a) $-15a^3b^3 + 5a^2by - 25a^2b$ b) 0 c) $16t^5 - 96t^4 + 16t^3$ d) $-x + 5a - 3ax + 1$
 2. a) $-2x^2 + 17x - 21$ b) $-6x^2 + 13x - 6$ c) $9 - 4x^2$ d) $4x^2 - 12x + 9$ e) $2a^2 - 4a + 2$
 3. a) $x - 3$ b) $-5x + 13$ c) $-12x + 31$ d) $-6x^2 + 31x - 40$
 4. a) $8x$ b) $-3x^2 + 13x - 3$ c) $-75m^2 + 30m - 3$ d) $-9x^2 + 6x - 1$
 5. a) 1 b) 2 c) 0 d) 1 e) 3 f) 4 g) -3 h) no solution i) -5 j) all real numbers k) 0
 6. 5cm 7. -5 8. 2 and 4

Discussion

1. According to the order of operations agreement, we perform exponentiations before multiplications. **Enrichment**
 Consider for example, $2 \cdot 3^2 = 2 \cdot 9 = 18$ and $(2 \cdot 3)^2 = 6^2 = 36$. They are not the same.

Similarly, $2(x-3)^2$ is different from $[2(x-3)]^2$.

$$2(x-3)^2 = 2(x^2 - 6x + 9) = 2x^2 - 12x + 18 \text{ and}$$

$$[2(x-3)]^2 = 2^2(x-3)^2 = 4(x-3)^2 = 4(x^2 - 6x + 9) = 4x^2 - 24x + 36$$

2. Notice that $3 - m$ is the opposite of $m - 3$, as

$$3 - m = -m + 3 = -(m - 3) = -1(m - 3)$$

If we square any number and its opposite, the result will be the same.

$$(3 - m)^2 = [-1(m - 3)]^2 = (-1)^2(m - 3)^2 = 1 \cdot (m - 3)^2 = (m - 3)^2$$

We can use this fact when it makes computations easier. For example, $(-x - 4)^2$ can be re-written as $(x + 4)^2$.

1. a) $(a + b)^2 = a^2 + 2ab + b^2$ b) $(a - b)^2 = a^2 - 2ab + b^2$ c) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
 d) $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ e) $(x - y)(x + y) = x^2 - y^2$
 f) $(x - y)(x^2 + xy + y^2) = x^3 - y^3$ g) $(x - y)(x^3 + x^2y + xy^2 + y^3) = x^4 - y^4$
 2. $5x = x \cdot 5 = x(2 + 3) = 2x + 3x$

3. Let $a = 5$ and $b = 3$. The expression $a^2 + b^2$ or $5^2 + 3^2$ is the combined region of the blue square and the purple square.

On the other hand, $(5 + 3)^2$ can be seen as the entire region, the 8 by 8 square. Clearly $34 \neq 64$.

$(a + b)^2$ is the entire image (Still $a = 5$ and $b = 3$)

a^2 is the blue region, b^2 is the purple region. The white rectangular dots represent ab and ba

So, $(a + b)^2 = a^2 + b^2 + 2ab$

8.2 – Basic Percent Problems

Sample Problems

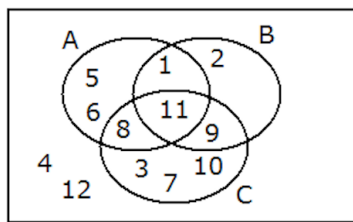
1. 576 2. 150% 3. 125 4. 1500 5. \$8560 6. 15% decrease 7. \$2400 per month 8. \$480

Practice Problems

1. $\frac{2}{17}$ 2. 261 3. 60 4. 700 5. 64% 6. 272 7. 320% 8. 700 9. 12000 10. \$80
 11. \$600 12. 260000 13. C 14. A 15. B 16. E 17. D
 18. a) \$1980 b) \$2178 c) 21% increase 19. 38% 20. 150%

Problem Set 8

1. a)



- b) $\{1, 8, 11\}$ c) $\{1, 5, 6, 8, 9, 11\}$
 d) $\{1, 5\}$ $\{5, 6\}$ $\{6, 8\}$ $\{8, 11\}$
 $\{1, 6\}$ $\{5, 8\}$ $\{6, 11\}$
 $\{1, 8\}$ $\{5, 11\}$
 $\{1, 11\}$

- e) 0–element subsets: $\emptyset$
 1–element subsets: $\{1\}, \{2\}, \{9\}, \{11\}$
 2–element subsets: $\{1, 2\}, \{1, 9\}, \{1, 11\}, \{2, 9\}, \{2, 11\}, \{9, 11\}$
 3–element subsets: $\{1, 2, 9\}, \{1, 2, 11\}, \{1, 9, 11\}, \{2, 9, 11\}$
 4–element subsets: $\{1, 2, 9, 11\}$
2. 1, 2, 6, 13, 39, 78
3. 701 ($201 = 3 \cdot 67$, $91 = 7 \cdot 13$, $2019 = 3 \cdot 673$, $901 = 17 \cdot 53$)
4. a) $\gcd(200, 360) = 40$ $\text{lcm}(200, 360) = 1800$ b) $\gcd(72, 35) = 1$ $\text{lcm}(72, 35) = 2520$
 c) $\gcd(120, 135) = 15$ $\text{lcm}(120, 135) = 1080$
5. a) $12^{100} = 2^{200} \cdot 3^{100}$ and $15^{50} = 3^{50} \cdot 5^{50}$ b) $\gcd(12^{100}, 15^{50}) = 3^{50}$ $\text{lcm}(12^{100}, 15^{50}) = 2^{200} \cdot 3^{100} \cdot 5^{50}$
6. a) true b) false
7. a) $A \cap B$ is the set of all integers divisible by 15. Using notation: $A \cap B = \{n \in \mathbb{Z} : n \text{ is a multiple of } 15\}$
 b) $S = \{-3, 0, 3, 6, 9\}$ c) $T = \{-10, -5, -0, 5, 10\}$
8. a) $(-8, 5)$ b) $P = 36$ unit, $A = 80$ unit²
9. a) 5 b) 16 c) undefined d) 19 e) -4 f) -15 g) -6 h) -10 i) 37 j) 51 k) 16
10. a) 76 b) $7\frac{3}{5}$ c) 760% 11. $\frac{3}{4}$ 12. \$1020
13. a) $\frac{3}{5}$ of 80 is 48 and $\frac{5}{8}$ of 80 is 50. Since $48 < 50$, we also have that $\frac{3}{5} < \frac{5}{8}$
 b) $\frac{3}{5} = \frac{24}{40}$ and $\frac{5}{8} = \frac{25}{40}$ Thus $\frac{3}{5} = \frac{24}{40} < \frac{25}{40} = \frac{5}{8}$
14. a) \$2060 b) \$2060 This is not a coincidence. If we take 103% of a quantity, that is the same as taking 3% of it and adding to the quantity itself.
15. a) $5 \cdot 10^{28}$ b) $2 \cdot 10^{41}$ c) $3.125 \cdot 10^3$ d) $1 \cdot 10^{70}$ e) $1.28 \cdot 10^9$
16. a) $-2x + 18$ b) $-10x + 38$ c) $x + 4$ d) $9y + 20$ e) $5n - 20$ f) $x - 9$
17. a) 16 b) -16 c) a^7 d) a^{12} e) $8a^5$ f) $\frac{1}{2a}$ g) $\frac{4}{a}$ h) $-8x^4$ i) $-\frac{x^2}{3}$ j) 2^{286}
 k) x^{16} l) $4x^8$ m) $-\frac{1}{8}$ n) $-12a^3b^4$

18. a) 8 b) 8^x c) 3^x d) $5 \cdot 5^x$
19. a) $3x^2 - 2x - 1$ b) $4x^2 - 4x + 1$ c) $8x^3 - 12x^2 + 6x - 1$ d) $x^2 - 2x + 1$
 e) $x^2 - 1$ f) $7x^2 - 3x + 9$ g) $16x^3$ h) $2x^6 + 32$ i) $a^3 - b^3$
20. a) 17 b) 2 c) 0 d) -1 e) 11 f) 4 g) no solution h) 11 21. 500
22. a) 600000 b) \$1260
23. a) -21 and 16 b) 110ft by 80ft c) 11 units d) 32° , 52° , and 96° e) 8 f) 5
24. 8 25. 2018 26. a) 10 units b) 84 unit² 27. 2^{2019} 28. there are 8 possibilities 29. 2^{1003}

9.1 – Integer Exponents

Sample Problems

1. $\frac{1}{9}$ 2. 8 3. $\frac{1}{m^4}$ 4. x^5 5. a^7 6. p^4 7. x^5 8. $5a^{15}$ 9. $\frac{1}{t^7}$ 10. 1
11. -1 12. 1 13. $\frac{1}{b^4}$ 14. b^4 15. m^3 16. $\frac{x^3z^4}{y^5}$ 17. $3q^6$ 18. $\frac{27}{8}$ 19. $\frac{2}{y^3}$
20. $\frac{1}{8y^3}$ 21. $\frac{25}{9}$ 22. $\frac{a^5}{b^8}$ 23. $\frac{1}{9m^6}$ 24. $-\frac{b^9}{8a^3}$ 25. k 26. $\frac{9}{4}a^3b$ 27. $-\frac{a^5}{8b^4}$
28. $18p^9q^{10}$ 29. $\frac{4a^{10}}{b^{12}}$ 30. $\frac{x^4}{y^6}$ 31. 1 32. $\frac{xy}{y-x}$ 33. $-\frac{b^8}{2}$ 34. $\frac{1}{b^8}$ 35. $\frac{2x^4}{y^3}$
36. a) $6.375 \cdot 10^{-4}$ b) $6.1413 \cdot 10^{-34}$ c) $4.7813 \cdot 10^4$ d) $1.1333 \cdot 10^{-19}$ e) $1.6903 \cdot 10^{63}$

9.2 – Graph of an Equation

1. Consider $y + 2 = |2x|$

$A(-3, 4)$ is on the graph, $6 = 6$ ✓

$B(0, -2)$ is on the graph, $0 = 0$ ✓

$C(7, 0)$ is not on the graph, $2 \neq 14$

Therefore, $y + 2 = |2x|$ is not the equation of the graph.

Consider $6 - y = |8 - |2x||$

$A(-3, 4)$ is on the graph, $2 = 2$ ✓

$B(0, -2)$ is on the graph, $8 = 8$ ✓

$C(7, 0)$ is not on the graph, $6 = 6$ ✓

Therefore, $6 - y = |8 - |2x||$ is the equation of the graph.

Consider $x^2 + y^2 = 1 + 4(x + y + 5)$

$A(-3, 4)$ is on the graph, $25 = 25$ ✓

$B(0, -2)$ is not on the graph, $4 \neq 13$

$C(7, 0)$ is on the graph, $49 = 49$ ✓

Therefore, $x^2 + y^2 = 1 + 4(x + y + 5)$ is not the equation of the graph.

2. Consider $2y = x + 9$

$A(-3, 3)$ is on the graph, $6 = 6$ ✓

$B(1, 5)$ is on the graph, $10 = 10$ ✓

$C(4, -2)$ is not on the graph, $-4 \neq 13$

Therefore, $2y = x + 9$ is not the equation of the graph.

Consider $|x| + |y| = 6$

$A(-3, 3)$ is on the graph, $6 = 6$ ✓

$B(1, 5)$ is on the graph, $6 = 6$ ✓

$C(4, -2)$ is on the graph, $6 = 6$ ✓

Therefore, $|x| + |y| = 6$ is probably the equation of the graph.

Consider $y + 3 = 9 - |x|$

$A(-3, 3)$ is on the graph, $6 = 6$ ✓

$B(1, 5)$ is on the graph, $8 = 8$ ✓

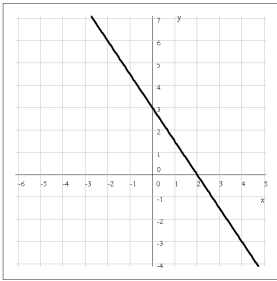
$C(4, -2)$ is not on the graph, $1 \neq 5$ ✓

Therefore, $y + 3 = 9 - |x|$ is not the equation of the graph.

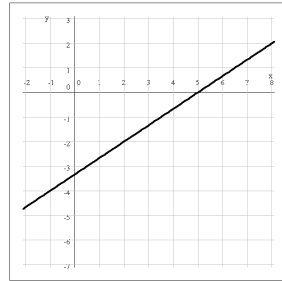
9.3 – Graphing a Line

Practice Problems

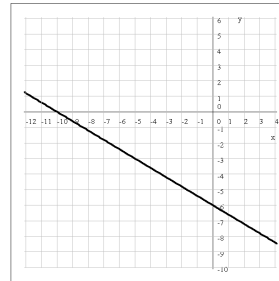
1. $3x + 2y = 6$



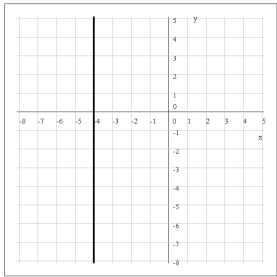
4. $2x - 3y = 10$



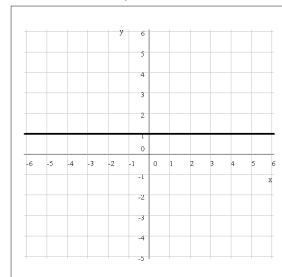
7. $3x + 5y = -30$



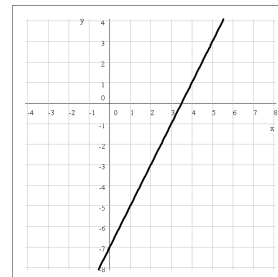
2. $x = -4$



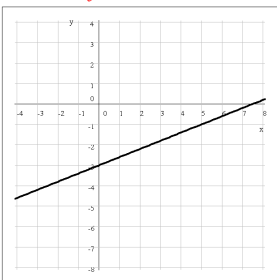
5. $y = 1$



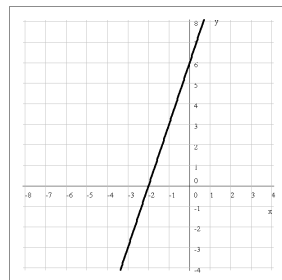
8. $2x - y = 7$



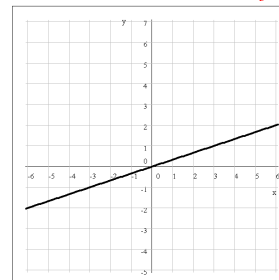
3. $y = \frac{2}{5}x - 3$



6. $y = 3x + 6$



9. $y = \frac{1}{3}x$

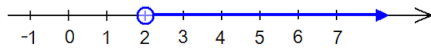


9.4 – Linear Inequalities

Sample Problems

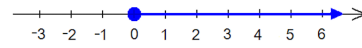
1. set-builder notation: $\{x|x > 2\}$

interval notation: $(2, \infty)$



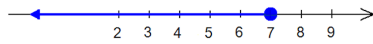
3. set-builder notation: $\{x|x \geq 0\}$

interval notation: $[0, \infty)$



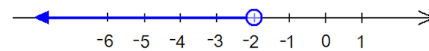
2. set-builder notation: $\{x|x \leq 7\}$

interval notation: $(-\infty, 7]$



4. set-builder notation: $\{x|x < -2\}$

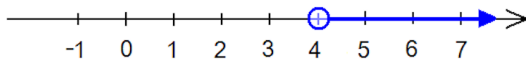
interval notation: $(-\infty, -2)$



Practice Problems

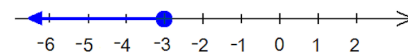
1. set-builder notation: $\{x|x > 4\}$

interval notation: $(4, \infty)$



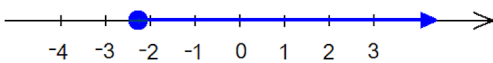
5. set-builder notation: $\{x|x \leq -3\}$

interval notation: $(-\infty, -3]$



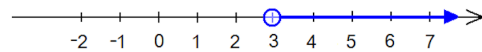
2. set-builder notation: $\left\{x|x \geq -\frac{7}{3}\right\}$

interval notation: $\left[-\frac{7}{3}, \infty\right)$



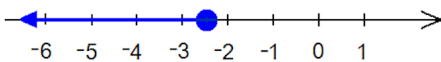
6. set-builder notation: $\{x|x > 3\}$

interval notation: $(3, \infty)$



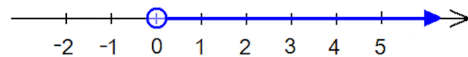
3. set-builder notation: $\left\{x|x < -\frac{5}{2}\right\}$

interval notation: $\left(-\infty, -\frac{5}{2}\right)$



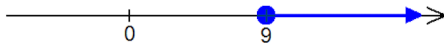
7. set-builder notation: $\{x|x > 0\}$

interval notation: $(0, \infty)$



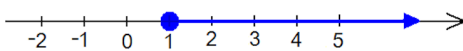
4. set-builder notation: $\{x|x \geq 9\}$

interval notation: $[9, \infty)$



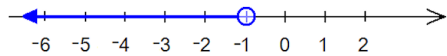
8. set-builder notation: $\{x|x \geq 1\}$

interval notation: $[1, \infty)$



9. set-builder notation: $\{x|x < -1\}$

interval notation: $(-\infty, -1)$



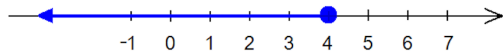
14. set-builder notation: $\{x|x > -8\}$

interval notation: $(-8, \infty)$



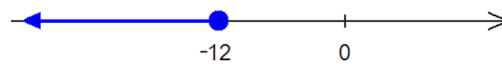
10. set-builder notation: $\{x|x \leq 4\}$

interval notation: $(-\infty, 4]$



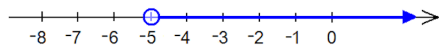
15. set-builder notation: $\{x|x \leq -12\}$

interval notation: $(-\infty, -12]$



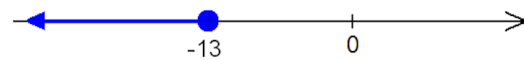
11. set-builder notation: $\{x|x > -5\}$

interval notation: $(-5, \infty)$



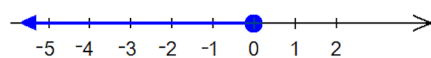
16. set-builder notation: $\{x|x \leq -13\}$

interval notation: $(-\infty, -13]$



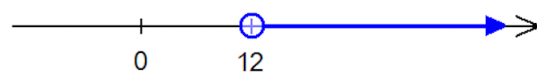
12. set-builder notation: $\{x|x \leq 0\}$

interval notation: $(-\infty, 0]$



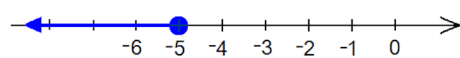
17. set-builder notation: $\{x|x > 12\}$

interval notation: $(12, \infty)$



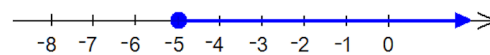
13. set-builder notation: $\{x|x \leq -5\}$

interval notation: $(-\infty, -5]$



18. set-builder notation: $\{x|x \geq -5\}$

interval notation: $[-5, \infty)$



Problem Set 9

1. a) false b) false c) true 2. a) $(-\infty, 6)$ b) $(3, 7]$ c) $(-\infty, \infty)$

3. a) $(3, 11]$ b) $(5, 8]$ c) $(5, 9)$ d) $(-\infty, \infty)$ e) $(2, \infty)$ f) $[5, \infty)$ 4. LCM = 16 and 192 5. $\frac{28}{5} = 5\frac{3}{5}$

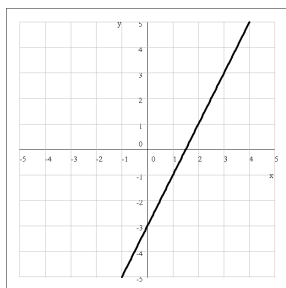
6. a) $\frac{1}{2}$ b) $\frac{3}{5}$ c) $\frac{31}{25}$ d) 1 7. 70 8. a) 3800cm b) 43in

9. a) $\frac{2}{x^3}$ b) $\frac{1}{8x^3}$ c) x^3 d) $\frac{y^3}{x^2}$ e) $\frac{1}{x^2y^3}$ f) $\frac{y}{x^3}$

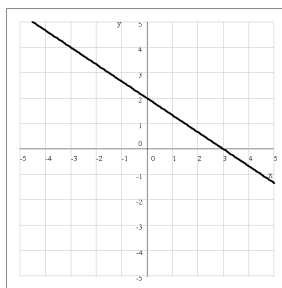
10. a) $2x^8$ b) $2x^2$ c) $8x^8$ d) $\frac{x^2}{8}$ e) $4x^2y^4$ f) $\frac{y^4}{4x^2}$ g) $-a^{14}b^5$ h) $-\frac{1}{a^2b}$ i) 1 j) $-\frac{xy^{12}}{18}$

11. a) $-4x - 8$ b) $-24x - 18$ c) $8x^3 - 125$ d) $\frac{17}{30}$ 12. a) $-\frac{2}{15}$ b) $\frac{3}{8}$ c) $\frac{3}{10}$ d) $-\frac{11}{2}$ e) 12

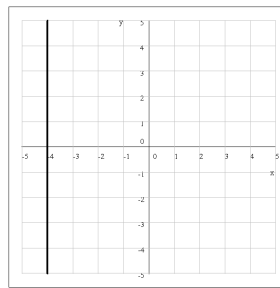
13. a) $y = 2x - 3$



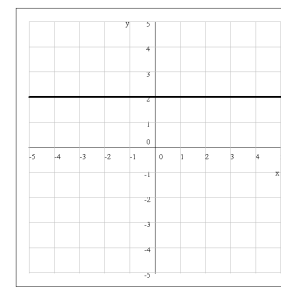
b) $2x + 3y = 6$



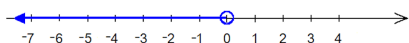
c) $x = -4$



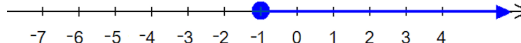
d) $y = 2$



14. a) $(-\infty, 0)$



b) $[-1, \infty)$



15. a) $D(-5, 3)$ b) $P = 24\text{unit}$ $A = 32\text{unit}^2$ c) $M(-1, 1)$ 16. a) 56% increase b) 16% decrease

17. 120 18. \$3200 19. Julia is 19 and Tom is 24

20. a) 16% b) 3 gallons c) 5 gallons d) 10 gallons 21. 22in and 38in 22. 7 unit by 12 unit

23. a) with Bank X: $10 + \frac{12}{100} \cdot 85 = \frac{101}{5} = 20\frac{1}{5} = 20.20$ and

with Bank Y: $14 + \frac{10}{100} \cdot 85 = \frac{45}{2} = 22\frac{1}{2} = 22.50 \implies$ so Bank X is better

b) with Bank X: $10 + \frac{12}{100} \cdot 300 = 46$ and

with Bank Y: $14 + \frac{10}{100} \cdot 300 = 44 \implies$ so Bank Y is better

c) 200

10.1 – Solving Systems of Linear Equations: Elimination

Practice Problems

1. a) $x = -5, y = -2$ b) $p = 1, q = -7$ c) $x = 3, y = -1$ d) $x = -6, y = 8$

e) $x = \frac{3}{2}, y = \frac{1}{2}$ f) $a = 12, b = -7$ g) $x = \frac{16}{13}, y = \frac{11}{13}$ h) $x = 5, y = 0$

i) $x = -0.5, y = 0.8$

2. a) $(-3, 7)$ b) $(3, 0)$ 3. 38 chickens, 22 cows 4. \$3500 at 7% and \$2500 at 11%

5. 38 dimes and 13 quarters 6. \$2000 at 9% and \$5600 at 13%

10.2 – Solving Systems of Linear Equations: Substitution

Practice Problems

1. a) $x = -2, y = 2$ b) $p = 3, q = -5$ c) $a = -2, b = 4$ d) $x = 12, y = -4$ e) $x = -1, y = -3$

f) $a = 1, b = -6$ g) $x = \frac{18}{19}, y = \frac{7}{19}$ h) $x = 0, y = 4$ i) $r = -0.5, s = 1.4$

2. a) $(-3, 7)$ b) $(-4, y - 1)$ c) $(8, 4)$ d) $(-6, 5)$ e) $(24, -9)$ 3. 37 chickens, 15 cows

4. \$3200 at 7% and \$6500 at 12% 5. 23 dimes and 31 quarters 6. \$3100 at 9% and \$4700 at 10%

10.3 – The Zero Product Rule

Practice Problems

1. $-2, 5$ 2. $-1, 0, 3$ 3. $-7, 0, 10$ 4. $-2, 4$ 5. $-6, -1, 0, 1$ 6. $-8, -\frac{3}{7}, 0, \frac{1}{2}$
8. $(x-3)(x+6) = 0$ 9. $x(x-8)(x+4) = 0$
10. yes, for example $x^7 = 0$ has only one solution: $x = 0$

10.4 – Factoring – Part 1 (GCF and Difference of Squares)

Discussion

The equation $x^2 = 9$ has two solutions, $x = 3$ and -3 . We can solve this equation by factoring:

$$\begin{array}{rcl} x^2 & = & 9 \\ x^2 - 9 & = & 0 \\ (x+3)(x-3) & = & 0 \\ x_1 & = & -3 \text{ and } x_2 = 3 \end{array} \qquad \begin{array}{rcl} x^2 & = & -9 \\ x^2 + 9 & = & 0 \\ (\quad ? \quad)(\quad ? \quad) & = & 0 \end{array}$$

If $x^2 + 9$ could be factored, then both linear factors would yield for a solution of the equation $x^2 + 9 = 0$. But the equation $x^2 = -9$ has no solution because the square of no real number is negative. Therefore, $x^2 + 9$ cannot be factored.

Sample Problems

1. a) $3(x-4)$ b) $(x+5y)(x-5y)$ c) $3(a+2)(a-2)$ d) $3a(a-4)$ e) $(x+1)(x-1)$ f) $x^2 + 1$
g) $(x^3 + 7)(x^3 - 7)$ h) $3a(a+3b)(a-3b)$ i) $2(p^2 + 9)(p+3)(p-3)$ j) $5x(x^2 + 4)$
2. a) $2, -3$, and $-\frac{1}{2}$ b) 0 and -7 c) -3 and 3 d) 0 and 9 e) 0 and $\frac{25}{4}$ f) $-\frac{5}{2}, 0$, and $\frac{5}{2}$
3. a) $0, 1$ b) $0, 2, -2$

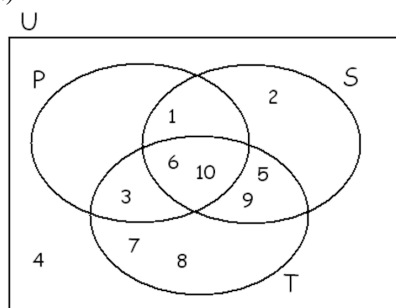
Practice Problems

1. a) $5ab^2(2a-3b+5abc)$ b) $3x^2(2x-5x^2-1)$ c) $a^2(a^2-a+1)$ d) $6a^2b(2a-5ab+1)$
e) $x^3(x^2-2x+4)$ f) $(a-3)(3xy+8t-200x^5)$
2. a) $-(-x^3+x^5-2)$ b) $-(x^2-3x+1)$ c) $-(x^2-3x+5)$
3. a) $(x+7)(x-7)$ b) $(3a+5)(3a-5)$ c) $(x+1)(x-1)$ d) $(y^3+10)(y^3-10)$
4. a) $5(a+3)(a-3)$ b) $-2(n-m)(m+n)(m^2+n^2)$ c) $2x^2(x+2)(x-2)$ d) $-3a(2b+1)(2b-1)$
e) $x(x+1)(x-1)$ f) $5x^3(y-2)(y+2)(y^2+4)$ g) $(a-3)(a+3)(x-1)$ h) $2x^2(3a-5)(3a+5)$
i) $(a+x-1)(a-x+1)$ j) $(a-2)(a+2)(a^2+4)$ k) $-6ab^2(b-10)(b+10)$ l) $4x^2y^3(x^2+9)$
m) $-2(x^2+9)(x+3)(x-3)$ n) $5ab(a^2b-3)$
5. a) $-5, 1$ b) $0, 2, -3$ c) $2, -3$ d) $2, -2$ e) $0, -6$ f) $0, 25$ g) $-5, 0, 5$ h) $-\frac{2}{3}, 0, \frac{2}{3}$
6. a) $0, 12$ b) $0, -5$ c) $-2, 0, 2$ 7. a) 200 b) 4001 c) 14000 d) 300

Problem Set 10

1. a) true b) true c) false d) true e) true f) true g) false h) true i) true
2. $\{1, 2\}$ $\{2, 3\}$ $\{3, 4\}$ $\{4, 5\}$ there are 10 such subsets
 $\{1, 3\}$ $\{2, 4\}$ $\{3, 5\}$
 $\{1, 4\}$ $\{2, 5\}$
 $\{1, 5\}$

3. a)

b) i) $\{6, 10\}$ ii) $\{1, 2, 3, 5, 6, 9, 10\}$ iii) $\{3, 5, 6, 9, 10\}$ iv) $\{1, 3, 5, 6, 9, 10\}$ c) $P \cap S$ d) true e) false4. a) $\{3, 6, 9, 11, 12, 13, 14, 15\}$ b) $\{12, 15\}$

5. a) true b) false c) true d) false

6. a) $2^4 \cdot 3 \cdot 5^2$ b) $2^5 \cdot 3^2 \cdot 7$ c) $2^2 \cdot 5 \cdot 7 \cdot 13$ 7. a) $2^{20} \cdot 3^{20} \cdot 5^{20}$ b) $2^{30} \cdot 3^{20}$ c) $15! = 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ 8. 19 9. $x = 20\text{in}$ 10. a) true b) true c) true d) false e) false f) true g) true11. a) $\frac{4}{15}$ b) $-\frac{31}{6}$ c) $\frac{4}{9}$ d) $-\frac{25}{36}$ 12. a) 37 b) 2 c) -513. a) $6.25 \cdot 10^{18}$ b) $4 \cdot 10^{12}$ c) $1.5625 \cdot 10^6$ d) $5 \cdot 10^4$ e) $4 \cdot 10^{-10}$ d) $6.4 \cdot 10^{-7}$ 14. a) \$2640 b) 25% increase c) 20% decrease d) 65 000 15. a) -1 b) 0 c) $\frac{3}{2}$ d) $\frac{7}{2}$ e) undefined16. a) $-\frac{1}{x^5}$ b) $(x(-x)^3)^{-4}$ c) x^8 d) x^6y e) $\frac{1}{18}$ f) $-3a$ g) $2x^2 - x - 10$ h) $-2x + 7$ i) $-5x^2 + 19x + 4$ j) $25x^2 - 49$ k) $11a - 6b$ l) $9x^2 - 12x + 4$ m) $-12x - 4$ n) $-80x - 16$ o) $6x - 9$ p) $x^3 - 6x^2 + 12x - 8$ 17. a) $-3x(x - 6)$ b) $-1(x^2 - x + 3)$ c) $4ab^2(ab - 5b^2 + 3a)$ d) $(2a - 1)(4a^2 - 3a + 5)$ 18. a) $(x + 6)(x - 6)$ b) $(5p - 7q^5)(5p + 7q^5)$ c) $(3x + 4)(2x - 5)$ d) $3x(4x^2 + 1)$ e) $2a^2x(x^2 + 4)(x + 2)(x - 2)$ f) $3(x - 2)(x - 4)$ 19. a) $-3a$ b) $-2x + 7$ c) $14m - 9$ d) $11a - 6b$ e) $-12x - 4$ f) $-80x - 16$ g) $2x^2 - x - 10$ h) $-5x^2 + 19x + 4$ i) $25x^2 - 49$ j) $9x^2 - 12x + 4$ k) $6x - 9$ l) $x^3 - 6x^2 + 12x - 8$

20. a) (4, 9) b) (-2, 1) c) (0, -4) d) (5, 11) e) (-3, 1) g) (4, -7)

21. a) -3 b) $\frac{2}{5}$ c) 24 d) $-\frac{3}{4}$ e) -4 f) 0 g) 4 h) $\frac{1}{2}$ i) 3 j) 12 l) 15 m) 1 n) -5 o) 0

22. a) -12, -8, -4, and 0 b) 12in by 43in c) 36 chickens, 57 cows d) 5 e) 18 years f) 13 g) 0, 9

h) -3, 0, 3 i) 6 nights j) 11m k) 118 dimes and 171 quarters l) 145 unit

11.1 – Completing the Square – Part 1

Practice Problems

1. $(x - 3)(x - 7)$ 2. $(x - 2)(x - 4)$ 3. $(y + 15)(y + 7)$ 4. $(b + 5)(b - 9)$ 5. $(a + 17)(a - 3)$ 6. can not be factored 7. $(x - 1)(x - 3)$ 8. can not be factored 9. $(x + 24)(x - 18)$ 10. can not be factored 11. $(m - 18)(m - 24)$ 12. $(x - 15)(x - 35)$ 13. $(y + 25)(y - 15)$ 14. $(x - 12)(x - 28)$ 15. can not be factored 16. $(q + 6)(q - 8)$ 17. $(x - 9)^2$ 18. $(t + 51)(t - 87)$ 19. $(x - 10)(x - 36)$ 20. $(q + 56)(q - 42)$

11.2 – Completing the Square – Part 2

Practice Problems

1. $2(x + 5)(x - 3)$ 2. $5a(a - 3)(a + 17)$ 3. $3b^2(b^2 - 10b + 26)$ 4. $2(x + 27)(x - 11)$ 5. $6c(c - 1)(c - 3)$ 6. $-d(d^2 + 2d + 2)$ 7. $-(x + 24)(x - 18)$ 8. can not be factored9. $5ac(b + 12)(b - 10)$ 10. $2y^3(y + 7)(y + 5)$ 11. $3x^2y(y + 12)(y - 6)$ 12. $-5x(x + 20)(x - 10)$

11.3 – Summation

Practice Problems

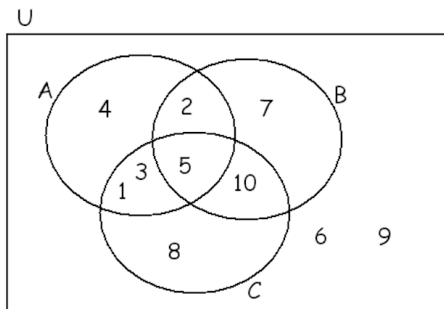
1. 757945 3. 103635 5. 14960 7. 44165 9. 1009
2. 14105 4. 27375 6. 86025 8. 86295 10. 1000
11. 20201

$$1 + 3 + 5 + \dots + 199 + 201 + 199 + 197 + \dots + 3 + 1$$

$$= 201 + 2(1 + 3 + 5 + \dots + 199) = 201 + 2 \cdot \frac{100(200)}{2} = 20201$$

Problem Set 11

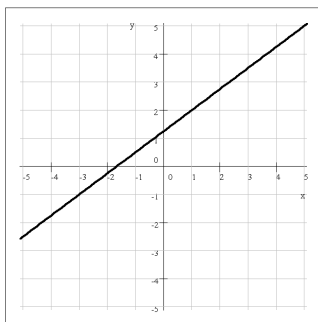
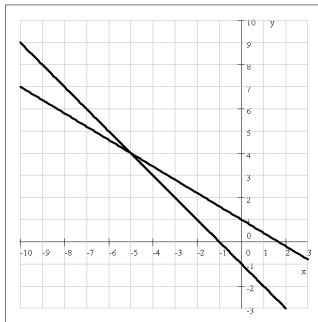
1. a) 1, 2, 4, 7, 8, 14, 28, 56 b) $2^4 \cdot 3 \cdot 5$ c) 60 and 2160 2. a) $2^{396} \cdot 3^{99}$ b) $2^9 \cdot 3^3 \cdot 5^3$ 3. 183 R 6
4. a) false b) true c) false d) true e) true f) true g) false h) false 5. 1
6. a) $P = 80\text{cm}$ $A = 240\text{cm}^2$ b) $A = 35 \text{ unit}^2$ 7. $A = 51\text{m}^2$
8. a) see below b) i) $\{2, 5\}$ ii) $\{1, 2, 3, 4, 5, 10\}$ iii) $\{1, 2, 3, 5\}$



9. a) $(0, 7]$ b) $[3, 5)$ c) $(-\infty, \infty)$ d) $(-4, 2)$
 e) $[1, 8]$ f) $(3, 5)$ g) $(-\infty, -3) \cup (-1, 12)$ h) $\emptyset$
 i) $[1, \infty)$ j) $(4, 6]$
10. a) \$3150 b) \$2040 c) \$648
11. a) 4 b) 9 c) 88 d) -10 e) -27 f) -2 g) 2 h) -4
12. a) 27 b) $\frac{9}{40}$ c) $\frac{2}{75}$ d) 7 e) undefined f) 5 g) $\frac{49}{2}$ h) $\frac{6}{5}$ i) $\frac{5}{16}$ j) $-\frac{5}{24}$
13. a) -5 b) undefined c) $-\frac{9}{2}$ d) $-\frac{59}{8}$ 14. a) 4A b) $\frac{A}{9}$ c) A^6 d) A e) $\sqrt{A}$
15. a) 5100 b) 5600 c) 6000 d) 11000 e) 15000
16. a) 50% increase b) 58% decrease c) 4% decrease d) 4% decrease
17. a) $x = 1.25 \cdot 10^{12}$ and $y = 2.5 \cdot 10^{-10}$ b) i) $3.125 \cdot 10^2$ ii) $3.90625 \cdot 10^{14}$ iii) $5 \cdot 10^{-5}$ iv) $4 \cdot 10^9$
18. a) 255024 b) 1593 c) 147620 d) 49 19. a) $\frac{a^2}{b^7}$ b) x^{40} c) $-\frac{3}{4} \frac{a}{x^5}$ d) $-8x^2y^{12}$
20. a) $x - 7$ b) $5x - 9$ c) $12x - 19$ d) $-6x^2 + 19x - 8$ e) $6x^2$ f) 10 g) $9x^4 - 25$ h) $4x^2 - 12x + 9$
 i) $8x^3 - 36x^2 + 54x - 27$ j) $x^3 - 7x + 6$ k) $x^4 - 8x^2 + 16$
21. a) $2(x+3)(x-3)$ b) $3x^2(2a-5)(2a+5)$ c) can not be factored d) $5a^3(a^2+1)(a+1)(a-1)$
 e) $-x(x-4)(x+4)$ f) $(x^8+5)(x^8-5)$ 22. a) $-2x^3(x+5)(x-15)$ b) $6(x-2)(x-8)$ c) $3a(a^2-10a+29)$
23. a) 0, 4 b) -2, 0, 2 c) -8 d) 7 e) -3 f) 0 g) all real numbers h) $\frac{5}{4}$ i) no solution j) 3 k) 0
 l) 0, -4 m) -8 n) -1, 0, 5 o) -1, 1 p) no solution

24.

a)

b) $(-5, 4)$ 25. a) $[5, \infty)$ b) $(-\infty, -5)$ c) $\left[\frac{3}{4}, \infty\right)$ d) $(-\infty, -2)$ 26. a) $(-4, -12)$ b) $(-2, 1)$ c) $(10, 14)$

27. a) 1536ft b) 15 seconds 28. 11ft by 21ft

29. 5 30. 60000

31. 33 ten-dollar bills and 56 five-dollar bills 32. -7 33. 6 and 27 34. 2 units 35. a) 3 units b) 126 unit²

36. 12 quarters, 58 dimes 37. 23 units

38. \$1250 39. 4418

40. 4 adult and 19 children tickets

12.2 – Fractions and Decimals

Practice Problems

1. a) i) 0.8 ii) $8.\overline{6}$ iii) 1.04 iv) $3.\overline{714285}$, the repeating block is 6 digits longb) i) $\frac{218}{100}$ ii) 3 iii) $\frac{641}{99}$ iv) $\frac{18687}{9990}$ 2. The decimal presentation of real numbers is not unique. 3. n can only have 2 and 5 in its prime-factorization4. It is repeating, only the repeating block is 6 digits long, $0.\overline{153846}$ 5. It is impossible

Problem Set 12

1. a) true b) true c) false d) false e) true f) true g) true h) false (5 is the only one though)

i) false (1 only has one) j) true k) false l) false (try $x = -2$ or -10) m) false

2. a) true b) false c) true d) true e) false f) true g) true h) false

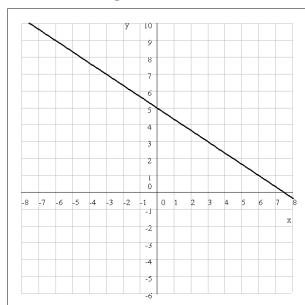
3. a) true b) false c) false d) true e) false

4. a) $\{2, 4\}$ b) $\{4\}$ c) $\{4, 7, 10\}$ d) $\{4\}$ e) $\{1, 2, 3, 4, 5, 6, 7, 9, 10\}$ f) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ g) $\{1, 2, 3, 4, 6, 7, 8, 9, 10\}$ h) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ i) Both set operations are associative, but not when both $\cap$ and $\cup$ appears.5. $\gcd = 40$ $\text{lcm} = 240$ 6. 16 7. a) 5^{60} b) 2^{100} c) 7^{60} d) 2^{50}

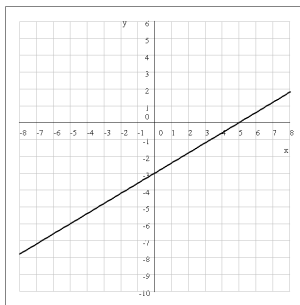
8. There are six possible values: 25, 50, 75, 100, 150, 300 9. a) 409839 b) 49 c) 155

10. a) $\frac{204}{100}$ b) $\frac{34}{90}$ c) $\frac{203}{990}$ d) $\frac{5420}{999}$ 11. a) 8% decrease b) 2400012. a) $\frac{1}{17}$ b) undefined c) 1 d) -13 e) 22 f) 3 g) 7 h) 3 i) 1 j) 6 k) 6 l) 3 m) $-\frac{3}{4}$ 13. a) $9a - b$ b) $16x^{20} - 8x^{10} + 1$ c) $125a^3 - 150a^2 + 60a - 8$ d) $-5x + \frac{7}{20}$ e) $\frac{b^2}{2a^3}$ 14. a) $3x^2y(x+5)(x-5)$ b) $2x(x^2+9)(x+3)(x-3)$ c) $5ab(a^2+4)$ d) $-5(x-2)(x+8)$ e) $4c(a+b)$ f) $8(m+1)(m-3)$ g) $3(m+1)(m-9)$ h) $5(b-1)^2(2a-3)$ i) $(x-5+y^5)(x-5-y^5)$ 15. a) 7 b) -13 c) -20 d) 0 e) 10 f) -1 g) -6 h) identity, all numbers are solution i) 5j) $-4, 2$ k) 0, 16 l) $-4, 0, 4$ m) $-2, 0, 2$ 16. a) $(-\infty, -2]$ b) $\left(\frac{3}{4}, \infty\right)$

17. a) $y = -\frac{2}{3}x + 5$



b) $3x - 5y = 15$



18. a) If x and y are rational numbers, then there exist integers a, b, c, d , with $b, d \neq 0$ such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$.

Then $x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$. Both numerator and denominator are integers, because the set of all integers is closed under addition and multiplication.

18. b) For example, $\sqrt{3}$ and $-\sqrt{3}$ are both irrational, but their sum, zero, is rational.

c) Suppose that $\sqrt{2} - 1$ is rational. Since 1 is rational, then $\sqrt{2} = (\sqrt{2} - 1) + 1$ would also be rational, as the sum of two rational numbers. But that is not possible, because $\sqrt{2}$ is irrational. Therefore, $\sqrt{2} - 1$ must be irrational.

19. a) There are two pairs, -45 with -23 and 23 with 45 . b) 8 c) 18, 20, 22, 24 d) 11 fives and 21 tens

e) 5 and 12 years f) Ann has \$5, Bonnie has \$15. g) -9 or 5 h) 36° , 56° , and 88° i) \$35 600 20. 6

Appendix B – Solutions

Solutions for 2.1 – The Order of Operations Agreement

1. $2 \cdot 3^2 - (6^2 - 2 \cdot 5) \div 2$

Solution: We start with the parentheses. We will work within the parentheses until the entire expression within it becomes one number. In the parentheses, there is an exponentiation, a subtraction, and a multiplication. Since it is stronger, we start with the exponent.

$$\begin{aligned} 2 \cdot 3^2 - (6^2 - 2 \cdot 5) \div 2 &= \text{exponent within parentheses} \\ 2 \cdot 3^2 - (36 - 2 \cdot 5) \div 2 &= \text{multiplication within parentheses} \\ 2 \cdot 3^2 - (36 - 10) \div 2 &= \text{subtraction within parentheses} \\ 2 \cdot 3^2 - (26) \div 2 &= \text{we may drop parentheses now} \\ 2 \cdot 3^2 - 26 \div 2 &= \end{aligned}$$

Now that there is no parentheses, we perform all exponents, left to right. There is only one, so we have

$$\begin{aligned} 2 \cdot 3^2 - 26 \div 2 &= \text{exponent} \\ 2 \cdot 9 - 26 \div 2 &= \end{aligned}$$

Now we execute all multiplications, divisions, left to right

$$\begin{aligned} 2 \cdot 9 - 26 \div 2 &= \text{multiplication} \\ 18 - 26 \div 2 &= \text{division} \\ 18 - 13 &= \text{subtraction} \\ &= \boxed{5} \end{aligned}$$

2. $18 - 7 - 3$

Solution: It is a common mistake to subtract 4 from 18. This is not what order of operations tell us to do. The two subtractions have to be performed left to right.

$$\begin{aligned} 18 - 7 - 3 &= \text{first subtraction from left} \\ 11 - 3 &= \text{subtraction} \\ &= \boxed{8} \end{aligned}$$

3. $5^2 - 2(10 - 2^2)$

Solution: We start with the parentheses

$$\begin{aligned}
 5^2 - 2(10 - 2^2) &= && \text{exponent in parentheses} \\
 5^2 - 2(10 - 4) &= && \text{subtraction in parentheses} \\
 5^2 - 2(6) &= && \text{drop parentheses} \\
 5^2 - 2 \cdot 6 &= && \text{exponents} \\
 25 - 2 \cdot 6 &= && \text{multiplication} \\
 25 - 12 &= && \text{subtraction} \\
 &= && \boxed{13}
 \end{aligned}$$

4. $8^2 - 3^2$

Solution: There are three operations, two exponents and a subtraction. We start with the exponents, left to right.

$$\begin{aligned}
 8^2 - 3^2 &= && \text{first exponent from left} \\
 64 - 3^2 &= && \text{exponent} \\
 64 - 9 &= && \text{subtraction} \\
 &= && \boxed{55}
 \end{aligned}$$

5. $(8 - 3)^2$

Solution: We start with the parentheses

$$\begin{aligned}
 (8 - 3)^2 &= && \text{subtraction in parentheses} \\
 (5)^2 &= && \text{drop parentheses} \\
 5^2 &= && \text{exponents} \\
 &= && \boxed{25}
 \end{aligned}$$

This problem and the previous one tells us a very important thing: $a^2 - b^2$ and $(a - b)^2$ are different expressions! In $a^2 - b^2$ we first square a and b and then subtract. In $(a - b)^2$ we first subtract b from a and then square the difference.

6. $(3^3 - 4 \cdot 5 + 2)^2$

Solution: We will work within the parentheses until it becomes a number. Within the parentheses, we start with the exponents.

$$\begin{aligned}
 (3^3 - 4 \cdot 5 + 2)^2 &= && \text{exponents within parentheses} \\
 (27 - 4 \cdot 5 + 2)^2 &= && \text{multiplication within parentheses} \\
 (27 - 20 + 2)^2 &= &&
 \end{aligned}$$

There is an addition and a subtraction in the parentheses. **It is not true that addition comes before subtraction!** Addition and subtraction are equally strong; we execute them left to right. We start with the subtraction within parentheses.

$$\begin{aligned}
 (27 - 20 + 2)^2 &= (7 + 2)^2 && \text{addition within parentheses} \\
 (9)^2 &= 9^2 = \boxed{81} && \text{drop parentheses, exponents}
 \end{aligned}$$

$$7. \frac{3 + 2(20 - 3^2 - 5)}{3^2 - 2^2} + 4^1$$

Solution: The division bar stretching over entire expressions is a case of the **invisible parentheses**. It instructs us to work out the top until we obtain a number, the bottom until we obtain a number, and finally divide. The invisible parentheses here means

$$\frac{3 + 2(20 - 3^2 - 5)}{3^2 - 2^2} = [3 + 2(20 - 3^2 - 5)] \div [3^2 - 2^2]$$

And now we see that the invisible parentheses was developed to simplify notation. We will start with the top. Naturally, we stay within the parentheses until they disappear.

$$\frac{3 + 2(20 - 3^2 - 5)}{3^2 - 2^2} + 4^1 = \text{exponent in parentheses}$$

$$\frac{3 + 2(20 - 9 - 5)}{3^2 - 2^2} + 4^1 = \text{first subtraction from left in parentheses}$$

$$\frac{3 + 2(11 - 5)}{3^2 - 2^2} + 4^1 = \text{subtraction in parentheses}$$

$$\frac{3 + 2(6)}{3^2 - 2^2} + 4^1 = \text{drop parentheses}$$

$$\frac{3 + 2 \cdot 6}{3^2 - 2^2} + 4^1 = \text{multiplication on top}$$

$$\frac{3 + 12}{3^2 - 2^2} + 4^1 = \text{addition on top}$$

$$\frac{15}{3^2 - 2^2} + 4^1 =$$

Now we work out the bottom, applying order of operations

$$\frac{15}{3^2 - 2^2} + 4^1 = \text{first exponent from left to right}$$

$$\frac{15}{9 - 2^2} + 4^1 = \text{exponent}$$

$$\frac{15}{9 - 4} + 4^1 = \text{subtraction}$$

$$\frac{15}{5} + 4^1 = \text{same as}$$

$$15 \div 5 + 4^1 =$$

We now have a division, an addition, and an exponent. We start with the exponent.

$$15 \div 5 + 4^1 = \text{exponent, } 4^1 = 4$$

$$15 \div 5 + 4 = \text{division}$$

$$3 + 4 = \text{addition}$$

$$= \boxed{7}$$

Solutions for 4.3 – Algebraic Expressions and Statements

1. Evaluate each of the following numerical expressions.

a) $2 - 5(3 - 7)$

Solution: We will apply order of operations. First we perform the subtraction in the parentheses.

$$\begin{aligned} 2 - 5(3 - 7) &= && \text{subtraction in parentheses} \\ 2 - 5(-4) &= && \text{multiplication} \\ 2 - (-20) &= && \text{subtraction} \\ 2 + 20 &= && \boxed{22} \end{aligned}$$

b) $24 - 10 + 2$

Solution: It is NOT true that addition comes before subtraction. Addition and subtraction are equally strong, so between those two, we perform them left to right. First come, first served.

$$24 - 10 + 2 = 14 + 2 = \boxed{16}$$

c) -4^2

Solution: as it was discussed before, -4^2 is quite different from $(-4)^2$. This is $-1 \cdot 4^2 = \boxed{-16}$.

d) $(-4)^2$

This is when -4 is squared. So $(-4)^2 = -4(-4) = \boxed{16}$

e) $|3| - |8|$

Solution: We subtract the absolute value of 8 from the absolute value of 3. So $|3| - |8| = 3 - 8 = \boxed{-5}$

f) $|3 - 8|$

Solution: This is the absolute value of the difference. Absolute value signs also function of grouping symbols (i.e. parentheses) to overwrite the usual order of operations. So $|3 - 8| = |-5| = \boxed{5}$

2. Evaluate each of the algebraic expressions when $p = -7$ and $q = 3$.

a) $15 - p$

Solution:

Step 1. We re-write the expression with one modification: we replace each variable by an empty pair of parentheses.

Step 2. We insert the values into the parentheses. Now the problem becomes an order of operations problem.

Step 3. We drop the unnecessary parentheses and work out the order of operations problem. (It may appear awkward to create these parentheses but they will later become extremely helpful.)

$$\begin{aligned} \text{Step 1.} \quad 15 - p &= 15 - (\quad) \\ \text{Step 2.} &= 15 - (-7) \\ \text{Step 3.} &= 15 + 7 \\ &= \boxed{22} \end{aligned}$$

b) $pq - |p - 2|$

Solution:

$$\begin{aligned}
 \text{Step 1.} \quad pq - |p - 2| &= () () - |() - 2| \\
 \text{Step 2.} \quad &= (-7)(3) - |-7 - 2| \\
 \text{Step 3.} \quad &= -7 \cdot 3 - |-9| = -7 \cdot 3 - 9 = -21 - 9 = \boxed{-30}
 \end{aligned}$$

c) $4p - q^3$

Solution:

$$\begin{aligned}
 \text{Step 1.} \quad 4p - q^3 &= 4() - ()^3 \\
 \text{Step 2.} \quad &= 4(-7) - (3)^3 \\
 \text{Step 3.} \quad &= 4 \cdot 7 - 3^3 && \text{exponentiation} \\
 &= 4 \cdot 7 - 27 && \text{multiplication} \\
 &= 28 - 27 && \text{subtraction} \\
 &= \boxed{1}
 \end{aligned}$$

d) $\frac{q^2 - p}{2q + p + 1}$

Solution:

$$\begin{aligned}
 \text{Step 1.} \quad \frac{q^2 - p}{2q + p + 1} &= \frac{()^2 - ()}{2() + () + 1} \\
 \text{Step 2.} \quad &= \frac{(3)^2 - (-7)}{2(3) + (-7) + 1} && \text{drop extra parentheses} \\
 &= \frac{3^2 - (-7)}{2 \cdot 3 + (-7) + 1} && \text{exponent upstairs} \\
 \text{Step 3.} \quad &= \frac{9 - (-7)}{2 \cdot 3 + (-7) + 1} && \text{subtraction upstairs; } 9 - (-7) = 9 + 7 \\
 &= \frac{16}{2 \cdot 3 + (-7) + 1} && \text{multiplication} \\
 &= \frac{16}{6 + (-7) + 1} && \text{additions, left to right} \\
 &= \frac{16}{-1 + 1} = \frac{16}{0} && \text{Division by zero is not allowed!} \\
 &= \boxed{\text{undefined}}
 \end{aligned}$$

e) $p^2 - q^2$

Solution:

$$\begin{aligned}
 p^2 - q^2 &= ()^2 - ()^2 \\
 &= (-7)^2 - (3)^2 = (-7)^2 - 3^2 && \text{exponents,} \\
 &= 49 - 3^2 && \text{left to right} \\
 &= 49 - 9 && \text{subtraction} \\
 &= \boxed{40}
 \end{aligned}$$

f) $(p - q)^2$

Solution:

$$\begin{aligned}
 (p-q)^2 &= [() - ()]^2 \\
 &= [(-7) - (3)]^2 = [-7 - 3]^2 && \text{subtraction in parentheses} \\
 &= (-10)^2 && \text{exponentiation} \\
 &= \boxed{100}
 \end{aligned}$$

g) $2q^2$

Solution:

$$\begin{aligned}
 2q^2 &= 2()^2 \\
 &= 2(3)^2 \\
 &= 2 \cdot 3^2 && \text{exponentiation} \\
 &= 2 \cdot 9 && \text{multiplication} \\
 &= \boxed{18}
 \end{aligned}$$

h) $(2q)^2$

Solution:

$$\begin{aligned}
 (2q)^2 &= [2()]^2 \\
 &= [2(3)]^2 \\
 &= (2 \cdot 3)^2 && \text{multiplication in parentheses} \\
 &= 6^2 && \text{exponents} \\
 &= \boxed{36}
 \end{aligned}$$

i) $15 - \frac{p+q}{|1-p|}$

Solution: From here on, we show computations **in the form they should appear**. Once you wrote down the expression with little parentheses instead of the letters, you can insert the values into it.

$$\begin{aligned}
 15 - \frac{p+q}{|1-p|} &= 15 - \frac{() + ()}{|1 - ()|} \\
 &= 15 - \frac{(-7) + (3)}{|1 - 3|} \\
 &= 15 - \frac{-7 + 3}{|1 - 3|} && \text{invisible parentheses! addition on top} \\
 &= 15 - \frac{-4}{|1 - 3|} && \text{subtraction in absolute value sign} \\
 &= 15 - \frac{-4}{|-2|} && \text{evaluate the absolute value of 2} \\
 &= 15 - \frac{-4}{2} && \text{division} \\
 &= 15 - (-2) && \text{subtraction} \\
 &= \boxed{17}
 \end{aligned}$$

j) $(p+q)^2 - (5q+2p)^4$

Solution:

$$\begin{aligned}
 (p+q)^2 - (5q+2p)^4 &= [(-7)+(3)]^2 - [5(3)+2(-7)]^4 \\
 &= (-7+3)^2 - (5 \cdot 3 + 2 \cdot (-7))^4 && \text{addition in first parentheses} \\
 &= (-4)^2 - (5 \cdot 3 + 2 \cdot (-7))^4 && \text{multiplications in parentheses} \\
 &= (-4)^2 - (15 + 2 \cdot (-7))^4 && \text{left to right} \\
 &= (-4)^2 - (15 + (-14))^4 && \text{addition in parentheses} \\
 &= (-4)^2 - 1^4 && \text{exponents, left to right} \\
 &= 16 - 1^4 && \text{careful! } 1^4 \neq 4 \\
 &= 16 - 1 && \text{subtraction} \\
 &= \boxed{15}
 \end{aligned}$$

k) $-p^2 - p + 8$

Solution:

$$\begin{aligned}
 -p^2 - p + 8 &= -()^2 - () + 8 \\
 &= -(-7)^2 - (-7) + 8 \\
 &= -49 + 7 + 8 \\
 &= -42 + 8 = \boxed{-34}
 \end{aligned}$$

3. Evaluate the expression $3x^2 - x + 5$ with the given values of x .

a) $x = 0$

Solution: We first copy the entire expression, replacing the letter x by little pairs of parentheses.

$$3x^2 - x + 5 = 3()^2 - () + 5$$

Then we insert the number 0 into each pair of parentheses.

$$3x^2 - x + 5 = 3(0)^2 - (0) + 5$$

Because we substituted zero, most parentheses are unnecessary. We will drop them:

$$3x^2 - x + 5 = 3(0)^2 - (0) + 5 = 3 \cdot 0^2 - 0 + 5$$

Then we solve the resulting order of operations problem. We start with the exponent.

$$\begin{aligned}
 3 \cdot 0^2 - 0 + 5 &= 3 \cdot 0 - 0 + 5 && \text{perform multiplication} \\
 &= 0 - 0 + 5 && \text{subtraction} \\
 &= 0 + 5 && \text{addition} \\
 &= \boxed{5}
 \end{aligned}$$

b) Evaluate $3x^2 - x + 5$ when $x = -1$.

Solution: We evaluate the expression with $x = -1$.

$$\begin{aligned}
 3x^2 - x + 5 &= 3(-1)^2 - (-1) + 5 \\
 &= 3 \cdot 1 + 1 + 5 = 3 + 1 + 5 = \boxed{9}
 \end{aligned}$$

4. We ejected a small object upward from the top of a 720ft tall building and started measuring time in seconds. We find that t seconds after launching, the vertical position of the object is $-16t^2 + 64t + 720$ feet.
- a) Where is the object 2 seconds after launch?
- b) Where is the object 8 seconds after launch?
- a) Solution: To find out where the object is after 2 seconds, we evaluate the expression with $t = 2$.

$$\begin{aligned} -16t^2 + 64t + 720 &= -16 \cdot 2^2 + 64 \cdot 2 + 720 \\ &= -16 \cdot 4 + 64 \cdot 2 + 720 \\ &= -64 + 128 + 720 = 64 + 720 = 784 \end{aligned}$$

So the object is at a height of 784 feet 2 seconds after launch.

- b) Solution: To find out where the object is after 8 seconds, we evaluate the expression with $t = 8$.

$$\begin{aligned} -16t^2 + 64t + 720 &= -16 \cdot 8^2 + 64 \cdot 8 + 720 \\ &= -16 \cdot 64 + 512 + 720 \\ &= -1024 + 512 + 720 = -512 + 720 = 208 \end{aligned}$$

So the object is at a height of 208 feet exactly 8 seconds after launch.

5. Let $a = -4$, $b = 2$, and $x = -3$. Evaluate each of the following expressions.

a) $a^2 - b^2$

Solution: First we re-write the expression with one change, we write little pairs of parentheses instead of the letters.

$$a^2 - b^2 = (\quad)^2 - (\quad)^2$$

We now write the values inside the parentheses. From here on this is an order of operations problem.

$$\begin{aligned} a^2 - b^2 &= (-4)^2 - (2)^2 && \text{drop extra parentheses} \\ &= (-4)^2 - 2^2 && \text{exponents} \\ &= 16 - 4 && \text{subtraction} \\ &= \boxed{12} \end{aligned}$$

b) $(a - b)^2$

Solution: First we re-write the expression with one modification: we write little pairs of parentheses instead of the letters.

$$(a - b)^2 = ((\quad) - (\quad))^2$$

We now write the values inside the parentheses. From here on this is an order of operations problem.

$$\begin{aligned} (a - b)^2 &= ((-4) - (2))^2 && \text{drop extra parentheses} \\ &= (-4 - 2)^2 && \text{subtraction in parentheses} \\ &= (-6)^2 && \text{exponent} \\ &= \boxed{36} \end{aligned}$$

This and the previous problem are here to remind you that $(a - b)^2$ and $a^2 - b^2$ are two different expressions.

c) $a^b - 2bx - x^2 - 2x$

Solution: First we re-write the expression with one modification only: we write little pairs of parentheses instead of the letters.

$$a^b - 2bx - x^2 - 2x = ()^{()} - 2()() - ()^2 - 2()$$

We now write the values inside the parentheses. From here on this is an order of operations problem.

$$a^b - 2bx - x^2 - 2x =$$

$$\begin{aligned} &= ()^{()} - 2()() - ()^2 - 2() \\ &= (-4)^{(2)} - 2(2)(-3) - (-3)^2 - 2(-3) && \text{drop extra parentheses} \\ &= (-4)^2 - 2 \cdot 2(-3) - (-3)^2 - 2(-3) && \text{exponents, left to right} \\ &= 16 - 2 \cdot 2(-3) - (-3)^2 - 2(-3) \\ &= 16 - 2 \cdot 2(-3) - 9 - 2(-3) && \text{multiplications, left to right} \\ &= 16 - 4(-3) - 9 - 2(-3) \\ &= 16 - (-12) - 9 - 2(-3) \\ &= 16 - (-12) - 9 - (-6) && \text{additions, subtractions, left to right} \\ &= 16 + 12 - 9 - (-6) \\ &= 28 - 9 - (-6) \\ &= 19 - (-6) = 19 + 6 = \boxed{25} \end{aligned}$$

d) $\frac{-x^2 + (x+2)^2}{(x-1)}$

Solution: First we re-write the expression with only one modification: we write little pairs of parentheses instead of the letters.

$$\frac{-x^2 + (x+2)^2}{(x-1)} = \frac{-()^2 + (() + 2)^2}{(() - 1)}$$

We now write the values inside the parentheses. From here on this is an order of operations problem.

$$\begin{aligned} \frac{-x^2 + (x+2)^2}{(x-1)} &= \frac{-(-3)^2 + ((-3) + 2)^2}{((-3) - 1)} && \text{drop parentheses} \\ &= \frac{-(-3)^2 + (-3 + 2)^2}{(-3 - 1)} && \text{addition in parentheses upstairs} \\ &= \frac{-(-3)^2 + (-1)^2}{(-3 - 1)} && \text{subtraction downstairs in parentheses} \\ &= \frac{-(-3)^2 + (-1)^2}{(-4)} && \text{drop parentheses} \\ &= \frac{-(-3)^2 + (-1)^2}{-4} && \text{exponents upstairs} \\ &= \frac{-9 + 1}{-4} && \text{addition} \\ &= \frac{-8}{-4} && \text{division} \\ &= \boxed{2} \end{aligned}$$

e) $\frac{x-1}{x+3}$

Solution: First we re-write the expression with only one modification: we write little pairs of parentheses instead of the letters.

$$\frac{x-1}{x+3} = \frac{()-1}{()+3}$$

We write the values inside the parentheses and evaluate the expression.

$$\frac{x-1}{x+3} = \frac{(-3)-1}{(-3)+3} = \frac{-4}{0} = \text{undefined}$$

6. Consider the equation $2x^2 + x + 34 = 21x - 8$. In case of each number given, determine whether it is a solution of the equation or not.

a) $x = 1$

Solution: We need to substitute 1 for x into both sides of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

If $x = 1$, then

$$\begin{aligned} \text{LHS} &= 2(1)^2 + (1) + 34 \\ &= 2 \cdot 1 + 1 + 34 \\ &= 2 + 1 + 34 = 37 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= 21(1) - 8 \\ &= 21 \cdot 1 - 8 \\ &= 13 \end{aligned}$$

$$\begin{aligned} 2x^2 + x + 34 &= 21x - 8 \\ \text{becomes } 37 &= 13 \\ \text{This is false.} \end{aligned}$$

So 1 is not a solution of the equation.

b) $x = 3$

Solution: We need to substitute 3 for x into both the left-hand side and right-hand side of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

If $x = 3$, then

$$\begin{aligned} \text{LHS} &= 2 \cdot 3^2 + 3 + 34 \\ &= 2 \cdot 9 + 3 + 34 \\ &= 18 + 3 + 34 \\ &= 21 + 34 = 55 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= 21 \cdot 3 - 8 \\ &= 63 - 8 \\ &= 55 \end{aligned}$$

$$\begin{aligned} 2x^2 + x + 34 &= 21x - 8 \\ \text{becomes } 55 &= 55 \\ \text{This is true!} \end{aligned}$$

So 3 is a solution of the equation.

c) $x = 4$

Solution: We will substitute 4 for x in both sides of the equation and compare the values.

If $x = 4$, then

$$\begin{aligned} \text{LHS} &= 2 \cdot 4^2 + 4 + 34 \\ &= 2 \cdot 16 + 4 + 34 \\ &= 32 + 4 + 34 \\ &= 36 + 34 = 70 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= 21 \cdot 4 - 8 \\ &= 84 - 8 \\ &= 76 \end{aligned}$$

$$\begin{aligned} 2x^2 + x + 34 &= 21x - 8 \\ \text{becomes } 70 &= 76 \\ \text{This is false.} \end{aligned}$$

Therefore 4 is not a solution of the equation.

d) $x = 7$

Solution: We will substitute 7 for x into both sides of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

If $x = 7$, then

$$\begin{aligned}\text{LHS} &= 2 \cdot 7^2 + 7 + 34 \\ &= 2 \cdot 49 + 7 + 34 \\ &= 98 + 7 + 34 \\ &= 105 + 34 = 139\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 21 \cdot 7 - 8 \\ &= 147 - 8 \\ &= 139\end{aligned}$$

$$\begin{aligned}2x^2 + x + 34 &= 21x - 8 \\ \text{becomes } 139 &= 139 \\ \text{This is true!}\end{aligned}$$

Therefore 7 is a solution of the equation.

7. Consider the equation $x^2 - 10x + x^3 - 4 = 4(x + 5)$. In each case, determine whether the number given is a solution of the equation or not.

a) $x = 0$

Solution: We simply evaluate both sides of the equation when $x = 0$.

If $x = 0$, then

$$\begin{aligned}\text{LHS} &= 0^2 - 10 \cdot 0 + 0^3 - 4 \\ &= 0 - 0 + 0 - 4 \\ &= -4\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 4(0 + 5) \\ &= 4 \cdot 5 \\ &= 20\end{aligned}$$

$$\begin{aligned}x^2 - 10x + x^3 - 4 &= 4(x + 5) \\ \text{becomes } -4 &= 20 \\ \text{This is false.}\end{aligned}$$

Therefore 0 is not a solution of the equation.

b) $x = -2$

Solution: We simply evaluate both sides of the equation when $x = -2$.

If $x = -2$,
then

$$\begin{aligned}\text{LHS} &= (-2)^2 - 10(-2) + (-2)^3 - 4 \\ &= 4 - 10(-2) + (-8) - 4 \\ &= 4 + 20 - 8 - 4 \\ &= 24 - 8 - 4 = 12\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 4(-2 + 5) \\ &= 4 \cdot 3 \\ &= 12\end{aligned}$$

$$\begin{aligned}x^2 - 10x + x^3 - 4 &= 4(x + 5) \\ \text{becomes } 12 &= 12 \\ \text{This is true.}\end{aligned}$$

Therefore -2 is a solution of the equation.

c) $x = -3$

Solution: We simply evaluate both sides of the equation when $x = -3$.

If $x = -3$,
then

$$\begin{aligned}\text{LHS} &= (-3)^2 - 10(-3) + (-3)^3 - 4 \\ &= 9 - 10(-3) + (-27) - 4 \\ &= 9 + 30 - 27 - 4 \\ &= 39 - 27 - 4 \\ &= 12 - 4 = 8\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 4(-3 + 5) \\ &= 4 \cdot 2 \\ &= 8\end{aligned}$$

$$\begin{aligned}x^2 - 10x + x^3 - 4 &= 4(x + 5) \\ \text{becomes } 8 &= 8 \\ \text{This is true.}\end{aligned}$$

Therefore -3 is a solution of the equation.

8. Consider the equation $3a - 2b - 1 = (a - b)^2 + 4$. In case of each pair of numbers given, determine whether it is a solution of the equation or not.

a) $a = 8$ and $b = 5$ or, as an ordered pair, $(8, 5)$

Solution: We will substitute $a = 8$ and $b = 5$ into both sides of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

If $a = 8$ and $b = 5$, then

$\begin{aligned} \text{LHS} &= 3 \cdot 8 - 2 \cdot 5 - 1 \\ &= 24 - 2 \cdot 5 - 1 \\ &= 24 - 10 - 1 \\ &= 14 - 1 = 13 \end{aligned}$	$\begin{aligned} \text{RHS} &= (8 - 5)^2 + 4 \\ &= 3^2 + 4 \\ &= 9 + 4 \\ &= 13 \end{aligned}$	$3a - 2b - 1 = (a - b)^2 + 4$ becomes $13 = 13$ This is true!
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Therefore $(8, 5)$ is a solution of the equation.

b) $a = 10$ and $b = 7$

Solution: We need to substitute $a = 10$ and $b = 7$ into both sides of the equation and evaluate those algebraic expressions to see whether the left-hand side equals to the right-hand side.

If $a = 10$ and $b = 7$, then

$\begin{aligned} \text{LHS} &= 3 \cdot 10 - 2 \cdot 7 - 1 \\ &= 30 - 2 \cdot 7 - 1 \\ &= 30 - 14 - 1 \\ &= 16 - 1 = 15 \end{aligned}$	$\begin{aligned} \text{RHS} &= (10 - 7)^2 + 4 \\ &= 3^2 + 4 \\ &= 9 + 4 \\ &= 13 \end{aligned}$	$3a - 2b - 1 = (a - b)^2 + 4$ becomes $15 = 13$ This is false.
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Therefore $(10, 7)$ is not a solution of the equation.

9. Consider the inequality $3(2y - 1) + 1 \leq 5y - 7$. In case of each number given, determine whether it is a solution of the inequality or not.

a) $y = -10$

Solution: We need to substitute $y = -10$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than or equal to the right-hand side.

If $y = -10$, then

$\begin{aligned} \text{LHS} &= 3(2(-10) - 1) + 1 \\ &= 3(-20 - 1) + 1 \\ &= 3(-21) + 1 \\ &= -63 + 1 = -62 \end{aligned}$	$\begin{aligned} \text{RHS} &= 5(-10) - 7 \\ &= -50 - 7 \\ &= -57 \end{aligned}$	$3(2y - 1) + 1 \leq 5y - 7$ becomes $-62 \leq -57$ This is true.
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Thus -10 is a solution of the inequality.

b) $y = 3$

Solution: We need to substitute $y = 3$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than or equal to the right-hand side.

If $y = 3$,
then

$$\begin{aligned}\text{LHS} &= 3(2 \cdot 3 - 1) + 1 \\ &= 3(6 - 1) + 1 \\ &= 3 \cdot 5 + 1 \\ &= 15 + 1 = 16\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 5 \cdot 3 - 7 \\ &= 15 - 7 \\ &= 8\end{aligned}$$

$$\begin{aligned}3(2y - 1) + 1 &\leq 5y - 7 \\ \text{becomes } 16 &\leq 8 \\ \text{This is false.}\end{aligned}$$

Thus 3 is not a solution of the inequality.

c) $y = -5$

Solution: We need to substitute $y = -5$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than or equal to the right-hand side.

If $y = -5$,
then

$$\begin{aligned}\text{LHS} &= 3(2(-5) - 1) + 1 \\ &= 3(-10 - 1) + 1 \\ &= 3(-11) + 1 \\ &= -33 + 1 = -32\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 5(-5) - 7 \\ &= -25 - 7 \\ &= -32\end{aligned}$$

$$\begin{aligned}3(2y - 1) + 1 &\leq 5y - 7 \\ \text{becomes } -32 &\leq -32 \\ \text{This is true.}\end{aligned}$$

Thus -5 is a solution of the inequality.

d) $y = 0$

Solution: We need to substitute $y = 0$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than or equal to the right-hand side.

If $y = 0$,
then

$$\begin{aligned}\text{LHS} &= 3(2 \cdot 0 - 1) + 1 \\ &= 3(0 - 1) + 1 \\ &= 3(-1) + 1 \\ &= -3 + 1 = -2\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 5 \cdot 0 - 7 \\ &= 0 - 7 \\ &= -7\end{aligned}$$

$$\begin{aligned}3(2y - 1) + 1 &\leq 5y - 7 \\ \text{becomes } -2 &\leq -7 \\ \text{This is false.}\end{aligned}$$

Thus 0 is not a solution of the inequality.

10. Consider the inequality $\frac{2x+1}{3} + 5 < \frac{3x-1}{2}$. In case of each number given, determine whether it is a solution of the inequality or not.

a) $x = 1$

Solution: We need to substitute $x = 1$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than the right-hand side.

$$\text{The left-hand side: } \text{LHS} = \frac{2 \cdot 1 + 1}{3} + 5 = \frac{2 + 1}{3} + 5 = \frac{3}{3} + 5 = 1 + 5 = 6$$

$$\text{The right-hand side: } \text{RHS} = \frac{3 \cdot 1 - 1}{2} = \frac{3 - 1}{2} = \frac{2}{2} = 1$$

So the statement $\frac{2x+1}{3} + 5 < \frac{3x-1}{2}$ becomes $6 < 1$. Since this is a false statement, $x = 1$ is not a solution of the inequality.

b) $x = 13$

Solution: We need to substitute $x = 13$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than the right-hand side. The left-hand side:

$$\text{LHS} = \frac{2(13)+1}{3} + 5 = \frac{26+1}{3} + 5 = \frac{27}{3} + 5 = 9 + 5 = 14$$

The right-hand side:

$$\text{RHS} = \frac{3(13)-1}{2} = \frac{39-1}{2} = \frac{38}{2} = 19$$

So the statement $\frac{2x+1}{3} + 5 < \frac{3x-1}{2}$ becomes $14 < 19$. Since this is a true statement, $x = 13$ is a solution of the inequality.

c) $x = 7$

Solution: We need to substitute $x = 7$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than the right-hand side. The left-hand side:

$$\text{LHS} = \frac{2(7)+1}{3} + 5 = \frac{14+1}{3} + 5 = \frac{15}{3} + 5 = 5 + 5 = 10$$

The right-hand side:

$$\text{RHS} = \frac{3(7)-1}{2} = \frac{21-1}{2} = \frac{20}{2} = 10$$

So the statement $\frac{2x+1}{3} + 5 < \frac{3x-1}{2}$ becomes $10 < 10$. Since this is a false statement, $x = 7$ is not a solution of the inequality.

d) $x = -5$

Solution: We need to substitute $x = -5$ into both sides of the inequality and evaluate those algebraic expressions to see whether the left-hand side is indeed less than the right-hand side. The left-hand side:

$$\text{LHS} = \frac{2(-5)+1}{3} + 5 = \frac{-10+1}{3} + 5 = \frac{-9}{3} + 5 = -3 + 5 = 2$$

The right-hand side:

$$\text{RHS} = \frac{3(-5)-1}{2} = \frac{-15-1}{2} = \frac{-16}{2} = -8$$

So the statement $\frac{2x+1}{3} + 5 < \frac{3x-1}{2}$ becomes $2 < -8$. Since this is a false statement, $x = -5$ is not a solution of the inequality.

11. Translate each of the following statements to an algebraic statement.

a) y is three less than four times x

Solution: "four times x " can be translated as $4x$.

Then we have: " y is three less than $4x$ ", which can be translated as $y = 4x - 3$.

b) The opposite of A is one greater than the difference of B and three times C .

Solution: "*The opposite of A* " can be translated as $-A$. "*three times C* " can be translated as $3C$.

Now we have: " $-A$ is one greater than the difference of B and $3C$ ".

"*The difference of B and $3C$* " can be translated as $B - 3C$. It is important to write them in the order they are mentioned; first come first served.

Now we have: " $-A$ is one greater than $B - 3C$ ". This can be translated as $-A = (B - 3C) + 1$.

The parentheses turns out to be unnecessary, but it does make sense here. We are adding 1 to the entire difference $B - 3C$, not just $-3C$.

c) Twice M is five less than the product of N and the opposite of M .

Solution: "*Twice M* " can be translated to $2M$ and "*the opposite of M* " can be translated as $-M$.

Now we have: " $2M$ is five less than the product of N and $-M$ ".

"*The product of N and $-M$* " can be translated as $N(-M)$ or $N \cdot (-M)$.

So now we have: " $2M$ is five less than $N(-M)$ ". This can be translated as $2M = N(-M) - 5$.

12. The longer side of a rectangle is three units shorter than five times the shorter side. If we label the shorter side by x , how can we express the longer side in terms of x ?

Solution: The longer side is three less than five times the shorter side. (The shorter side is x .)

So, the longer side is three less than five times x .

Or, the longer side is three less than $5x$.

Or, the longer side is $5x - 3$.

13. Suppose we have three consecutive integers.

a) Express them in terms of x if x denotes the smallest number.

Solution: To get from one consecutive integer to the next one, we simply need to add 1. So the middle number is $x + 1$ and the largest number we obtain by adding 1 again, so we get $x + 2$. The answer is $x, x + 1, x + 2$.

b) Express them in terms of y if y denotes the number in the middle.

Solution: then the smallest number is one less than y , which is $y - 1$, and the largest number is one greater than y , which is $y + 1$. So the answer is $y - 1, y, y + 1$.

c) Express them in terms of L if L denotes the greatest number.

Solution: Then we have to subtract one to get to the middle number, and two to get to the smallest number. So the answer is $L - 2, L - 1, L$.

Solutions for 5.3 – Linear Equations 1

Solve each of the following equations. Make sure to check your solutions.

1. $2x - 5 = 17$

Solution:

$$\begin{array}{rcll} 2x - 5 & = & 17 & \text{add 5 to both sides} \\ 2x & = & 22 & \text{divide by 2} \\ x & = & 11 & \end{array}$$

We check: if $x = 11$, then

$$\text{RHS} = 2(11) - 5 = 22 - 5 = 17 = \text{LHS}$$

Thus our solution, $x = 11$ is correct.

2. $\frac{a - 10}{5} = -3$

Solution:

$$\begin{array}{rcll} \frac{a - 10}{5} & = & -3 & \text{multiply both sides by 5} \\ a - 10 & = & -15 & \text{add 10 to both sides} \\ a & = & -5 & \end{array}$$

We check: if $a = -5$, then

$$\text{LHS} = \frac{-5 - 10}{5} = \frac{-15}{5} = -3 = \text{RHS}$$

Thus our solution, $a = -5$ is correct.

3. $\frac{t}{4} - 10 = -4$

Solution:

$$\begin{array}{rcll} \frac{t}{4} - 10 & = & -4 & \text{add 10 to both sides} \\ \frac{t}{4} & = & 6 & \text{multiply both sides by 4} \\ t & = & 24 & \end{array}$$

We check: if $t = 24$, then

$$\text{RHS} = \frac{t}{4} - 10 = \frac{24}{4} - 10 = 6 - 10 = -4 = \text{LHS}$$

Thus our solution, $t = 24$ is correct.

4. $\frac{t-5}{12} = 4$

Solution:

$$\begin{array}{rcll} \frac{t-5}{12} & = & 4 & \text{multiply both sides by 12} \\ t-5 & = & 48 & \text{add 5 to both sides} \\ t & = & 53 & \end{array}$$

We check: if $t = 53$, then $\text{LHS} = \frac{53-5}{12} = \frac{48}{12} = 4 = \text{RHS}.$

Thus our solution, $\boxed{t = 53}$ is correct.

5. $2x - 7 = -3$

Solution: We apply all operations to both sides.

$$\begin{array}{rcll} 2x-7 & = & -3 & \text{add 7} \\ 2x & = & 4 & \text{divide by 2} \\ x & = & 2 & \end{array}$$

We check: if $x = 2$, then

$$\text{LHS} = 2(2) - 7 = 4 - 7 = -3 = \text{RHS}$$

Thus our solution, $\boxed{x = 2}$ is correct.

6. $\frac{x+8}{3} = -2$

Solution: We apply all operations to both sides.

$$\begin{array}{rcll} \frac{x+8}{3} & = & -2 & \text{multiply by 3} \\ x+8 & = & -6 & \text{subtract 8} \\ x & = & -14 & \end{array}$$

We check: $\text{LHS} = \frac{-14+8}{3} = \frac{-6}{3} = -2 = \text{RHS}$

Thus our solution, $\boxed{x = -14}$ is correct.

7. $\frac{x}{3} + 8 = -2$

Solution: We apply all operations to both sides.

$$\begin{array}{rcll} \frac{x}{3} + 8 & = & -2 & \text{subtract 8} \\ \frac{x}{3} & = & -10 & \text{multiply by 3} \\ x & = & -30 & \end{array}$$

We check:

$$\text{LHS} = \frac{-30}{3} + 8 = -10 + 8 = -2 = \text{RHS}$$

Thus our solution, $\boxed{x = -30}$ is correct.

8. $-2x + 3 = 3$

Solution: We apply all operations to both sides.

$$\begin{array}{rcl} -2x + 3 & = & 3 \quad \text{subtract 3} \\ -2x & = & 0 \quad \text{divide by } -2 \\ x & = & 0 \end{array}$$

We check: if $x = 0$, then

$$\text{LHS} = -2 \cdot 0 + 3 = 0 + 3 = 3 = \text{RHS}$$

Thus our solution, $\boxed{x = 0}$ is correct.

9. $3(x + 7) = 36$

Solution: We apply all operation to both sides,

$$\begin{array}{rcl} 3(x + 7) & = & 36 \quad \text{divide by 3} \\ x + 7 & = & 12 \quad \text{subtract 7} \\ x & = & 5 \end{array}$$

We check: if $x = 5$, then

$$\text{LHS} = 3(5 + 7) = 3 \cdot 12 = 36 = \text{RHS}$$

Thus our solution, $\boxed{x = 5}$ is correct.

10. $3x - 10 = -10$

Solution:

$$\begin{array}{rcl} 3x - 10 & = & -10 \quad \text{add 10 to both sides} \\ 3x & = & 0 \quad \text{divide by 3} \\ x & = & 0 \end{array}$$

We check: if $x = 0$, then

$$\text{LHS} = 3 \cdot 0 - 10 = 0 - 10 = -10 = \text{RHS}$$

Thus our solution, $\boxed{x = 0}$ is correct.

11. $-4x + 6 = -18$

Solution:

$$\begin{array}{rcl} -4x + 6 & = & -18 \quad \text{subtract 6} \\ -4x & = & -24 \quad \text{divide by } -4 \\ x & = & 6 \end{array}$$

We check:

$$\text{RHS} = -4x + 6 = -4 \cdot 6 + 6 = -24 + 6 = -18 = \text{LHS}$$

Thus our solution, $\boxed{x = 6}$ is correct.

$$12. \frac{\frac{5x-1}{7} + 3}{\frac{5}{3}} - 10 = -4$$

Soluton: This is one of those many-step equations where we have more than two steps, but we simply perform them in the reverse order like we did in the case of two-step equations.

$$\begin{aligned} \frac{\frac{5x-1}{7} + 3}{\frac{5}{3}} - 10 &= -4 && \text{multiply by 3} \\ \frac{5x-1}{7} + 3 - 10 &= -12 && \text{add 10} \\ \frac{5x-1}{7} + 3 &= -2 && \text{multiply by 5} \\ \frac{5x-1}{7} + 3 &= -10 && \text{subtract 3} \\ \frac{5x-1}{7} &= -13 && \text{multiply by 7} \\ 5x - 1 &= -91 && \text{add 1} \\ 5x &= -90 && \text{divide by 5} \\ x &= -18 \end{aligned}$$

We check: if $x = -18$, then

$$\begin{aligned} \text{LHS} &= \frac{\frac{5(-18)-1}{7} + 3}{\frac{5}{3}} - 10 = \frac{\frac{-90-1}{7} + 3}{\frac{5}{3}} - 10 = \frac{\frac{-91}{7} + 3}{\frac{5}{3}} - 10 = \frac{\frac{-13+3}{5}}{\frac{5}{3}} - 10 \\ &= \frac{\frac{-10}{5}}{\frac{5}{3}} - 10 = \frac{-2-10}{3} = \frac{-12}{3} = -4 = \text{RHS } \checkmark \end{aligned}$$

Thus our solution, $x = -18$ is correct.

13. Paul invested his money on the stock market. First he bet on a risky stock and lost half of his money. Then he became a bit more careful and invested money in more conservative stocks that involved less risk but also less profit. His investments made him 80 dollars. If he has 250 dollars in the stock market today, with how much money did he start investing?

Solution: Let us denote the amount of money with which Paul started to invest by x . First he lost half of his money, so he had $\frac{x}{2}$. Then he gained 80 dollars and ended up with 250 dollars. So, we can write the equation $\frac{x}{2} + 80 = 250$. We will solve this two-step equation for x . What happened to the unknown was first division by 2 and then addition of 80. To reverse these operations, we will first subtract 80 and then multiply by 2.

$$\begin{aligned} \frac{x}{2} + 80 &= 250 && \text{subtract 80} \\ \frac{x}{2} &= 170 && \text{multiply by 2} \\ x &= 340 \end{aligned}$$

So Paul started with 340 dollars. We check: If we lose half of 340, dollars we have 170 dollars left. Then when we add 80, dollars we indeed end up with 250 dollars. So our solution is correct,

Paul started with 340 dollars.

14. In a hotel, the first night costs 45 dollars, and all additional nights cost 35 dollars. How long did Mr. Williams stay in the hotel if his bill was 325 dollars?

Solution: Suppose that Mr. Williams stayed for the first night and then an additional x many nights. Then the bill would be $45 + x \cdot 35$ or $35x + 45$. So we write and then solve the equation $35x + 45 = 325$.

$$\begin{array}{rcll} 35x + 45 & = & 325 & \text{subtract 45} \\ 35x & = & 280 & \text{divide by 35} \\ x & = & 8 & \end{array}$$

Thus Mr. Williams stayed in the hotel for 9 nights. Why not 8 if we got $x = 8$? Remember, the first night was counted separately; there was the first night and then $x = 8$ additional nights. This is why it is a good idea to read the text of the problem one more time before we state our final answer. So Mr. Williams stayed 9 nights in the hotel. We check: the bill for 9 nights would be $45 + 8(35) = 325$, and so our solution is correct.

15. Solve each of the following formulas.

a) $3x - 4y = z$ for x

Solution: Let us look at the formula from the point of view of x . What happened to x was first a multiplication by 3 and then subtraction of $4y$. We will perform the inverse operations, in a reverse order: add first $4y$ and then divide by 3.

$$\begin{array}{rcll} 3x - 4y & = & z & \text{add } 4y \\ 3x & = & z + 4y & \text{divide by 3} \\ x & = & \frac{z + 4y}{3} & \end{array}$$

Our solution is $x = \frac{z + 4y}{3}$.

b) $3x - 4y = z$ for y

Solution: The fact that y is apparently subtracted can be handled in two different ways.

Method 1. Let us swap the two terms to have $-4y + 3x$. Then we can interpret it as multiplication by -4 and then addition of $3x$. To undo those, we will subtract $3x$ and divide by -4 . When simplifying the answer, we need to watch out for the distributive law.

$$\begin{array}{rcll} -4y + 3x & = & z & \text{subtract } 3x \\ -4y & = & z - 3x & \text{divide by } -4 \\ y & = & \frac{z - 3x}{-4} & \end{array}$$

This result is not simplified because we have a negative sign in the denominator. We can fix this by multiplying both numerator and denominator by -1 . With the numerator, we have to apply the distributive law.

$$y = \frac{-1(z - 3x)}{-1(-4)} = \frac{-z + 3x}{4}$$

Method 2. This is a frequently used trick if we want to avoid division by a negative number. Our first step is to add $4y$ to both sides. Then $4y$ disappears on the left-hand side and appears on the right-hand side. We will solve for y there (or swap the two sides).

$$\begin{array}{rcll}
 3x - 4y & = & z & \text{add } 4y \\
 3x & = & z + 4y & \text{subtract } z \\
 3x - z & = & 4y & \text{divide by } 4 \\
 \frac{3x - z}{4} & = & y &
 \end{array}$$

Naturally, our result is the same, $y = \frac{3x - z}{4}$.

c) $AB + PQ = S$ for Q

Solution: Let us look at the formula from the point of view of Q . What happened to x was first a multiplication by P and then addition of AB . We will perform the inverse operations, in a reverse order: subtract AB and then divide by Q .

$$\begin{array}{rcll}
 AB + PQ & = & S & \text{subtract } AB \\
 PQ & = & S - AB & \text{divide by } P \\
 Q & = & \frac{S - AB}{P} &
 \end{array}$$

Notice that it is essential that P is not zero, because division by zero is not allowed. What if P IS zero?

If P is zero, then the original equation $AB + PQ = S$ becomes $AB + 0 \cdot Q = S$ or $AB = S$ and it is not possible to solve for Q .

Solutions for 6.1 – Simplifying Algebraic Expressions

1. Evaluate each of the following algebraic expressions with the value(s) given.

a) $-x^2 - 5x + 2$ if $x = -2$

Solution: We substitute -2 into the expression. Please note that if the value of x is negative, we will need to place parentheses around it.

$$-x^2 - 5x + 2 = -(-2)^2 - 5(-2) + 2$$

According to order of operations, we perform the exponentiation first.

$$-(-2)^2 - 5(-2) + 2 = -4 - 5(-2) + 2 = -4 - (-10) + 2 = -4 + 10 + 2 = 6 + 2 = \boxed{8}$$

Why don't the two minuses make a plus in $-(-2)^2$? They do, it's just that there are three minus signs and not two: $-(-2)^2 = -1(-2)(-2)$.

b) $-16t^2 + 32t + 240$ if $t = 3$

Solution: We substitute 3 into the expression. Please note that if the value of x is negative, we will need to place parentheses around it.

$$\begin{aligned} -16t^2 + 32t + 240 &= -16(3)^2 + 32(3) + 240 \\ &= -16 \cdot 9 + 32 \cdot 3 + 240 \\ &= -144 + 96 + 240 = -48 + 240 = \boxed{192} \end{aligned}$$

2. Add the algebraic expressions as indicated.

a) $(3x - 5y) + (2x + 4y)$

Solution: We drop the parentheses and combine like terms.

$$\begin{aligned} (3x - 5y) + (2x + 4y) &= \\ 3x - 5y + 2x + 4y &= \text{organize like terms together} \\ 3x + 2x - 5y + 4y &= 3 + 2 = 5 \text{ and } -5 + 4 = -1 \\ &= \boxed{5x - y} \end{aligned}$$

b) $(2a - 5b + 3) + (-a - 8b + 3)$

Solution: We drop the parentheses and combine like terms.

$$\begin{aligned} (2a - 5b + 3) + (-a - 8b + 3) &= \\ 2a - 5b + 3 - a - 8b + 3 &= \\ 2a - a - 5b - 8b + 3 + 3 &= 2 - 1 = 1, \quad -5 - 8 = -13, \text{ and } 3 + 3 = 6 \\ 1a - 13b + 6 &= \boxed{a - 13b + 6} \end{aligned}$$

3. Multiply the algebraic expressions by a number as indicated.

a) $3(2x + 4y - 5)$

Solution: We distribute 3.

$$3(2x + 4y - 5) = \boxed{6x + 12y - 15}$$

b) $-5(2a - b + 8)$

Solution: We distribute -5 .

$$-5(2a - b + 8) = \boxed{-10a + 5b - 40}$$

c) $-1(-p + 3q - 8m + 6)$

Solution: We distribute -1 .

$$-1(-p + 3q - 8m + 6) = \boxed{p - 3q + 8m - 6}$$

d) $-(-a + 3b - 7)$

Solution: The notation here indicates multiplication by -1 , which is the same as taking the opposite of a quantity. We distribute -1 .

$$-1(-a + 3b - 7) = \boxed{a - 3b + 7}$$

4. Subtract the algebraic expressions as indicated.

a) $(2a + b) - (a - b)$

Solution: To subtract is to add the opposite. The opposite of $a - b$ is $-a + b$ since

$$-1(a - b) = -a + b$$

$$\begin{array}{llll} \text{Thus} & (2a + b) - (a - b) & = & (2a + b) + (-a + b) & \text{drop parentheses} \\ & & = & 2a + b - a + b & \text{combine like terms} \\ & & = & \boxed{a + 2b} & \end{array}$$

b) $(3x - 5y) - (2x + 4y)$

Solution: To subtract is to add the opposite. The opposite of $2x + 4y$ is $-2x - 4y$ since

$$-1(2x + 4y) = -2x - 4y$$

$$\begin{array}{llll} (3x - 5y) - (2x + 4y) & = & (3x - 5y) + (-2x - 4y) & \text{to subtract is to add the opposite} \\ & = & 3x - 5y - 2x - 4y & \text{drop parentheses, combine like terms} \\ & = & \boxed{x - 9y} & \end{array}$$

c) $(2m - 5n + 3) - (-m - 8n + 3)$

Solution: To subtract is to add the opposite. The opposite of $-m - 8n + 3$ is $m + 8n - 3$ since

$$-1(-m - 8n + 3) = m + 8n - 3$$

$$\begin{aligned}(2m - 5n + 3) - (-m - 8n + 3) &= && \text{to subtract is to add the opposite} \\(2m - 5n + 3) + (m + 8n - 3) &= && \text{drop parentheses, combine like terms} \\2m - 5n + 3 + m + 8n - 3 &= && \boxed{3m + 3n}\end{aligned}$$

d) $(2a - 2b) - (b - a)$

Solution: To subtract is to add the opposite. The opposite of $b - a$ is $-b + a$ since

$$-1(b - a) = -b + a$$

$$\begin{aligned}(2a - 2b) - (b - a) &= (2a - 2b) + (-b + a) && \text{to subtract is to add the opposite} \\&= 2a - 2b - b + a && \text{drop parentheses, combine like terms} \\&= \boxed{3a - 3b}\end{aligned}$$

e) $(3a - 2) - (1 - 4a)$

Solution: To subtract is to add the opposite. The opposite of $1 - 4a$ is $-1 + 4a$ since

$$-1(1 - 4a) = -1 + 4a$$

$$\begin{aligned}(3a - 2) - (1 - 4a) &= && \text{to subtract is to add the opposite} \\(3a - 2) + (-1 + 4a) &= && \text{drop parentheses, combine like terms} \\3a - 2 - 1 + 4a &= && \boxed{7a - 3}\end{aligned}$$

5. Simplify each of the following.

a) $3(x - 5) - 5(x - 1)$

Solution: We apply the law of distributivity and combine like terms. Notice that the last term is $-5(-1) = 5$.

$$\begin{aligned}3(x - 5) - 5(x - 1) &= && \text{apply the distributive law} \\3x - 15 - 5x + 5 &= && \text{combine like terms} \\&= && \boxed{-2x - 10}\end{aligned}$$

b) $4(2a - b) - 3(5a - 2b)$

Solution: We apply the law of distributivity and combine like terms

$$\begin{aligned}4(2a - b) - 3(5a - 2b) &= 8a - 4b - 15a + 6b \\&= \boxed{-7a + 2b}\end{aligned}$$

c) $2(a - 2b) + 3(5b - 2a) - 4(2b - a)$

Solution: We apply the law of distributivity and combine like terms

$$2(a - 2b) + 3(5b - 2a) - 4(2b - a) = 2a - 4b + 15b - 6a - 8b + 4a = \boxed{3b}$$

d) $-(3a - 2) - (1 - 4a)$

Solution:

$$\begin{aligned} -(3a - 2) - (1 - 4a) &= -1(3a - 2) - 1(1 - 4a) && \text{multiplication} \\ &= -3a + 2 - 1 + 4a && \text{combine like terms} \\ &= \boxed{a + 1} \end{aligned}$$

6. Simplify the given expression. $3(-2(-(6x - 1) + 3(2x - 5) - 5x + 7) - 8x - 10)$

Solution: We first need to investigate the parentheses and pair them off. There is no point starting the computation without understanding them; it will only get worse later. We will denote the different parentheses with differently shaped grouping symbols. There are two innermost parentheses, one after the other. We start there, left to right.

$$\begin{aligned} 3\{-2[-(6x - 1) + 3(2x - 5) - 5x + 7] - 8x - 10\} &= && \text{distribute } -1 \\ = 3\{-2[-6x + 1 + 3(2x - 5) - 5x + 7] - 8x - 10\} &&& \text{combine like terms} \\ = 3\{-2[-11x + 8 + 3(2x - 5)] - 8x - 10\} &&& \text{distribute } 3 \\ = 3\{-2[-11x + 8 + 6x - 15] - 8x - 10\} &&& \text{combine like terms} \\ = 3\{-2(-5x - 7) - 8x - 10\} &&& \text{distribute } -2 \\ = 3\{10x + 14 - 8x - 10\} &&& \text{combine like terms} \\ = 3(2x + 4) &&& \text{distribute } 3 \\ = \boxed{6x + 12} \end{aligned}$$

7. Solve each of the given equations. Make sure to check your solutions.

a) $5(3x - 8) - 2(8x - 1) = -33$

Solution: We need to simplify the left-hand side. After that, the equation will be a simple two-step equation.

$$\begin{aligned} 5(3x - 8) - 2(8x - 1) &= -33 && \text{distribute } 5 \text{ and } -2 \\ 15x - 40 - 16x + 2 &= -33 && \text{combine like terms} \\ -x - 38 &= -33 && \text{add } 38 \\ -x &= 5 && \text{divide (or multiply) by } -1 \\ x &= -5 \end{aligned}$$

We check: if $x = -5$, then

$$\begin{aligned} \text{LHS} &= 5[3(-5) - 8] - 2[8(-5) - 1] = 5(-15 - 8) - 2(-40 - 1) = 5(-23) - 2(-41) \\ &= -115 + 82 = -33 = \text{RHS } \checkmark \end{aligned}$$

Thus our solution, $\boxed{x = -5}$, is correct.

$$\text{b) } 4(5x - 3(2x - 1) + 2x) = 52$$

Solution: We need to simplify the left-hand side. After that, the equation will be a simple two-step equation. We start with the innermost parentheses.

$$\begin{array}{rcll} 4(5x - 3(2x - 1) + 2x) & = & 52 & \text{distribute } -3 \\ 4(5x - 6x + 3 + 2x) & = & 52 & \text{combine like terms} \\ 4(x + 3) & = & 52 & \text{distribute 4} \\ 4x + 12 & = & 52 & \text{subtract 12} \\ 4x & = & 40 & \text{divide by 4} \\ x & = & 10 & \end{array}$$

We check: if $x = 10$, then

$$\begin{aligned} \text{LHS} &= 4(5 \cdot 10 - 3(2 \cdot 10 - 1) + 2 \cdot 10) = 4(5 \cdot 10 - 3(20 - 1) + 2 \cdot 10) = 4(5 \cdot 10 - 3 \cdot 19 + 2 \cdot 10) \\ &= 4(50 - 57 + 20) = 4(-7 + 20) = 4 \cdot 13 = 52 = \text{RHS } \checkmark \end{aligned}$$

$$\text{c) } 2(3(4(5x - 1) - 17x + 2) - 3x + 1) = 14$$

Solution: After we simplified the expression on the left-hand side, we will again have an easy two-step equation. The three pairs of parentheses are nested inside each other. We start with the innermost one.

$$\begin{array}{rcll} 2\{3[4(5x - 1) - 17x + 2] - 3x + 1\} & = & 14 & \text{distribute 4} \\ 2\{3(20x - 4 - 17x + 2) - 3x + 1\} & = & 14 & \text{combine like terms} \\ 2\{3(3x - 2) - 3x + 1\} & = & 14 & \text{distribute 3} \\ 2(9x - 6 - 3x + 1) & = & 14 & \text{combine like terms} \end{array} \quad \begin{array}{rcll} 2(6x - 5) & = & 14 & \text{distribute 2} \\ 12x - 10 & = & 14 & \text{add 10} \\ 12x & = & 24 & \text{divide by 12} \\ x & = & 2 & \end{array}$$

We check: if $x = 2$, then

$$\begin{aligned} \text{LHS} &= 2\{3[4(5 \cdot 2 - 1) - 17 \cdot 2 + 2] - 3 \cdot 2 + 1\} = 2(3(4 \cdot 9 - 17 \cdot 2 + 2) - 3 \cdot 2 + 1) \\ &= 2(3(36 - 34 + 2) - 3 \cdot 2 + 1) = 2(3(2 + 2) - 3 \cdot 2 + 1) = 2(3 \cdot 4 - 3 \cdot 2 + 1) \\ &= 2(12 - 6 + 1) = 2(6 + 1) = 2 \cdot 7 = 14 = \text{RHS } \checkmark \end{aligned}$$

So our solution, $x = 2$ is correct.

8. Solve each of the following application problems.

a) Ann and Betty dine together. The total bill is \$38. Ann paid \$2 more than Betty. How much did Betty pay?

Solution: Let us denote the amount paid by Betty paid by x . Then Ann paid $x + 2$. The equation expresses the total amount paid:

$$\begin{array}{rcll} x + x + 2 & = & 38 & \text{combine like terms} \\ 2x + 2 & = & 38 & \text{subtract 2} \\ 2x & = & 36 & \text{divide by 2} \\ x & = & 18 & \end{array}$$

Thus Betty paid \$18 and Ann paid \$20.

b) 55 people showed up on the party. There were 3 less women than men. How many men were there?

Solution: Let us denote the number of women by x . Then $x + 3$ men showed up. The equation expresses the total number of people:

$$x + x + 3 = 55 \quad \text{combine like terms}$$

$$2x + 3 = 55 \quad \text{subtract 3}$$

$$2x = 52 \quad \text{divide by 2}$$

$$x = 26$$

Thus there were 26 women and 29 men on the party.

c) One side of a rectangle is 7 cm shorter than five times the other side. Find the length of the sides if the perimeter of the rectangle is 118 cm.

Solution: Let us denote the shorter side by x . Then the longer side is $5x - 7$. We obtain the equation for the perimeter:

$$2x + 2(5x - 7) = 118 \quad \text{distribute}$$

$$2x + 10x - 14 = 118 \quad \text{combine like terms}$$

$$12x - 14 = 118 \quad \text{add 14}$$

$$12x = 132 \quad \text{divide by 12}$$

$$x = 11$$

Thus the shorter side is 11 cm, the longer side is $5(11\text{cm}) - 7\text{cm} = 48\text{cm}$. We check: the perimeter is

$$2(11\text{cm}) + 2(48\text{cm}) = 118\text{ cm and } 48 \text{ is indeed } 7 \text{ shorter than five times } 11. \text{ Thus the solution is: } \boxed{11 \text{ cm by } 48 \text{ cm}.}$$

d) One side of a rectangle is 3cm longer than four times the other side. Find the sides if the perimeter of the rectangle is 216cm.

Solution: Let us denote the shorter side by x . Then the longer side is $4x + 3$. We obtain the equation for the perimeter:

$$2x + 2(4x + 3) = 216 \quad \text{distribute}$$

$$2x + 8x + 6 = 216 \quad \text{combine like terms}$$

$$10x + 6 = 216 \quad \text{subtract 6}$$

$$10x = 210 \quad \text{divide by 10}$$

$$x = 21$$

Thus the shorter side is 21 cm, the longer side is $4 \cdot 21 + 3 = 87$ cm. We check: the perimeter is $2 \cdot 21 + 2 \cdot 87 = 42 + 174 = 216\text{cm}$ and 87 is indeed 3 longer than four times 21. Thus the solution is: 21 cm by 87 cm.

e) The sum of two consecutive even integers is -170 . Find these numbers.

Solution: Let us denote the smaller number by x . Then the larger number is $x + 2$. The equation expresses the sum of the numbers.

$$x + x + 2 = -170 \quad \text{combine like terms}$$

$$2x + 2 = -170 \quad \text{subtract 2}$$

$$2x = -172 \quad \text{divide by 2}$$

$$x = -86$$

Then the larger number must be $-86 + 2 = -84$. Thus the numbers are -86 and -84 .

f) The sum of three consecutive odd integers is 57. Find these numbers.

Solution: Let us denote the smallest number by x . Then the other two numbers must be $x + 2$ and $x + 4$. The equation expresses the sum of the three numbers.

$$\begin{aligned} x + x + 2 + x + 4 &= 57 && \text{combine like terms} \\ 3x + 6 &= 57 && \text{subtract 6} \\ x &= 51 && \text{divide by 3} \\ x &= 17 \end{aligned}$$

Thus the three numbers are 17, and $17 + 2 = 19$, and $17 + 4 = 21$. We check: indeed, $17 + 19 + 21 = 57$. Thus the solution is 17, 19, and 21.

g) Small ones weigh 3 lb, big ones weigh 4 lb. The number of small ones is 3 more than twice the number of big ones. All together, they weigh 79 lb. How many small ones are there?

Solution: Let us denote the number of big ones by x . Then the number of small ones is $2x + 3$. We obtain the equation expressing the total weight:

$$\begin{aligned} 3(2x + 3) + 4x &= 79 && \text{distribute} \\ 6x + 9 + 4x &= 79 && \text{combine like terms} \\ 10x + 9 &= 79 && \text{subtract 9} \\ 10x &= 70 && \text{divide by 10} \\ x &= 7 \end{aligned}$$

The number of big ones is then 7, and so the number of small ones is $2(7) + 3 = 17$. We check: the number of small ones, 17 is indeed 3 more than twice the number of big ones, 7. The total weight is $7(4) + 17(3) = 28 + 51 = 79$. Thus the solution is 7 big, 17 small.

h) We have a jar of coins, all quarters and dimes. All together, they are worth \$17.60 We have 13 more quarters than dimes. How many quarters, how many dimes?

Solution: Let us denote the number of dimes by x . Then the number of quarters must be $x + 13$. We obtain the equation by expressing the total value, **in pennies**.

$$\begin{aligned} 10x + 25(x + 13) &= 1760 && \text{distribute} \\ 10x + 25x + 325 &= 1760 && \text{combine like terms} \\ 35x + 325 &= 1760 && \text{subtract 325} \\ 35x &= 1435 && \text{divide by 35} \\ x &= 41 \end{aligned}$$

Thus we have 41 dimes and $41 + 13 = 54$ quarters. We check: $41(0.10) + 54(0.25) = 4.10 + 13.50 = 17.6$. Thus the solution is 41 dimes and 54 quarters.

i) Red pens cost \$2 each, blue ones cost \$3 each. We bought some pens. The number of red pens is 7 less than five times the number of blue pens. How many of each did we buy if we paid \$116?

Solution: Let us denote the number of blue pens by x . Then the number of red pens is $5x - 7$. The equation will express the total cost of the pens:

$$\begin{aligned} 2(5x - 7) + 3(x) &= 116 && \text{distribute} \\ 10x - 14 + 3x &= 116 && \text{combine like terms} \\ 13x - 14 &= 116 && \text{add 14} \\ 13x &= 130 && \text{divide by 13} \\ x &= 10 \end{aligned}$$

Thus we bought 10 blue and $5(10) - 7 = 43$ red pens. We check:

$$\begin{aligned} 43 &= 5(10) - 7 \checkmark \\ 2(43) + 3(10) &= 86 + 30 = 116 \checkmark \end{aligned}$$

Thus our solution is correct; we bought 10 blue and 43 red pens.

Solutions for 6.2 – The Greatest Common Factor and Least Common Multiple

3. Solution: $\gcd(6, 120, n) = 6$ implies that 6 is a divisor of n .

Similarly, $\text{lcm}(6, 120, n) = 120$ implies that n is a divisor of 120

all divisors of 120 are: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120

all divisors of 120 that are divisible by 6 are: 6, 12, 24, 30, 60, 120 - and all these numbers work.

Solutions for 7.2 – Rules of Exponents

Let us recall the rules of exponents.

$$1. a^n \cdot a^m = a^{n+m}$$

$$4. (ab)^n = a^n b^n$$

$$2. \frac{a^n}{a^m} = a^{n-m}$$

$$5. \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$3. (a^n)^m = a^{nm}$$

1. Simplify each of the following.

a) $(2x^5)(x^4)$

Solution: $(2x^5)(x^4) = 2x^5x^4 = 2x^{5+4} = \boxed{2x^9}$ by rule 1.

b) $(2x)^5(x^4)$

Solution:

$$\begin{aligned} (2x)^5(x^4) &= 2^5x^5x^4 && \text{by rule 4} \\ &= 32x^{5+4} && \text{by rule 1} \\ &= \boxed{32x^9} \end{aligned}$$

c) $(2x^5)^4$

Solution:

$$\begin{aligned} (2x^5)^4 &= 2^4(x^5)^4 && \text{by rule 4} \\ &= \boxed{16x^{20}} && \text{by rule 3} \end{aligned}$$

d) $(-xy)^2(-xy^2)^3$

Solution:

$$\begin{aligned} (-xy)^2(-xy^2)^3 &= (-1xy)^2(-1xy^2)^3 && \text{the 1s will help with signs} \\ &= (-1)^2x^2y^2(-1)^3x^3(y^2)^3 && \text{by rule 4} \\ &= 1 \cdot x^2y^2(-1)x^3y^6 && \text{by rule 3} \\ &= 1(-1)x^2x^3y^2y^6 && \text{multiplication is commutative} \\ &= -1x^{2+3}y^{2+6} && \text{by rule 1} \\ &= \boxed{-x^5y^8} \end{aligned}$$

e) $-2a^3(-2a^4)^2$

Solution:

$$\begin{aligned}
 -2a^3(-2a^4)^2 &= -2a^3(-2)^2(a^4)^2 && \text{rule 4} \\
 &= -2a^3(4)a^8 && \text{rule 3} \\
 &= -2(4)a^3a^8 && \text{multiplication is commutative} \\
 &= -8a^{3+8} = \boxed{-8a^{11}} && \text{rule 1}
 \end{aligned}$$

f) $2a^3(-2ab^2)^3ab^2$

Solution:

$$\begin{aligned}
 2a^3(-2ab^2)^3ab^2 &= 2a^3(-2)^3a^3(b^2)^3ab^2 && \text{rule 4} \\
 &= 2a^3(-8)a^3b^6ab^2 && \text{rule 3} \\
 &= 2(-8)a^3a^3ab^6b^2 && \text{multiplication is commutative} \\
 &= -16a^{3+3+1}b^{6+2} = \boxed{-16a^7b^8} && \text{rule 1}
 \end{aligned}$$

g) $\frac{(-2x^5)^2y^3}{2x^3y^2}$

Solution:

$$\begin{aligned}
 \frac{(-2x^5)^2y^3}{2x^3y^2} &= \frac{(-2)^2(x^5)^2y^3}{2x^3y^2} && \text{rule 4} \\
 &= \frac{4x^{10}y^3}{2x^3y^2} && \text{rule 3} \\
 &= \frac{4x^{10-3}y^{3-2}}{2} && \text{rule 2} \\
 &= \frac{4x^7y^1}{2} = \boxed{2x^7y}
 \end{aligned}$$

h) $\frac{(2ab)^3(-3a^2b)^2}{-a(6ab^2)^2}$

Solution:

$$\begin{aligned}
 \frac{(2ab)^3(-3a^2b)^2}{-a(6ab^2)^2} &= \frac{(2ab)^3(-3a^2b)^2}{-1a(6ab^2)^2} && \text{the 1 will help with signs} \\
 &= \frac{2^3a^3b^3(-3)^2(a^2)^2b^2}{-1 \cdot a \cdot 6^2 \cdot a^2(b^2)^2} && \text{by rule 4} \\
 &= \frac{8a^3b^3 \cdot 9 \cdot a^4b^2}{-1 \cdot a \cdot 36 \cdot a^2b^4} = \frac{8 \cdot 9 \cdot a^3a^4b^3b^2}{-1 \cdot 36 \cdot a \cdot a^2 \cdot b^4} && \text{by rule 3} \\
 &= \frac{72a^{3+4}b^{3+2}}{-36a^{1+2}b^4} = \frac{72a^7b^5}{-36a^3b^4} && \text{by rule 1} \\
 &= \frac{-2a^7b^5}{a^3b^4} && \text{simplify numbers: } \frac{72}{-36} = \frac{-2}{1} \\
 &= \frac{-2a^{7-3}b^{5-4}}{1} = -2a^4b^1 = \boxed{-2a^4b} && \text{rule 2}
 \end{aligned}$$

2. Write each of the following expressions in terms of a fixed number or a single exponential expression.

a) $\frac{3^{2x+1}}{9^{x-1}}$

Solution: We will re-write the denominator in terms of base 3. After that, we can apply $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{3^{2x+1}}{9^{x-1}} = \frac{3^{2x+1}}{(3^2)^{x-1}} \stackrel{\text{Rule 3}}{=} \frac{3^{2x+1}}{3^{2(x-1)}} = \frac{3^{2x+1}}{3^{2x-2}} \stackrel{\text{Rule 2}}{=} 3^{2x+1-(2x-2)} = 3^{2x+1-2x+2} = 3^3 = \boxed{27}$$

b) $\frac{(8^{b-2})(2^{b+1})}{4^{2b-3}}$

Solution: We will re-write each exponential expressions in terms of base 2.

$$\begin{aligned} \frac{(8^{b-2})(2^{b+1})}{4^{2b-3}} &= \frac{8^{b-2} \cdot 2^{b+1}}{4^{2b-3}} = \frac{(2^3)^{b-2} \cdot 2^{b+1}}{(2^2)^{2b-3}} \stackrel{\text{Rule 3}}{=} \frac{2^{3(b-2)} (2^{b+1})}{2^{2(2b-3)}} = \frac{2^{3b-6} \cdot 2^{b+1}}{2^{4b-6}} \stackrel{\text{Rule 1}}{=} \frac{2^{3b-6+(b+1)}}{2^{4b-6}} \\ &= \frac{2^{4b-5}}{2^{4b-6}} \stackrel{\text{Rule 2}}{=} 2^{(4b-5)-(4b-6)} = 2^{4b-5-4b+6} = 2^1 = \boxed{2} \end{aligned}$$

c) $5^{2x-1} \cdot 25^{3-x}$

$$5^{2x-1} \cdot 25^{3-x} = 5^{2x-1} \cdot (5^2)^{3-x} \stackrel{\text{Rule 3}}{=} 5^{2x-1} \cdot 5^{2(3-x)} = 5^{2x-1} \cdot 5^{6-2x} \stackrel{\text{Rule 1}}{=} 5^{2x-1+6-2x} = 5^5 = \boxed{3125}$$

3. Let us denote 3^{100} by M . Express each of the following in terms of M .

a) 3^{101}

Solution: Using rule 1, we write $3^{101} = 3^{100+1} = 3^{100} \cdot 3^1 = M \cdot 3 = \boxed{3M}$

b) $3^{100} - 2 \cdot 3^{101} + 3^{102}$

Solution: Using rule 1, we re-write 3^{101} and 3^{102}

$$\begin{aligned} 3^{101} &= 3^{100+1} = 3^{100} \cdot 3^1 = M \cdot 3 = 3M \\ 3^{102} &= 3^{100+2} = 3^{100} \cdot 3^2 = M \cdot 9 = 9M \end{aligned}$$

Then our expression becomes

$$3^{100} - 2 \cdot 3^{101} + 3^{102} = M - 2 \cdot (3M) + 9M = M - 6M + 9M = -5M + 9M = \boxed{4M}$$

c) 3^{99}

Solution: Using rule 2, we write $3^{99} = 3^{100-1} = \frac{3^{100}}{3^1} = \boxed{\frac{M}{3}}$

d) 9^{100}

Solution: This time we will use rule 3 in a novel way: $(a^n)^m = (a^m)^n$

$$9^{100} = (3^2)^{100} = (3^{100})^2 = M^2$$

We can also solve this problem using rule 4

$$9^{100} = (3 \cdot 3)^{100} = 3^{100} \cdot 3^{100} = M \cdot M = \boxed{M^2}$$

4. Find the prime-factorization for each of the following numbers.

a) 10^{2018}

Solution: We will find the prime-factorization of the base and then apply rules of exponents.

$$10^{2018} = (2 \cdot 5)^{2018} = \boxed{2^{2018} \cdot 5^{2018}}$$

b) 18^{1000}

Solution: We will find the prime-factorization of the base and then apply rules of exponents.

$$18^{1000} = (2 \cdot 9)^{1000} = (2 \cdot 3^2)^{1000} = 2^{1000} \cdot (3^2)^{1000} = 2^{1000} \cdot 3^{2 \cdot 1000} = \boxed{2^{1000} \cdot 3^{2000}}$$

c) 360^{50}

Solution: We will find the prime-factorization of the base and then apply rules of exponents.

$$360 = 36 \cdot 10 = (4 \cdot 9) \cdot (2 \cdot 5) = 2^3 \cdot 3^2 \cdot 5. \text{ So the prime factorization of 360 is } 2^3 \cdot 3^2 \cdot 5.$$

$$360^{50} = (2^3 \cdot 3^2 \cdot 5)^{50} = (2^3)^{50} (3^2)^{50} (5)^{50} = 2^{3 \cdot 50} \cdot 3^{2 \cdot 50} \cdot 5^{50} = \boxed{2^{150} \cdot 3^{100} \cdot 5^{50}}$$

d) The product $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ comes up a lot in mathematics, so there is notation for it.

$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$ We pronounce $10!$ as ten factorial.

Now we just find the prime-factorization of each factor and collect the prime-factorization that way.

$$\begin{aligned} 10! &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 && \text{we drop 1 at the end} \\ &= (2 \cdot 5) (3^2) (2^3) \cdot 7 \cdot (2 \cdot 3) \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 \end{aligned}$$

We collect the two-factors: one from 10, three from 8, one from 6, two from 4, and one from two. Similarly, the three-factors: two from 9, one from 6 and one from 3. Five factors come out only from 10 and 5, one from each, and the greatest prime factor is 7.

$$\begin{aligned} 10! &= (2 \cdot 5) (3^2) (2^3) \cdot 7 \cdot (2 \cdot 3) \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 \\ &= 2^{1+3+1+2+1} \cdot 3^{2+1+1} \cdot 5^{1+1} \cdot 7 = \boxed{2^8 \cdot 3^4 \cdot 5^2 \cdot 7} \end{aligned}$$

5. Re-write each of the given numbers using scientific notation.

a) 3800000000

Solution: 3800000000 ends in eight zeroes. This can be translated as

$3800000000 = 38 \cdot 10^8$. We are not done yet: the first factor is too big, it must be between 1 and 10. So we re-write 38 as $3.8 \cdot 10$ and use the rules of exponents:

$$3800000000 = 38 \cdot 10^8 = (3.8 \cdot 10) \cdot 10^8 = \boxed{3.8 \cdot 10^9}$$

b) 625000000000

Solution: We count ten trailing zeroes, so

$625000000000 = 625 \cdot 10^{10}$. We re-write 625 as $6.25 \cdot 10^2$. So

$$625000000000 = 625 \cdot 10^{10} = (6.25 \cdot 10^2) \cdot 10^{10} = \boxed{6.25 \cdot 10^{12}}$$

6. Suppose that $A = 5 \cdot 10^{18}$, and $B = 8 \cdot 10^7$. Compute each of the following. Present your answer using scientific notation.

a) AB b) A^2 c) $2A$ d) $\frac{4A}{B}$

Solution: $AB = (5 \cdot 10^{18})(8 \cdot 10^7) = (5 \cdot 8)(10^{18} \cdot 10^7) = 40 \cdot 10^{25} = \boxed{4 \cdot 10^{26}}$

b) A^2

Solution: $A^2 = (5 \cdot 10^{18})^2 = 5^2 \cdot (10^{18})^2 = 25 \cdot 10^{18 \cdot 2} = 25 \cdot 10^{36} = 2.5 \cdot 10 \cdot 10^{36} = \boxed{2.5 \cdot 10^{37}}$

c) $2A$

Solution: $2A = 2(5 \cdot 10^{18}) = (2 \cdot 5) \cdot 10^{18} = 10 \cdot 10^{18} = \boxed{1 \cdot 10^{19}}$

d) $\frac{4A}{B}$

Solution: $\frac{4A}{B} = \frac{4(5 \cdot 10^{18})}{8 \cdot 10^7} = \frac{20 \cdot 10^{18}}{8 \cdot 10^7}$

Unfortunately, 8 is not a divisor of 20, but it is a divisor of 200. So, we borrow a ten from the ten-power.

$$\frac{20 \cdot 10^{18}}{8 \cdot 10^7} = \frac{200 \cdot 10^{17}}{8 \cdot 10^7} = 25 \cdot 10^{10} = \boxed{2.5 \cdot 10^{11}}$$

Solutions for 7.4 – Linear Equations 2

1. $2x + 3 = 4x + 9$

Solution:

$$\begin{array}{rcll} 2x + 3 & = & 4x + 9 & \text{subtract } 2x \text{ from both sides} \\ 3 & = & 2x + 9 & \text{subtract } 9 \text{ from both sides} \\ -6 & = & 2x & \text{divide both sides by 2} \\ -3 & = & x & \end{array}$$

We check: if $x = -3$, then

$$\begin{array}{lcl} \text{LHS} & = & 2(-3) + 3 = -6 + 3 = -3 \\ \text{RHS} & = & 4(-3) + 9 = -12 + 9 = -3 \end{array}$$

Thus our solution, $x = -3$ is correct. (Note: LHS is short for the left-hand side and RHS is short for the right-hand side.)

2. $3w - 5 = 5(w + 1)$

Solution: we first apply the law of distributivity to simplify the right-hand side.

$$\begin{array}{rcll} 3w - 5 & = & 5(w + 1) & \\ 3w - 5 & = & 5w + 5 & \text{subtract } 3w \text{ from both sides} \\ -5 & = & 2w + 5 & \text{subtract } 5 \text{ from both sides} \\ -10 & = & 2w & \text{divide both sides by 2} \\ -5 & = & w & \end{array}$$

We check. If $w = -5$, then

$$\begin{array}{lcl} \text{LHS} & = & 3(-5) - 5 = -15 - 5 = -20 \\ \text{RHS} & = & 5((-5) + 1) = 5(-4) = -20 \end{array}$$

Thus our solution, $w = -5$ is correct.

3. $7x - 2 = 5x - 2$

Solution:

$$\begin{array}{rcll} 7x - 2 & = & 5x - 2 & \text{subtract } 5x \text{ from both sides} \\ 2x - 2 & = & -2 & \text{add } 2 \text{ to both sides} \\ 2x & = & 0 & \text{divide both sides by 2} \\ x & = & 0 & \end{array}$$

We check: if $x = 0$, then

$$\begin{array}{lcl} \text{LHS} & = & 7(0) - 2 = 0 - 2 = -2 \\ \text{RHS} & = & 5(0) - 2 = 0 - 2 = -2 \end{array}$$

Thus our solution, $x = 0$ is correct.

4. $3y - 9 = -2y + 4$

Solution:

$$\begin{array}{rcll} 3y - 9 & = & -2y + 4 & \text{add } 2y \text{ to both sides} \\ 5y - 9 & = & 4 & \text{add } 9 \text{ to both sides} \\ 5y & = & 13 & \text{divide both sides by } 5 \\ y & = & \frac{13}{5} & \end{array}$$

We check. If $y = \frac{13}{5}$, then

$$\begin{aligned} \text{LHS} &= 3\left(\frac{13}{5}\right) - 9 = \frac{3}{1} \cdot \frac{13}{5} - 9 = \frac{39}{5} - \frac{9}{1} = \frac{39}{5} - \frac{45}{5} = \frac{-6}{5} = -\frac{6}{5} \\ \text{RHS} &= -2\left(\frac{13}{5}\right) + 4 = \frac{-2}{1} \cdot \frac{13}{5} + \frac{4}{1} = \frac{-26}{5} + \frac{20}{5} = \frac{-6}{5} = -\frac{6}{5} \end{aligned}$$

Thus $y = \frac{13}{5}$ is the correct solution.

5. $2w + 1 = 2w - 9$

Solution:

$$\begin{array}{rcll} 2w + 1 & = & 2w - 9 & \text{subtract } 2w \text{ from both sides} \\ 1 & = & -9 & \end{array}$$

The statement $1 = -9$ is false no matter what the value of w is. Such a statement is called an **unconditionally false statement**, or **contradiction**. This equation has **no solution**.

6. $\frac{1}{2}x - 1 = \frac{2}{3}x + 4$

Solution: Structurally, this equation is no different from the previous equations. However, because the coefficients of x are fractions, each step will take a bit more work.

$$\begin{array}{rcll} \frac{1}{2}x - 1 & = & \frac{2}{3}x + 4 & \text{subtract } \frac{1}{2}x \text{ from both sides} \\ -1 & = & \frac{1}{6}x + 4 & \text{subtract } 4 \text{ from both sides} \\ -5 & = & \frac{1}{6}x & \text{divide both sides by } \frac{1}{6} \\ -30 & = & x & \end{array}$$

Here are the computations for each step. To subtract $\frac{1}{2}x$ from the right-hand side:

$$\frac{2}{3}x - \frac{1}{2}x = \left(\frac{2}{3} - \frac{1}{2}\right)x = \left(\frac{4}{6} - \frac{3}{6}\right)x = \frac{4-3}{6}x = \frac{1}{6}x$$

We divide both sides by $\frac{1}{6}$. To divide is to multiply by the reciprocal:

$$-5 \div \frac{1}{6} = \frac{-5}{1} \div \frac{1}{6} = \frac{-5}{1} \cdot \frac{6}{1} = \frac{-30}{1} = -30$$

We check: if $x = -30$, then

$$\begin{aligned} \text{LHS} &= \frac{1}{2}(-30) - 1 = -15 - 1 = -16 \\ \text{RHS} &= \frac{2}{3}(-30) + 4 = \frac{2}{3} \cdot \frac{-30}{1} + 4 = \frac{-60}{3} + 4 = -20 + 4 = -16 \end{aligned}$$

Thus our solution, $x = -30$ is correct.

$$7. -\frac{2}{5}x + \frac{1}{3} = \frac{4}{15}x$$

Solution:

$$\begin{aligned} -\frac{2}{5}x + \frac{1}{3} &= \frac{4}{15}x && \text{add } \frac{2}{5}x \text{ to both sides} \\ \frac{1}{3} &= \frac{2}{3}x && \text{divide both sides by } \frac{2}{3} \\ \frac{1}{2} &= x \end{aligned}$$

The computation for each step are as follows. To add $\frac{2}{5}x$ to the right-hand side:

$$\frac{4}{15}x + \frac{2}{5}x = \left(\frac{4}{15} + \frac{2}{5}\right)x = \left(\frac{4}{15} + \frac{6}{15}\right)x = \frac{10}{15}x = \frac{2}{3}x$$

To divide by $\frac{2}{3}$ is to multiply by its reciprocal:

$$\frac{1}{3} \div \frac{2}{3} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

We check: if $x = \frac{1}{2}$, then

$$\begin{aligned} \text{LHS} &= -\frac{2}{5}\left(\frac{1}{2}\right) + \frac{1}{3} = -\frac{1}{5} + \frac{1}{3} = \frac{-3}{15} + \frac{5}{15} = \frac{-3+5}{15} = \frac{2}{15} \\ \text{RHS} &= \frac{4}{15}\left(\frac{1}{2}\right) = \frac{4}{30} = \frac{2}{15} \end{aligned}$$

Thus our solution, $x = \frac{1}{2}$ is correct.

8. $4 - x = 3(x - 7)$

Solution: We first apply the law of distributivity to simplify the right-hand side.

$$\begin{aligned}
 4 - x &= 3(x - 7) && \text{distribute } 3 \\
 4 - x &= 3x - 21 && \text{add } x \text{ to both sides} \\
 4 &= 4x - 21 && \text{add } 21 \text{ to both sides} \\
 25 &= 4x && \text{divide both sides by } 4 \\
 \frac{25}{4} &= x
 \end{aligned}$$

We check. If $x = \frac{25}{4}$, then

$$\begin{aligned}
 \text{LHS} &= 4 - x = 4 - \frac{25}{4} = \frac{4}{1} - \frac{25}{4} = \frac{16}{4} - \frac{25}{4} = \frac{16 - 25}{4} = \frac{-9}{4} = -\frac{9}{4} \\
 \text{RHS} &= 3(x - 7) = 3\left(\frac{25}{4} - 7\right) = 3\left(\frac{25}{4} - \frac{7}{1}\right) = 3\left(\frac{25}{4} - \frac{28}{4}\right) = 3\left(\frac{25 - 28}{4}\right) \\
 &= 3\left(\frac{-3}{4}\right) = \frac{3}{1} \cdot \frac{-3}{4} = \frac{-9}{4} = -\frac{9}{4}
 \end{aligned}$$

Thus our solution, $x = \frac{25}{4}$ is correct.

9. $7(j - 5) + 9 = 2(-2j + 5) + 5j$

Solution:

$$\begin{aligned}
 7(j - 5) + 9 &= 2(-2j + 5) + 5j && \text{distribute on both sides} \\
 7j - 35 + 9 &= -4j + 10 + 5j && \text{combine like terms} \\
 7j - 26 &= j + 10 && \text{subtract } j \\
 6j - 26 &= 10 && \text{add } 26 \\
 6j &= 36 && \text{divide by } 6 \\
 j &= 6
 \end{aligned}$$

We check: if $j = 6$, then

$$\begin{aligned}
 \text{LHS} &= 7(6 - 5) + 9 = 7 \cdot 1 + 9 = 7 + 9 = 16 \\
 \text{RHS} &= 2(-2 \cdot 6 + 5) + 5 \cdot 6 = 2(-12 + 5) + 30 = 2(-7) + 30 = -14 + 30 = 16
 \end{aligned}$$

Thus our solution, $j = 6$ is correct.

10. $3(x - 5) - 5(x - 1) = -2x + 1$

Solution:

$$\begin{aligned}
 3(x - 5) - 5(x - 1) &= -2x + 1 && \text{multiply out parentheses} \\
 3x - 15 - 5x + 5 &= -2x + 1 && \text{combine like terms} \\
 -2x - 10 &= -2x + 1 && \text{add } 2x \\
 -10 &= 1
 \end{aligned}$$

Since x disappeared from the equation and we are left with an unconditionally false statement, $\boxed{\text{there is no solution}}$ for this equation. This type of an equation is called a **contradiction**.

11. $\frac{2}{3}(x-7) = \frac{4}{5}(x+1)$

Solution:

$$\begin{aligned}\frac{2}{3}(x-7) &= \frac{4}{5}(x+1) \\ \frac{2}{3} \cdot \frac{x-7}{1} &= \frac{4}{5} \cdot \frac{x+1}{1} \\ \frac{2(x-7)}{3} &= \frac{4(x+1)}{5} && \text{bring fractions to common denominator} \\ \frac{5 \cdot 2(x-7)}{15} &= \frac{3 \cdot 4(x+1)}{15} && \text{multiply both sides by 15}\end{aligned}$$

$$\begin{aligned}10(x-7) &= 12(x+1) && \text{multiply out parentheses} \\ 10x - 70 &= 12x + 12 && \text{subtract } 10x \\ -70 &= 2x + 12 && \text{subtract 12} \\ -82 &= 2x && \text{divide by 2} \\ -41 &= x\end{aligned}$$

We check:

$$\begin{aligned}\text{LHS} &= \frac{2}{3}(-41-7) = \frac{2}{3}(-48) = -32 \\ \text{RHS} &= \frac{4}{5}(-41+1) = \frac{4}{5}(-40) = -32\end{aligned}$$

Thus our solution, $x = -41$ is correct.

12. Five times the opposite of a number is one less than seven times the sum of the number and seven.

Solution: Let us denote the number by x . Then five times the the opposite of this number can be translated as $5(-x)$, and that can be simplified as $-5x$. Seven times the sum of the number and seven is $7(x+7)$. The equation will express the comparison between the two.

$$\begin{aligned}-5x &= 7(x+7) - 1 && \text{distribute 7} \\ -5x &= 7x + 49 - 1 && \text{combine like terms} \\ -5x &= 7x + 48 && \text{add } 5x \\ 0 &= 12x + 48 && \text{subtract 48} \\ -48 &= 12x && \text{divide by 12} \\ -4 &= x\end{aligned}$$

Therefore, this number is -4 . We check: Five times the opposite of our number is $5(4) = 20$, and seven times the sum of the number and seven is $7(-4+7) = 21$. Indeed, 20 is one less than 21, so our solution is correct.

Solutions for 8.1 – Multiplying Algebraic Expressions

1. Multiply the algebraic expressions as indicated.

a) $3xy(2x^2 + 4y - 5)$

Solution: We distribute $3xy$. $3xy(2x^2 + 4y - 5) = \boxed{6x^3y + 12xy^2 - 15xy}$

b) $-5x^3(2x^2 - x + 8)$

Solution: We distribute $-5x^3$. $-5x^3(2x^2 - x + 8) = \boxed{-10x^5 + 5x^4 - 40x^3}$

c) $-(-x + 3y - 8z^2 + 6)$

Solution: The notation here indicates multiplication by -1 , which is the same as taking the opposite of a quantity. We distribute -1 .

$$-1(-x + 3y - 8z^2 + 6) = \boxed{x - 3y + 8z^2 - 6}$$

d) $-2a(-a + 3b^2 - 2ab + 7)$

Solution: $-2a(-a + 3b^2 - 2ab + 7) = \boxed{2a^2 - 6ab^2 + 4a^2b - 14a}$

2. Multiply the algebraic expressions as indicated.

a) $(x + 3)(5x - 3)$

Solution: We expand the expression using the distributive law. In the simplest case, when both expressions have only two terms, we use FOIL (F - first term with first term, O - outer terms, I - inner terms, L - last terms)

$$\begin{aligned} (x + 3)(5x - 3) &= \text{FOIL} \\ 5x^2 - 3x + 15x - 9 &= \text{combine like terms} \\ &= \boxed{5x^2 + 12x - 9} \end{aligned}$$

b) $(5 - 2x)^2$

Solution: To square something means to write it down twice and multiply.

$$\begin{aligned} (5 - 2x)^2 &= (5 - 2x)(5 - 2x) && \text{FOIL} \\ &= 25 - 10x - 10x + 4x^2 && \text{combine like terms} \\ &= \boxed{4x^2 - 20x + 25} \end{aligned}$$

c) $(x + 4)(1 - 2x)$

Solution: We expand the expression and combine like terms.

$$(x + 4)(1 - 2x) = x - 2x^2 + 4 - 8x = \boxed{-2x^2 - 7x + 4}$$

d) $-2(x - 3)^2$

Solution: We have two operations: multiplication by -2 and exponentiation. Order of operations still apply, thus we start with exponentiation.

$$-2(x - 3)^2 = -2((x - 3)(x - 3)) = -2(x^2 - 3x - 3x + 9) = -2(x^2 - 6x + 9) = \boxed{-2x^2 + 12x - 18}$$

3. Simplify each of the following expressions.

a) $(x - 5)^2 - (2x - 1)(x + 3)$

Solution: If we consider this as operations on algebraic expressions, then we are faced with an exponentiation, a multiplication, and a subtraction. We will execute them exactly in this order.

$$\begin{aligned} (x - 5)^2 &= (x - 5)(x - 5) = x^2 - 5x - 5x + 25 = x^2 - 10x + 25 \\ (2x - 1)(x + 3) &= 2x^2 + 6x - x - 3 = 2x^2 + 5x - 3 \\ (x - 5)^2 - (2x - 1)(x + 3) &= x^2 - 10x + 25 - (2x^2 + 5x - 3) = \\ &= x^2 - 10x + 25 + (-2x^2 - 5x + 3) = \boxed{-x^2 - 15x + 28} \end{aligned}$$

b) $-(m-3)^2$

Solution: We are asked to take the opposite of a complete square.

$$-(m-3)^2 = -1((m-3)(m-3)) = -1(m^2 - 3m - 3m + 9) = -1(m^2 - 6m + 9) = \boxed{-m^2 + 6m - 9}$$

c) $-2(3x-5) - (2x-1)^2$

Solution: We start with exponentiation

$$\begin{aligned} -2(3x-5) - (2x-1)^2 &= & (2x-1)^2 &= (2x-1)(2x-1) = 4x^2 - 2x - 2x + 1 \\ & & &= 4x^2 - 4x + 1 \end{aligned}$$

$$= -2(3x-5) - (4x^2 - 4x + 1) \quad \text{distribute } -2, \text{ to subtract is to add the opposite}$$

$$= -6x + 10 + (-4x^2 + 4x - 1) \quad \text{drop parentheses, combine like terms}$$

$$= \boxed{-4x^2 - 2x + 9}$$

4. a) $(x-3)^2 - (2x-5)(x+1) = 5 - (x-1)^2$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned} (x-3)^2 - (2x-5)(x+1) &= 5 - (x-1)^2 \\ x^2 - 3x - 3x + 9 - (2x^2 + 2x - 5x - 5) &= 5 - (x^2 - x - x + 1) && \text{combine like terms} \\ x^2 - 6x + 9 - (2x^2 - 3x - 5) &= 5 - (x^2 - 2x + 1) && \text{distribute} \\ x^2 - 6x + 9 - 2x^2 + 3x + 5 &= 5 - x^2 + 2x - 1 && \text{combine like terms} \\ -x^2 - 3x + 14 &= -x^2 + 2x + 4 && \text{add } x^2 \\ -3x + 14 &= 2x + 4 && \text{add } 3x \\ 14 &= 5x + 4 && \text{subtract } 4 \\ 10 &= 5x && \text{divide by } 5 \\ 2 &= x \end{aligned}$$

We check. If $x = 2$, then

$$\begin{aligned} \text{LHS} &= (2-3)^2 - (2 \cdot 2 - 5)(2+1) = (-1)^2 - (4-5)(2+1) = (-1)^2 - (-1) \cdot 3 \\ &= 1 - (-3) = 4 \end{aligned}$$

$$\text{RHS} = 5 - (2-1)^2 = 5 - 1^2 = 5 - 1 = 4$$

Thus $\boxed{x=2}$ is indeed the solution.

b) $(x+1)^2 - (2x-1)^2 + (3x)^2 = 6x(x-2)$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned} (x+1)^2 - (2x-1)^2 + (3x)^2 &= 6x(x-2) \\ x^2 + x + x + 1 - (4x^2 - 2x - 2x + 1) + 9x^2 &= 6x^2 - 12x \\ x^2 + 2x + 1 - (4x^2 - 4x + 1) + 9x^2 &= 6x^2 - 12x && \text{distribute} \\ x^2 + 2x + 1 - 4x^2 + 4x - 1 + 9x^2 &= 6x^2 - 12x && \text{combine like terms} \\ 6x^2 + 6x &= 6x^2 - 12x && \text{subtract } 6x^2 \\ 6x &= -12x && \text{add } 12x \\ 18x &= 0 && \text{divide by } 18 \\ x &= 0 \end{aligned}$$

We check. If $x = 0$, then

$$\text{LHS} = (0+1)^2 - (2 \cdot 0 - 1)^2 + (3 \cdot 0)^2 = 1^2 - (-1)^2 + (0)^2 = 1 - 1 + 0 = 0$$

$$\text{RHS} = 6 \cdot 0 \cdot (0-2) = 6 \cdot 0 \cdot (-2) = 0$$

Thus $\boxed{x=0}$ is indeed the solution.

$$\text{c) } 12 - (2p - 1)(p + 1) = -2(-p + 5)^2$$

Solution: We first multiply the polynomials as indicated. If the product is subtracted or further multiplied, we must keep the parentheses.

$$\begin{aligned}
 12 - (2p - 1)(p + 1) &= -2(-p + 5)^2 \\
 12 - (2p^2 + 2p - p - 1) &= -2(p^2 - 5p + 25) && \text{combine like terms} \\
 12 - (2p^2 + p - 1) &= -2(p^2 - 10p + 25) && \text{distribute} \\
 12 - 2p^2 - p + 1 &= -2p^2 + 20p - 50 && \text{combine like terms} \\
 -2p^2 - p + 13 &= -2p^2 + 20p - 50 && \text{add } 2p^2 \\
 -p + 13 &= 20p - 50 && \text{add } p \\
 13 &= 21p - 50 && \text{add } 50 \\
 63 &= 21p && \text{divide by } 21 \\
 3 &= p
 \end{aligned}$$

We check. If $p = 3$, then

$$\begin{aligned}
 \text{LHS} &= 12 - (2 \cdot 3 - 1)(3 + 1) = 12 - (6 - 1)(3 + 1) = 12 - 5 \cdot 4 = 12 - 20 = -8 \\
 \text{RHS} &= -2(-3 + 5)^2 = -2 \cdot 2^2 = -2 \cdot 4 = -8
 \end{aligned}$$

Thus $\boxed{p = 3}$ is indeed the solution.

$$\text{d) } (x + 3)^2 - (x + 1)(2x - 1) = 5 - (x - 2)^2$$

Solution: We first expand the products.

$$\begin{aligned}
 (x + 3)^2 &= (x + 3)(x + 3) \\
 &= x^2 + 3x + 3x + 9 \\
 &= x^2 + 6x + 9 \\
 (x + 1)(2x - 1) &= 2x^2 - x + 2x - 1 \\
 &= 2x^2 + x - 1 \\
 (x - 2)^2 &= (x - 2)(x - 2) \\
 &= x^2 - 2x - 2x + 4 \\
 &= x^2 - 4x + 4
 \end{aligned}$$

We are now ready to solve the equation. We need to be careful, though: because of the subtraction, we still need some of the parentheses.

$$\begin{aligned}
 (x + 3)^2 - (x + 1)(2x - 1) &= 5 - (x - 2)^2 \\
 x^2 + 6x + 9 - (2x^2 + x - 1) &= 5 - (x^2 - 4x + 4) && \text{distribute } -1 \text{ on both sides} \\
 x^2 + 6x + 9 - 2x^2 - x + 1 &= 5 - x^2 + 4x - 4 && \text{combine like terms} \\
 -x^2 + 5x + 10 &= -x^2 + 4x + 1 && \text{add } x^2 \\
 5x + 10 &= 4x + 1 && \text{subtract } 4x \\
 x + 10 &= 1 && \text{subtract } 10 \\
 x &= -9
 \end{aligned}$$

We check: if $x = -9$, then

$$\begin{aligned}
 \text{LHS} &= ((-9) + 3)^2 - ((-9) + 1)(2(-9) - 1) = (-6)^2 - (-8)(-19) = 36 - 152 = -116 \\
 \text{RHS} &= 5 - (-9 - 2)^2 = 5 - (-11)^2 = 5 - 121 = -116
 \end{aligned}$$

and so our solution, $\boxed{x = -9}$ is correct.

5. If we increase each side of a square by 3cm, its area increases by 51cm^2 . How long are the sides before the increase?

Solution: Let us denote the length of the sides of the original square by x . Then the area of the square is x^2 . The side of the larger square is $x + 3$. Therefore, the area of the larger square is $(x + 3)^2$. The equation will express the comparison between the two areas. expand complete square on the left-hand side

$$\begin{array}{rcll} (x+3)^2 & = & x^2 + 51 & \text{expand complete square on the left-hand side} \\ x^2 + 6x + 9 & = & x^2 + 51 & \text{subtract } x^2 \\ 6x + 9 & = & 51 & \text{subtract 9} \\ 6x & = & 42 & \text{divide by 6} \\ x & = & 7 & \end{array}$$

Thus the original square has sides 7cm long. The area is 49cm^2 . If we increased each side by 3cm, the new side is 10cm, and the new area 100cm^2 . Indeed, the two areas differ by $100\text{cm}^2 - 49\text{cm}^2 = 51\text{cm}^2$. Thus our solution is correct: the original square has sides that are 7cm long.

6. If we add two to a number and multiply that by one less than that number, the product is 6 less than the square of the number. Find this number.

Solution: Let x denote this number. Then one quantity compared is $(x + 2)(x - 1)$, the other quantity compared is x^2 , and we express the comparison when setting up the equation.

$$(x + 2)(x - 1) = x^2 - 6$$

We then solve for x . We expand the product as margin-work.

$$\begin{aligned} (x + 2)(x - 1) &= x^2 - x + 2x - 2 \\ &= x^2 + x - 2 \end{aligned}$$

$$\begin{array}{rcll} x^2 + x - 2 & = & x^2 - 6 & \text{subtract } x^2 \\ x - 2 & = & -6 & \text{add 2} \\ x & = & -4 & \end{array}$$

So our number is -4 . We check against the conditions stated in the problem. The product of two more and one less of our number is then $-2(-5) = 10$. This number is indeed 6 less than 16, which is the square of our number, $\overline{-4}$. So our solution is correct.

7. One side of a rectangle is three less than twice another side. If we increase both sides by 1 unit, the area of the rectangle will increase by 10 square-units. Find the sides of the original rectangle.

Solution: Let us denote one side of the original rectangle by x . Then the other side in the rectangle is $2x - 3$. The area of this rectangle is then $x(2x - 3)$.

If we increase both sides, then the sides become $x + 1$ and $2x - 2$. This rectangle has an area $(x + 1)(2x - 2)$. The equation will express a comparison between the two areas.

$$\begin{array}{rcll} (x+1)(2x-2) & = & x(2x-3) + 10 & \\ 2x^2 - 2x + 2x - 2 & = & 2x^2 - 3x + 10 & \text{subtract } x^2, \text{ combine like terms} \\ -2 & = & -3x + 10 & \text{add } 3x \\ 3x - 2 & = & 10 & \text{add 2} \\ 3x & = & 12 & \text{divide by 3} \\ x & = & 4 & \end{array}$$

Our result means that the original rectangle had sides 4 and $2 \cdot 4 - 3 = 5$. This rectangle has sides 4 and 5 units resulting in an area of 20 unit^2 . Let us increase both sides by 1 unit. The new rectangle has sides 5 and 6 units. That is an area of 30 unit^2 , and so the increase was indeed 10 square-unit.

**Discussion:**

1. Explain why re-writing $2(x-3)^2$ as $(2x-6)^2$ would be an incorrect step.

Solution: If we don't immediately see what is going on with abstract algebraic expressions, it might be useful to make things concrete by looking at the numbers this expressions become for a value of x . If $x = 0$, then the value of $2(x-3)^2 = 2(-3)^2 = 2 \cdot 9 = 18$ and the value of $(2x-6)^2 = (-6)^2 = 36$ so the two expressions can not be equivalent. We

can check a few more values using any numbers for x , we see that the number in the third column is stubbornly twice the number in the second column.

x	$2(x-3)^2$	$(2x-6)^2$
0	18	36
5	8	16
-1	32	64
10	98	196

$(2x-6)^2$ can be re-written as follows.

$$(2x-6)^2 = (2x-6)(2x-6) = [2(x-3)][2(x-3)] = 4(x-3)(x-3) = 4(x-3)^2$$

So, $2(x-3)^2$ can not be simplified as $(2x-6)^2$. We would be multiplying by 4 instead of 2.

2. Expand $(m-3)^2$ and $(3-m)^2$. Explain what you notice.

Notice that $3-m$ is the opposite of $m-3$, as

$$3-m = -m+3 = -(m-3) = -1(m-3)$$

If we square any number and its opposite, the result will be the same.

$$(3-m)^2 = [-1(m-3)]^2 = (-1)^2(m-3)^2 = 1 \cdot (m-3)^2 = (m-3)^2$$

We can use this fact when it makes computations easier. For example, $(-x-4)^2$ can be re-written as $(x+4)^2$.

Solutions for 8.2 – Basic Percent Problems

1. Find 16% of 3600.

Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x . In this case,

$$(\text{is}) = x$$

$$F = 0.16$$

$$(\text{of}) = 3600$$

We will substitute these into the formula and solve for x . $0.16 \cdot 3600 = 576$

$$(\text{is}) = F \cdot (\text{of})$$

$$x = 0.16 \cdot 3600$$

$$x = 576$$

Thus 16% of 3600 is 576.

2. 27 is what percent of 18?

Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x . Because the is-number is smaller than the of-number, we should expect a percentage larger than 100%.

$$(\text{is}) = 27$$

$$F = x$$

$$(\text{of}) = 18$$

We will substitute these into the formula and solve for x .

$$(\text{is}) = F \cdot (\text{of})$$

$$27 = x \cdot 18 \quad \text{divide by 18}$$

$$\frac{27}{18} = x$$

$$x = 1.5 = 150\%$$

Thus 27 is 150% of 18. We can check by computing 150% of 18:

$$(\text{is}) = F \cdot (\text{of})$$

$$(\text{is}) = 1.5 \cdot 18 = 27$$

Thus our solution, 150% is correct.

3. 120% of what number is 150?

Solution: We first write a table, listing the three quantities, is-number, fraction, and of-number. We need to identify the two quantities given, and call the third one x . In this case,

$$\begin{aligned}(\text{is}) &= 150 \\ F &= 1.2 \\ (\text{of}) &= x\end{aligned}$$

We will substitute these into the formula and solve for x .

$$\begin{aligned}(\text{is}) &= F \cdot (\text{of}) \\ 150 &= 1.2 \cdot x && \text{divide by 1.2} \\ \frac{150}{1.2} &= x \\ 125 &= x\end{aligned}$$

We can check by computing 120% of 125:

$$\begin{aligned}(\text{is}) &= F \cdot (\text{of}) \\ (\text{is}) &= 1.2 \cdot 125 = 150\end{aligned}$$

Thus our solution, 125 is correct.

The following examples all boil down to one of the basic problems shown above. One advice: before starting computations, re-write the problem to a basic question. Once we have this question, the problem is easy to solve.

4. What do we get if we increase 600 by 150%?

Solution 1: We compute 150% of 600 and then we add it to 600. To compute 150% of 600 is a type 1 problem:

$$\begin{aligned}(\text{is}) &= x \\ F &= \frac{150}{100} = 1.5 \\ (\text{of}) &= 600\end{aligned}$$

Now we use our formula to find x .

$$\begin{aligned}(\text{is}) &= F \cdot (\text{of}) \\ x &= 1.5 \cdot 600 = 900\end{aligned}$$

So after the increase, we have $600 + 900 = 1500$.

Solution 2: This is a neat shortcut that will become very important in other problems. If a quantity is increased by 150%, then it "grew up" from 100% of itself to $100\% + 150\% = 250\%$ of itself. We can find the answer quickly if we simply compute 250% of 600. The basic question is: What is 250% of 600? (Type 1)

$$\begin{aligned}(\text{is}) &= x \\ F &= 250\% = 2.5 \\ (\text{of}) &= 600\end{aligned}$$

We substitute the data into the formula:

$$\begin{aligned}(\text{is}) &= F \cdot (\text{of}) \\ x &= 2.5 \cdot 600 \\ x &= 1500\end{aligned}$$

Thus the answer is 1500.

5. We placed \$8000 into a bank account with an annual 7% of interest rate. How much money do we have in the account a year later?

Solution 1: We compute 7% of 8000 and add the result to 8000. The basic question is: What is 7% of 8000? (Type 1)

$$\begin{aligned}(\text{is}) &= x \\ F &= 0.07 \\ (\text{of}) &= 8000\end{aligned}$$

We substitute the data into the formula:

$$\begin{aligned}(\text{is}) &= F \cdot (\text{of}) \\ x &= 0.07 \cdot 8000 \\ x &= 560\end{aligned}$$

Thus we earned \$ 560 in interest, and now we have $\$ 8000 + \$ 560 = \$ 8560$.

Solution 2: This is a neat shortcut that will become very important in other problems. If a quantity is increased by 7%, then it "grew up" from 100% of itself to 107% of itself. We can find the amount in the bank, if we simply compute 107% of 8000. The basic question is: What is 107% of 8000?

$$\begin{aligned}(\text{is}) &= x \\ F &= 1.07 \\ (\text{of}) &= 8000\end{aligned}$$

We substitute the data into the formula:

$$\begin{aligned}(\text{is}) &= F \cdot (\text{of}) \\ x &= 1.07 \cdot 8000 \\ x &= 8560\end{aligned}$$

Thus we now have \$8560.

6. The population of a town has decreased from 80 000 to 68 000. What percent of a decrease does this represent?

Solution 1: We subtract 68 000 from 80 000 to determine the change. $80\,000 - 68\,000 = 12\,000$. Now the question is: 12 000 is what percent of 80 000? (Type 2)

$$(\text{is}) = 12\,000$$

$$F = x$$

$$(\text{of}) = 80\,000$$

We substitute the data into the formula:

$$\begin{aligned} (\text{is}) &= F \cdot (\text{of}) \\ 12\,000 &= x \cdot 80\,000 \\ \frac{12\,000}{80\,000} &= x \\ 0.15 &= x \end{aligned}$$

Thus

$$x = 0.15 = \frac{0.15}{1} = \frac{0.15(100)}{1(100)} = \frac{15}{100} = 15\%$$

This is a 15% decrease.

Solution 2: Instead of comparing the change to the original condition, we can compare the new condition to the original condition and interpret the result. The question may be re-phrased as: 68 000 is what percent of 80 000? Then

$$(\text{is}) = 68\,000$$

$$F = x$$

$$(\text{of}) = 80\,000$$

We substitute the data into the formula:

$$\begin{aligned} (\text{is}) &= F \cdot (\text{of}) \\ 68\,000 &= x \cdot 80\,000 \\ \frac{68\,000}{80\,000} &= x \\ 0.85 &= x \end{aligned}$$

Thus

$$x = 0.85 = \frac{0.85}{1} = \frac{0.85(100)}{1(100)} = \frac{85}{100} = 85\%$$

Since the population has decreased from 100% of itself to 85% of itself, our result reflects a 15% decrease.

7. Tom got a 4% raise in his job. Now he makes 2496 per month. How much was he making before the raise?

Solution: It would be a mistake to simply subtract 4% of 2496 from 2496, because the raise was 4% of a smaller, unknown number! The trick is to realize that Tom is now making exactly 104% of his original pay. The question is thus (Type 3) 104% of what number is 2496?

$$(\text{is}) = 2496$$

$$F = 1.04$$

$$(\text{of}) = x$$

We substitute the data into the formula:

$$(\text{is}) = F \cdot (\text{of})$$

$$2496 = 1.04 \cdot x \quad \text{divide by 1.04}$$

$$\frac{2496}{1.04} = x$$

$$2400 = x$$

Thus his original pay was \$ 2400 per month. We can check by taking 104% of 2400 and find that it is indeed 2496.

8. A TV set went on a 35% sale. The sale price is \$ 312. Find the original price.

Solution: This problem is similar to the previous one in the sense that it is type 3, and the method used previously is essential. It would be a mistake to simply add 35% of 312 from 312, because the deduction was 35% of a larger, unknown number! The trick is to realize that the current price is 65% of the original price. The difference is that we subtract 35% this time. The question is thus (Type 3) 65% of what number is 312?

$$(\text{is}) = 312$$

$$F = 0.65$$

$$(\text{of}) = x$$

We substitute the data into the formula:

$$(\text{is}) = F \cdot (\text{of})$$

$$312 = 0.65 \cdot x \quad \text{divide by 0.65}$$

$$\frac{312}{0.65} = x$$

$$480 = x$$

Thus the original price was \$ 480. We can check by taking 65% of 480 and find that it is indeed 312.

Solutions for Problem Set 8

$$27. 2^{2018} + 2^{2018} = 2 \cdot 2^{2018} = 2^{2018+1} = 2^{2019}$$

$$28. \text{Solution: } 1440 = 2^5 \cdot 3^2 \cdot 5 \quad \text{and } 6 = 2 \cdot 3$$

Both x and y must be divisible by 6. Neither x nor y can have any prime factor besides 2, 3, and 5. If say x is divisible by 7, then the least common multiple would be divisible by 7. So, we can assign $2 \cdot 3$ to the prime factorization to both x and y . We have four more 2-factors. However, we can not split it between x and y , because if we give one more 2-factor to both x and y , then the greatest common factor will be at least 12. So, all four 2-powers must be assigned either to x or to y .

x	y	x	y
$2 \cdot 3$	$2^5 \cdot 3^2 \cdot 5$	$2 \cdot 3 \cdot 5$	$2^5 \cdot 3^2$
$2^5 \cdot 3$	$2 \cdot 3^2 \cdot 5$	$2^5 \cdot 3 \cdot 5$	$2 \cdot 3^2$
$2 \cdot 3^2$	$2^5 \cdot 3 \cdot 5$	$2 \cdot 3^2 \cdot 5$	$2^5 \cdot 3$
$2^5 \cdot 3^2$	$2 \cdot 3 \cdot 5$	$2^5 \cdot 3^2 \cdot 5$	$2 \cdot 3$

$$29. \text{ Let us write both expressions in terms of } x = 2^{1000}.$$

Then $2^{1000} = x$, and $2^{1001} = 2^{1000+1} = 2^{1000} \cdot 2^1 = x \cdot 2 = 2x$, and $2^{1002} = 2^{1000+2} = 2^{1000} \cdot 2^2 = x \cdot 4 = 4x$, and $2^{1003} = 2^{1000+3} = 2^{1000} \cdot 2^3 = x \cdot 8 = 8x$

Thus $2^{1000} + 2^{1001} + 2^{1002} = x + 2x + 4x = 7x$ and $2^{1003} = 8x$. Since x is positive, $8x > 7x$. Thus 2^{1003} is greater.

Solutions for 9.1 – Integer Exponents

Simplify each of the following. Assume that all variables represent positive numbers. Present your answer without negative exponents.

$$1. 3^{-2}$$

Solution: We just apply the rule $a^{-n} = \frac{1}{a^n}$.

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$2. \frac{1}{2^{-3}}$$

Solution: We apply the rule $a^{-n} = \frac{1}{a^n}$.

$$\frac{1}{2^{-3}} = \frac{1}{\frac{1}{2^3}} = \frac{1}{\frac{1}{8}}$$

To divide is to multiply by the reciprocal:

$$\frac{1}{\frac{1}{8}} = 1 \cdot \frac{8}{1} = 8$$

This is true in general: $\frac{1}{a^{-n}} = a^n$

$$\frac{1}{a^{-n}} = \frac{1}{\frac{1}{a^n}} = 1 \cdot \frac{a^n}{1} = a^n$$

3. m^{-4}

Solution: We apply the rule $a^{-n} = \frac{1}{a^n}$.

$$m^{-4} = \frac{1}{m^4}$$

4. $\frac{1}{x^{-5}}$

Solution: We have already proven that $\frac{1}{a^{-n}} = a^n$

$$\frac{1}{x^{-5}} = x^5$$

5. $a^8 \cdot a^{-1}$

Solution 1: We can apply the rule $a^n \cdot a^m = a^{n+m}$

$$a^8 \cdot a^{-1} = a^{8+(-1)} = a^7$$

Solution 2: We can apply the rule $a^{-n} = \frac{1}{a^n}$ and then the rule $\frac{a^n}{a^m} = a^{n-m}$.

$$a^8 \cdot a^{-1} = a^8 \cdot \frac{1}{a^1} = \frac{a^8}{1} \cdot \frac{1}{a} = \frac{a^8}{a} = \frac{a^8}{a^1} = a^{8-1} = a^7$$

6. $p^3(p^{-7})p^8$

Solution 1: We can apply the rule $a^n \cdot a^m = a^{n+m}$

$$p^3(p^{-7})p^8 = p^{3+(-7)+8} = p^4$$

Solution 2: We can apply the rules $a^{-n} = \frac{1}{a^n}$ and $a^n \cdot a^m = a^{n+m}$ and $\frac{a^n}{a^m} = a^{n-m}$.

$$p^3(p^{-7})p^8 = p^3 \cdot \frac{1}{p^7} \cdot p^8 = \frac{p^3}{1} \cdot \frac{1}{p^7} \cdot \frac{p^8}{1} = \frac{p^3 \cdot p^8}{p^7} = \frac{p^{3+8}}{p^7} = \frac{p^{11}}{p^7} = p^{11-7} = p^4$$

7. $\frac{x^{-4}}{x^{-9}}$

Solution 1: We can apply the rule $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{x^{-4}}{x^{-9}} = x^{-4-(-9)} = x^{-4+9} = x^5$$

Solution 2: We can apply the rules $a^{-n} = \frac{1}{a^n}$ and $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{x^{-4}}{x^{-9}} = \frac{x^9}{x^4} = x^{9-4} = x^5$$

8. $\frac{50a^{12}}{10a^{-3}}$

Solution 1: We can apply the rule $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{50a^{12}}{10a^{-3}} = 5a^{12-(-3)} = 5a^{12+3} = 5a^{15}$$

Solution 2: We can apply the rules $a^{-n} = \frac{1}{a^n}$ and $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{50a^{12}}{10a^{-3}} = \frac{50a^{12}a^3}{10} = 5a^{12+3} = 5a^{15}$$

9. $\frac{t^{-3}}{t^4}$

Solution 1: We can apply the rules $\frac{a^n}{a^m} = a^{n-m}$ and then $a^{-n} = \frac{1}{a^n}$.

$$\frac{t^{-3}}{t^4} = t^{-3-4} = t^{-7} = \frac{1}{t^7}$$

Solution 2: We can apply the rule $a^{-n} = \frac{1}{a^n}$ and then $a^n \cdot a^m = a^{n+m}$.

$$\frac{t^{-3}}{t^4} = \frac{1}{t^4 \cdot t^3} = \frac{1}{t^7}$$

10. x^0

Solution: There is a separate rule stating that as long as x is not zero, then $x^0 = 1$. So the answer is 1.

11. $-x^0$

Solution: This is the opposite of x^0 and so the answer is -1 .

$$-x^0 = -1 \cdot x^0 = -1 \cdot 1 = -1$$

12. $(-x)^0$

Solution: This is again 1 because any non-zero raised to the power zero is 1.

13. $(b^{-5})(b^2)(b^{-1})$

Solution 1: We can apply the rules $a^n \cdot a^m = a^{n+m}$ and then $a^{-n} = \frac{1}{a^n}$.

$$(b^{-5})(b^2)(b^{-1}) = b^{-5+2+(-1)} = b^{-4} = \frac{1}{b^4}$$

Solution 2: We can apply the rule $a^{-n} = \frac{1}{a^n}$ and then just cancel.

$$(b^{-5})(b^2)(b^{-1}) = \frac{1}{b^5} \cdot b^2 \cdot \frac{1}{b^1} = \frac{1}{b^5} \cdot \frac{b^2}{1} \cdot \frac{1}{b^1} = \frac{b^2}{b^6} = \frac{-b \cdot -b}{-b \cdot -b \cdot b \cdot b \cdot b \cdot b} = \frac{1}{b^4}$$

14. $\frac{1}{(b^{-5})(b^2)(b^{-1})}$

Solution 1: We can apply the rules $a^n \cdot a^m = a^{n+m}$ and then $a^{-n} = \frac{1}{a^n}$.

$$\frac{1}{(b^{-5})(b^2)(b^{-1})} = \frac{1}{b^{-5+2+(-1)}} = \frac{1}{b^{-4}} = \frac{1}{\frac{1}{b^4}} = 1 \cdot \frac{b^4}{1} = b^4$$

Solution 2: We can apply the rule $a^{-n} = \frac{1}{a^n}$ and then $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{1}{(b^{-5})(b^2)(b^{-1})} = \frac{b^5 \cdot b^1}{b^2} = \frac{b^6}{b^2} = b^{6-2} = b^4$$

15. $\frac{m^{-2}}{m^{-5}}$

Solution 1: We can apply the rules $\frac{a^n}{a^m} = a^{n-m}$ and then $a^{-n} = \frac{1}{a^n}$.

$$\frac{m^{-2}}{m^{-5}} = m^{-2-(-5)} = m^{-2+5} = m^3$$

Solution 2: We can apply the rule $a^{-n} = \frac{1}{a^n}$ and then $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{m^{-2}}{m^{-5}} = \frac{m^5}{m^2} = m^{5-2} = m^3$$

16. $\frac{x^3y^{-5}}{z^{-4}}$

Solution: Each variable occurs only once and so this problem is just about bringing it to the form required.

We can apply the rule $a^{-n} = \frac{1}{a^n}$. We have already shown that $\frac{1}{a^{-n}} = a^n$.

$$\frac{x^3y^{-5}}{z^{-4}} = \frac{x^3z^4}{y^5}$$

17. $\frac{18q^3}{6q^{-3}}$

Solution 1: We can apply the rule $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{18q^3}{6q^{-3}} = \frac{-6 \cdot 3q^{3-(-3)}}{-6 \cdot 1} = \frac{3q^{3+3}}{1} = 3q^6$$

Solution 2: We can apply the rules $a^{-n} = \frac{1}{a^n}$ and then $a^n \cdot a^m = a^{n+m}$.

$$\frac{18q^3}{6q^{-3}} = \frac{-6 \cdot 3q^3 q^3}{-6 \cdot 1} = 3q^6$$

18. $\left(\frac{2}{3}\right)^{-3}$

Solution: We can apply the rule $a^{-n} = \frac{1}{a^n}$.

$$\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}} = \frac{1}{\frac{8}{27}} = 1 \cdot \frac{27}{8} = \frac{27}{8}$$

Note that we basically proved here that $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.

19. $2y^{-3}$

Solution: We can apply the rule $a^{-n} = \frac{1}{a^n}$. It is important to note that the base of exponentiation is y and not $2y$.

$$2y^{-3} = 2 \cdot \frac{1}{y^3} = \frac{2}{1} \cdot \frac{1}{y^3} = \frac{2}{y^3}$$

20. $(2y)^{-3}$

Solution: We can apply the rule $a^{-n} = \frac{1}{a^n}$. This time the base of exponentiation is $2y$. So we will apply the rule $(ab)^n = a^n b^n$.

$$(2y)^{-3} = \frac{1}{(2y)^3} = \frac{1}{2^3 y^3} = \frac{1}{8y^3}$$

21. $\left(-\frac{3}{5}\right)^{-2}$

Solution 1: We can apply the rule $a^{-n} = \frac{1}{a^n}$.

$$\left(-\frac{3}{5}\right)^{-2} = \frac{1}{\left(-\frac{3}{5}\right)^2} = \frac{1}{\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right)} = \frac{1}{\frac{-3}{5} \cdot \frac{-3}{5}} = \frac{1}{\frac{9}{25}} = 1 \cdot \frac{25}{9} = \frac{25}{9}$$

Solution 2: We proved previously that $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$. Using that,

$$\left(-\frac{3}{5}\right)^{-2} = \left(-\frac{5}{3}\right)^2 = \left(-\frac{5}{3}\right)\left(-\frac{5}{3}\right) = \frac{25}{9}$$

22. $\frac{a^3b^{-5}}{a^{-2}b^3}$

Solution 1: We can apply the rule $\frac{a^n}{a^m} = a^{n-m}$ and then $a^{-n} = \frac{1}{a^n}$.

$$\frac{a^3b^{-5}}{a^{-2}b^3} = a^{3-(-2)}b^{-5-3} = a^{3+2}b^{-5-3} = a^5b^{-8} = a^5 \cdot \frac{1}{b^8} = \frac{a^5}{1} \cdot \frac{1}{b^8} = \frac{a^5}{b^8}$$

Solution 2: We can apply the rules $a^{-n} = \frac{1}{a^n}$ and $a^n \cdot a^m = a^{n+m}$.

$$\frac{a^3b^{-5}}{a^{-2}b^3} = \frac{a^3a^2}{b^3b^5} = \frac{a^5}{b^8}$$

23. $(3m^3)^{-2}$

Solution: We can apply the rule $a^{-n} = \frac{1}{a^n}$ and then $(ab)^n = a^n b^n$ and also $(a^n)^m = a^{nm}$.

$$(3m^3)^{-2} = \frac{1}{(3m^3)^2} = \frac{1}{3^2(m^3)^2} = \frac{1}{9m^{3 \cdot 2}} = \frac{1}{9m^6}$$

24. $(-2ab^{-3})^{-3}$

Solution: We can apply the rule $(ab)^n = a^n b^n$ and then $(a^n)^m = a^{nm}$.

$$(-2ab^{-3})^{-3} = (-2)^{-3} a^{-3} (b^{-3})^{-3} = (-2)^{-3} a^{-3} b^{-3(-3)} = (-2)^{-3} a^{-3} b^9$$

We now apply $a^{-n} = \frac{1}{a^n}$.

$$(-2)^{-3} a^{-3} b^9 = \frac{1}{(-2)^3} \cdot \frac{1}{a^3} \cdot b^9 = \frac{1}{-8} \cdot \frac{1}{a^3} \cdot \frac{b^9}{1} = \frac{b^9}{-8a^3} = -\frac{b^9}{8a^3}$$

25. $\frac{(k^3)^{-3}}{(k^{-5})^2}$

Solution: We can apply the rule $(a^n)^m = a^{nm}$ and then $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{(k^3)^{-3}}{(k^{-5})^2} = \frac{k^{3(-3)}}{k^{-5 \cdot 2}} = \frac{k^{-9}}{k^{-10}} = k^{-9-(-10)} = k^{-9+10} = k^1 = k$$

$$26. \left(\frac{2a^{-3}b^5}{-3a^3b^{-2}} \right)^{-2} (a^3b^{-5})^{-3}$$

Solution:

$$\begin{aligned}
 E &= \left(\frac{2a^{-3}b^5}{-3a^3b^{-2}} \right)^{-2} (a^3b^{-5})^{-3} = \left(\frac{2a^{-3-3}b^{5-(-2)}}{-3} \right)^{-2} (a^3b^{-5})^{-3} && \text{apply } \frac{a^n}{a^m} = a^{n-m} \\
 &= \left(\frac{2a^{-6}b^{5+2}}{-3} \right)^{-2} (a^3b^{-5})^{-3} \\
 &= \left(\frac{2a^{-6}b^7}{-3} \right)^{-2} (a^3b^{-5})^{-3} && \text{apply } \left(\frac{a}{b} \right)^{-n} = \left(\frac{b}{a} \right)^n \\
 &= \left(\frac{-3}{2a^{-6}b^7} \right)^2 (a^3b^{-5})^{-3} && \text{apply } \left(\frac{a}{b} \right)^n = \frac{a^n}{b^n} \\
 &= \frac{(-3)^2}{(2a^{-6}b^7)^2} (a^3b^{-5})^{-3} && \text{apply } (ab)^n = a^n b^n \text{ and } a^{-n} = \frac{1}{a^n} \\
 &= \frac{9}{2^2(a^{-6})^2(b^7)^2} \cdot \frac{1}{(a^3b^{-5})^3} && \text{apply } (a^n)^m = a^{nm} \text{ and } (ab)^n = a^n b^n \\
 &= \frac{9}{4a^{-12}b^{14}} \cdot \frac{1}{(a^3)^3(b^{-5})^3} && \text{apply } a^{-n} = \frac{1}{a^n} \text{ and } (ab)^n = a^n b^n \\
 &= \frac{9a^{12}}{4b^{14}} \cdot \frac{1}{a^{3 \cdot 3}b^{(-5)^3}} \\
 &= \frac{9a^{12}}{4b^{14}} \cdot \frac{1}{a^9b^{-15}} && \text{apply } a^{-n} = \frac{1}{a^n} \\
 &= \frac{9a^{12}}{4b^{14}} \cdot \frac{b^{15}}{a^9} = \frac{9a^{12}b^{15}}{4b^{14}a^9} && \text{apply } \frac{a^n}{a^m} = a^{n-m} \\
 &= \frac{9a^{12-9}b^{15-14}}{4} = \frac{9a^3b^1}{4} = \frac{9}{4}a^3b
 \end{aligned}$$

$$27. (-2a^{-3})(-2a^{-2}b)^{-4}$$

Solution:

$$\begin{aligned}
 E &= (-2a^{-3})(-2a^{-2}b)^{-4} && \text{apply } a^{-n} = \frac{1}{a^n} \\
 &= \left(-2 \cdot \frac{1}{a^3}\right) \frac{1}{(-2a^{-2}b)^4} && \text{apply } (ab)^n = a^n b^n \\
 &= \left(\frac{-2}{1} \cdot \frac{1}{a^3}\right) \frac{1}{(-2)^4 (a^{-2})^4 b^4} && \text{apply } (a^n)^m = a^{nm} \\
 &= \frac{-2}{a^3} \cdot \frac{1}{16a^{-8}b^4} && \text{apply } a^{-n} = \frac{1}{a^n} \\
 &= \frac{-2}{a^3} \cdot \frac{a^8}{16b^4} \\
 &= \frac{-2a^8}{a^3 \cdot 16b^4} = \frac{-2a^8}{16a^3b^4} && \text{apply } \frac{a^n}{a^m} = a^{n-m} \\
 &= \frac{-1 \cdot -2a^{8-3}}{8 \cdot -2b^4} = \frac{-a^5}{8b^4}
 \end{aligned}$$

$$28. \frac{(-3p^3q^5)^2}{(2q^0p^3)^{-1}}$$

Solution:

$$\begin{aligned}
 E &= \frac{(-3p^3q^5)^2}{(2q^0p^3)^{-1}} && \text{apply } q^0 = 1 \text{ and } \frac{1}{a^{-n}} = a^n \\
 &= (-3p^3q^5)^2 (2 \cdot 1p^3)^1 \\
 &= (-3p^3q^5)^2 \cdot 2p^3 && \text{apply } (ab)^n = a^n b^n \\
 &= (-3)^2 (p^3)^2 (q^5)^2 \cdot 2p^3 && \text{apply } (a^n)^m = a^{nm} \\
 &= 9p^{3 \cdot 2} q^{5 \cdot 2} \cdot 2p^3 \\
 &= 18p^6 q^{10} p^3 && \text{apply } a^n \cdot a^m = a^{n+m} \\
 &= 18p^{6+3} q^{10} = 18p^9 q^{10}
 \end{aligned}$$

$$29. \left(\frac{2a^{-2}b^3}{-2^2(a^{-1}b)^{-3}} \right)^{-2}$$

Solution:

$$\begin{aligned}
 E &= \left(\frac{2a^{-2}b^3}{-2^2(a^{-1}b)^{-3}} \right)^{-2} && \text{apply } \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \\
 &= \left(\frac{-2^2(a^{-1}b)^{-3}}{2a^{-2}b^3} \right)^2 && \text{apply } (ab)^n = a^n b^n \\
 &= \left(\frac{-4(a^{-1})^{-3}b^{-3}}{2a^{-2}b^3} \right)^2 && \text{apply } (a^n)^m = a^{nm}
 \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{-4a^{-1(-3)}b^{-3}}{2a^{-2}b^3} \right)^2 \\
&= \left(\frac{-2a^3b^{-3}}{a^{-2}b^3} \right)^2 && \text{apply } \frac{a^n}{a^m} = a^{n-m} \\
&= \left(-2a^{3-(-2)}b^{-3-3} \right)^2 \\
&= \left(-2a^{3+2}b^{-3-3} \right)^2 \\
&= \left(-2a^5b^{-6} \right)^2 && \text{apply } (ab)^n = a^n b^n \\
&= (-2)^2 (a^5)^2 (b^{-6})^2 && \text{apply } (a^n)^m = a^{nm} \\
&= 4a^{5 \cdot 2} b^{-6 \cdot 2} = 4a^{10} b^{-12} && \text{apply } a^{-n} = \frac{1}{a^n} \\
&= 4a^{10} \cdot \frac{1}{b^{12}} \\
&= \frac{4a^{10}}{1} \cdot \frac{1}{b^{12}} = \frac{4a^{10}}{b^{12}}
\end{aligned}$$

$$30. \left(-\frac{x^3 y^0 x^{-5}}{y^{-3}} \right)^{-2}$$

Solution:

$$\begin{aligned}
E &= \left(-\frac{x^3 y^0 x^{-5}}{y^{-3}} \right)^{-2} && y^0 = 1 \text{ and } a^n \cdot a^m = a^{n+m} \\
&= \left(-\frac{x^{3+(-5)}}{y^{-3}} \right)^{-2} = \left(\frac{-1x^{-2}}{y^{-3}} \right)^{-2} && \text{apply } \left(\frac{a}{b} \right)^n = \frac{a^n}{b^n} \\
&= \frac{(-1x^{-2})^{-2}}{(y^{-3})^{-2}} && \text{apply } (ab)^n = a^n b^n \\
&= \frac{(-1)^{-2} (x^{-2})^{-2}}{(y^{-3})^{-2}} && \text{apply } (a^n)^m = a^{nm} \text{ and } a^{-n} = \frac{1}{a^n} \\
&= \frac{x^{-2(-2)}}{(-1)^2 y^{-3(-2)}} = \frac{x^4}{1y^6} = \frac{x^4}{y^6}
\end{aligned}$$

$$31. \left(-\frac{x^3 y^7 x^{-5}}{y^{-3}} \right)^0$$

Solution: Any non-zero quantity raised to the power zero is 1. So the answer is 1.

$$32. \frac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}}$$

Solution: This problem is very different because there are addition and subtraction involved. Because of that, we can not simply move the expressions with negative exponents. Instead, this will be a problem involving complex fractions.

$$\begin{aligned}
 E &= \frac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}} = \frac{\frac{1}{x^1} + \frac{1}{y^1}}{\frac{1}{x^2} - \frac{1}{y^2}} = \frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x^2} - \frac{1}{y^2}} && \text{bring fractions to the common denominator} \\
 &= \frac{\frac{1 \cdot y}{x \cdot y} + \frac{1 \cdot x}{y \cdot x}}{\frac{1 \cdot y^2}{x^2 \cdot y^2} - \frac{1 \cdot x^2}{y^2 \cdot x^2}} = \frac{\frac{y}{xy} + \frac{x}{xy}}{\frac{y^2}{x^2 y^2} - \frac{x^2}{x^2 y^2}} = \frac{\frac{y+x}{xy}}{\frac{y^2 - x^2}{x^2 y^2}} && \text{to divide is to multiply by the reciprocal} \\
 &= \frac{y+x}{xy} \cdot \frac{x^2 y^2}{y^2 - x^2} && \text{cancel out } xy \\
 &= \frac{y+x}{1} \cdot \frac{xy}{y^2 - x^2} = \frac{xy(x+y)}{y^2 - x^2} && \text{factor } y^2 - x^2 \text{ via the difference of squares theorem, cancel out } x+y \\
 &= \frac{xy(x+y)}{(y-x)(y+x)} = \frac{xy}{y-x}
 \end{aligned}$$

$$33. \frac{(-2a^{-2})^{-2} b^3 a^0 (-aba^{-2}b^{-2})^{-3}}{2a^2 (-2a^{-2}b)^{-2} ab^0}$$

Solution:

$$\begin{aligned}
 E &= \frac{(-2a^{-2})^{-2} b^3 a^0 (-aba^{-2}b^{-2})^{-3}}{2a^2 (-2a^{-2}b)^{-2} ab^0} && a^0 = b^0 = 1 \text{ and } x^n x^m = x^{n+m} \\
 &= \frac{(-2a^{-2})^{-2} b^3 (-a^{1+(-2)} b^{1+(-2)})^{-3}}{2a^{2+1} (-2a^{-2}b)^{-2}} = \frac{(-2a^{-2})^{-2} b^3 (-1a^{-1}b^{-1})^{-3}}{2a^3 (-2a^{-2}b)^{-2}} && \text{apply } (xy)^n = x^n y^n \\
 &= \frac{(-2)^{-2} (a^{-2})^{-2} b^3 (-1)^{-3} (a^{-1})^{-3} (b^{-1})^{-3}}{2a^3 (-2)^{-2} (a^{-2})^{-2} b^{-2}} && \text{apply } (x^n)^m = x^{nm} \\
 &= \frac{(-2)^{-2} a^{-2(-2)} b^3 (-1)^{-3} a^{-1(-3)} b^{-1(-3)}}{2a^3 (-2)^{-2} a^{-2(-2)} b^{-2}} = \frac{(-2)^{-2} a^4 b^3 (-1)^{-3} a^3 b^3}{2a^3 (-2)^{-2} a^4 b^{-2}} && \text{cancel out } a^4 \text{ and } a^3 \text{ and } (-2)^{-2} \\
 &= \frac{b^3 (-1)^{-3} b^3}{2b^{-2}} && \text{apply } x^n x^m = x^{n+m} \\
 &= \frac{(-1)^{-3} b^{3+3}}{2b^{-2}} = \frac{(-1)^{-3} b^6}{2b^{-2}} && \text{apply } x^{-n} = \frac{1}{x^n} \\
 &= \frac{b^6 b^2}{(-1)^3 2} && \text{apply } x^n x^m = x^{n+m} \\
 &= \frac{b^{6+2}}{-1 \cdot 2} = \frac{b^8}{-2} = -\frac{b^8}{2}
 \end{aligned}$$

$$34. \left(\frac{-a^2 (b^{-1}a)^{-5}}{b^7 (-ab^2)^{-3}} \right)^{-2}$$

Solution:

$$\begin{aligned}
 E &= \left(\frac{-a^2 (b^{-1}a)^{-5}}{b^7 (-ab^2)^{-3}} \right)^{-2} && \text{apply } (xy)^n = x^n y^n \\
 &= \left(\frac{-a^2 (b^{-1})^{-5} a^{-5}}{b^7 (-1)^{-3} a^{-3} (b^2)^{-3}} \right)^{-2} && \text{apply } (x^n)^m = x^{nm} \\
 &= \left(\frac{-a^2 b^{-1(-5)} a^{-5}}{b^7 (-1)^{-3} a^{-3} b^{2(-3)}} \right)^{-2} = \left(\frac{-a^2 b^5 a^{-5}}{b^7 (-1)^{-3} a^{-3} b^{-6}} \right)^{-2} && \text{apply } x^n x^m = x^{n+m} \\
 &= \left(\frac{-a^{2+(-5)} b^5}{(-1)^{-3} b^{7+(-6)} a^{-3}} \right)^{-2} = \left(\frac{-1 \cdot a^{-3} b^5}{(-1)^{-3} b^1 a^{-3}} \right)^{-2} && \text{cancel out } a^{-3} \\
 &= \left(\frac{-1 b^5}{(-1)^{-3} b^1} \right)^{-2} && \text{apply } a^{-n} = \frac{1}{a^n} \\
 &= \left(\frac{-1 (-1)^3 b^5}{b^1} \right)^{-2} = \left(\frac{-1 (-1) b^5}{b^1} \right)^{-2} = \left(\frac{1 b^5}{b^1} \right)^{-2} && \text{apply } \frac{x^n}{x^m} = x^{n-m} \\
 &= (b^{5-1})^{-2} = (b^4)^{-2} && \text{apply } (x^n)^m = x^{nm} \\
 &= b^{4(-2)} = b^{-8} = \frac{1}{b^8}
 \end{aligned}$$

$$35. \frac{(x^{-2})^{-2} y^3 x^0 (-2yx^0 y^{-2} x^{-2})^0}{yx^5 (y^{-2}x)^{-3} (2x^{-1}yx^3)^{-1}}$$

Solution:

$$\begin{aligned}
 E &= \frac{(x^{-2})^{-2} y^3 x^0 (-2yx^0 y^{-2} x^{-2})^0}{yx^5 (y^{-2}x)^{-3} (2x^{-1}yx^3)^{-1}} && \text{apply } a^0 = 1 \text{ and } a^n a^m = a^{n+m} \\
 &= \frac{(x^{-2})^{-2} y^3}{yx^5 (y^{-2}x)^{-3} (2x^{-1+3}y)^{-1}} && \text{apply } (ab)^n = a^n b^n \\
 &= \frac{(x^{-2})^{-2} y^3}{yx^5 (y^{-2})^{-3} x^{-3} (2x^2y)^{-1}} && \text{apply } (ab)^n = a^n b^n \text{ and } a^n a^m = a^{n+m} \\
 &= \frac{(x^{-2})^{-2} y^3}{yx^{5+(-3)} (y^{-2})^{-3} (2)^{-1} (x^2)^{-1} y^{-1}} && \text{apply } (a^n)^m = a^{nm} \\
 &= \frac{x^{-2(-2)} y^3}{yx^2 y^{-2(-3)} 2^{-1} x^{2(-1)} y^{-1}} = \frac{x^4 y^3}{2^{-1} yx^2 y^6 x^{-2} y^{-1}} && \text{apply } a^n a^m = a^{n+m} \\
 &= \frac{x^4 y^3}{2^{-1} y^{1+6+(-1)} x^{2+(-2)}} = \frac{x^4 y^3}{2^{-1} y^6 x^0} && x^0 = 1 \text{ and } \frac{a^n}{a^m} = a^{n-m} \\
 &= \frac{x^4 y^{3-6}}{2^{-1}} = \frac{x^4 y^{-3}}{2^{-1}} && \text{apply } a^{-n} = \frac{1}{a^n} \\
 &= \frac{2^1 x^4}{y^3} = \frac{2x^4}{y^3}
 \end{aligned}$$

Solutions for 9.4 – Linear Inequalities

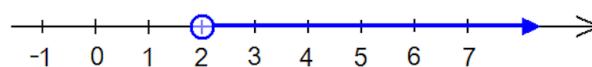
1. $-7 > -5x + 3$

Solution: Solving linear inequalities requires almost the same techniques as solving linear equations. There is only one difference: **when multiplying or dividing an inequality by a negative number, the inequality sign must be reversed.**

$$\begin{array}{rcl} -7 & > & -5x + 3 \quad \text{subtract 3} \\ -10 & > & -5x \quad \text{divide by } -5 \\ 2 & < & x \end{array}$$

When we divided both sides by -5 , we reversed the inequality sign. The final answer is all real numbers greater than 2. This set of numbers can be presented in numerous ways:

- 1) set-builder notation: $\{x|x > 2\}$
- 2) interval notation: $(2, \infty)$
- 3) graphing the solution set on the number line:



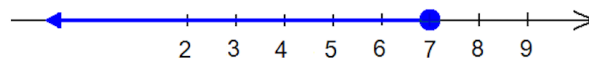
2. $3(x - 2) \leq 2x + 1$

Solution:

$$\begin{array}{rcl} 3(x - 2) & \leq & 2x + 1 \quad \text{distribute} \\ 3x - 6 & \leq & 2x + 1 \quad \text{subtract } 2x \\ x - 6 & \leq & 1 \quad \text{add 6} \\ x & \leq & 7 \end{array}$$

The final answer is all real numbers less than or equal to 7. This set of numbers can be presented in numerous ways:

- 1) set-builder notation: $\{x|x \leq 7\}$
- 2) interval notation: $(-\infty, 7]$
- 3) graphing the solution set on the number line:

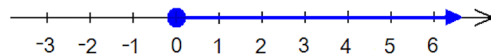


$$3. \ 5(4x - 1) - (x - 3) \geq -x - 2$$

Solution:

$$\begin{array}{rcl}
 5(4x - 1) - (x - 3) & \geq & -x - 2 & \text{distribute} \\
 20x - 5 - x + 3 & \geq & -x - 2 & \text{combine like terms} \\
 19x - 2 & \geq & -x - 2 & \text{add 2} \\
 19x & \geq & -x & \text{add } x \\
 20x & \geq & 0 & \text{divide by 20} \\
 x & \geq & 0 &
 \end{array}$$

The final answer is all real numbers greater than or equal to 0. This set of numbers can be presented in numerous ways: in set-builder notation: $\{x|x \geq 0\}$, in interval notation: $[0, \infty)$, or by graphing the solution set on the number line:

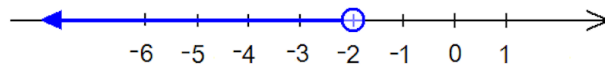


$$4. \ \frac{m+4}{2} - \frac{4m+3}{5} > 2$$

Solution:

$$\begin{array}{rcl}
 \frac{m+4}{2} - \frac{4m+3}{5} & > & 2 & \text{make everything a fraction} \\
 \frac{m+4}{2} - \frac{4m+3}{5} & > & \frac{2}{1} & \text{bring to common denominator} \\
 \frac{5(m+4)}{10} - \frac{2(4m+3)}{10} & > & \frac{20}{10} & \text{multiply by 15} \\
 5(m+4) - 2(4m+3) & > & 20 & \text{distribute} \\
 5m + 20 - 8m - 6 & > & 20 & \text{combine like terms} \\
 -3m + 14 & > & 20 & \text{subtract 14} \\
 -3m & > & 6 & \text{divide by } -3 \\
 m & < & -2 &
 \end{array}$$

When we divided both sides by -3 , we reversed the inequality sign. The final answer is all real numbers less than -2 . This set of numbers can be presented in numerous ways: in set-builder notation: $\{x|x < -2\}$, in interval notation: $(-\infty, -2)$, or by graphing the solution set on the number line:



Solutions for 10.4 – Factoring – Part 1

1. Completely factor each of the following.

a) $3x - 12$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 3.

$$3x - 12 = \boxed{3(x - 4)}$$

What is in the parentheses, $x - 4$ can not be further factored. We can easily check our work by multiplication.

b) $x^2 - 25y^2$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. However, $x^2 - 25y^2$ can be factored via the difference of squares theorem.

$$x^2 - 25y^2 = x^2 - (5y)^2 = (x + 5y)(x - 5y)$$

The expressions in neither parentheses can be further factored and so we are done. We check our work by multiplication:

$$(x + 5y)(x - 5y) = x^2 - 5xy + 5xy - 25 = x^2 - 25y^2$$

and so our answer, $\boxed{(x + 5y)(x - 5y)}$ is correct.

c) $3a^2 - 12$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 3.

$$3a^2 - 12 = 3(a^2 - 4)$$

What is in the parentheses, $a^2 - 4$ can be further factored via the difference of squares theorem.

$$3(a^2 - 4) = 3(a^2 - 2^2) = 3(a + 2)(a - 2)$$

The expressions in neither parentheses can be further factored and so we are done. We check our work by multiplication:

$$3(a + 2)(a - 2) = 3(a^2 - 2a + 2a - 4) = 3(a^2 - 4) = 3a^2 - 12$$

and so our answer, $\boxed{3(a + 2)(a - 2)}$ is correct.

d) $3a^2 - 12a$

Solution: This problem, together with the previous one, illustrates that two problems might look very similar, those small differences are quite significant when it comes to the solution and to the techniques we need to use to solve them. We start with the greatest common factor (or GCF). In this case, the GCF is $3a$.

$$3a^2 - 12a = 3a(a - 4)$$

What is in the parentheses, $a - 4$ can not be further factored and so we are done. We can easily check our work by multiplication:

$$3a(a - 4) = 3a^2 - 12a$$

and so our answer, $\boxed{3a(a - 4)}$ is correct.

e) $x^2 - 1$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. However, $x^2 - 1$ can be factored via the difference of squares theorem.

$$x^2 - 1 = x^2 - 1^2 = (x + 1)(x - 1)$$

The expressions in neither parentheses can not be further factored and so we are done. We check our work by multiplication:

$$(x + 1)(x - 1) = x^2 - x + x - 1 = x^2 - 1$$

and so our answer, $(x + 1)(x - 1)$ is correct. This is probably the most commonly occurring difference of two squares.

f) $x^2 + 1$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. In addition, $x^2 + 1$ can NOT be factored via the difference of squares theorem. **The sum of two squares can never be factored.** So, there is nothing that can be done here, and the final answer is $x^2 + 1$.

g) $-49 + x^6$

Solution: We start with the greatest common factor (or GCF). In this case, the GCF is 1, so we can not factor out any common factor. Before we proceed any further, we rearrange the terms so that the difference of squares becomes easier to observe.

$$-49 + x^6 = x^6 - 49$$

This factors via the difference of squares theorem. It is x^3 that we need to square to obtain x^6 .

$$x^6 - 49 = (x^3)^2 - 7^2 = (x^3 + 7)(x^3 - 7)$$

What is in both parentheses, $x^3 + 7$ and $x^3 - 7$ can not be further factored and so we are done. We can easily check our work by multiplication:

$$(x^3 + 7)(x^3 - 7) = x^6 - 7x^3 + 7x^3 - 49 = x^6 - 49$$

and so our answer, $(x^3 + 7)(x^3 - 7)$ is correct.

h) $3a^3 - 27ab^2$

Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned} 3a^3 - 27ab^2 &= \text{factor out GCF} \\ 3a(a^2 - 9b^2) &= \text{re-write } 9b^2 \text{ as } (3b)^2 \\ 3a(a^2 - (3b)^2) &= \text{factor via the difference of squares theorem} \\ &= 3a(a + 3b)(a - 3b) \end{aligned}$$

We check by multiplication:

$$3a(a + 3b)(a - 3b) = 3a(a^2 - 3ab + 3ab - 9b^2) = 3a(a^2 - 9b^2) = 3a^3 - 27ab^2$$

Thus our solution, $3a(a + 3b)(a - 3b)$ is correct.

i) $2p^4 - 162$

Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned}
 2p^4 - 162 &= \text{factor out GCF} \\
 2(p^4 - 81) &= \text{re-write both quantities as squares} \\
 2((p^2)^2 - 9^2) &= \text{factor via the difference of squares theorem} \\
 2(p^2 + 9)(p^2 - 9) &= \text{second factor will factor again} \\
 2(p^2 + 9)(p^2 - 3^2) &= \text{factor via the difference of squares theorem} \\
 &= 2(p^2 + 9)(p + 3)(p - 3)
 \end{aligned}$$

We check by multiplication:

$$\begin{aligned}
 2(p^2 + 9) \underbrace{(p + 3)(p - 3)}_{\text{FOIL}} &= 2(p^2 + 9)(p^2 - 3p + 3p - 9) = 2 \underbrace{(p^2 + 9)(p^2 - 9)}_{\text{FOIL}} \\
 &= 2(p^4 - 9p^2 + 9p^2 - 81) = 2(p^4 - 81) = 2p^4 - 162
 \end{aligned}$$

Thus our solution, $\boxed{2(p^2 + 9)(p + 3)(p - 3)}$ is correct.

j) $20x + 5x^3$

Solution: We rearrange the terms by degree first and then factor out the GCF.

$$20x + 5x^3 = 5x^3 + 20x = 5x(x^2 + 4)$$

Since the sum of squares does not factor, the final answer is $\boxed{5x(x^2 + 4)}$. We can easily check the result by multiplication.

2. Solve each of the following equations. Make sure to check your solution.

a) $(x - 2)(x + 3)(2x + 1) = 0$

Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule. Most of these were already done for us as the right-hand side is zero and the left-hand side is completely factored. All we need to do is apply the zero product rule. **A product can only be zero if one of its factors is zero.** $(x - 2)(x + 3)(2x + 1) = 0$ means that either $x - 2 = 0$ or $x + 3 = 0$ or $2x + 1 = 0$. We solve these linear equations separately:

$$\begin{array}{lll}
 x - 2 = 0 & \text{or} & x + 3 = 0 & \text{or} & 2x + 1 = 0 \\
 x = 2 & & x = -3 & & 2x = -1 \\
 & & & & x = -\frac{1}{2}
 \end{array}$$

We check all three solutions. If $x = 2$, then $\text{LHS} = (2 - 2)(2 + 3)(2(2) + 1) = 0 \cdot 5 \cdot 5 = 0 = \text{RHS} \checkmark$

If $x = -3$, then $\text{LHS} = (-3 - 2)(-3 + 3)(2(-3) + 1) = -5 \cdot 0 \cdot (-5) = 0 = \text{RHS} \checkmark$

and if $x = -\frac{1}{2}$, then

$$\left(-\frac{1}{2} - 2\right)\left(-\frac{1}{2} + 3\right)\left(2\left(-\frac{1}{2}\right) + 1\right) = -\frac{3}{2} \cdot \frac{5}{2} \cdot 0 = 0$$

and so all three numbers, $\boxed{2, -3, \text{ and } -\frac{1}{2}}$ are correct.

b) $m(m+7) = 0$

Solution: We will apply the zero product rule. **A product can only be zero if one of its factors is zero.** $m(m+7) = 0$ means that either $m = 0$. We solve these linear equations separately and obtain $m = 0$ and $m = -7$. We check: If $m = 0$, then

$$0(0+7) = 0 \cdot 7 = 0$$

and if $m = -7$, then

$$-7(-7+7) = -7 \cdot 0$$

and so both numbers, 0 and -7 are correct.

c) $x^2 = 9$

Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} x^2 &= 9 && \text{subtract 9} \\ x^2 - 9 &= 0 && \text{factor via the difference of squares theorem} \\ x^2 - 3^2 &= 0 \\ (x+3)(x-3) &= 0 \end{aligned}$$

A product can only be zero if one of its factors is zero. $(x+3)(x-3) = 0$ means that either $x-3 = 0$ or $x+3 = 0$. We solve these linear equations separately and obtain 3 and -3. We check: $3^2 = 9$ and $(-3)^2 = 9$.

Note: one could ask why the four steps if we could just conclude from $x^2 = 9$ that then $x = \pm 3$. This shortcut (called the square root property) is perfectly fine, as long as we remember that there are two numbers whose square is 9: 3 and -3. It is a common and serious error to go from $x^2 = 9$ to $x = 3$. One advantage of the difference of squares theorem that it will not allow for this mistake.

d) $x^2 = 9x$

Solution: Since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} x^2 &= 9x && \text{subtract } 9x \\ x^2 - 9x &= 0 && \text{factor out the GCF} \\ x(x-9) &= 0 \end{aligned}$$

A product can only be zero if one of its factors is zero. $x(x-9) = 0$ means that either $x = 0$ or $x-9 = 0$. We solve these linear equations separately and obtain 0 and 9. We check: $0^2 = 9 \cdot 0$ and $9^2 = 9 \cdot 9$ and so our solution is correct.

e) $8x^3 = 50x^2$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{array}{rcll} 8x^3 & = & 50x^2 & \text{subtract } 50x^2 \\ 8x^3 - 50x^2 & = & 0 & \text{the GCF is } 2x^2 \\ 2x^2(4x - 25) & = & 0 & \end{array}$$

We now apply the zero product rule. If this product is zero, then either $2x^2 = 0$ or $4x - 25 = 0$. We solve these equations for x .

$$\begin{array}{rcll} 2x^2 & = & 0 & \text{or} & 4x - 25 = 0 \\ 2 \cdot x \cdot x & = & 0 & \text{or} & 4x = 25 \\ x & = & 0 & \text{or} & x = \frac{25}{4} \end{array}$$

We check both solutions. If $x = 0$, then $\text{LHS} = 8 \cdot 0^3 = 8 \cdot 0 = 0$ and $\text{RHS} = 50 \cdot 0^2 = 50 \cdot 0 = 0 \checkmark$

If $x = \frac{25}{4}$, then

$$\text{LHS} = 8 \left(\frac{25}{4} \right)^3 = \frac{8}{1} \cdot \frac{15625}{64} = \frac{15625}{8} \quad \text{and} \quad \text{RHS} = 50 \left(\frac{25}{4} \right)^2 = \frac{50}{1} \cdot \frac{625}{16} = \frac{15625}{8} \checkmark$$

Thus both solutions, $\boxed{0 \text{ and } \frac{25}{4}}$ are correct.

f) $8p^3 = 50p$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{array}{rcll} 8p^3 & = & 50p & \text{subtract } 50p \\ 8p^3 - 50p & = & 0 & \text{the GCF is } 2p \\ 2p(4p^2 - 25) & = & 0 & \\ 2p((2p)^2 - 5^2) & = & 0 & \text{factor via difference of squares theorem} \\ 2p(2p + 5)(2p - 5) & = & 0 & \end{array}$$

We now apply the special zero property. If this product is zero, then either $2p = 0$ or $2p + 5 = 0$ or $2p - 5 = 0$. We solve these equations for p .

$$\begin{array}{rcll} 2p + 5 & = & 0 & \text{or} & 2p - 5 = 0 & \text{or} & 2p = 0 \\ 2p & = & -5 & \text{or} & 2p = 5 & \text{or} & p = 0 \\ p & = & -\frac{5}{2} & \text{or} & p = \frac{5}{2} & & \end{array}$$

We check all three solutions. If $p = -\frac{5}{2}$, then

$$\text{LHS} = 8 \left(-\frac{5}{2} \right)^3 = \frac{8}{1} \cdot \frac{-125}{8} = -125 \quad \text{and} \quad \text{RHS} = 50 \left(-\frac{5}{2} \right) = \frac{50}{1} \cdot \frac{-5}{2} = \frac{-250}{2} = -125 \checkmark$$

And if $p = \frac{5}{2}$, then $\text{LHS} = 8 \left(\frac{5}{2} \right)^3 = \frac{8}{1} \cdot \frac{125}{8} = 125$ and $\text{RHS} = 50 \left(\frac{5}{2} \right) = \frac{50}{1} \cdot \frac{5}{2} = \frac{250}{2} = 125 \checkmark$

And if $p = 0$, then $\text{LHS} = 8 \cdot 0^3 = 8 \cdot 0 = 0$ and $\text{RHS} = 50 \cdot 0 = 0 \checkmark$

Thus all three solutions, $\boxed{-\frac{5}{2}, 0, \text{ and } \frac{5}{2}}$ are correct.

3. Word Problems

a) Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

Solution: Let us denote the number by x . The equation is

$$\begin{array}{ll} x^2 &= x && \text{reduce one side to zero} \\ x^2 - x &= 0 && \text{factor} \\ x(x-1) &= 0 && \text{apply the zero property} \end{array}$$

$$\begin{array}{lll} x &= 0 & \text{or} \quad x-1=0 \\ x &= 0 & \text{or} \quad x=1 \end{array}$$

Thus there are two numbers, 0 and 1, satisfying the property. We check: $0^2 = 0$ and $1^2 = 1$.

Thus our answer is: $\boxed{0 \text{ and } 1}$.

b) Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.

Solution: Let us denote the number by x . The equation is

$$\begin{array}{ll} x^3 &= 4x && \text{reduce one side to zero} \\ x^3 - 4x &= 0 && \text{factor out the GCF} \\ x(x^2 - 4) &= 0 && \text{factor via the difference of squares theorem} \\ x(x+2)(x-2) &= 0 && \text{apply the zero property} \end{array}$$

$$\begin{array}{llll} x &= 0 & \text{or} & x+2=0 & \text{or} & x-2=0 \\ x &= 0 & \text{or} & x=-2 & \text{or} & x=2 \end{array}$$

Thus there are three numbers, 0, 2 and -2 , satisfying the property. We check: $0^3 = 4 \cdot 0$, $2^3 = 4 \cdot 2$, and $-2^3 = 4(-2)$. Thus our answer is: $\boxed{0, 2, \text{ and } -2}$.