

Beginning and Intermediate Algebra with Geometry - Part 2

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Marta Hidegkuti

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Chapter 13

13.1 Radical Expressions 1

Caution! Currently, square roots are defined differently in different textbooks. In conversations about mathematics, approach the concept with caution and first make sure that everyone in the conversation uses the same definition of square roots.

Definition: Let N be a non-negative number. Then the **square root of N** (notation: $\sqrt{N}$) is the non-negative number that, if we square, the result is N . If N is negative, then $\sqrt{N}$ is undefined.

For example, $\sqrt{25} = 5$. On the other hand, $\sqrt{-25}$ is undefined.

Example 1. Evaluate each of the given numerical expressions.

- a) $\sqrt{49}$ b) $-\sqrt{49}$ c) $\sqrt{-49}$ d) $-\sqrt{-49}$

Solution: a) $\sqrt{49} = \boxed{7}$

b) $-\sqrt{49} = \boxed{-7}$

c) $\sqrt{-49} = \boxed{\text{undefined}}$

d) $-\sqrt{-49} = \boxed{\text{undefined}}$

Square roots, when stretched over entire expressions, also serve as grouping symbols.

Example 2. Evaluate each of the following expressions.

- a) $\sqrt{25} - \sqrt{16}$ b) $\sqrt{25 - 16}$ c) $\sqrt{144 + 25}$ d) $\sqrt{144} + \sqrt{25}$

Solution: a) $\sqrt{25} - \sqrt{16} = 5 - 4 = \boxed{1}$

b) $\sqrt{25 - 16} = \sqrt{9} = \boxed{3}$

c) $\sqrt{144 + 25} = \sqrt{169} = \boxed{13}$

d) $\sqrt{144} + \sqrt{25} = 12 + 5 = \boxed{17}$

Not all square roots are as nice as $\sqrt{9}$ or $\sqrt{16}$. In a previous section we have seen that $\sqrt{2}$ is a real number but not rational.

Definition: A number is **rational** if it can be written as a quotient of two integers. A number is **irrational** if it can not be written as a quotient of two integers.

The set of all real numbers is the union of rational and irrational numbers. There are many types of irrational numbers, but our first example was $\sqrt{2}$. If a non-negative number is not a perfect square, then its square root exists but is an irrational real number. So, all of $\sqrt{3}$, $\sqrt{8}$, and $\sqrt{10}$ are also irrational.

So how can we deal with numbers like $\sqrt{2}$? By definition, irrational numbers can not be written as a fraction of integers. We have also seen before that irrational numbers have terrible decimal presentations: they are non-terminating and non-repeating. Therefore, we can not write the exact value of $\sqrt{2}$ as a decimal either: all attempts to do so would result in *approximations* as opposed to *exact value*.

In order to write the exact value of $\sqrt{2}$, we can not use fractions or decimals. Our only tool is to write it as a radical, as $\sqrt{2}$. This is an elusive positive real number that, when we square, the result is 2. Our only option is to include these radical expressions in our computations. Until now we had numbers and letters. From now on, we will have nice numbers (i.e rational numbers), yucky numbers (i.e. irrational numbers), and the variables. In what follows, we will explore how we can compute with these new radical expressions in our algebra.

By definition, $\sqrt{2}$ is the unique positive number that, when squared, the result is 2. Therefore,

$$(\sqrt{2})^2 = 2$$

One useful skill is to quickly estimate the value of a radical.

Example 3. Estimate each of the following radical expressions by placing them between two consecutive integers.

a) $\sqrt{20}$ b) $\sqrt{50}$ c) $\sqrt{5}$

Solution: a) If we are looking at positive numbers only, the greater the number, the greater is its square. In order to estimate $\sqrt{20}$ we need to find two perfect squares, one less than 20, and the other greater than 20. In this case, 16 and 25 are two consecutive perfect squares such that 20 is locked between them.

$$16 < 20 < 25, \text{ therefore } 4 < \sqrt{20} < 5$$

Our conclusion is that $\sqrt{20}$ is a number between 4 and 5. We can check this using our calculator:
 $\sqrt{20} \approx 4.4721$.

b) We need to find two consecutive squares such that 50 is between them. Since $49 < 50 < 64$, we have that $7 < \sqrt{50} < 8$ that $\sqrt{50}$ is a number between 7 and 8.

c) Since 5 is a number between 4 and 9, we conclude that $\sqrt{5}$ is a number between 2 and 3.

Square roots are defined in terms of squaring, which is multiplication. As a result, radical expressions will have nice properties with respect to multiplication and division, but not addition and subtraction.

Theorem: For all positive numbers a and b ,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \quad \text{but} \quad \sqrt{a} + \sqrt{b} \neq \sqrt{a+b} \quad \text{and} \quad \sqrt{a} - \sqrt{b} \neq \sqrt{a-b}$$

We can see these properties using perfect squares.

Example 4. Evaluate each of the following expressions.

a) $\sqrt{4}\sqrt{9}$ b) $\sqrt{4 \cdot 9}$ c) $\frac{\sqrt{100}}{\sqrt{25}}$ d) $\sqrt{\frac{100}{25}}$
e) $\sqrt{9} + \sqrt{16}$ f) $\sqrt{9+16}$ g) $\sqrt{25} - \sqrt{9}$ h) $\sqrt{25-9}$

Solution: a) $\sqrt{4}\sqrt{9} = 2 \cdot 3 = \boxed{6}$

e) $\sqrt{9} + \sqrt{16} = 3 + 4 = \boxed{7}$

b) $\sqrt{4 \cdot 9} = \sqrt{36} = \boxed{6}$

f) $\sqrt{9+16} = \sqrt{25} = \boxed{5}$

c) $\frac{\sqrt{100}}{\sqrt{25}} = \frac{10}{5} = \boxed{2}$

g) $\sqrt{25} - \sqrt{9} = 5 - 3 = \boxed{2}$

d) $\sqrt{\frac{100}{25}} = \sqrt{4} = \boxed{2}$

h) $\sqrt{25-9} = \sqrt{16} = \boxed{4}$

One fundamental property of square roots is the result after squaring. Irrational numbers are elusive, but one thing we know about $\sqrt{5}$ is that once we square it, the result is 5. Also recall the rule $(a^n)^m = a^{nm}$.

Example 5. Evaluate each of the following expressions.

a) $(\sqrt{7})^2$ b) $(\sqrt{2})^6$ c) $(-\sqrt{3})^2$ d) $\sqrt{2} \cdot \sqrt{18}$ e) $(2\sqrt{5})^2$

Solution: a) By definition of square roots, $(\sqrt{7})^2 = 7$.

b) We know that $(\sqrt{2})^2 = 2$. We will use the rule $(a^n)^m = a^{nm}$ to extract a 2 from the exponent.

$$(\sqrt{2})^6 = (\sqrt{2})^{2 \cdot 3} = \left[(\sqrt{2})^2 \right]^3 = 2^3 = \boxed{8}$$

c) $(-\sqrt{3})^2 = (-\sqrt{3})(-\sqrt{3}) = (\sqrt{3})^2 = \boxed{3}$

d) We will use the rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

$$\sqrt{2} \cdot \sqrt{18} = \sqrt{2 \cdot 18} = \sqrt{36} = \boxed{6}$$

e) Recall the rule of exponents $(ab)^n = a^n b^n$.

$$(2\sqrt{5})^2 = 2^2 (\sqrt{5})^2 = 4 \cdot 5 = \boxed{20}$$

This last example suggests that $2\sqrt{5}$ and $\sqrt{20}$ are the same number, because they are both positive and when squared, the result is 20. Indeed, that is true, $\sqrt{20} = 2\sqrt{5}$. We can use the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ to verify this and also, to simplify more complicated square roots.

Example 6. Simplify each of the following radicals.

a) $\sqrt{20}$ b) $\sqrt{50}$ c) $\sqrt{72}$ d) $\sqrt{12}$

Solution: a) We will express 20 as a product in which one factor is a square, the greatest square factor of 20 that we can find. Then we separate the square part from the rest via the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and simplify.

$$\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4}\sqrt{5} = \boxed{2\sqrt{5}}$$

b) $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = \boxed{5\sqrt{2}}$

We can check that this is true by squaring both sides. $(\sqrt{50})^2 = 50$ by definition of square roots and

$$(5\sqrt{2})^2 = 5^2 (\sqrt{2})^2 = 25 \cdot 2 = 50.$$

c) $\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36}\sqrt{2} = \boxed{6\sqrt{2}}$

d) $\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4}\sqrt{3} = \boxed{2\sqrt{3}}$

Consider the last example, $\sqrt{12} = 2\sqrt{3}$. These two are equal, and officially, $\sqrt{12}$ is not simplified. Theoretically, a number such as $\sqrt{12}$ must be presented in its simplified form, as $2\sqrt{3}$. This does not necessarily mean that the form $2\sqrt{3}$ is always better than $\sqrt{12}$. Depending on the computation, sometimes one form is more suitable than the other. We will see later that both forms have their own advantages.

Now we will begin to perform operations that involve radical expressions. When it comes to addition and subtraction, radicals are unlike terms to rational numbers. Their behavior is like that of a variable. Just as much as $3x + 2$ cannot be simplified, $3\sqrt{5} + 2$ can also not be simplified. Just as much as $3x + 4x$ can be simplified to $7x$, $3\sqrt{5} + 4\sqrt{5}$ can be simplified. Worse yet, $\sqrt{2}$ and $\sqrt{3}$ are also unlike terms to each other, like the variables x and y . Two radicals can only be combined if they are the square root of the same number.

Example 7. Simplify each of the following expressions.

a) $\sqrt{3} - 2 + 7\sqrt{3} + 4$ b) $\sqrt{50} - 3\sqrt{8}$ c) $3\sqrt{5} - \sqrt{5} + \sqrt{20}$

Solution: a) $\sqrt{3}$ and $7\sqrt{3}$ are like terms and -2 and 4 are like terms, just like in the expression $x - 2 + 7x + 4$. If we use this analogy, $x - 2 + 7x + 4 = (1x + 7x) + (-2 + 4) = 8x + 2$. Similarly,

$$\sqrt{3} - 2 + 7\sqrt{3} + 4 = \boxed{8\sqrt{3} + 2}$$

b) We will first simplify the radicals to see if they can be combined.

$$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2} \text{ and } \sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$$

These expressions are like terms and so they can be combined:

$$\sqrt{50} - 3\sqrt{8} = 5\sqrt{2} - 3(2\sqrt{2}) = 5\sqrt{2} - 6\sqrt{2} = -1\sqrt{2} = \boxed{-\sqrt{2}}$$

c) We simplify $\sqrt{20}$ first, to see if it can be combined with the other terms.

$$\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$$

Now we can combine all three terms:

$$3\sqrt{5} - \sqrt{5} + \sqrt{20} = 3\sqrt{5} - \sqrt{5} + 2\sqrt{5} = (3 - 1 + 2)\sqrt{5} = \boxed{4\sqrt{5}}$$

As our algebraic language is growing to include radicals, we pay attention to preserve the rules we already have. One important example is the distributive law.

Example 8. Perform each of the multiplications as indicated.

$$\text{a) } \sqrt{2}(3\sqrt{2}-1) \quad \text{b) } \sqrt{3}(4-\sqrt{12}) \quad \text{c) } \sqrt{5}(3+\sqrt{2})$$

Solution: a) We apply the distributive law.

$$\sqrt{2}(3\sqrt{2}-1) = \sqrt{2} \cdot 3\sqrt{2} - \sqrt{2} \cdot 1 = 3(\sqrt{2})^2 - \sqrt{2} = 3 \cdot 2 - \sqrt{2} = \boxed{6 - \sqrt{2}}$$

$$\text{b) } \sqrt{3}(4-\sqrt{12}) = \sqrt{3} \cdot 4 - \sqrt{3}\sqrt{12} = 4\sqrt{3} - \sqrt{3 \cdot 12} = 4\sqrt{3} - \sqrt{36} = \boxed{4\sqrt{3} - 6}$$

$$\text{c) } \sqrt{5}(3+\sqrt{2}) = \sqrt{5} \cdot 3 + \sqrt{5}\sqrt{2} = 3\sqrt{5} + \sqrt{5 \cdot 2} = 3\sqrt{5} + \sqrt{10} = \boxed{3\sqrt{5} + \sqrt{10}}$$

As a special case of the distributive law, when we multiply two binomial expressions, FOIL still applies, where F is for first with first, O for outer terms, I for inner terms, and L for last term with last term.

Example 9. Perform each of the multiplications as indicated.

$$\text{a) } (\sqrt{3}-2)(2\sqrt{3}+1) \quad \text{b) } (\sqrt{2}-1)(5\sqrt{2}-3) \quad \text{c) } (\sqrt{5}-1)(3\sqrt{5}+2)$$

Solution: a) We apply the distributive law. Consider the multiplication $(x-2)(2x+1)$. When we work this out, we get

$$\begin{array}{cccc} & \text{F} & & \text{O} & & \text{I} & & \text{L} \\ (x-2)(2x+1) & = & x \cdot 2x & + & x \cdot 1 & - & 2 \cdot 2x & - & 2 \cdot 1 \\ & = & 2x^2 & + & x & - & 4x & - & 2 \\ & = & 2x^2 & - & 3x & - & 2 \end{array}$$

combine like terms O and I

This situation is similar to ours, as $\sqrt{3}$ behaves like x does, with one exception: that its square is 3 and not x^2 .

$$\begin{array}{cccc} & \text{F} & & \text{O} & & \text{I} & & \text{L} \\ (\sqrt{3}-2)(2\sqrt{3}+1) & = & \sqrt{3} \cdot 2\sqrt{3} & + & \sqrt{3} \cdot 1 & - & 2 \cdot 2\sqrt{3} & - & 2 \cdot 1 \\ & = & 2(\sqrt{3})^2 & + & \sqrt{3} & - & 4\sqrt{3} & - & 2 \\ & = & 2 \cdot 3 & + & \sqrt{3} & - & 4\sqrt{3} & - & 2 \\ & = & 6 & - & 3\sqrt{3} & - & 2 & = & \boxed{4 - 3\sqrt{3}} \end{array}$$

$$\begin{array}{cccc} & \text{F} & & \text{O} & & \text{I} & & \text{L} \\ \text{b) } (\sqrt{2}-1)(5\sqrt{2}-3) & = & \sqrt{2} \cdot 5\sqrt{2} & - & \sqrt{2} \cdot 3 & - & 1 \cdot 5\sqrt{2} & + & 1 \cdot 3 \\ & = & 5(\sqrt{2})^2 & - & 3\sqrt{2} & - & 5\sqrt{2} & + & 3 \\ & = & 5 \cdot 2 & - & 8\sqrt{2} & + & 3 \\ & = & 10 & - & 8\sqrt{2} & + & 3 & = & \boxed{13 - 8\sqrt{2}} \end{array}$$

- c) The previous examples showed many details so that the reader can understand. In this example, we will show as much notation as is needed to be shown by students. Writing too much can make reading computations as difficult as not writing enough.

$$\begin{aligned}
 (\sqrt{5} - 1)(3\sqrt{5} + 2) &= 3(\sqrt{5})^2 + 2\sqrt{5} - 3\sqrt{5} - 2 \\
 &= 3 \cdot 5 - \sqrt{5} - 2 \\
 &= 15 - \sqrt{5} - 2 = \boxed{13 - \sqrt{5}}
 \end{aligned}$$

The following examples show special products such as complete squares. Notice that if we square an expression such as $2\sqrt{5} - 1$, the result will still have radical expressions in it.

Example 10. Perform each of the multiplications as indicated.

$$\text{a) } (2\sqrt{5} - 1)^2 \quad \text{b) } (\sqrt{3} - 2)^3 \quad \text{c) } (\sqrt{5} - \sqrt{2})^2$$

Solution: a) $(2\sqrt{5} - 1)^2 = (2\sqrt{5} - 1)(2\sqrt{5} - 1)$

$$\begin{aligned}
 &= (2\sqrt{5})^2 - 2\sqrt{5} - 2\sqrt{5} + 1 \\
 &= 2^2(\sqrt{5})^2 - 4\sqrt{5} + 1 \\
 &= 4 \cdot 5 - 4\sqrt{5} + 1 = 20 - 4\sqrt{5} + 1 = \boxed{21 - 4\sqrt{5}}
 \end{aligned}$$

- b) We will first work out $(\sqrt{3} - 2)^2$ and then $(\sqrt{3} - 2)^3$.

$$\begin{aligned}
 (\sqrt{3} - 2)^2 &= (\sqrt{3} - 2)(\sqrt{3} - 2) \\
 &= (\sqrt{3})^2 - 2\sqrt{3} - 2\sqrt{3} + 4 \\
 &= 3 - 4\sqrt{3} + 4 = \boxed{7 - 4\sqrt{3}}
 \end{aligned}$$

We are now ready to work out the third power:

$$\begin{aligned}
 (\sqrt{3} - 2)^3 &= (\sqrt{3} - 2)(\sqrt{3} - 2)^2 \\
 &= (\sqrt{3} - 2)(7 - 4\sqrt{3}) \\
 &= \sqrt{3} \cdot 7 - 4(\sqrt{3})^2 - 2 \cdot 7 + 8\sqrt{3} \\
 &= 7\sqrt{3} - 4 \cdot 3 - 14 + 8\sqrt{3} \\
 &= 7\sqrt{3} - 12 - 14 + 8\sqrt{3} \\
 &= \boxed{15\sqrt{3} - 26}
 \end{aligned}$$

c) $(\sqrt{5} - \sqrt{2})^2 = (\sqrt{5} - \sqrt{2})(\sqrt{5} - \sqrt{2})$

$$\begin{aligned}
 &= (\sqrt{5})^2 - \sqrt{5}\sqrt{2} - \sqrt{2}\sqrt{5} + (\sqrt{2})^2 \\
 &= 5 - \sqrt{10} - \sqrt{10} + 2 \\
 &= \boxed{7 - 2\sqrt{10}}
 \end{aligned}$$

The previous examples show that if we want to get rid of radicals in a sum or difference such as $2\sqrt{5} - 3$, squaring the expression will not eliminate the radicals. However, conjugates do. Whenever we multiply two conjugates, O and I cancel out each other, and we are left with F and L. This is the difference of squares theorem.

Example 11. Perform each of the multiplications as indicated.

$$\text{a) } (2\sqrt{7} - 3)(2\sqrt{7} + 3) \quad \text{b) } (3\sqrt{2} + 2\sqrt{6})(3\sqrt{2} - 2\sqrt{6}) \quad \text{c) } (\sqrt{10} + 3)(\sqrt{10} - 3)$$

Solution: a) $(2\sqrt{7} - 3)(2\sqrt{7} + 3) = 2^2(\sqrt{7})^2 + 6\sqrt{7} - 6\sqrt{7} - 9$
 $= 4 \cdot 7 - 9 = 28 - 9 = \boxed{19}$

b) $(3\sqrt{2} + 2\sqrt{6})(3\sqrt{2} - 2\sqrt{6}) = (3\sqrt{2})^2 - (3\sqrt{2})(2\sqrt{6}) + (3\sqrt{2})(2\sqrt{6}) - (2\sqrt{6})^2$
 $= 9 \cdot 2 - 4 \cdot 6 = 18 - 24 = \boxed{-6}$

c) $(\sqrt{10} + 3)(\sqrt{10} - 3) = (\sqrt{10})^2 + 3\sqrt{10} - 3\sqrt{10} - 9$
 $= 10 - 9 = \boxed{1}$

This last example shows that $\sqrt{10} + 3$ and $\sqrt{10} - 3$ are not only conjugates of each other, but also reciprocals. (Recall that x and y are reciprocals if their product is 1). We will see that this new algebra that now includes radical expressions is very rich in connections.

Example 12. Find the exact value of $-3x^2 - x + 4$ if $x = 2\sqrt{3} - 1$

Solution: Let us first work out x^2 separately first.

$$x^2 = (2\sqrt{3} - 1)^2 = (2\sqrt{3} - 1)(2\sqrt{3} - 1) = (2\sqrt{3})^2 - 2\sqrt{3} - 2\sqrt{3} + 1 = 4 \cdot 3 - 4\sqrt{3} + 1$$

$$= 12 - 4\sqrt{3} + 1 = 13 - 4\sqrt{3}$$

Now we are ready to compute the value of the entire expression.

$$-3x^2 - x + 4 = -3(13 - 4\sqrt{3}) - (2\sqrt{3} - 1) + 4 = -39 + 12\sqrt{3} - 2\sqrt{3} + 1 + 4 = \boxed{-34 + 10\sqrt{3}}$$



Sample Problems

Simplify each of the following expressions. Assume that a represents a positive number.

1. $\sqrt{32}$

2. $\sqrt{45}$

3. $\sqrt{48x^5y^3}$

4. $\sqrt{125} - 3\sqrt{80} + \sqrt{45}$

5. $\frac{\sqrt{24}}{\sqrt{54}}$

6. $\sqrt{80a^{11}} - 2\sqrt{180a^{11}} + 3\sqrt{245a^{11}}$

7. $(\sqrt{7} + 2)(\sqrt{7} - 2)$

8. $(\sqrt{7} - 2)^2$

9. $(\sqrt{3} - 1)^3$

10. $(\sqrt{5x} - 2)(\sqrt{5x} + 3)$

11. $(2 - \sqrt{x})(3 + 2\sqrt{x})$

12. $(\sqrt{x} - \sqrt{2})^2$

13. Find the exact value of $x^2 - 4x + 6$ if $x = 2 - \sqrt{3}$.



Practice Problems

Simplify each of the following expressions. Assume that a represents a positive number.

1. $\sqrt{20}$

2. $\sqrt{300} - 2\sqrt{75} + \sqrt{12}$

3. $3\sqrt{20} - 4\sqrt{45} + 2\sqrt{245}$

4. $5\sqrt{18a^5} - 7\sqrt{32a^5} + 4\sqrt{50a^5}$

5. $(\sqrt{5} + 2)(\sqrt{5} - 2)$

6. $(\sqrt{5} - 2)^2$

7. $(\sqrt{5} - 2)^3$

8. $(3\sqrt{5} - 1)(7\sqrt{5} + 2)$

9. Find the exact value of each of the following.

a) $x^2 - 6x + 1$ if $x = 3 - \sqrt{10}$

c) $x^2 - 10x + 16$ if $x = \sqrt{6} + 5$

b) $b^2 + 8b - 20$ if $b = \sqrt{5} - 2$

13.2 Smallest Value of a Quadratic Expression

Linear expression such as $2x + 1$ have no smallest or greatest values. Substituting greater and greater positive numbers (as x) into $2x + 1$ produces greater and greater values. Similarly, substituting large negative numbers (as x) into $2x + 1$ produces negative values with increasing absolute value.

Quadratic expressions are fundamentally different, as no square of any real number is negative. Consequently, most quadratic expressions have a smallest possible value: zero.

Example 1. What is the smallest value of each of the following quadratic expressions?

- a) x^2 b) $(x - 4)^2$ c) $(2x + 1)^2$ d) $(x + 7)^2$

Solution: a) No real number can have a negative square. Consequently, if x^2 could take the value zero, that would be the smallest value. That is possible, when we square $x = 0$. Thus, the lowest value of x^2 is zero, when x is zero.

b) Recall that $(x - 4)^2$ is a complete square; a difference squared. No matter what the value of x is, $x - 4$ is a real number and so it can NOT have a negative square. Consequently, if $(x - 4)^2$ could take the value zero, that would be the smallest value. That is possible, but only when we square zero. Thus, the lowest value of $(x - 4)^2$ is zero, when $x - 4$ is zero, that is, when $x = 4$. (Just solve the linear equation $x - 4 = 0$). In short: the lowest value of $(x - 4)^2$ is zero, when $x = 4$.

c) Recall that $(2x + 1)^2$ is a square, thus no value for x will ever result in a negative value. Consequently, if $(2x + 1)^2$ could take the value zero, that would be the smallest value. That is possible, but only when we square zero. Thus, the lowest value of $(2x + 1)^2$ is zero, when $2x + 1 = 0$. We solve the linear equation $2x + 1 = 0$ for x and obtain $x = -\frac{1}{2}$. So, the lowest value of $(2x + 1)^2$ is zero, when $x = -\frac{1}{2}$.

d) Since $(x + 7)^2$ is a square, no value for x will ever result in a negative value. Consequently, if $(x + 7)^2$ could take the value zero, that would be the smallest value. That is possible, but only when we square zero. Thus, the lowest value of $(x + 7)^2$ is zero, when $x + 7 = 0$. We solve the linear equation $x + 7 = 0$ for x and obtain $x = -7$. So, the lowest value of $(x + 7)^2$ is zero, when $x = -7$.

Example 2. Find the smallest value of each of the following quadratic expressions.

- a) $x^2 + 25$ b) $(x - 4)^2 - 1$ c) $(2x + 1)^2 + 12$ d) $(x + 7)^2 - 100$

Solution: First imagine a room where some people gathered. Everyone empties their vallets and pockets and count all their cash and then compare. As it turns out, Mr. X has the least amount of money on them, only \$1.50. Then another hour later, everyone in the room receives \$20. Who has the least amount of money now? The answer is clearly Mr. X, this time with \$21.50.

a) Consider the expression $x^2 + 25$. Recall that the lowest value of x^2 is zero, when x is zero. Then the lowest value of $x^2 + 25$ is 25, when x is zero. (Imagine that the 'poorest' person in the room was Ms. Y, with no money on her. If then everyone in the room receives \$25, she would still be the one with the smallest amount of money, exactly \$20.)

b) Consider now the expression $(x - 4)^2 - 1$. The lowest value of $(x - 4)^2$ is zero, when $x = 4$. Then the lowest value of $(x - 4)^2 - 1$ is -1 , when $x = 4$.

c) Consider now the expression $(2x + 1)^2 + 12$. Because it is a square, the lowest value of $(2x + 1)^2$ is zero, when $x = -\frac{1}{2}$. Then the lowest value of $(2x + 1)^2 + 12$ is 12, when $x = -\frac{1}{2}$.

d) Consider now the expression $(x + 7)^2 - 100$. Because it is a square, the lowest value of $(x + 7)^2$ is zero, when $x = -7$. Then the lowest value of $(x + 7)^2 - 100$ is -100 , when $x = -7$.

Example 3. Find the smallest value of each of the following quadratic expressions.

a) $x^2 + 2x - 5$ b) $x^2 - 8x + 15$ c) $x^2 - 12x$ d) $x^2 - 100$

Solution: a) We will apply the same ideas, but first we need to transform this expression into a more suitable form. For that, We simply complete the square. Half of the linear coefficient is 1, so our suitable complete square is $(x+1)^2$.

$$\begin{aligned} E &= x^2 + 2x - 5 && \text{helper line: } (x+1)^2 = x^2 + 2x + 1 \\ &= \underbrace{x^2 + 2x + 1}_{(x+1)^2} - 1 - 5 && \text{so we smuggle in 1} \\ &= (x+1)^2 - 6 \end{aligned}$$

This form of a quadratic expression is often called **the standard form**. Once we brought the expression to the standard form, we can easily determine the smallest value. The smallest value of $(x+1)^2 - 6$ is -6 , when $x = -1$.

- b) Consider the expression $x^2 - 8x + 15$. We complete the square to bring the expression to its standard form.

$$\begin{aligned} E &= x^2 - 8x + 15 && \text{helper line: } (x-4)^2 = x^2 - 8x + 16 \\ &= \underbrace{x^2 - 8x + 16}_{(x-4)^2} - 16 + 15 && \text{so we smuggle in 16} \\ &= (x-4)^2 - 1 \end{aligned}$$

Once we brought the expression to the standard form, we can easily determine the smallest value. The smallest value of $(x-4)^2 - 1$ is -1 , when $x = 4$.

- c) Consider the expression $x^2 - 12x$. We complete the square to bring the expression to its standard form.

$$\begin{aligned} E &= x^2 - 12x && \text{helper line: } (x-6)^2 = x^2 - 12x + 36 \\ &= \underbrace{x^2 - 12x + 36}_{(x-6)^2} - 36 && \text{so we smuggle in 36} \\ &= (x-6)^2 - 36 \end{aligned}$$

The smallest value of $x^2 - 12x$ is -36 , when $x = 6$.

- d) Consider the expression $x^2 - 100$. We do not complete the square as the expression is already in its standard form. If it helps, we can think of $x^2 - 100$ as $(x-0)^2 - 100$. Either way, the smallest value of $x^2 - 100$ is -100 , when $x = 0$.

Completing the square is not just a factoring technique. It is really our only way (at this point) to determine the smallest value that the expression takes. Completing the square is a way to understand quadratic expressions.

Example 4v. Find the smallest value of each of the following quadratic expressions.

a) $x^2 - 10x + 14$ b) $(2a-1)^2 + 8$ c) $x^2 - 8x - 180$ [Solution - Youtube link](#)

Example 5v. One number is 10 greater than another number. Find the smallest possible value of the product of these numbers. [Solution - Youtube link](#)

Solution: If we label the smaller number by x , then the other number is $x + 10$. The product of these numbers is $x(x + 10) = x^2 + 10x$, a quadratic expression. We complete the square to find the smallest possible value.

$$\begin{aligned} x(x + 10) &= x^2 + 10x & (x + 5)^2 &= x^2 + 10x + 25 \\ &= x^2 + 10x + 25 - 25 \\ &= (x + 5)^2 - 25 \end{aligned}$$

The smallest possible value of the complete square, $(x + 5)^2$ is zero. Then the smallest value of the entire expression is $\boxed{-25}$.

We can check: to achieve this smallest value, x must be -5 . Then the greater number is $-5 + 10 = 5$. Indeed, the product of 5 and -5 is -25 . Can we check if it is the smallest value? If we select another number for x , say -2 , then the greater number must be 8, and therefore the product is $-2 \cdot 8 = -16$. If x is positive, the other number is also positive, therefore, so is the product of the two numbers. So it does look like -25 is indeed the smallest value that this expression can achieve.

Part 2 - Optional Topic - The Arithmetic and Geometric Means

The fact that squares are non-negative and therefore have a smallest value pops up many times in mathematics. The following is an example for that. What we call average, is just one of several types of averages used in mathematics. The simplest one is the one we know simply as average, and it is called the arithmetic mean. There is also the geometric mean. As the name suggests, this mean shows up a lot in geometry. The harmonic means is an average that often shows up in physics.

Definition: Let a and b represent non-negative numbers. The arithmetic and geometric, means of a and b are defined as follows.

$$\text{arithmetic mean} = \frac{a+b}{2} \quad \text{and} \quad \text{geometric mean} = \sqrt{ab}$$

The harmonic average of a and b is defined as follows. Take the reciprocal of a and b . Take the 'regular' average of these reciprocals and then take the reciprocal of that. So the harmonic average of a and b is the reciprocal of $\frac{\frac{1}{a} + \frac{1}{b}}{2}$, which is $\frac{2}{\frac{1}{a} + \frac{1}{b}}$. Today we will only focus on the arithmetic and geometric means.

Example 6. Find the arithmetic, geometric, and harmonic means of 2 and 18.

Solution: The arithmetic mean is $\frac{2+18}{2} = 10$. The geometric mean is $\sqrt{2 \cdot 18} = 6$. The harmonic mean is (the reciprocal of the average of the reciprocals) is $\frac{2}{\frac{1}{2} + \frac{1}{18}} = \frac{2}{\frac{9}{18} + \frac{1}{18}} = \frac{2}{\frac{10}{18}} = 2 \cdot \frac{18}{10} = 3.6$

For any a and b , these means have a natural order. The arithmetic mean is always greater than or equal to the geometric mean: $\frac{a+b}{2} \geq \sqrt{ab}$ and the equality holds if and only if $a = b$. Why would this be so? Because squares are non-negative, and only zero when we square zero.

Theorem: (The arithmetic and geometric means). Suppose that a and b are positive numbers. Then

$$\frac{a+b}{2} \geq \sqrt{ab}$$

and the equality holds if and only if $a = b$.

Proof: Recall that a and b are both positive, so $\sqrt{a}$ and $\sqrt{b}$ both exist. For all a and b , $(\sqrt{a} - \sqrt{b})^2 \geq 0$ and the equality holds if and only if $a = b$.

$$\begin{aligned} (\sqrt{a} - \sqrt{b})^2 &\geq 0 \\ (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 &\geq 0 \\ a - 2\sqrt{ab} + b &\geq 0 && \text{add } 2\sqrt{ab} \\ a + b &\geq 2\sqrt{ab} && \text{divide by 2} \\ \frac{a+b}{2} &\geq \sqrt{ab} \end{aligned}$$

We prove inequalities too, not just solve them. In geometry, many inequalities we prove boil down to either the triangle inequality, or the arithmetic-geometric means theorem. When we look at an inequality we want to prove, the type of the inequality itself is an important clue. Strict inequalities (i.e. $>$ or greater) suggest the triangle inequality. When we see $\geq$ greater than or equal in the relationship to be proved, chances are that we can prove it using the arithmetic-geometric means.

Example 7. Prove that for all positive numbers x , $x + \frac{1}{x} \geq 2$.

Solution: Let us apply the arithmetic-geometric means to the positive numbers x and $\frac{1}{x}$. The arithmetic mean of

these two numbers is $\frac{x + \frac{1}{x}}{2}$, and their geometric mean is $\sqrt{x \cdot \frac{1}{x}} = \sqrt{1} = 1$. Now we state the inequality:

$$\begin{aligned} \frac{x + \frac{1}{x}}{2} &\geq 1 && \text{multiply by 2} \\ x + \frac{1}{x} &\geq 2 \end{aligned}$$

This completes our proof.



Practice Problems

Find the smallest value of each of the following expressions.

1. $x^2 - 4x + 7$
2. $x^2 + 10x + 14$
3. $x^2 - 2x$
4. $x^2 + 6x + 42$
5. $x^2 - 100x + 60$
6. $x^2 + 8x + 20$
7. $x^2 - 16$
8. $x^2 - 16x + 4$
9. $x^2 - 4x + 4$
10. $x^2 + 18x + 81$
11. One number is 24 less than another. Find the smallest possible value of the product of these numbers.



Enrichment

1. Not all squares produce zero as their smallest value. Consider the expression $(x^2 + 3)^2$. What is the smallest value of this expression? How about $(x^2 + 3)^2 - 1$? And $x^4 + 10x^2 + 14$?
2. Until now, the quadratic expressions we saw all had a leading coefficient of 1. How is our argument modified with different leading coefficients? Discuss the smallest value of each of the given expressions.
 - a) $3(x - 5)^2 + 8$
 - b) $-2(x + 1)^2 - 49$
 - c) $5((x + 4)^2 - 25)$
3. The product of two positive numbers is 6. Find the smallest possible value of the sum of these two numbers.

Problem Set 13

- Label each of the following statements as true or false.
 - If p is a prime number, then the number p^2 has exactly three positive divisors.
 - If an integer is divisible by 4 and by 9, then it is also divisible by 36.
 - If an integer is divisible by 2 and by 18, then it is also divisible by 36.
- Simplify each of the following expressions.
 - $m^2 \cdot m^3$
 - $(m^2)^3$
 - $\frac{(a^3b^2)^3}{ab^4}$
 - $\frac{3^{2x-1}}{9^{x-1}}$
 - $\frac{(-p^2)^3 q^5}{(-p)^2 q^4}$
 - $\left(-\frac{2x^3y}{y^4}\right)^2 \left(\frac{2x^3y^2}{-8x^5y^2}\right)^3$
- Simplify each of the following expressions.
 - $\left(\frac{2}{3^{-1}}\right)^{-2}$
 - $(2 - 3^{-1})^{-2}$
 - $\left(\frac{-a^{-2}b}{b^{-2}}\right)^{-3} (-a)^{-2}$
 - $\left(\frac{-2x^{-2}y^3}{(-3)^3 x^{-5}y^7}\right)^0$
- Compute the sum $49 + 54 + 59 + \dots + 2019$
- Re-write the decimal $0.1\overline{74}$ as a fraction of two integers.
- Simplify each of the following expressions.
 - $(5a - 1)^2$
 - $(3x^5 + 4y)(3x^5 - 4y)$
 - $\sqrt{125} - 3\sqrt{80} + \sqrt{45}$
 - $(2 - \sqrt{3})(5\sqrt{3} + 1)$
 - $(\sqrt{7} - 2)^2$
 - $(\sqrt{7} - 2)^3$
 - $(\sqrt{5} - \sqrt{2})^2$
- Find the exact value of $-x^2 + 4x + 6$ if
 - $x = \sqrt{2} - 1$
 - $x = 2 - \sqrt{3}$
 - $x = 2\sqrt{3} + 5$
- Completely factor each of the following.
 - $9x^2 - 16y^{10}$
 - $5x^2 + 10x$
 - $720x + 24x^2 - 3x^3$
 - $32x^4 - 2$
- Solve each of the following equations. Make sure to check your solution(s).
 - $2x^3 = 20x^2 + 1750x$
 - $\frac{3x+17}{2} = x - 1 + \frac{x+19}{2}$
 - $\frac{2}{3}(x-7) = \frac{4}{5}(x+1)$
 - $7x^2 + (x+3)(2x-1) = (3x+1)^2$
 - $8a + 2a^2 = 42$
 - $8x^3 = 50x^2$
 - $8p^3 = 50p$
 - $2 - (3-x)(2x+5) = (x-1)(2x-1)$
 - $2x^2 = 2x$
- Let A be the solution set of the inequality $\frac{2x-7}{3} - \frac{x-4}{2} > 9 - \frac{1}{2}x$ and B be the solution set of the inequality $3(2x-7) + 4(5-x) \leq 6x-1$. Find each of the following sets.
 - A
 - B
 - $A \cup B$
 - $A \cap B$
- Solve each of the following systems of linear equations.
 - $\begin{cases} 3x - 2y = 12 \\ 2x - y = 4 \end{cases}$
 - $\begin{cases} x - 7y = 2 \\ \frac{1}{3}(x-1) - \frac{3}{2}(y+1) = -2 \end{cases}$
 - $\begin{cases} 3y - (2x-1)^2 + 4x^2 - 7 = 8 \\ 4x - (3y-2)^2 + 9y^2 = 12 \end{cases}$

12. Graph the line $2x - 3y = -12$.
13. Express two consecutive decrease by 30% as a single change. What percent of a change would that be?
14. A farmer has some chickens and cows. One day he was asked: "*How many chickens and how many cows do you have?*" His answer was: "*All together, there are 73 heads and 188 legs*". How many chickens and how many cows does the farmer have?
15. Kendall is thinking of two real numbers. If one number is 20 less than the other, then what is the smallest possible value of the product of the two numbers?
16. a) One side of a rectangle is 4ft shorter than three times another side. Find the sides if the perimeter is 64ft.
b) One side of a rectangle is 24ft shorter than three times another side. Find the sides if the area is 315ft^2 .
17. Today a cab ride costs a flat fee of \$3.25 and then \$1.8 per mile. We take a ride that costs \$23.05. How far was our destination?
18. A total of \$20000 is to be invested in bonds and stocks. The value of the bonds increased by 3%. The value of the stocks increased by 8%. How much was invested in bonds and how much in in stocks if our portfolio is now worth. \$20980?
19. We throw a small object upward from the top of a 1024ft tall building. The height of the object, (measured in feet) t seconds after we threw it is $h = -16t^2 + 192t + 1024$
 - a) How long does it take for the object to hit the ground?
 - b) What is the greatest hight that the obect reaches?
20. Sally worked 50 hours last week and made \$660 for the week. For every hour worked over 40 her job pays time and a half. What is Sally's regular hourly pay rate?

Chapter 14

14.1 Completing the Square – Part 3

It is strongly recommended to use fractions and not decimals. The steps are identical as before (see Factoring by Completing the Square Parts 1 and 2), the only complications are that we need to perform the same computations with fractions.

Example 1. Factor $54x - 6x^2 + 60$ by completing the square.

$$\begin{aligned} 54x - 6x^2 + 60 &= && \text{rearrange terms} \\ -6x^2 + 54x + 60 &= && \text{factor out } -6 \\ -6(x^2 - 9x - 10) &= && \text{the "magic number" is } \frac{-9}{2} = -\frac{9}{2} \end{aligned}$$

We work out $\left(x - \frac{9}{2}\right)^2$ on the margin:

$$\left(x - \frac{9}{2}\right)^2 = \left(x - \frac{9}{2}\right)\left(x - \frac{9}{2}\right) = x^2 - \frac{9}{2}x - \frac{9}{2}x + \frac{81}{4} = x^2 - \frac{18}{2}x + \frac{81}{4} = x^2 - 9x + \frac{81}{4}$$

So we know to smuggle in the missing third term, $\frac{81}{4}$.

$$\text{our helper line: } \left(x - \frac{9}{2}\right)^2 = x^2 - 9x + \frac{81}{4}$$

$$-6(x^2 - 9x - 10) =$$

$$= -6\left(\underbrace{x^2 - 9x + \frac{81}{4}}_{\text{complete square}} - \frac{81}{4} - \frac{10 \cdot 4}{1 \cdot 4}\right)$$

realize complete square, re-write 10 as $\frac{10}{1} = \frac{40}{4}$

$$= -6\left(\left(x - \frac{9}{2}\right)^2 - \frac{81}{4} - \frac{40}{4}\right)$$

combine like terms

$$= -6\left(\left(x - \frac{9}{2}\right)^2 - \frac{121}{4}\right)$$

re-write $\frac{121}{4}$ as $\left(\frac{11}{2}\right)^2$

$$= -6\left(\left(x - \frac{9}{2}\right)^2 - \left(\frac{11}{2}\right)^2\right)$$

$$\begin{aligned}
&= -6 \left(\left(x - \frac{9}{2} \right)^2 - \left(\frac{11}{2} \right)^2 \right) && \text{factor via the difference of squares theorem} \\
&= -6 \left(x - \frac{9}{2} + \frac{11}{2} \right) \left(x - \frac{9}{2} - \frac{11}{2} \right) && \text{combine like terms} \\
&= -6 \left(x + \frac{2}{2} \right) \left(x - \frac{20}{2} \right) && \text{simplify fractions} \\
&= \boxed{-6(x+1)(x-10)}
\end{aligned}$$

We check by multiplication:

$$-6(x+1)(x-10) = -6(x^2 - 10x + x - 10) = -6(x^2 - 9x - 10) = -6x^2 + 54x + 60$$

Thus our result, $-6(x+1)(x-10)$ is correct.

Example 2. Factor $5p + p^2 + 6$ by completing the square.

$$\begin{aligned}
5p + p^2 + 6 &= && \text{rearrange terms} \\
p^2 + 5p + 6 &= && \text{there is no GCF}
\end{aligned}$$

The magic number is $\frac{5}{2}$. We work out $\left(p + \frac{5}{2}\right)^2$ on the margin:

$$\left(p + \frac{5}{2}\right)^2 = \left(p + \frac{5}{2}\right) \left(p + \frac{5}{2}\right) = p^2 + \frac{5}{2}p + \frac{5}{2}p + \frac{25}{4} = p^2 + \frac{10}{2}p + \frac{25}{4} = p^2 + 5p + \frac{25}{4}$$

So we know to smuggle in the missing third term, $\frac{25}{4}$.

$$p^2 + 5p + 6 = \quad \text{our helper line: } \left(p + \frac{5}{2}\right)^2 = p^2 + 5p + \frac{25}{4}$$

$$\begin{aligned}
&= \underbrace{p^2 + 5p + \frac{25}{4}} - \frac{25}{4} + 6 && \text{realize comple square} \\
&= \left(p + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{6 \cdot 4}{1 \cdot 4} && \text{re-write 6 as } \frac{6}{1} = \frac{24}{4} \\
&= \left(p + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{24}{4} && \text{combine like terms} \\
&= \left(p + \frac{5}{2}\right)^2 - \frac{1}{4} && \text{re-write } \frac{1}{4} \text{ as } \left(\frac{1}{2}\right)^2 \\
&= \left(p + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2 && \text{factor via the difference of squares theorem} \\
&= \left(p + \frac{5}{2} + \frac{1}{2}\right) \left(p + \frac{5}{2} - \frac{1}{2}\right) && \text{combine like terms} \\
&= \left(p + \frac{6}{2}\right) \left(p + \frac{4}{2}\right) && \text{simplify fractions} \\
&= \boxed{(p+3)(p+2)}
\end{aligned}$$

We check by multiplication:

$$(p+3)(p+2) = p^2 + 2p + 3p + 6 = p^2 + 5p + 6$$

Thus our result, $(p+3)(p+2)$ is correct.

Note: These polynomials can easily be factored using trial and error and other methods. Why should we use completing the square? All other methods will break down once the numbers are less friendly. Then ONLY completing the square will work. Since those computations will be more difficult, you should learn this method while numbers are easy.

Example 3. Factor $3x^2 - 4x - 319$ by completing the square.

If the leading coefficient is not 1, we will factor it out before completing the square.

$$3x^2 - 4x - 319 = 3 \left(x^2 - \frac{4}{3}x - \frac{319}{3} \right)$$

Half of the linear coefficient is $-\frac{4}{3} \div 2 = -\frac{4}{3} \cdot \frac{1}{2} = -\frac{4}{6} = -\frac{2}{3}$, thus we work out $\left(x - \frac{2}{3}\right)^2$.

$$\left(x - \frac{2}{3}\right)^2 = \left(x - \frac{2}{3}\right) \left(x - \frac{2}{3}\right) = x^2 - \frac{2}{3}x - \frac{2}{3}x + \frac{4}{9} = x^2 - \frac{4}{3}x + \frac{4}{9}$$

Thus we smuggle in $\frac{4}{9}$. The computation:

$$3x^2 - 4x - 319 =$$

$$\begin{aligned} &= 3 \left(x^2 - \frac{4}{3}x - \frac{319}{3} \right) & \left(x - \frac{2}{3}\right)^2 &= x^2 - \frac{4}{3}x + \frac{4}{9} \\ &= 3 \left(\underbrace{x^2 - \frac{4}{3}x + \frac{4}{9}}_{\left(x - \frac{2}{3}\right)^2} - \frac{4}{9} - \frac{319 \cdot 3}{3 \cdot 3} \right) \\ &= 3 \left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{957}{9} \right) \\ &= 3 \left(\left(x - \frac{2}{3}\right)^2 - \frac{961}{9} \right) & \frac{961}{9} &= \left(\frac{31}{3}\right)^2 \\ &= 3 \left(\left(x - \frac{2}{3}\right)^2 - \left(\frac{31}{3}\right)^2 \right) \\ &= 3 \left(x - \frac{2}{3} + \frac{31}{3} \right) \left(x - \frac{2}{3} - \frac{31}{3} \right) \\ &= 3 \left(x + \frac{29}{3} \right) \left(x - \frac{33}{3} \right) \\ &= 3 \left(x + \frac{29}{3} \right) (x - 11) = \boxed{(3x + 29)(x - 11)} \end{aligned}$$

We check: $(3x+29)(x-11) = 3x^2 - 33x + 29x - 319 = 3x^2 - 4x - 319$. Thus our answer is correct.

Example 4. Factor $11x + 6x^2 - 10$ by completing the square.

We first rearrange the terms by degree and then factor out the leading coefficient.

$$11x + 6x^2 - 10 = 6x^2 + 11x - 10 = 6 \left(x^2 + \frac{11}{6}x - \frac{5}{3} \right)$$

Half of the linear coefficient is $\frac{11}{6} \div 2 = \frac{11}{6} \left(\frac{1}{2} \right) = \frac{11}{12}$ and so we FOIL $\left(x + \frac{11}{12} \right)^2$.

$$\left(x + \frac{11}{12} \right)^2 = \left(x + \frac{11}{12} \right) \left(x + \frac{11}{12} \right) = x^2 + \frac{11}{12}x + \frac{11}{12}x + \left(\frac{11}{12} \right)^2 = x^2 + \frac{11}{6}x + \frac{121}{144} \qquad \frac{11}{12} + \frac{11}{12} = \frac{22}{12} = \frac{11}{6}$$

Thus we know to smuggle in $\frac{121}{144}$

$$6 \left(x^2 + \frac{11}{6}x - \frac{5}{3} \right) =$$

$$= 6 \left(x^2 + \frac{11}{6}x + \frac{121}{144} - \frac{121}{144} - \frac{5}{3} \right) \qquad \frac{5}{3} = \frac{5 \cdot 48}{3 \cdot 48} = \frac{240}{144}$$

$$= 6 \left(\left(x + \frac{11}{12} \right)^2 - \frac{121}{144} - \frac{240}{144} \right)$$

$$= 6 \left(\left(x + \frac{11}{12} \right)^2 - \frac{361}{144} \right)$$

$$\sqrt{361} = 19 \quad \text{and} \quad \sqrt{144} = 12$$

$$= 6 \left(\left(x + \frac{11}{12} \right)^2 - \left(\frac{19}{12} \right)^2 \right)$$

$$= 6 \left(x + \frac{11}{12} + \frac{19}{12} \right) \left(x + \frac{11}{12} - \frac{19}{12} \right)$$

$$= 6 \left(x + \frac{30}{12} \right) \left(x - \frac{8}{12} \right)$$

$$= 6 \left(x + \frac{5}{2} \right) \left(x - \frac{2}{3} \right) = 2 \left(x + \frac{5}{2} \right) 3 \left(x - \frac{2}{3} \right) = \boxed{(2x + 5)(3x - 2)}$$

We check: $(2x + 5)(3x - 2) = 6x^2 - 4x + 15x - 10 = 6x^2 + 11x - 10$. Thus our answer, $(2x + 5)(3x - 2)$ is correct.



Practice Problems

Completely factor each of the following by completing the square.

1. $x + x^2 - 12$

5. $3x + 3x^2 - 60$

9. $5a^2 - 14a - 3$

13. $10p^2 - 11p - 6$

2. $x^2 - x - 90$

6. $2x^2 - 6x + 17$

10. $33c^4 - 270c^3 - c^5$

14. $15x^2 - 34x + 15$

3. $11x + x^2 + 30$

7. $x + 2x^2 - 1$

11. $m + 6m^2 - 2$

4. $x^2 - 17x + 72$

8. $112x + 2x^2 - 2x^3$

12. $6x^2 - 7x - 3$

14.2 Completing the Square – Part 4

One important achievement of our studies is the ability to solve quadratic equations with any real coefficients. In what follows, we will master this important skill. As a matter of fact, we can apply the method of completing the square to any quadratic expression. The steps are still the same - it's just that each step takes a bit more work.

Recall that a **perfect square** is the square of an integer. For example, 4, 25, or 49 are perfect squares.

Recall that if N is a non-negative number, then $\sqrt{N}$ is the non-negative number x such that $x^2 = N$. For example, $\sqrt{7}$ is the non-negative number, that, when we square, we get 7. In short, $(\sqrt{7})^2 = 7$. In what follows, we start interpreting numbers that are not perfect squares as squares anyway. For example, we can interpret 7 as $(\sqrt{7})^2$ or 5 as $(\sqrt{5})^2$. This will be useful when we want to apply the difference of squares theorem.

Example 1. Solve the equation $x^2 + 13 = 8x$ over the real numbers. Make sure to check your solutions.

Solution: Since the equation is quadratic, we reduce one side to zero and factor the other side by completing the square.

$$\begin{aligned}
 x^2 + 13 &= 8x \\
 x^2 - 8x + 13 &= 0 & (x-4)^2 = x^2 - 8x + 16 \quad \text{so we smuggle in 16} \\
 \underbrace{x^2 - 8x + 16}_{(x-4)^2} - 16 + 13 &= 0 \\
 (x-4)^2 - 3 &= 0 & 3 = (\sqrt{3})^2 \\
 (x-4)^2 - (\sqrt{3})^2 &= 0 & \text{factor via difference of squares theorem} \\
 (x-4+\sqrt{3})(x-4-\sqrt{3}) &= 0
 \end{aligned}$$

We apply the zero product rule, and solve for the zeroes of each linear factor.

$$\begin{aligned}
 x-4+\sqrt{3} &= 0 & \text{or} & & x-4-\sqrt{3} &= 0 \\
 x+\sqrt{3} &= 4 & & & x-\sqrt{3} &= 4 \\
 x_1 &= 4-\sqrt{3} & & & x_2 &= 4+\sqrt{3} \\
 x_1 &= 4-\sqrt{3} & x_2 &= 4+\sqrt{3} & \text{or in shorter notation: } & x_{1,2} = 4 \pm \sqrt{3}
 \end{aligned}$$

We check: if $x = 4 - \sqrt{3}$, then

$$\begin{aligned}
 \text{LHS} &= (4 - \sqrt{3})^2 + 13 = 16 - 4\sqrt{3} - 4\sqrt{3} + 3 + 13 = 32 - 8\sqrt{3} \\
 \text{RHS} &= 8(4 - \sqrt{3}) = 32 - 8\sqrt{3}
 \end{aligned}$$

If $x = 4 + \sqrt{3}$, then

$$\begin{aligned}
 \text{LHS} &= (4 + \sqrt{3})^2 + 13 = 16 + 4\sqrt{3} + 4\sqrt{3} + 3 + 13 = 32 + 8\sqrt{3} \\
 \text{RHS} &= 8(4 + \sqrt{3}) = 32 + 8\sqrt{3}
 \end{aligned}$$

Thus our solution, $\boxed{4 + \sqrt{3} \text{ and } 4 - \sqrt{3}}$ is correct.

Our solutions are irrational numbers. When a quadratic equation has irrational solutions, all other methods break

down, and only completing the square works. If the reader is familiar with methods such as grouping or the AC-method, or trial and error, try solving $x^2 + 13 = 8x$ using any of those methods!

Being able to handle irrational solutions is a great advantage of the method of completing the square.

Please also note that in case of irrational solutions, writing the exact value necessitates the use of radicals. Recall that the exact value of an irrational number can not be captured as a fraction or as a terminating or repeating decimal, so radical form is our only option.

Example 2. Solve the equation $x^2 = 4x + 1$ over the real numbers. Make sure to check your solutions.

Solution: We complete the square.

$$\begin{aligned}
 x^2 &= 4x + 1 \\
 x^2 - 4x - 1 &= 0 & (x-2)^2 &= x^2 - 4x + 4 \\
 \underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4 - 1 &= 0 & 5 &= (\sqrt{5})^2 \\
 (x-2)^2 - 5 &= 0 \\
 (x-2)^2 - (\sqrt{5})^2 &= 0 \\
 (x-2+\sqrt{5})(x-2-\sqrt{5}) &= 0
 \end{aligned}$$

We apply the zero product rule to the product of the two linear factors.

$$\begin{aligned}
 x-2+\sqrt{5} &= 0 & \text{or} & & x-2-\sqrt{5} &= 0 \\
 x+\sqrt{5} &= 2 & & & x-\sqrt{5} &= 2 \\
 x_1 &= 2-\sqrt{5} & & & x_2 &= 2+\sqrt{5} \\
 x_1 = 2-\sqrt{5} \text{ and } x_2 = 2+\sqrt{5} & \text{ or in shorter notation: } x_{1,2} = 2 \pm \sqrt{5}
 \end{aligned}$$

We check: if $x = 2 - \sqrt{5}$, then

$$\begin{aligned}
 \text{LHS} &= (2 - \sqrt{5})^2 = 4 - 2\sqrt{5} - 2\sqrt{5} + 5 = 9 - 4\sqrt{5} \\
 \text{RHS} &= 4(2 - \sqrt{5}) + 1 = 8 - 4\sqrt{5} + 1 = 9 - 4\sqrt{5}
 \end{aligned}$$

If $x = 2 + \sqrt{5}$, then

$$\begin{aligned}
 \text{LHS} &= (2 + \sqrt{5})^2 = 4 + 2\sqrt{5} + 2\sqrt{5} + 5 = 9 + 4\sqrt{5} \\
 \text{RHS} &= 4(2 + \sqrt{5}) + 1 = 8 + 4\sqrt{5} + 1 = 9 + 4\sqrt{5}
 \end{aligned}$$

Thus our solution, $\boxed{2 + \sqrt{5} \text{ and } 2 - \sqrt{5}}$ is correct.

Example 3. Solve the equation $2x^2 + 10x + 9 = 0$ over the real numbers. Make sure to check your solutions.

Solution: We will factor by completing the square. We first factor out the leading coefficient.

$$\begin{aligned} 2x^2 + 10x + 9 &= 0 \\ 2\left(x^2 + 5x + \frac{9}{2}\right) &= 0 \end{aligned}$$

Half of the linear coefficient is $\frac{5}{2}$. Thus the complete square is $\left(x + \frac{5}{2}\right)^2$

$$\left(x + \frac{5}{2}\right)^2 = \left(x + \frac{5}{2}\right)\left(x + \frac{5}{2}\right) = x^2 + \frac{5}{2}x + \frac{5}{2}x + \frac{25}{4} = x^2 + 5x + \frac{25}{4}$$

So we smuggle in $\frac{25}{4}$.

$$\begin{aligned} 2\left(\underbrace{x^2 + 5x + \frac{25}{4}}_{\left(x + \frac{5}{2}\right)^2} - \frac{25}{4} + \frac{9}{2}\right) &= 0 & \frac{9}{2} &= \frac{18}{4} \\ 2\left(\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{18}{4}\right) &= 0 \\ 2\left(\left(x + \frac{5}{2}\right)^2 - \frac{7}{4}\right) &= 0 \end{aligned}$$

We next apply the difference of squares theorem. $\frac{7}{4}$ is the square of a number - its own square root.

$$\begin{aligned} 2\left(\left(x + \frac{5}{2}\right)^2 - \frac{7}{4}\right) &= 0 & \frac{7}{4} &= \left(\sqrt{\frac{7}{4}}\right)^2 = \left(\frac{\sqrt{7}}{\sqrt{4}}\right)^2 = \left(\frac{\sqrt{7}}{2}\right)^2 \\ 2\left(\left(x + \frac{5}{2}\right)^2 - \left(\frac{\sqrt{7}}{2}\right)^2\right) &= 0 \\ 2\left(x + \frac{5}{2} + \frac{\sqrt{7}}{2}\right)\left(x + \frac{5}{2} - \frac{\sqrt{7}}{2}\right) &= 0 \end{aligned}$$

We solve for the zeroes of both linear factors:

$$\begin{aligned} x + \frac{5}{2} + \frac{\sqrt{7}}{2} &= 0 & \text{or} & & x + \frac{5}{2} - \frac{\sqrt{7}}{2} &= 0 \\ x + \frac{\sqrt{7}}{2} &= -\frac{5}{2} & & & x - \frac{\sqrt{7}}{2} &= -\frac{5}{2} \\ x_1 &= -\frac{5}{2} - \frac{\sqrt{7}}{2} = \frac{-5 - \sqrt{7}}{2} & & & x_2 &= -\frac{5}{2} + \frac{\sqrt{7}}{2} = \frac{-5 + \sqrt{7}}{2} \end{aligned}$$

So the answer is $\boxed{x_1 = \frac{-5 - \sqrt{7}}{2} \text{ and } x_2 = \frac{-5 + \sqrt{7}}{2}}$ or, in a more compact form, $x_{1,2} = \frac{-5 \pm \sqrt{7}}{2}$.

We check: if $x = \frac{-5 - \sqrt{7}}{2}$, then

$$\begin{aligned} \text{LHS} &= 2x^2 + 10x + 9 = 2 \left(\frac{-5 - \sqrt{7}}{2} \right)^2 + 10 \left(\frac{-5 - \sqrt{7}}{2} \right) + 9 = \frac{2}{1} \cdot \frac{(-5 - \sqrt{7})^2}{2^2} + \frac{10}{1} \cdot \frac{-5 - \sqrt{7}}{2} + 9 \\ &= \frac{2(-5 - \sqrt{7})^2}{4} + \frac{10(-5 - \sqrt{7})}{2} + \frac{9}{1} = \frac{(-5 - \sqrt{7})^2}{2} + \frac{10(-5 - \sqrt{7})}{2} + \frac{18}{2} = \\ &= \frac{25 + 7 + 10\sqrt{7} - 50 - 10\sqrt{7} + 18}{2} = \frac{32 + 10\sqrt{7} - 50 - 10\sqrt{7} + 18}{2} = 0 = \text{RHS} \end{aligned}$$

and if $x = \frac{-5 + \sqrt{7}}{2}$, then

$$\begin{aligned} \text{LHS} &= 2x^2 + 10x + 9 = 2 \left(\frac{-5 + \sqrt{7}}{2} \right)^2 + 10 \left(\frac{-5 + \sqrt{7}}{2} \right) + 9 = \frac{2}{1} \cdot \frac{(-5 + \sqrt{7})^2}{2^2} + \frac{10}{1} \cdot \frac{-5 + \sqrt{7}}{2} + 9 \\ &= \frac{2(-5 + \sqrt{7})^2}{4} + \frac{10(-5 + \sqrt{7})}{2} + \frac{9}{1} = \frac{(-5 + \sqrt{7})^2}{2} + \frac{10(-5 + \sqrt{7})}{2} + \frac{18}{2} = \\ &= \frac{25 + 7 - 10\sqrt{7} - 50 + 10\sqrt{7} + 18}{2} = \frac{32 - 10\sqrt{7} - 50 + 10\sqrt{7} + 18}{2} = 0 = \text{RHS} \end{aligned}$$

Thus our solution, $\frac{-5 - \sqrt{7}}{2}$ and $\frac{-5 + \sqrt{7}}{2}$ is correct.

Example 4. Is there a real number that is exactly one less than its own square?

Solution: Let us denote this number by x . The square of x is then x^2 . We set up and solve the equation comparing the number to its own square.

$$x = x^2 - 1$$

Since the equation is quadratic, we reduce one side to zero and factor the other side.

$$\begin{aligned} x &= x^2 - 1 \\ 0 &= x^2 - x - 1 \\ x^2 - x - 1 &= 0 & \left(x - \frac{1}{2} \right)^2 &= x^2 - x + \frac{1}{4} \\ \underbrace{x^2 - x + \frac{1}{4}}_{\left(x - \frac{1}{2} \right)^2} - \frac{1}{4} - 1 &= 0 & \frac{5}{4} &= \left(\sqrt{\frac{5}{4}} \right)^2 = \left(\frac{\sqrt{5}}{\sqrt{4}} \right)^2 = \left(\frac{\sqrt{5}}{2} \right)^2 \\ \left(x - \frac{1}{2} \right)^2 - \frac{1}{4} - \frac{4}{4} &= 0 \\ \left(x - \frac{1}{2} \right)^2 - \frac{5}{4} &= 0 \\ \left(x - \frac{1}{2} \right)^2 - \left(\frac{\sqrt{5}}{2} \right)^2 &= 0 \\ \left(x - \frac{1}{2} + \frac{\sqrt{5}}{2} \right) \left(x - \frac{1}{2} - \frac{\sqrt{5}}{2} \right) &= 0 \end{aligned}$$

$$\begin{array}{ll} \text{Either } x - \frac{1}{2} + \frac{\sqrt{5}}{2} = 0 & \text{or } x - \frac{1}{2} - \frac{\sqrt{5}}{2} = 0 \\ x + \frac{\sqrt{5}}{2} = \frac{1}{2} & x - \frac{\sqrt{5}}{2} = \frac{1}{2} \\ x_1 = \frac{1}{2} - \frac{\sqrt{5}}{2} & x_2 = \frac{1}{2} + \frac{\sqrt{5}}{2} \end{array}$$

The greater number, $x_2 = \frac{1+\sqrt{5}}{2} \approx 1.618034$ is a famous number. It is called the **golden mean** or **golden ratio**. $\frac{1+\sqrt{5}}{2}$ is often denoted by φ and has very interesting properties. For example, it is one less than its own square.

$$\begin{aligned} \varphi^2 - 1 &= \left(\frac{1+\sqrt{5}}{2} \right)^2 - 1 = \frac{(\sqrt{5}+1)^2}{2^2} - 1 = \frac{5+1+2\sqrt{5}}{4} - 1 = \frac{6+2\sqrt{5}}{4} - 1 = \frac{2(3+\sqrt{5})}{2 \cdot 2} - 1 \\ &= \frac{3+\sqrt{5}}{2} - \frac{2}{2} = \frac{3+\sqrt{5}}{2} = \varphi \quad \checkmark \quad \text{and so } \varphi = \frac{1+\sqrt{5}}{2} \text{ works.} \end{aligned}$$

Let us check the other number. Let us denote $\frac{1-\sqrt{5}}{2}$ by ψ .

$$\begin{aligned} \psi^2 - 1 &= \left(\frac{1-\sqrt{5}}{2} \right)^2 - 1 = \frac{(\sqrt{5}-1)^2}{2^2} - 1 = \frac{5+1-2\sqrt{5}}{4} - 1 = \frac{6-2\sqrt{5}}{4} - 1 = \frac{2(3-\sqrt{5})}{2 \cdot 2} - 1 \\ &= \frac{3-\sqrt{5}}{2} - \frac{2}{2} = \frac{3-\sqrt{5}}{2} = \psi \quad \checkmark \quad \text{and so } \psi = \frac{1-\sqrt{5}}{2} \text{ also works.} \end{aligned}$$



Sample Problems

1. Solve the equation $3x^2 - 5x - 7 = 0$ over the real numbers. Make sure to check your solutions.
2. Find all numbers that are exactly three less than their own square.



Practice Problems

Solve each of the following equations over the real numbers. Make sure to check your solutions.

- | | | |
|----------------------|--------------------------|------------------------|
| 1. $x^2 - 13 = 6x$ | 4. $3x^2 - 11x + 5 = 0$ | 7. $x^2 + 8x - 10 = 0$ |
| 2. $x^2 = 10x + 3$ | 5. $-2x^2 + 8x - 29 = 0$ | 8. $3x^2 + x - 1 = 0$ |
| 3. $x^2 + x - 1 = 0$ | 6. $5x^2 - 4x = 20$ | 9. $x^2 + x + 1 = 0$ |

10. Find all numbers with the following property: twice the number is two less than the square of the number.
11. Find all numbers with the following property: the number is four less than the square of the number.

14.3 Rational Expressions 1

Definition: A **rational expression** is a quotient of two polynomials.

Rational expressions are abstract fractions, such as $\frac{1}{x}$, $\frac{x}{x+1}$, or $\frac{2x-5}{x^2-4x+3}$. Because they are fractions, we can do the same thing to rational expressions as we did with fractions formed from numbers. To present fractions as final results, we always use the reduced form in which numerator and denominator do not share any common factors.

Recall the fundamental property of fractions: the only thing we can do to a fraction without changing its value is multiplying or dividing both numerator and denominator by the same number. In order to bring rational expressions to lowest terms, we factor both numerator and denominator and cancel out the common factors. It is important to remember that we can only cancel out common factors; we can not cancel out anything unless both numerator and denominator are factored.

Example 1. Simplify each of the given rational expressions.

$$\text{a) } \frac{18ab^2cx}{12a^3b^2c^3x} \quad \text{b) } \frac{x^2-25}{x^2-6x+5} \quad \text{c) } \frac{2x^4+4x^3-6x^2}{4x^5-4x^4} \quad \text{d) } \frac{a-3}{3-a}$$

Solution: a) Between 18 and 12, the common factor is 6. Between a and a^3 , the common factor is a . Between c and c^3 , the common factor is c . The factors b^2 and x can be completely cancelled out.

$$\frac{18ab^2cx}{12a^3b^2c^3x} = \frac{6ab^2cx(3)}{6ab^2cx(2a^2c^2)} = \frac{\cancel{6ab^2cx}(3)}{\cancel{6ab^2cx}(2a^2c^2)} = \boxed{\frac{3}{2a^2c^2}}$$

b) We first factor both numerator and denominator. $x^2-25 = (x+5)(x-5)$ by the difference of squares theorem. We can factor the denominator by trial and error and get that $x^2-6x+5 = (x-5)(x-1)$. Then we cancel out the common factors.

$$\frac{x^2-25}{x^2-6x+5} = \frac{(x+5)(x-5)}{(x-1)(x-5)} = \frac{(x+5)\cancel{(x-5)}}{(x-1)\cancel{(x-5)}} = \boxed{\frac{x+5}{x-1}}$$

c) We first factor both numerator and denominator and then cancel out factors common to both numerator and denominator. In both cases, we start with the greatest common factor or GCF.

$$\begin{aligned} 2x^4+4x^3-6x^2 &= 2x^2(x^2+2x-3) = 2x^2(x+3)(x-1) \\ 4x^5-4x^4 &= 4x^4(x-1) \end{aligned}$$

The factors common to numerator and denominator are $2x^2$ and $x-1$. We can cancel those out.

$$\frac{2x^4+4x^3-6x^2}{4x^5-4x^4} = \frac{2x^2(x+3)(x-1)}{4x^4(x-1)} = \frac{2x^2(x+3)\cancel{(x-1)}}{4x^4\cancel{(x-1)}} = \boxed{\frac{x+3}{2x^2}}$$

d) This problem often presents a troubling situation because neither numerator nor denominator can be factored. But if we substitute a few values into a , we find that in each case, the result is -1 . The expressions $a-3$ and $3-a$ are opposites. We can factor out from one of the two and then we can cancel out the common linear factors.

$$\frac{a-3}{3-a} = \frac{a-3}{-1(-3+a)} = \frac{a-3}{-1(a-3)} = \frac{\cancel{(a-3)}}{-1\cancel{(a-3)}} = \frac{1}{-1} = \boxed{-1}$$

We can also perform operations on rational expressions. We can add and subtract them, just like we do with fractions but that will be a topic later. For now, we will perform much easier operations: multiplications and divisions. When we multiply two fractions, terms common to the numerator of one and the denominator of the other can also be cancelled out, just like in $\frac{5}{8} \cdot \frac{3}{10} = \frac{5}{8} \cdot \frac{3}{5 \cdot 2} = \frac{3}{8 \cdot 2} = \frac{3}{16}$.

Example 2. Perform the given operations and simplify.

$$\text{a) } \frac{2a^2 - 18}{3a^2 - 6a - 9} \cdot \frac{a^2 + 2a + 1}{6a^2 + 6a} \quad \text{b) } \frac{5y^2 + 5y}{8y - 16} \div \frac{10y^3 + 10y^2}{4y^2 - 16}$$

Solution: a) First we factor numerators and denominators in both rational expressions.

$$\begin{aligned} 2a^2 - 18 &= 2(a^2 - 9) = 2(a + 3)(a - 3) \\ 3a^2 - 6a - 9 &= 3(a^2 - 2a - 3) = 3(a - 3)(a + 1) \\ a^2 + 2a + 1 &= (a + 1)(a + 1) \\ 6a^2 + 6a &= 6a(a + 1) \end{aligned}$$

Now we are ready to perform the multiplication.

$$\begin{aligned} \frac{2a^2 - 18}{3a^2 - 6a - 9} \cdot \frac{a^2 + 2a + 1}{6a^2 + 6a} &= \frac{2(a + 3)(a - 3)}{3(a - 3)(a + 1)} \cdot \frac{(a + 1)(a + 1)}{6a(a + 1)} = \frac{2(a + 3)\cancel{(a - 3)}}{3\cancel{(a - 3)}(a + 1)} \cdot \frac{(a + 1)\cancel{(a + 1)}}{6a\cancel{(a + 1)}} \\ &= \frac{2(a + 3)(a + 1)}{3(6a)(a + 1)} = \frac{2(a + 3)\cancel{(a + 1)}}{3(6a)\cancel{(a + 1)}} = \frac{2(a + 3)}{3(6a)} \\ &= \frac{2(a + 3)}{3(2 \cdot 3a)} = \frac{\cancel{2}(a + 3)}{3(\cancel{2} \cdot 3a)} = \frac{a + 3}{3(3a)} = \boxed{\frac{a + 3}{9a}} \end{aligned}$$

- f) This problem indicates a division. Recall that to divide is to multiply by the reciprocal. After we re-write the problem as multiplication of rational expressions, this problem becomes very similar to the previous one.

$$\frac{5y^2 + 5y}{8y - 16} \div \frac{10y^3 + 10y^2}{4y^2 - 16} = \frac{5y^2 + 5y}{8y - 16} \cdot \frac{4y^2 - 16}{10y^3 + 10y^2}$$

As before, we start with four factoring exercises.

$$5y^2 + 5y = 5y(y + 1) \quad \text{and} \quad 8y - 16 = 8(y - 2)$$

$$4y^2 - 16 = 4(y^2 - 4) = 4(y + 2)(y - 2) \quad \text{and} \quad 10y^3 + 10y^2 = 10y^2(y + 1)$$

$$\begin{aligned} \text{Therefore, } \frac{5y^2 + 5y}{8y - 16} \cdot \frac{4y^2 - 16}{10y^3 + 10y^2} &= \frac{5y(y + 1)}{8(y - 2)} \cdot \frac{4(y + 2)(y - 2)}{10y^2(y + 1)} = \frac{5y\cancel{(y + 1)}}{8(y - 2)} \cdot \frac{4(y + 2)\cancel{(y - 2)}}{10y^2\cancel{(y + 1)}} \\ &= \frac{5y}{8(y - 2)} \cdot \frac{4(y + 2)}{10y^2} = \frac{\cancel{5}y}{2 \cdot \cancel{4}} \cdot \frac{\cancel{4}(y + 2)}{\cancel{5} \cdot 2y \cdot y} = \boxed{\frac{y + 2}{4y}} \end{aligned}$$

**Discussion:**

Consider the expression $\frac{a-3}{3-a}$ from example 1d). If we substitute a few values into a (say 2, -5 , 5, and 0), the value of the expression is -1 . This is not true for every number. Find a number that, when substituted into the expression, does not result in -1 .

**Practice Problems**

1. Simplify each of the following.

$$\begin{array}{lll} \text{a) } \frac{2b-5}{10-4b} & \text{c) } \frac{4t^2-9}{4t-6} & \text{e) } \frac{6m+m^2-216}{6m-m^2+72} \end{array} \quad \text{g) } \frac{-7x-11-3(x-2)}{5x-11+3(x+5)}$$

$$\begin{array}{lll} \text{b) } \frac{x^2-1}{x+1} & \text{d) } \frac{p^2-p}{p^2-1} & \text{f) } \frac{x^2-144x-900}{x^2-158x+1200} \end{array}$$

2. Perform the indicated operations and simplify.

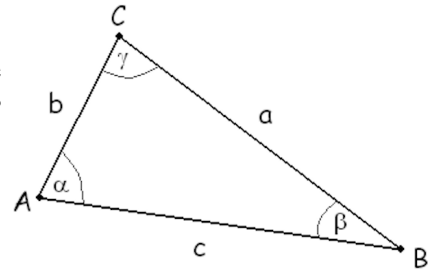
$$\begin{array}{lll} \text{a) } \frac{x}{5yz} \cdot \frac{10x^2y^3z}{4xy^2} & \text{c) } \frac{x^2-5x}{x^2-2x-15} \cdot \frac{x^2-9}{x^2-3x} & \text{e) } \frac{5y-35}{y^2-2y-35} \cdot \frac{3y+15}{5y-5} \end{array}$$

$$\begin{array}{lll} \text{b) } \frac{a^2-8a+16}{a} \cdot \frac{a^3}{4-a} & \text{d) } \frac{x^2-4x-21}{x^2-49} \div \frac{8x+x^2+15}{2x+x^2-35} & \text{f) } \frac{2x^2-98}{x^2-6x-7} \div \frac{21+3x}{6x^2-6} \end{array}$$

14.4 The Pythagorean Theorem

The Pythagorean theorem is a fundamental theorem about the connection between the three sides of a right triangle. Let us recall first a few things we will need for this topic.

Standard labeling simplifies notation. According to standard labeling, side a is opposite point A and angle α . Similarly, side b is opposite of point B and angle β , and side c is opposite point C and angle γ .



Also recall that the three angles in any triangle add up to 180° .

In any triangle, the order between sides is the same as the order between the angles opposite them. For example, the longest side is always opposite the greatest angle, and vica versa: the greatest angle is opposite the longest side. The shortest side is opposite the smallest angle, and vica versa: the smallest angle is opposite of the shortest side. If two sides are equally long, the angles opposite them are also equal.

In a right triangle, the right angle is the single greatest angle in the triangle, because the other two angles must add up to 90° so they both are smaller than 90° .

Therefore, right triangles have a single longest side and it is located opposite of the right angle.

Definition: In a right triangle, the side opposite the right angle is the longest side. We call this side the **hypotenuse** of the triangle.

We are now ready to state the Pythagorean theorem. It actually has two parts.

Theorem: (The Pythagorean theorem)

Part 1. If ABC is a right triangle with shorter sides a and b and hypotenuse c , then

$$a^2 + b^2 = c^2$$

Part 2. In any triangle with sides a , b , and c , if $a^2 + b^2 = c^2$, then the angle opposite side c measures 90° .

The first part tells us how right triangles behave. The second part tells us that *only* right triangles have this behavior.

Example 1. In each case, determine whether the three sides given are sides in a right triangle or not.

- a) 5cm, 7cm, and 9cm b) 5ft, 13ft and 12ft c) 53 units, 28 units, and 45 units.

Solution: We will use the second part of the Pythagorean theorem. Let us denote the longest side by c and the other two sides by a and b . If the statement $a^2 + b^2 = c^2$, is true, the triangle is a right triangle. If not, the triangle has no right angle in it.

- a) The only side here that could be the hypotenuse, is the longest one, measuring 9cm. Therefore, the two quantities we need to compare are $(5\text{cm})^2 + (7\text{cm})^2$ and $(9\text{cm})^2$.

$$(5\text{cm})^2 + (7\text{cm})^2 = 25\text{cm}^2 + 49\text{cm}^2 = 74\text{cm}^2 \quad \text{and} \quad (9\text{cm})^2 = 81\text{cm}^2$$

Since $74\text{cm}^2 \neq 81\text{cm}^2$, this triangle is not a right triangle.

- b) The only side here that could be the hypotenuse, is the longest one, measuring 13ft. Therefore, the two quantities we need to compare are $(5\text{ft})^2 + (12\text{ft})^2$ and $(13\text{ft})^2$.

$$(5\text{ft})^2 + (12\text{ft})^2 = 25\text{ft}^2 + 144\text{ft}^2 = 169\text{ft}^2 \quad \text{and} \quad (13\text{ft})^2 = 169\text{ft}^2$$

Since $169\text{ft}^2 = 169\text{ft}^2$, this triangle is a right triangle with hypotenuse 13ft long.

- c) The only side here that could be the hypotenuse, is the longest one, measuring 53 units. Therefore, the two quantities we need to compare are $28^2 + 45^2$ and 53^2 .

$$28^2 + 45^2 = 784 + 2025 = 2809 \quad \text{and} \quad 53^2 = 2809$$

Since $2809 = 2809$, this triangle is a right triangle.

If we know two sides of a right triangle, the Pythagorean theorem enables us to find the third side.

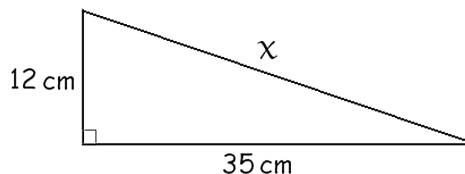
Example 2. In each case, find the missing side of the right triangle.

- a) the shortest two sides are 12cm and 35cm c) the shortest two sides are both 1 unit long
b) the longest two sides are 15ft and 17ft

Solution: a) We will use the first part of the Pythagorean theorem. Let us denote the hypotenuse by x .

We state the Pythagorean theorem for this right triangle.

$$\begin{aligned} 12^2 + 35^2 &= x^2 \\ 144 + 1225 &= x^2 \\ 1369 &= x^2 \\ \pm 37 &= x \end{aligned}$$

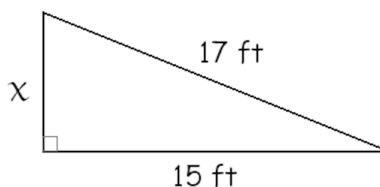


Since x represents a distance, and distances can never be negative, $x = -37$ is ruled out as a possible solution. Thus the hypotenuse of this triangle is 37cm.

- b) We will use the first part of the Pythagorean theorem. Let us denote the missing shortest side by x .

We state the Pythagorean theorem for this right triangle.

$$\begin{aligned} x^2 + 15^2 &= 17^2 \\ x^2 + 225 &= 289 \\ x^2 &= 64 \\ x &= \pm 8 \end{aligned}$$

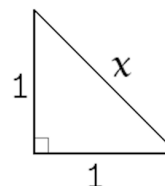


Since distances cannot be negative, $x = -8$ is ruled out as a possible solution. Thus the missing side is 8ft long.

- c) We will use the first part of the Pythagorean theorem. Let us denote the hypotenuse by x .

We state the Pythagorean theorem for this right triangle.

$$\begin{aligned} 1^2 + 1^2 &= x^2 \\ 1 + 1 &= x^2 \\ 2 &= x^2 \\ \pm\sqrt{2} &= x \end{aligned}$$



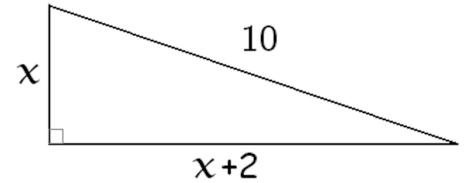
Since x represents a distance, and distances can never be negative, $x = -\sqrt{2}$ is ruled out as a possible solution. Thus the hypotenuse of this triangle is $\sqrt{2}$ unit.

This last example shows that right triangles often have side lengths that are irrational numbers. This particular triangle is responsible for bringing $\sqrt{2}$ into the mathematics of the ancient Greeks. They did not have the concept of irrational numbers yet, and so couldn't really deal with $\sqrt{2}$.

In order to find missing sides, we do not always need to know the lengths of two sides if other information is given.

Example 3. The hypotenuse of a right triangle is 10 units long. The difference between the other two sides is 2 units. Find the missing sides of the right triangle.

Solution: If we label the shortest side by x , then the other missing side can be denoted by $x + 2$. We state the Pythagorean theorem for this right triangle and solve the equation for x .



$$x^2 + (x + 2)^2 = 10^2 \quad \text{expand complete square}$$

$$x^2 + x^2 + 4x + 4 = 100 \quad \text{combine like terms}$$

$$2x^2 + 4x + 4 = 100 \quad \text{subtract 100}$$

$$2x^2 + 4x - 96 = 0 \quad \text{factor out 2}$$

$$2(x^2 + 2x - 48) = 0 \quad \text{factor (we will complete the square)} \quad (x + 1)^2 = x^2 + 2x + 1$$

$$2(\underbrace{x^2 + 2x + 1}_{(x+1)^2} - 1 - 48) = 0$$

$$2((x + 1)^2 - 49) = 0$$

$$2((x + 1)^2 - 7^2) = 0$$

$$2(x + 1 + 7)(x + 1 - 7) = 0$$

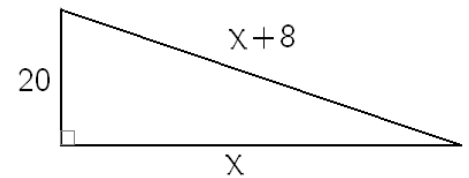
$$2(x + 8)(x - 6) = 0 \implies x_1 = -8 \quad x_2 = 6$$

Since distances can never be negative, -8 is ruled out as a possible solution. The other solution, $x = 6$ means that the other side, denoted by $x + 2$ must be $6 + 2 = 8$ units long. Thus, this right triangle has sides 6, 8, and 10 units long. We check: $6^2 + 8^2 = 36 + 64 = 100 = 10^2$, thus the triangle is indeed right. Also, $8 - 6 = 2$, so our answer is correct.

The following example will be a similar application problem, but the computation will be entirely different.

Example 4. The shortest side of a right triangle is 20 units long. The difference between the other two sides is 8 units. Find the missing sides of the right triangle.

Solution: If we label the shorter missing side by x , then the other missing side, the hypotenuse can be denoted by $x + 8$. We state the Pythagorean theorem for this right triangle and solve the equation for x .



$$20^2 + x^2 = (x + 8)^2 \quad \text{expand complete square}$$

$$400 + x^2 = x^2 + 16x + 64 \quad \text{subtract } x^2$$

$$400 = 16x + 64 \quad \text{subtract 64}$$

$$336 = 16x \quad \text{divide by 16}$$

$$21 = x$$

Therefore, the shorter missing side, denoted by x , is 21 units long, and the hypotenuse, denoted by $x + 8$ must be $21 + 8 = 29$ units long. So the three sides of this right triangle are 20, 21, and 29 units long. We check: the difference between 29 and 21 is indeed 8.

For the Pythagorean theorem, $20^2 + 21^2 = 400 + 441 = 841$ and $29^2 = 841$, and so $20^2 + 21^2 = 29^2$. This means that this is indeed a right triangle, and so our answer is correct.

This problem turned out to be much easier because the equation was linear after x^2 was subtracted from both sides.

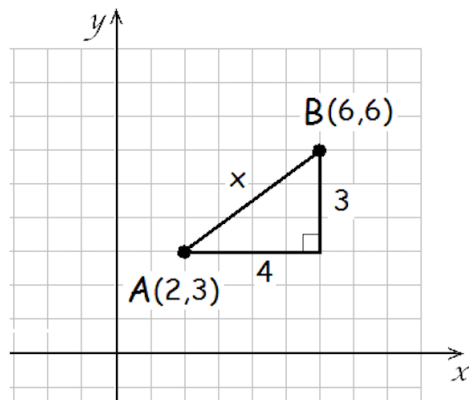
The following example is an extremely important application of the Pythagorean theorem.

Example 5. Find the distance between the given points.

- a) $A(2,3)$ and $B(6,6)$ b) $P(-3,2)$ and $Q(1,4)$

Solution: a) Let us plot the given points in a coordinate system as shown. The right triangle we created has shorter sides 4 and 3 units long. We denote the line segment connecting A and B (the hypotenuse) by x and state the Pythagorean theorem.

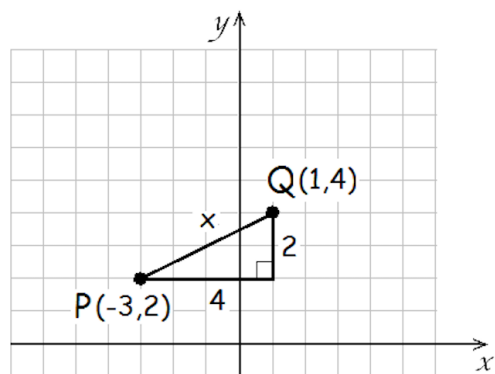
$$\begin{aligned} 4^2 + 3^2 &= x^2 \\ 16 + 9 &= x^2 \\ 25 &= x^2 \\ \pm 5 &= x \end{aligned}$$



Since distances can never be negative, -5 is ruled out and so our answer is 5 units.

- b) Let us plot the given points in a coordinate system as shown. The right triangle has shorter sides 2 and 4 units long. We denote the line segment connecting P and Q (the hypotenuse) by x and state the Pythagorean theorem for the triangle.

$$\begin{aligned} 4^2 + 2^2 &= x^2 \\ 16 + 4 &= x^2 \\ 20 &= x^2 \\ \pm\sqrt{20} &= x \end{aligned}$$



Since distances can never be negative, $-\sqrt{20}$ is ruled out and so our answer is $\sqrt{20}$ units. We might simplify $\sqrt{20}$ and write it as $2\sqrt{5}$. Then our final answer is $2\sqrt{5}$ units.

If the two points are $A(x_1, y_1)$ and $B(x_2, y_2)$, then the horizontal side of the triangle is the difference between the x -coordinates of the points, and can be expressed as $x_2 - x_1$. Similarly, the vertical side of the triangle is the difference between the y -coordinates of the points, and can be expressed as $y_2 - y_1$. If we pack all the steps in a single formula, **we get the distance formula:**

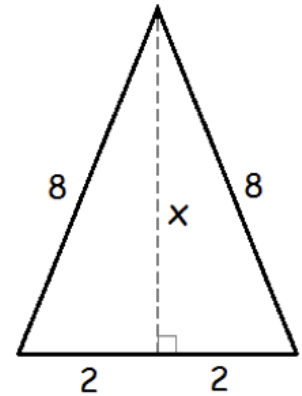
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 6. The sides of an isosceles triangle are 4 units, 8 units, and 8 units long. Find the exact value of the length of the height drawn to the 4 units long side.

Solution: In case of isosceles triangles, the height drawn to the base splits the triangle into two identical right triangles. The height now can be easily computed via the Pythagorean theorem applied to one of these right triangles. It is important to recognize that the missing side in the right triangle is not the hypotenuse.

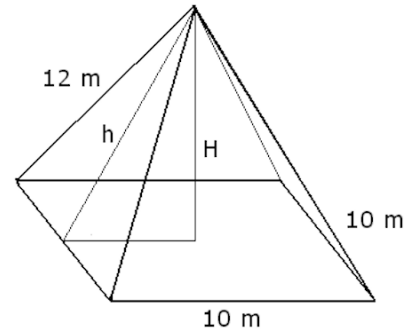
$$\begin{aligned} 2^2 + x^2 &= 8^2 \\ 4 + x^2 &= 64 \\ x^2 &= 60 \\ x &= \pm\sqrt{60} = \pm 2\sqrt{15} \implies x = 2\sqrt{15} \end{aligned}$$

Again, the negative solution of the equation is ruled out because distances can not be negative. The height belonging to the base is $2\sqrt{15}$ units long. Because we are asked for the exact value, we must present our answer in radical form, as $2\sqrt{15}$ units.

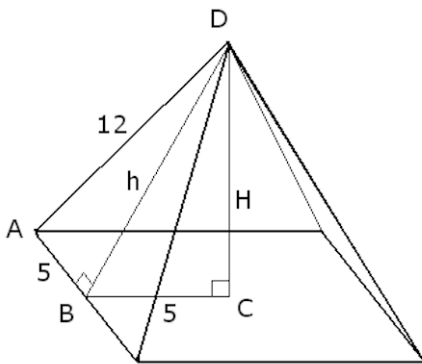


Example 7. Consider the square-based pyramid shown on the picture. Its base has sides 10m (meters) long. The other faces of the pyramid are isosceles triangles with sides 10m, 12m, and 12m long.

- Find the exact value of h , the length of the height in a triangular face.
- Find the exact value of H , the length of the height of the pyramid.



Solution: a) Let us label some of the vertices as shown on the figure below. We can find the value of h by stating the Pythagorean theorem on the right triangle ABD . The hypotenuse is the side opposite the right angle.



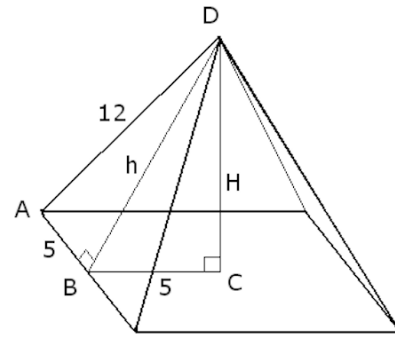
$$\begin{aligned} (5\text{m})^2 + h^2 &= (12\text{m})^2 \\ 25\text{m}^2 + h^2 &= 144\text{m}^2 \\ h^2 &= 119\text{m}^2 \\ h &= \pm\sqrt{119\text{m}^2} \implies h = \sqrt{119}\text{m} \end{aligned}$$

Therefore, the height of the triangular face is $\sqrt{119}\text{m}$. This is also called **the slant height** of the pyramid.

- b) Let use the labels shown on the figure above. We can find the value of H by stating the Pythagorean theorem on the right triangle BCD . The hypotenuse is the side opposite the right angle.

$$\begin{aligned}(5\text{m})^2 + H^2 &= h^2 \\(5\text{m})^2 + H^2 &= (\sqrt{119}\text{m})^2 \\25\text{m}^2 + H^2 &= 119\text{m}^2 \\H^2 &= 94\text{m}^2 \\H &= \pm\sqrt{94\text{m}^2} \implies H = \sqrt{94}\text{m}\end{aligned}$$

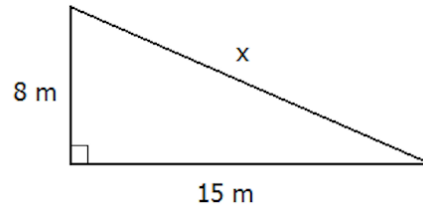
Thus the height of the pyramid is $\sqrt{94}\text{m}$ long.



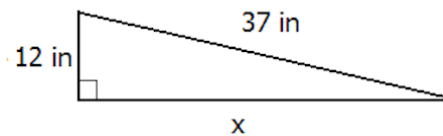
Sample Problems

- Could the three line segments given below be the three sides of a right triangle? Explain your answer.
a) 6cm, 10cm, and 8cm b) 7ft, 15ft, and 50ft c) 4m, 5m, and 6m

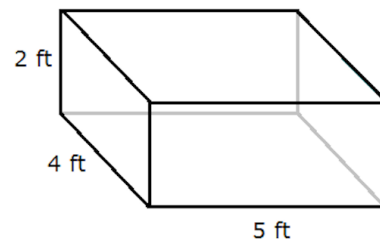
- Find the hypotenuse of the triangle shown on the figure.



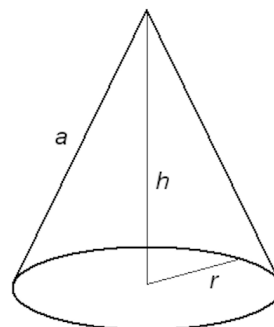
- Find the missing leg of the right triangle shown on the picture.
- Find the distance between the points $(3, 8)$ and $(8, -4)$.



- The sides of an isosceles triangle are 42 units, 29 units, and 29 units long. Find the length of the height drawn to the 42 units long side.
- The hypotenuse of a right triangle is 20cm. The difference between the other two sides is 4cm. Find the sides of the triangle.
- Find the length of the longest line segment inside the rectangular prism shown on the picture. (This line segment is called the **main diagonal** of the prism.)



- Find the height h of the cone shown on the picture, if the base has a radius of $r = 10\text{m}$ and $a = 26\text{m}$.
- An arch is in the shape of a semicircle. At a point along the base 1 foot from an end of the arch, the height of the arch is 7 feet. Find the maximum height of the arch.





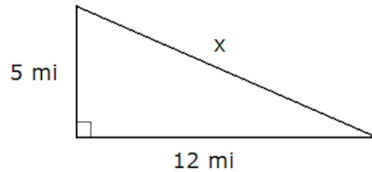
Practice Problems

1. Could the three line segments given below be the three sides of a right triangle? Explain your answer.

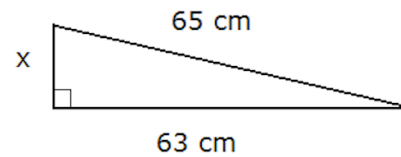
a) 3 cm, 7 cm, and 8cm c) 6m, 7m, and 8m

b) 37ft, 12ft, and 35ft

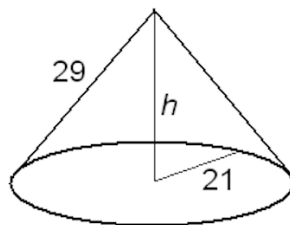
2. Find the hypotenuse of the triangle shown on the figure.



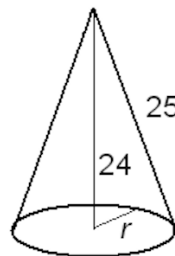
3. Find the missing leg of the right triangle shown on the picture.



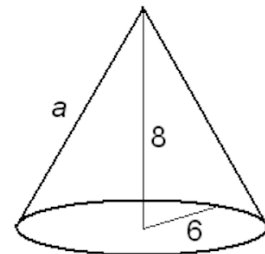
4. Find the length of the diagonal in a rectangle with sides 20 ft and 21 ft long.
5. The sides of an isosceles triangle are 25m, 25m, and 14m long. Find the length of the height drawn to the 14m long side.
6. Two sides of a right triangle are 8cm and 17cm long. Find the length of the missing side.
7. Find the distance between the given points. a) $(-2, -3)$ and $(6, 3)$ b) $(-9, -3)$ and $(15, 4)$.
8. One leg of a right triangle is 9cm. The difference between the other two sides is 1cm. Find the length of all sides.
9. The hypotenuse of a right triangle is 50 in. The difference between the other two sides is 34 in. Find the length of all sides.
10. The shortest side of a right triangle is 16 units long. The difference between the other two sides is 4 units. Find the sides of this right triangle.
11. Find the missing lengths indicated on the picture below.



a)



b)

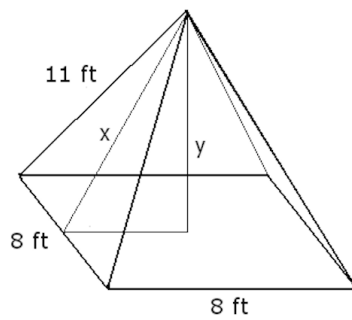


c)

12. Find the length of the diagonal of a square with sides 1 unit long.
13. Find the length of the main diagonal in a rectangular prism with sides
- a) 2m, 10m, and 11m b) 5ft, 7ft, and 1ft

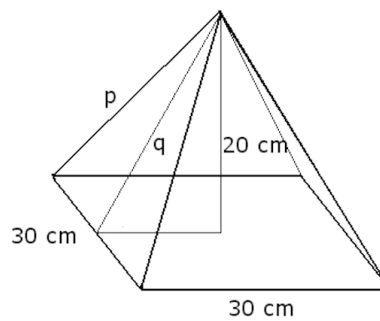
14. Find the exact value of the missing lengths, labeled

a) x and y



(a)

b) p and q



(b)

Problem Set 14

1. a) Find the prime factorization of each of the following. i) 180 ii) 1575 iii) 80^{100}
 b) Compute the greatest common factor and least common multiple of 180 and 1575.
 c*) The greatest common factor of 48 and x is 6. The least common multiple of 48 and x is 720. What values are possible for x ?
 d*) Is it possible for two numbers to have their greatest common factor be equal to their least common multiple?
2. Factor each of the following over the real numbers by completing the square.
 a) $x^2 - 6x + 7$ b) $6x^2 - 7x - 5$ c) $9x^2 - 12x - 1$
3. a) Solve the equation $3x^2 - 10x = -6$ by completing the square.
 b) Check your answer using exact values.
4. Find the smallest value of each of the given expressions.
 a) $x^2 - 10x + 21$ b) $x^2 + 2x + 7$ c) $x^2 + x + 1$
5. Prove that the decimal $0.\overline{372} = 0.372727272\dots$ represents a rational number by converting it into a fraction of two integers.
6. Suppose that $M = 3^{2018}$. Write each of the following in terms of M .
 a) $3^{2020} - 3^{2018}$ b) 9^{2017} c) $\frac{3^{2021} - 3^{2019}}{3^{2018}}$ d) 3^{1009}

7. Simplify each of the following.

a) $\frac{\sqrt{40}}{2}$

e) $(\sqrt{5}-2)^2$

i) $(\sqrt{2}-1)^3$

b) $\sqrt{18}-\sqrt{2}$

f) $(2\sqrt{5}-3)(2\sqrt{5}+3)$

j) $(\sqrt{2}-1)^4$

c) $\sqrt{3}(2\sqrt{3}-1)$

g) $\frac{12-\sqrt{20}}{-2}$

d) $(3\sqrt{2}-1)(2\sqrt{2}+3)$

h) $(\sqrt{2}-1)^2$

k) $\left(\frac{1-\sqrt{13}}{2}\right)^2 - \left(\frac{1-\sqrt{13}}{2}\right)$

8. Compute the exact value of the expression x^2+2x-5 if

a) $x = \sqrt{10}$ b) $x = \sqrt{2}+1$ c) $x = 2\sqrt{3}-1$

9. Simplify each of the following.

a) $\frac{18x^2-2}{12x+4}$ b) $\frac{x^2-1}{x^2-4x-5}$ c) $\frac{12-x}{x-12}$ d) $\frac{x^4-1}{x^2-1}$

10. Solve each of the given equations.

a) $x^2 = 2x$ c) $x^2 = 2x-5$ e) $2x^2+9x = -8$
b) $x^2 = 2x+5$ d) $2x^2+9x = 5$ f) $2x^2+9x = -12$

11. Solve each of the given system of equations.

a) $\begin{cases} (x+2)^2 - (x-3)^2 = 5y \\ 4x-3y = 7 \end{cases}$ b) $\begin{cases} (x-2)(y+4) = (x+1)(y+6) \\ \frac{2}{3}(x+1) = 3 - \frac{1}{5}(y+3) \end{cases}$

12. Graph each of the given equations.

a) $y = -x+1$ b) $y = -\frac{2}{3}x+2$ c) $3x-5y = -1$ d*) $(x-2)(y+1) = 0$

13. Compute the perimeter of the parallelogram determined by the points $A(-3,4)$, $B(21,14)$, $C(21,30)$, and $D(-3,20)$.

14. Compute the exact value of the distance between the points given.

a) $A(2,-9)$ and $B(5,-5)$ b) $P(-4,6)$ and $Q(3,5)$
c) Find the value(s) of k for which the distance between the points $A(-3,1)$ and $B(3,k)$ is 10 units.

15. a) The shortest side of a right triangle is 12cm. The difference between the other two sides is 2cm. Find the sides of the triangle. Use exact values.

b) The hypotenuse of a right triangle is 82cm. The difference between the other two sides is 62cm. Find the sides of the triangle. Use exact values.

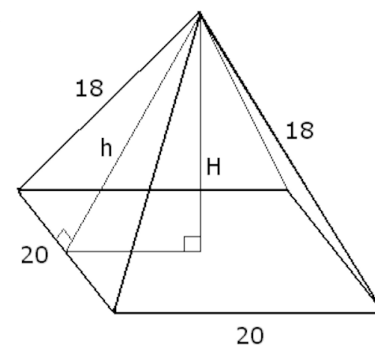
16. a) One number is four less than another. Find these numbers if their product is 780.

b) Allie and Bethany are sisters. When asked about their ages, Allie says: "*I am four years younger than Bethany.*", and Bethany says: "*The product of our ages is 780.*" How old are the sisters?

c) Why do the previous problems have different answers?

17. Find the length of the main diagonal in a rectangular prism with sides 7m, 8m, and 10m long. Use exact values.

18. Find the length of the line segments denoted by h and H , based on the picture shown. Use exact values.



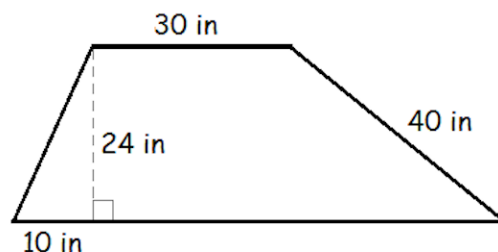
19. Consider the isosceles triangle with sides 7m, 7m, and 10m long.
a) Compute the exact value of the height belonging to the 10m long sides.
b) Compute the exact value of the area of the triangle.

20. Compute the exact value of the perimeter and area of a right triangle if its shorter sides are

a) 10cm and 24cm long b) 5ft and 10ft long

21. Find the perimeter and area of the trapezoid shown on the picture.

22. There were 75 bills in the drawer, all ten and twenty-dollar bills. How many of each were there if the value of all bills was \$1090 dollars?



23. The hypotenuse of a right triangle is 74m long. The difference between the other two sides is 46m. Find the sides of this triangle.

24. One number is 32 greater than another. What is the smallest possible value of their product?

25. Compute the sum of all numbers shown on the figure.

26. a) Find all real numbers that are exactly 2 less than their own square.
b) Find all real numbers that are exactly 3 less than their own square.

27. The longer side of a rectangle is 15ft long. The diagonal is 17ft long. Compute the area of the rectangle.

28. The hypotenuse of a right triangle is 8m shorter than twice the shortest side. The second longest side is 8m longer than the shortest side. Find the sides of this triangle.

29. Compute the given sum:

$$\frac{1}{2018} + \frac{2}{2018} + \frac{3}{2018} + \dots + \frac{2017}{2018}$$

30. Find the sum of the digits of the number $\frac{10^{2018} - 1}{3}$.

31. *Find the last two digits of 2^{2018} .

32. *a) Prove that the sum of two rational numbers is always rational.

b) We have proved that $\sqrt{2}$ is irrational. Use that and part a) to prove that $\sqrt{2} - 1$ is an irrational number.

1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18
10	11	12	13	14	15	16	17	18	19

Chapter 15

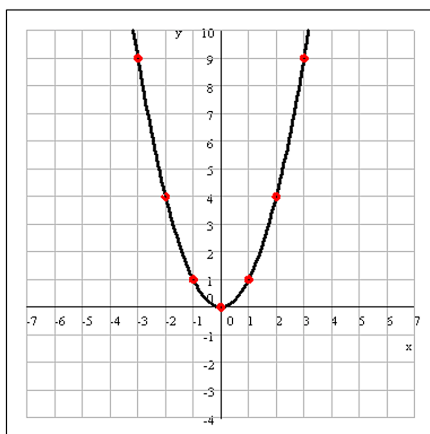
15.1 Graphing a Parabola – Part 1

We have been graphing equations in x , y , or x and y . Until now, we have only graphed linear equations - and the graph was a line. We will see that the graphs of quadratic equations are very different.

Example 1. Graph the equation $y = x^2$.

Solution: Just like in case of quadratic equations, we can find points on the graph by selecting a value for x and computing the y belonging to it using the equation $y = x^2$. We collect the points we found in a table and connect the points.

x	y
0	0
1	1
2	4
3	9
-1	1
-2	4
-3	9



We can always obtain more points. For example, if $x = \frac{1}{2}$, then $y = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Therefore, the point $\left(\frac{1}{2}, \frac{1}{4}\right)$ is also on the graph.

The shape we obtain is called a **parabola**. If the equation is linear in y and quadratic in x , the shape of its graph will be a parabola. Notice that there are significant differences from lines.

There is no point below the x -axis. This is because there is no point on the parabola with a negative y -value. This is because $y = x^2$ and squares can never be negative.

There is a lowest point. As we have discussed before, quadratic expressions have a smallest possible value. This is because zero can be obtained as the square of zero, but no square is a negative number. The lowest point on the parabola is called the **vertex**.

We observe a new symmetry that was not present in lines. The parabola $y = x^2$ is symmetrical to the y -axis. This is because a number and its opposite have the same square. Therefore, the y -value assigned to 2 is the same as the y -value assigned to -2 .

The parabola is not straight, it is curved. As the value of x is increased by 1, the y -values no longer increase by the same amount, causing the graph to be curved. Notice that as x increases by 1 from 0 to 1 to 2 to 3, the corresponding y -values increase by more and more. The differences are 0, 1, 3, 5, and so on. Most quadratic equations will have slightly more complicated graphs than that of $y = x^2$.

Example 2. Graph the parabola $y = x^2 - 8x + 7$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: We start with algebra. We obtain all three form of the equation first. These three forms are the polynomial form, the standard form, and the factored form. The polynomial form was given.

$$y = x^2 - 8x + 7 \implies \text{polynomial form}$$

We then complete the square. Half of the linear coefficient is -4 , thus we work out $(x - 4)^2$.

$$(x - 4)^2 = (x - 4)(x - 4) = x^2 - 4x - 4x + 16 = x^2 - 8x + 16$$

We are now ready to complete the square:

$$\begin{aligned} y &= x^2 - 8x + 7 & (x - 4)^2 &= x^2 - 8x + 16 \\ y &= \underbrace{x^2 - 8x + 16}_{(x-4)^2} - 16 + 7 \\ y &= (x - 4)^2 - 9 \implies \text{standard form} \end{aligned}$$

We factor via the difference of squares theorem

$$\begin{aligned} y &= (x - 4)^2 - 3^2 \quad \text{since } 3^2 = 9 \\ y &= (x - 4 + 3)(x - 4 - 3) \\ y &= (x - 1)(x - 7) \implies \text{factored form} \end{aligned}$$

From the polynomial form, we easily obtain the y -intercept, $(0, 7)$, since

$$\text{If } x = 0, \text{ then } y = 0^2 - 8(0) + 7 = 7 \implies \text{found } (0, 7)$$

From the standard form, we obtain the coordinates of the vertex.

$$y = (x - 4)^2 - 9$$

The trick is to think of the vertex as the point on the graph where the quadratic expression achieves its lowest possible value. Let us start with $(x - 4)^2$. Because it is a square, $(x - 4)^2$ will be non-negative for every value of x . The lowest possible value of $(x - 4)^2$ is zero, when we square zero. This means that $x - 4 = 0$ must hold, thus $x = 4$.

In short, the lowest value of $(x - 4)^2$ is zero, when $x = 4$.

Consequently, the lowest possible value of $(x - 4)^2 - 9$ is -9 , when $x = 4$. Thus the coordinates of the vertex are $(4, -9)$.

The factored form tells us the coordinates of the x -intercepts. For the x -intercepts, we have to solve the equation

$$x = ? \text{ so that } y = 0$$

$$x = ? \text{ so that } x^2 - 8x + 7 = 0$$

$$0 = x^2 - 8x + 7$$

$$0 = (x - 1)(x - 7) \implies x_1 = 1 \quad x_2 = 7$$

Thus $y = 0$ when $x = 1$ and $x = 7$. Thus, there are two x -intercepts, $(1, 0)$ and $(7, 0)$.

We will compute a few more points before graphing the parabola. To avoid finding points with too large coordinates, we will work with x -coordinates, close to the vertex. The y -coordinate can be found by substituting values for x into any of the three forms of the equations to find y . This time we will work with the polynomial form.

$$\text{if } x = 2, \text{ then } y = (2)^2 - 8(2) + 7 = 4 - 16 + 7 = -5 \implies \text{found } (2, -5)$$

$$\text{if } x = 3, \text{ then } y = (3)^2 - 8(3) + 7 = 9 - 24 + 7 = -8 \implies \text{found } (3, -8)$$

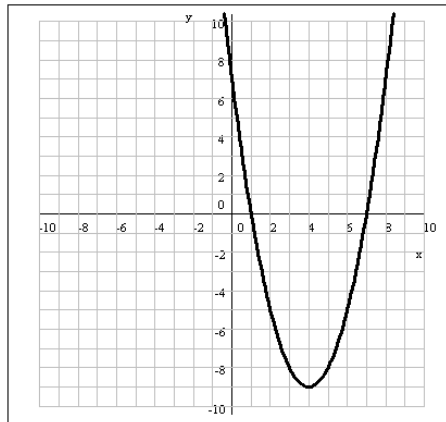
$$\text{if } x = 4, \text{ then } y = -9 \text{ already found } (4, -9)$$

$$\text{if } x = 5, \text{ then } y = (5)^2 - 8(5) + 7 = 25 - 40 + 7 = -8 \implies \text{found } (5, -8)$$

$$\text{if } x = 6, \text{ then } y = (6)^2 - 8(6) + 7 = 36 - 48 + 7 = -5 \implies \text{found } (6, -5)$$

We are ready to graph: we have the following points, listed left to right.

y-intercept	(0, 7)
x-intercept	(1, 0)
	(2, -5)
	(3, -8)
vertex	(4, -9)
	(5, -8)
	(6, -5)
x-intercept	(7, 0)



Example 3. Graph the parabola $y = x^2 + 6x + 5$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: We first obtain all three forms of the equation.

$$y = x^2 + 6x + 5 \implies \text{polynomial form} \implies y\text{-intercept: } (0, 5)$$

$$y = x^2 + 6x + 5 \qquad (x+3)^2 = x^2 + 6x + 9$$

$$y = \underbrace{x^2 + 6x + 9}_{(x+3)^2} - 9 + 5$$

$$y = (x+3)^2 - 4 \implies \text{standard form} \implies \text{vertex: } (-3, -4)$$

$$y = (x+3)^2 - 2^2$$

$$y = (x+3+2)(x+3-2)$$

$$y = (x+5)(x+1) \implies \text{factored form} \implies x\text{-intercepts } (-5, 0), (-1, 0)$$

We find a few points close to the vertex, (i.e. x is close to -3). This time we will use the standard square form of the equation, $y = (x+3)^2 - 4$.

$$\text{if } x = -5, \text{ then } y = 0 \text{ already found } (-5, 0)$$

$$\text{if } x = -4, \text{ then } y = (-4+3)^2 - 4 = (-1)^2 - 4 = 1 - 4 = -3 \implies \text{found } (-4, -3)$$

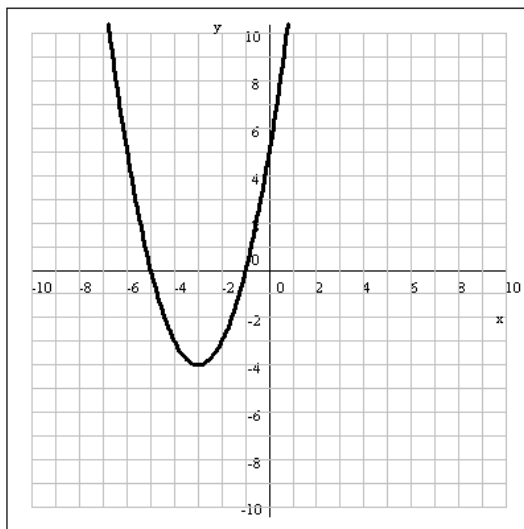
$$\text{if } x = -3, \text{ then } y = -4 \text{ already found } (-3, -4)$$

$$\text{if } x = -2, \text{ then } y = (-2+3)^2 - 4 = 1^2 - 4 = 1 - 4 = -3 \implies \text{found } (-2, -3)$$

$$\text{if } x = -1, \text{ then } y = 0 \text{ already found } (-1, 0)$$

We are ready to graph: we have the following points, listed left to right:

x -intercept	$(-5, 0)$
	$(-4, -3)$
vertex	$(-3, -4)$
	$(-2, -3)$
x -intercept	$(-1, 0)$
y -intercept	$(0, 5)$



Example 4. Graph the parabola $y = x^2 - 2x - 15$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: We first obtain all three forms of the equation.

$$\begin{aligned}
 y &= x^2 - 2x - 15 && \implies y\text{-intercept: } (0, -15) \\
 y &= x^2 - 2x - 15 && (x-1)^2 = x^2 - 2x + 1 \\
 y &= \underbrace{x^2 - 2x + 1}_{(x-1)^2} - 1 - 15 \\
 y &= (x-1)^2 - 16 && \implies \text{vertex: } (1, -16) \\
 y &= (x-1)^2 - 4^2 \\
 y &= (x-1+4)(x-1-4) \\
 y &= (x+3)(x-5) && \implies x\text{-intercepts } (-3, 0), (5, 0)
 \end{aligned}$$

We find a few points close to the vertex, (i.e. x is close to -3). This time we will use the factored form of the equation, $y = (x+3)(x-5)$.

$(-3, 0)$ already found

if $x = -1$, then

$$y = (-1+3)(-1-5) = 2(-6) = -12 \implies (-1, -12)$$

$(0, -15)$ already found

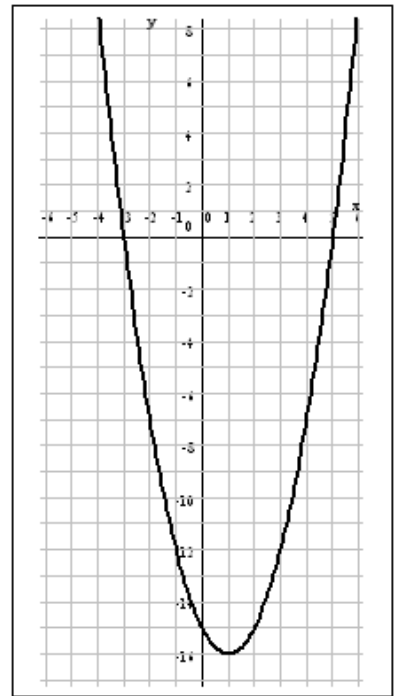
$(1, -16)$ already found

$$\text{if } x = 2, \text{ then } y = (2+3)(2-5) = 5(-3) = -15 \implies (2, -15)$$

$$\text{if } x = 3, \text{ then } y = (3+3)(3-5) = 6(-2) = -12 \implies (3, -12)$$

$(5, 0)$ already found

more additional points can be similarly found: $(-4, 9)$ and $(6, 9)$.



Practice Problems

Graph each of the parabolas given below. In each case, state the coordinates of five points of the parabola, including vertex and intercepts.

1. $y = x^2 - 10x + 21$
2. $y = x^2 + 4x - 5$
3. $y = x^2 - 4x + 3$

15.2 Compound Inequalities

As we have seen recently, solution sets are becoming more and more complicated as we learn new types of algebraic problems such as inequalities and systems of equations. Recall that a compound statement is one that was formed from two statements, using the word *and* or *or* to connect the two statements

Example 1. Label each of the given statements as true or false.

- a) 51 is a prime number.
- b) Every square is a rectangle.
- c) 51 is a prime number and every square is a rectangle.
- d) 51 is a prime number or every square is a rectangle.

Solution: a) If we add the digits of 51, the result is 6. This indicates that 51 is divisible by 3. Indeed, $51 = 3 \cdot 17$ and so 51 has at least four divisors: 1, 3, 17, and 51. prime numbers have exactly two divisors. Therefore, 51 is not a prime number and so the given statement is false.

b) Every square has four right angles. Having four right angles is how we define rectangles. Therefore, the given statement is true.

c) A true and a false statements are connected with the word *and*. Such a statement is true only if both statements are true. Therefore, the given statement is false.

d) A true and a false statements are connected with the word *or*. Such a statement is true if one or both statements are true. Because one of the two statements is true, the given statement is true.

Example 2. Consider the set $S = \{1, 4, 7, 8, 13\}$. Find all elements of S for which the given statement is true.

- a) $x < 8$ and x is odd
- b) $x < 8$ or x is odd

Solution: a) Consider each of the following statements. Recall that a compound statement connected with the word *and* is true if both statements are true.

- i) $1 < 8$ and 1 is odd This compound statement is true because both statements in the compound statement are true.
- ii) $4 < 8$ and 4 is odd This compound statement is false because the second statement is false.
- iii) $7 < 8$ and 7 is odd This compound statement is true because both statements in the compound statement are true.
- iv) $8 < 8$ and 8 is odd This compound statement is false because both statements in the compound statement are false.
- v) $13 < 8$ and 13 is odd This compound statement is false because the first statement is false.

Therefore, only 1 and 7 are the elements of S for which the given compound statement is true.

b) A compound statement connected with the word *or* is true if at least one of the statements is true.

- i) $1 < 8$ or 1 is odd This compound statement is true because both statements in the compound statement are true.
- ii) $4 < 8$ or 4 is odd This compound statement is true because the first statement is true.
- iii) $7 < 8$ or 7 is odd This compound statement is true because both statements in the compound statement are true.
- iv) $8 < 8$ or 8 is odd This compound statement is false because both statements in the compound statement are false.
- v) $13 < 8$ or 13 is odd This compound statement is true because the second statement is true.

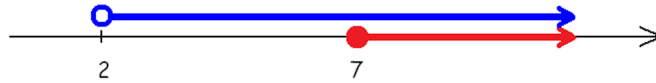
Therefore, 1, 4, 7 and 13 are the elements of S for which the given compound statement is true.

Compound inequalities will be just that: two statements of inequality, connected with the word *and* or *or*.

Example 3. Find all numbers that satisfy the given compound inequalities.

- a) $x > 2$ and $x \geq 7$ b) $x > 2$ or $x \geq 7$

Solution: Before we would solve the compound inequalities, let us depict on a number line the set of all numbers for which $x > 2$ and also, the set of all numbers for which $x \geq 7$.



- a) The numbers greater than 2 have the blue line above them. The numbers greater than or equal to 7 are marked with red. Because the two statements are connected with the word *and*, we are looking for numbers for which both statements are true. This means that the numbers we are looking for are marked red and have the blue line above them. That set is clearly $[7, \infty)$.
- b) Now the two statements are connected with the word *or*. This means that we are looking for numbers for which at least one of the statements are true. This means that the numbers we are looking for are marked red or have the blue line above them, or both. That set is clearly $(2, \infty)$.

Example 4. Find all numbers that satisfy the given compound inequalities.

- a) $x < 1$ and $x \geq 0$ b) $x < 1$ or $x \geq 0$

Solution: Before we would solve the compound inequalities, let us depict on a number line the set of all numbers for which $x < 1$ and also, the set of all numbers for which $x \geq 0$.

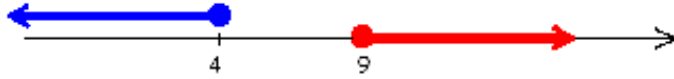


- a) The numbers less than 1 are marked with red. The numbers greater than or equal to 0 have the blue line above them. Because the two statements are connected with the word *and*, we are looking for numbers for which both statements are true. This means that the numbers we are looking for are marked red and have the blue line above them. That set is clearly $[0, 1)$. In case of a compound inequality of this type, we can re-write $x < 1$ and $x \geq 0$ in a shorter form, as $0 \leq x < 1$.
- b) Now the two statements are connected with the word *or*. This means that we are looking for numbers for which at least one of the statements are true. This means that the numbers we are looking for are marked red or have the blue line above them, or both. That set is clearly the set of all real numbers, $\mathbb{R}$.

Example 5. Find all numbers that satisfy the given compound inequalities.

a) $x \leq 4$ and $x \geq 9$ b) $x \leq 4$ or $x \geq 9$

Solution: Before we would solve the compound inequalities, let us depict on a number line the set of all numbers for which $x \leq 4$ and also, the set of all numbers for which $x \geq 9$.



a) The numbers less than or equal to 4 have the blue line above them. The numbers greater than or equal to 9 are marked with red. Because the two statements are connected with the word *and*, we are looking for numbers for which both statements are true. This means that the numbers we are looking for are marked red and have the blue line above them. There is no such number and so the solution set is the empty set, $\emptyset$.

b) Now the two statements are connected with the word *or*. This means that we are looking for numbers for which at least one of the statements are true. This means that the numbers we are looking for are marked red or have the blue line above them, or both. That set is $(-\infty, 4] \cup [9, \infty)$.



Practice Problems

Solve each of the given compound inequalities. Present your answer in interval form.

1. a) $5x - 1 > -16$ and $-x + 8 \geq 3$

b) $5x - 1 > -16$ or $-x + 8 \geq 3$

2. a) $3((2x - 5) - (x - 4)) \leq 2x - 7$ or $(x - 6) - 2(x - 4) \leq -x - 10$

b) $3((2x - 5) - (x - 4)) \leq 2x - 7$ and $(x - 6) - 2(x - 4) \leq -x - 10$

3. a) $\frac{2}{5}x - 4 > -\frac{1}{2}x + 5$ and $4(x - 3) - 2(x - 3) < -2x + 14$

b) $\frac{2}{5}x - 4 > -\frac{1}{2}x + 5$ or $4(x - 3) - 2(x - 3) < -2x + 14$

4. a) $\frac{3}{4}x - 2 > x - 3$ or $2x + 3 > 5x - 21$

b) $\frac{3}{4}x - 2 > x - 3$ and $2x + 3 > 5x - 21$

5. a) $(x + 3)^2 - (x - 1)^2 \leq -5(x + 1)$ and $\frac{2}{3}x - \frac{3}{5} > \frac{1}{5}x - \frac{2}{15}$

b) $(x + 3)^2 - (x - 1)^2 \leq -5(x + 1)$ or $\frac{2}{3}x - \frac{3}{5} > \frac{1}{5}x - \frac{2}{15}$

Problem Set 15

Chapter 16

16.1 Slope of a Line

Let us first recall a definition.

Definition: Equations that are in x , or in y , or in x and y can be graphed. **The graph of such an equation is the set of all points $P(x,y)$ for which the coordinates x and y form a solution of the equation.**

The slope of a line is a very important concept.

Definition: The slope of a line (usually denoted by m) is a signed ratio expressing the steepness of the line. Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the slope of the line connecting these points is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

This formula is called the **slope formula**.

Example 1. In each case, find the slope of the line determined by the two points given.

- a) $(5, -1)$ and $(1, 3)$ b) $(2, 1)$ and $(6, 3)$ c) $(8, -1)$ and $(-1, -1)$ d) $(7, 3)$ and $(7, -4)$

Solution: a) We find the slope determined by the points. $(5, -1) = (x_1, y_1)$ and $(1, 3) = (x_2, y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-1)}{1 - 5} = \frac{4}{-4} = -1$$

So the slope is -1 .

- b) We find the slope determined by the points. $(2, 1) = (x_1, y_1)$ and $(6, 3) = (x_2, y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{6 - 2} = \frac{2}{4} = \frac{1}{2}$$

So the slope is $\frac{1}{2}$.

- c) We find the slope determined by the points. $(8, -1) = (x_1, y_1)$ and $(-1, -1) = (x_2, y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - (-1)}{-1 - 8} = \frac{0}{-9} = 0$$

So the slope is 0. Note that **the slope of all horizontal lines is zero**.

- d) We find the slope determined by the points. $(7, 3) = (x_1, y_1)$ and $(7, -4) = (x_2, y_2)$ using the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 3}{7 - 7} = \frac{-7}{0} = \text{undefined}$$

So this line has no slope. Note that **vertical lines have no slope**.



Discussion: Suppose we are finding the slope of a line connecting $A(-5, 2)$ and $B(1, 10)$. Ann is using point A as (x_1, y_1) and point B as (x_2, y_2) . Beth is using point B as (x_1, y_1) and point A as (x_2, y_2) . Compute the slope both ways. Do we get the same answer? Why or why not?

Theorem: Suppose that the equation of a line is $y = mx + b$. Then the slope of this line is m , the coefficient of x in the equation.

For example, the line $y = 2x - 3$ has slope $m = 2$, the line $y = -\frac{2}{3}x + 1$ has slope $m = -\frac{2}{3}$.

Caution! If the line's equation is not in this form, the coefficient of x is NOT the slope. For example, the slope of the line $2x - 3y = -12$ is NOT 2.

The equation $y = mx + b$ is called the **slope-intercept form** of the line.

Example 2. In each case, find the slope of the line given.

a) $y = -3x + 8$ b) $y = \frac{3}{4}x - 2$ c) $y = -2$ d) $x = 7$

Solution: a) Since the equation is in the slope-intercept form, the slope of the line is the coefficient of x , -3 . So $m = -3$.

b) Since the equation is in the slope-intercept form, the slope of the line is the coefficient of x , $\frac{3}{4}$. So $m = \frac{3}{4}$.

c) This equation is horizontal, so its slope is zero. There are two ways to remind ourselves of this fact: when in doubt, find any two points on the line, say $(1, -2)$ and $(5, -2)$ and then apply the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - (-2)}{5 - 1} = \frac{0}{4} = 0$$

Another way is to think of the equation $y = -2$ as $y = 0x - 2$ and then the coefficient of x is zero.

- d) This line is vertical, so it does not have a slope. There are two ways to remind ourselves of this fact: when in doubt, find any two points on the line, say $(7, 1)$ and $(7, 2)$ and then apply the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 1}{7 - 7} = \frac{0}{0} = \text{undefined}$$

So this line has no slope. (Another way is to think of the equation $x = 7$ as an equation in which we can not solve for y , because it doesn't even appear in the equation. Or, $x = 0y + 7$ and then solving for y would result in having to divide by zero.)

The slope-intercept form of a line's equation enables us to plot a line quickly and with very little computation. Every non-vertical line has a slope-intercept form. In case of an integer m , we can think of m as a fraction: $\frac{m}{1}$. If m is already a fraction, great. If $m = \frac{p}{q}$ where p and q are integers, then from one nice lattice point to another, we will make q steps to the right, and p steps up (if p is positive) or down (if p is negative).

$$m = 4 = \frac{4}{1} \implies 1 \text{ step to the right, 4 steps up}$$

$$m = -1 = \frac{-1}{1} \implies 1 \text{ step to the right, 1 step down}$$

$$m = \frac{2}{3} \implies 3 \text{ steps to the right, 2 steps up}$$

$$m = -\frac{5}{6} = \frac{-5}{6} \implies 6 \text{ steps to the right, 5 steps down}$$

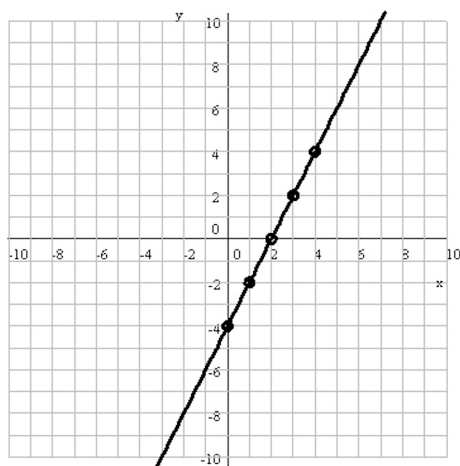
Example 3. Graph the line $y = 2x - 4$

Solution: This line is given in slope-intercept form. The slope of this line is the coefficient of x . It is $m = 2$.

We graph the y -intercept of the line. We obtain the y -intercept by substituting $x = 0$ into the equation. Clearly, the y -intercept is $(0, -4)$. In general, the y -intercept of the line $y = mx + b$ is $(0, b)$. We graph that point first.

We graph additional points using the slope of the line, $m = 2$. If the slope is a not fraction, we can always divide it by 1 to create one.

$$2 = \frac{2}{1} \implies \frac{2 \text{ up}}{1 \text{ to the right}}$$



We start at the y -intercept, $(0, -4)$ and to plot additional points, we step 1 to the right, 2 up. We repeat this process several times to plot enough points.

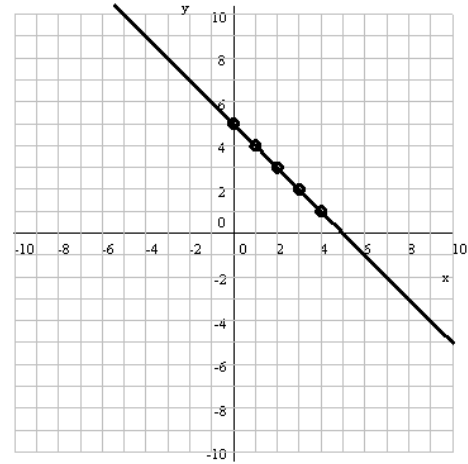
Example 4. Graph the line $y = -x + 5$.

Solution: The slope of this line is $m = -1$, and the y -intercept of it is $(0, 5)$. We graph the y -intercept first and then graph additional points using the slope of the line.

The slope of this line is $m = -1$. If the slope is a not fraction, we can always divide it by 1 to create one.

$$-1 = \frac{-1}{1} \implies \frac{1 \text{ down}}{1 \text{ to the right}}$$

We start at the y -intercept, $(0, 5)$ and to plot additional points, we step 1 to the right, 1 down. We repeat this process several times to plot enough points.



Example 5. Graph the line $x + 2y = 4$

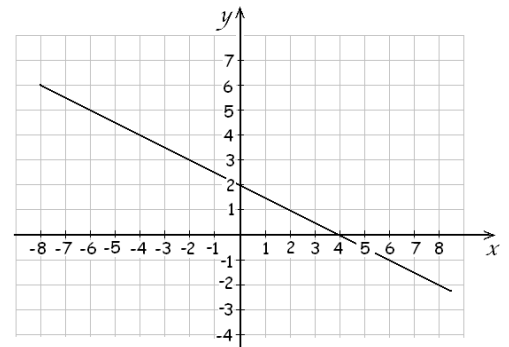
Solution: Step 1. We bring the equation to its slope-intercept form, $y = mx + b$ by solving for y in $x + 2y = 4$.

$$\begin{aligned} x + 2y &= 4 && \text{subtract } x \\ 2y &= -x + 4 && \\ y &= \frac{-x + 4}{2} = \frac{-x}{2} + \frac{4}{2} = -\frac{1}{2}x + 2 && \text{divide by 2} \end{aligned}$$

The slope-intercept form of the equation is $y = -\frac{1}{2}x + 2$.

Step 2. We graph first the y -intercept of the line, which we obtain by substituting $x = 0$ into the slope-intercept form of the line, $y = -\frac{1}{2}x + 2$. Clearly, the y -intercept is $(0, 2)$.

Step 3. Graph additional points using the slope of the line using the slope.



The slope of this line is $m = -\frac{1}{2}$. The denominator tells us how many units we step to the right, and the numerator tells us how many units we step down (since the slope is negative).

$$-\frac{1}{2} = \frac{-1}{2} \implies \frac{1 \text{ down}}{2 \text{ to the right}}$$

We start at the y -intercept, $(0, 2)$ and to plot additional points, we step 2 to the right, 1 down. We repeat this process several times to plot enough points.

Example 6. Graph the line $3x - 4y = 12$.

Solution: Step 1. We bring the equation to its slope-intercept form, $y = mx + b$ by solving for y in $3x - 4y = 12$.

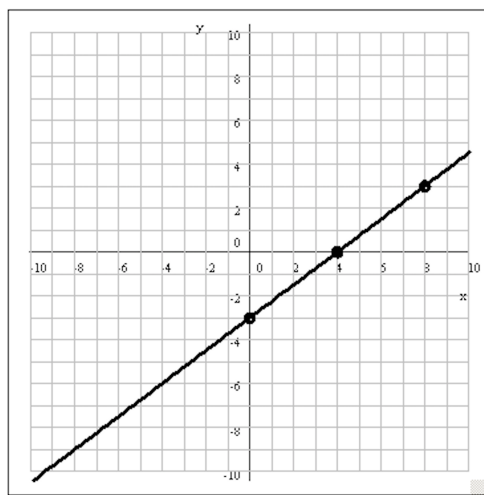
$$\begin{aligned}
 3x - 4y &= 12 && \text{add } 4y \\
 3x &= 4y + 12 && \text{subtract } 12 \\
 3x - 12 &= 4y && \text{divide by } 4 \\
 \frac{3x - 12}{4} &= y \\
 y &= \frac{3x - 12}{4} = \frac{3x}{4} - \frac{12}{4} = \frac{3}{4}x - 3
 \end{aligned}$$

The slope-intercept form of the equation is $y = \frac{3}{4}x - 3$.

Step 2. We graph the y -intercept of the line. We obtain the y -intercept by substituting $x = 0$ into the equation. Clearly, the y -intercept is $(0, -3)$.

Step 3. Graph additional points using the slope of the line.

$$m = \frac{3}{4} \implies \frac{3 \text{ up}}{4 \text{ to the right}}$$



We start at the y -intercept, $(0, -3)$ and to plot additional points, we step 4 to the right, and 3 up. We repeat this process several times to plot enough points.

Theorem: Two lines are parallel if and only if they are both vertical or they have the same slope.

In other words, two lines, if they have slopes and are parallel, then the slopes must be the same.

Theorem: Two lines are perpendicular if and only if one is vertical and the other is horizontal, or they both have slopes and they are negative reciprocals, $m_1 m_2 = -1$.



Practice Problems

1. In each case, determine the slope of the line connecting the points given.

- a) $(-2, 7)$ and $(3, -3)$ b) $(7, -5)$ and $(3, -8)$ c) $(3, -2)$ and $(3, 6)$ d) $(2, 7)$ and $(10, 7)$

2. Graph each of the following lines.

a) $3x + 2y = 6$

d) $2x - 3y = 10$

g) $3x + 5y = -30$

b) $x = -4$

e) $y = 1$

h) $2x - y = 7$

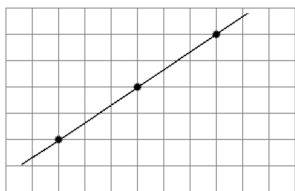
c) $y = \frac{2}{5}x - 3$

f) $y = 3x + 6$

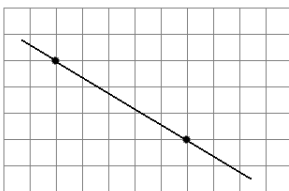
i) $y = \frac{1}{3}x$

3. Determine the slope of each of the following lines, based on their graphs.

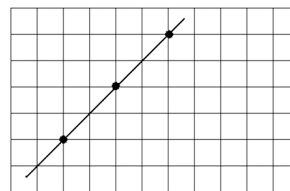
a)



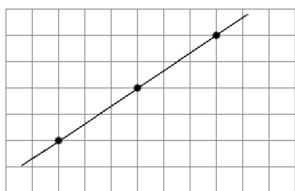
b)



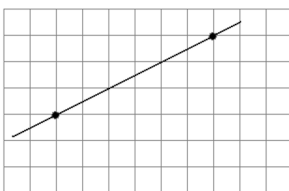
c)



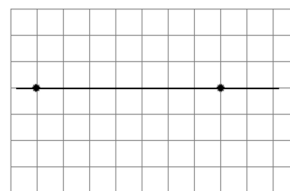
d)



e)



f)



16.2 Combinatorics 1

Part 1 - What is Combinatorics?

Combinatorics is a branch of mathematics that studies counting techniques. Combinatorial questions usually have non-negative integer answers. No negative numbers, no fractions. To illustrate the study of combinatorics, we start with a few examples of famous questions from combinatorics. A warning to the reader: do not expect that we will immediately know how to solve all of these problems.

Question 1. There are 100 people in a large conference room. If everyone greeted everyone else in the room with a handshake, how many handshakes would take place?

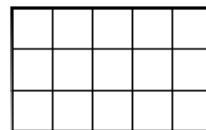
Question 2. There are 50 married couple in a large conference room. If everyone greeted everyone else in the room - except for their own spouse - with a handshake, how many handshakes would take place?

Question 3. We have a group of 9 students, 3 females and 6 males. In how many different ways can we form three groups of three students?

Question 4. We have a group of 9 students, 3 females and 6 males. In how many different ways can we form three groups of three students where each group contains exactly one female?

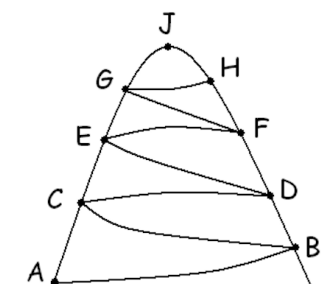
Question 5. We have 20 letters to mail to different addresses, with different content. We also have 20 addressed envelopes. Somehow the letters get all mixed up. If we place 1 letter into each envelope, in how many different ways can we do that?

Question 6. How many rectangles are there on the picture? How many squares?



Question 7. In Illinois, the Mega Lottery's rules are that we randomly select six numbers out of $\{1, 2, 3, \dots, 45\}$. If we wanted to fill out a lottery ticket for every possible outcome, how many tickets do we need to purchase?

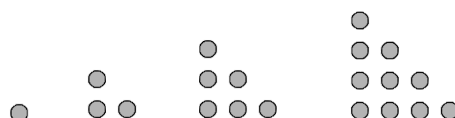
Question 8. As the picture shows, a mountain has two different tracks leading to its top. There is a mild, comfortable path, and also a steep, difficult track. People can choose at each intersection which path to take. Given that walking downward is not allowed, how many different pathways are there from point A to point J? (For example, ABDFGHJ and ABCEFHJ are two different pathways.)



Question 9. In how many different ways can we spell the word LOGIC based on the picture? Assume that we start in the upper left corner and only step to the right or down.

L	O	G	I	C
O	G	I	C	
G	I	C		
I	C			
C				

Question 10. The first few triangular numbers are 1, 3, 6, and 10. What is the 100th triangular number?



Part 2 - The First Counting Technique: Systematic Listing

The first technique is **systematic listing**. We want to know how many cases are there? We write them all down and count them. We have to be systematic to ensure that the count is correct - the two evils to avoid are skipping something and counting something more than once. For this reason, the greatest challenge in the problem is to develop a system or language for the counting. The rest is then easy.

We should never, never feel 'too good' for systematic listing. In many cases, when faced with a combinatorial problem and there is no immediate insight, we simply start with systematic listing. As we do that, we often get the key idea(s) that help us selecting and setting the more complex counting methods.

Example 1. List all different 3-digit numbers that can be formed using only the digits 1, 2, 3, and 4.

Solution: We are systematic if we simply insist on counting the numbers in an increasing order. The smallest number we can create is 111. To avoid skipping them, we change just the end of the number: we 'roll up' to 112. Then 113. Then 114.

111, 112, 113, 114

The next number can not be 115 because we can not use the digit 5. So what is the smallest next number? The answer is: 121. We list all possible numbers with the first digit is 1 and the second digit is 2.

111	121
112	122
113	123
114	124

Then we roll that 2 in the tenth place up to 3 and then to 4.

111	121	131	141
112	122	132	142
113	123	133	143
114	124	134	144

What do we have at this point? Well, we found 16 such numbers, but this is the complete list of all possible numbers that start with 1. We will have a similar table for numbers starting with 2, 3, and 4, giving us

111	121	131	141
112	122	132	142
113	123	133	143
114	124	134	144

16 numbers, starting with 1

211	221	231	241
212	222	232	242
213	223	233	243
214	224	234	244

16 numbers, starting with 2

311	321	331	341
312	322	332	342
313	323	333	343
314	324	334	344

16 numbers, starting with 3

411	421	431	441
412	422	432	442
413	423	433	443
414	424	434	444

16 numbers, starting with 4

And so we listed all such numbers. Please notice that without a system to follow, (in this case we listed the numbers in an increasing order) gathering these 64 numbers would have been much more difficult.

Example 2. List all different 3-digit numbers with three different digits that can be formed using the digits 1, 2, 3, and 4.

Solution: This problem is similar to the previous one. The only difference is: this time repetition of digits is not allowed. So numbers such as 133 or 141 needed to be counted before, in this case those can not be counted. We can ensure that we are systematic if, again, just list these numbers in an increasing order. The first (i.e. smallest) few such numbers are

123 132 142
124 134 143

and we already listed all such numbers that start with the digit 1. Clearly, this list will be shorter than the previous problem's list. We will have a similar table for numbers starting with 2, 3, and 4, giving us

123	132	142
124	134	143

6 numbers, starting with 1

213	231	241
214	234	243

6 numbers, starting with 2

312	321	341
314	324	342

6 numbers, starting with 3

412	421	431
413	423	432

6 numbers, starting with 4

And so we listed all such numbers. Please notice that without a system to follow, (in this case we listed the numbers in an increasing order) gathering these 24 numbers would have been much more difficult.

Example 3. In a miniature lottery game, we randomly pull three numbers out of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. List all possible outcomes.

Solution: It doesn't matter in what order the numbers come in, at the end, we organize and announce the winning numbers in an increasing order. So, we will do the same. Notice the spatial arrangement helping in counting.

1, 2, 3
1, 2, 4 1, 3, 4
1, 2, 5 1, 3, 5 1, 4, 5
1, 2, 6 1, 3, 6 1, 4, 6 1, 5, 6
1, 2, 7 1, 3, 7 1, 4, 7 1, 5, 7 1, 6, 7
1, 2, 8 1, 3, 8 1, 4, 8 1, 5, 8 1, 6, 8 1, 7, 8

These are only the outcomes where the smallest winning number is 1. What if the smallest winning number is 2?

2, 3, 4
2, 3, 5 2, 4, 5
2, 3, 6 2, 4, 6 2, 5, 6
2, 3, 7 2, 4, 7 2, 5, 7 2, 6, 7
2, 3, 8 2, 4, 8 2, 5, 8 2, 6, 8 2, 7, 8

These are all the possible outcomes in which the smallest number is 2. What if the smallest number is 3?

3, 4, 5
3, 4, 6 3, 5, 6
3, 4, 7 3, 5, 7 3, 6, 7
3, 4, 8 3, 5, 8 3, 6, 8 3, 7, 8

As we proceed, we find less and less outcomes with higher and higher numbers as their smallest number.

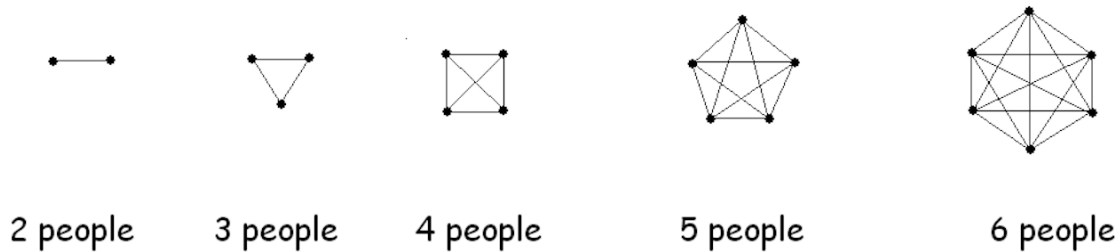
4, 5, 6
4, 5, 7 4, 6, 7
4, 5, 8 4, 6, 8 4, 7, 8
5, 6, 7
5, 6, 8 5, 7, 8
6, 7, 8

So we listed all 56 possible outcomes.

Part 3 - The Handshake Problem

Example 4. There are 100 people in a large conference room. If everyone greeted everyone else in the room with a handshake, how many handshakes would take place?

The problem is worth exploring. We can investigate the question using smaller crowds and systematic listing. However, this method eventually becomes too laborious for numbers such as 100. One representation could be using pictures representing people with points and the handshakes with a line connecting them.



So in case of 2 people, there is just 1 handshake. With 3 people, there are 3 handshakes. And so on, but we notice that counting the line segments is becoming more and more difficult as the numbers increase. Such a picture is called a graph, and graph theory is a branch of combinatorics that studies the properties of such geometric objects.

So we should look for another way to systematically list them. Let us label the people by numbers 1,2,3,4, and 5. Then one handshake between persons 2 and 5 could be denoted by 2,5 or even 25. We do not need to list 5,2 as they refer to the same handshake. So, systematic listing of handshakes for 4 people would be

for 4 people:	1,2 1,3 2,3 1,4 2,4 3,4 6 handshakes	for 5 people:	1,2 1,3 2,3 1,4 2,4 3,4 1,5 2,5 3,5 4,5 10 handshakes	for 6 people:	1,2 1,3 2,3 1,4 2,4 3,4 1,5 2,5 3,5 4,5 1,6 2,6 3,6 4,6 5,6 15 handshakes
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Systematic listing gets us only a slick technique for counting, but the numbers we obtained don't seem to follow an easy pattern.

people	handshakes
2	1
3	3
4	6
5	10
6	21

Attention! Never feel 'too good' for systematic listing. Even if you do not get immediate answers to your current question, there is much to discover, (even in this particular case) and those patterns and connections reveal themselves as we apply and improve systematic listing.

The craft becomes art when we move beyond listing. Many combinatorial ideas are based on a reorganization of how we count; on creating a new choreography.

Imagine the 100 people in the room. You are an extra person, from the media. You interview every single person in the room and ask everyone how many times they shook hands today in the room. Every single person would reply the same number: 99 times. So, we have a 100 answers, each of them 99. Does that mean that we have

$100 \cdot 99 = 9900$ many handshakes? Well, we are not sure. One importance of listing is that they provide the correct answers for the first smallest few cases. Therefore, we can use them to test our idea.

Suppose we have three people in the room. We interview all three of them. They all claim to have shaken two hands. $3 \cdot 2 = 6$ but the correct number (we know from listing) should be 3.

Suppose we have four people in the room. We interview all four of them. They all claim to have shaken three hands. $4 \cdot 3 = 12$ but the correct number (we know from listing) should be 6.

Suppose we have five people in the room. We interview all five of them. They all claim to have shaken four hands. $5 \cdot 4 = 20$ but the correct number (we know from listing) should be 10.

We notice now that our formula produces exactly twice the correct answer. Why would this be? Our method of multiplying the number of people by the handshakes results in counting **each** handshake **twice**. So, we adjust for this error and divide our number by 2. Let us first see if this idea produces the right results.

2 people	$\frac{2 \cdot 1}{2} = 1$	✓	5 people	$\frac{5 \cdot 4}{2} = 10$	✓
3 people	$\frac{3 \cdot 2}{2} = 3$	✓	6 people	$\frac{6 \cdot 5}{2} = 15$	✓
4 people	$\frac{4 \cdot 3}{2} = 6$	✓			

and so we have found the correct counting method that can now be applied to 100.

$$\text{number of handshakes} = \frac{100 \cdot 99}{2} = 4950$$

The development of such choreographies or counting arguments is a central technique in combinatorics.



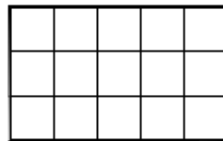
Practice Problems

1. There are 50 married couple in a large conference room. If everyone greeted everyone else in the room - except for their own spouse - with a handshake, how many handshakes would take place?
2. There are 50 married couple in a large conference room. If every man greeted every other man in the room with a handshake and also every woman greeted every other woman in the room with a handshake, how many handshakes would take place?

3. List all 4—element subsets of the set $\{1, 2, 3, 4, 5, 6\}$.

4. List all subsets of $A = \{1, 2, 3, 4, 5\}$

5. How many rectangles are there on the picture shown?
How many squares?



Problem Set 16

- Which number has more divisors, 24 or 36?
- Suppose that $A = 2^{99}$. Express each of the following in terms of A .
 - 2^{100}
 - $2^{99} + 2^{100} - 3 \cdot 2^{101}$
 - 8^{99}
- Find the smallest value of each of the following expressions.
 - $x^2 - 6x + 4$
 - $x^2 + 10x + 7$
 - $x^2 + 6x$
- Find the prime factorization for each of the following.
 - 1200
 - 10^{100}
 - $75 \cdot 80$
- Use the prime factorization to find the greatest common factor and least common multiple of 90 and 135.
- Simplify each of the given expressions.
 - $\frac{3}{8} \cdot (-3)^{-2}$
 - $\frac{2^{-1} + 5^{-1}}{2^{-2} - 5^{-2}}$
 - $\frac{2^{-1} \cdot 5^{-1}}{2^{-2} \cdot 5^{-2}}$
 - $\frac{(2x^{-1}y^3)^{-5}}{(2xy^{-2})^{-3}}$
 - $\frac{(-2xy^{-2})^3 (-2^{-2}x^{-3}y)^4}{(-4x^3y)^{-2}}$
- Compute each of the following sums.
 - $42 + 50 + 58 + \dots + 482$
 - $-35 + (-25) + \dots + 755$
 - $31 + 40 + 49 + \dots + 1102$
- Perform the indicated operations and simplify.
 - $(2\sqrt{7} - 1)(\sqrt{7} + 3)$
 - $(3\sqrt{2} - 1)^2$
 - $(\sqrt{2} - 1)^4$
 - $(3\sqrt{7} - 8)(3\sqrt{7} + 8)$
 - $\sqrt{50} - \sqrt{18} + 2\sqrt{72}$
 - $(2\sqrt{5} - 4)^3 (2\sqrt{5} + 4)^3$
 - $\frac{12 - \sqrt{20}}{2}$
 - $\frac{\sqrt{80}}{\sqrt{45}}$
 - $(-\sqrt{2})^8$
 - $(-\sqrt{2})^9$
- Simplify each of the following.
 - $(1, 4) \cup (2, 7)$
 - $(-\infty, 3) \cup (-\infty, 5]$
 - $(-\infty, 10] \cup (7, \infty)$
 - $(1, 5) \cup [12, \infty)$
 - $(1, 4) \cap (2, 7)$
 - $(-\infty, 3) \cap (-\infty, 5]$
 - $(-\infty, 10] \cap (7, \infty)$
 - $(1, 5) \cap [12, \infty)$
- Evaluate each of the given expressions with the values given for the variable(s).
 - $2x^2 - x - 3$ if $x = 2\sqrt{3} - 5$
 - $-a^2 + 5a - 1$ if $a = 4 - \sqrt{3}$
 - $x^2 - 8x + 6$ if $x = 4 - \sqrt{10}$
- Graph each of the following.
 - $y = -\frac{2}{3}x + 1$
 - $x + y = 3$
 - $2x - 3y = -1$
- Find the slope of each of the given lines.
 - the line connecting $(-2, 5)$ and $(2, 7)$
 - the line connecting $(3, 8)$ and $(8, 3)$
 - the line $3x - 5y = -30$
 - a line perpendicular to $2x - 3y = -12$
- Graph $y = x^2 - 4x - 5$. State the coordinates of at least five points, including vertex and intercepts.
- Find the perimeter and area of the quadrilateral determined by the points $A(-3, 5)$, $B(-3, -2)$, $C(2, -2)$, and $D(2, 5)$.

15. Perform the following operations. Show all steps.

a) $18 - 2(-5) - 2(11 - 2(-5))$

d) $\left| -\frac{1}{2} - 2 \right| - \left| \frac{3}{4} - 3 \right|$

g) $\left| -\frac{1}{2} - \left| 2 - \frac{3}{4} - 3 \right| \right|$

b) $\frac{-3^2 + (-1)^3}{7 - 3(-1)^3}$

e) $\left| -\frac{1}{2} - 2 - \left| \frac{3}{4} - 3 \right| \right|$

h) $\left| -\frac{1}{2} \right| - 2 - \frac{3}{4} - 3$

c) $-2^2(24 - 2(-3) - 5(-2)^2) - 12$

f) $\left| -\frac{1}{2} - 2 \left| \frac{3}{4} - 3 \right| \right|$

i) $\sqrt{3\sqrt{49} - \sqrt{25}}$

16. Completely factor each of the following over the integers.

a) $(3a + 1)^2 - 49$

c) $5x^6 - 80x^2$

e) $405 - 5x^4$

g) $(2x^2 - x + 2)^2 - (2x^2 + x - 4)^2$

b) $200x + 2x^3$

d) $x^{10} - 49$

f) $12x^2 - 110x + 2x^3$

17. If A is the set of all integers divisible by 5 and B is the set of all integers divisible by 3, what is $A \cap B$?

18. Evaluate $\frac{3xy + 2x^2 - 2y^2}{x + 2y}$ if

a) $x = \frac{1}{2}$ and $y = -3$

b) $x = -1$ and $y = -2$

c) $x = -6$ and $y = 3$

19. Simplify each of the following.

a) $(3x^2 - 5x + 8) + (-3x^2 - 5x + 12)$

f) $(2x - 3)(2x + 3)$

j) $\frac{x^2 - 25}{x^2 - 12x + 35}$

b) $(3x^2 - 5x + 8) - (-3x^2 - 5x + 12)$

g) $2(m - 3)^2 - (2m + 5)(m - 3)$

k) $\frac{3x - 8}{8 - 3x}$

c) $3(x - 5) - 4(x - 2)$

h) $(a - 4)(2a + 1) - (a - 3)^2$

d) $5(2a + 1) - 2(2 - a) - 7(a + 8)$

i) $(3x - 1)^3$

l) $\frac{3a + 24}{a^2 + 2a - 48}$

e) $(x - 5)(x - 2)$

20. Solve each of the following equations.

a) $-\frac{2}{5}x + \frac{1}{2} = \frac{1}{5}$

f) $20x^5 = 5x^3$

b) $\frac{3x + 1}{5} - \frac{2x - 4}{3} = 2x - 15$

g) $(2x - 5)^2 - (2x + 6)^2 = 4x - 11$

c) $x(3x - 1) - 2x - 3 = 3(x - 1)^2$

h) $60x - 20x^2 = 5x^3$

d) $\frac{2}{3}\left(x - \frac{1}{2}\right) - \frac{1}{2}\left(x + \frac{2}{3}\right) = \frac{1}{6}(x + 2)$

i) $x^4(x^2 + 1)(x^4 + 9) = 0$

j) $4(x - 3) - 3(x + 1) = 2x - 10 - (x + 5)$

e) $20x^5 = 5x^4$

k) $(2x - 1)^2 - (x - 2)^2 = 2x^2 + 1$

21. Solve each of the following compound inequalities.

a) $-2x - 3 < 7$ and $3x - 8 \geq 7$

b) $-2x - 3 < 7$ or $3x - 8 \geq 7$

c) $(x - 3)^2 \leq (x + 5)^2$ or $\frac{2}{3}x - \frac{5}{6} < \frac{1}{2}x - \frac{3}{4}$

d) $(x - 3)^2 \leq (x + 5)^2$ and $\frac{2}{3}x - \frac{5}{6} < \frac{1}{2}x - \frac{3}{4}$

e) $3(x - 2) - 5(6x - 1) \leq 3x - 1$ and $\frac{3x - 8}{4} < \frac{5x - 17}{3}$

$$\text{f) } 3(x-2) - 5(6x-1) \leq 3x-1 \text{ or } \frac{3x-8}{4} < \frac{5x-17}{3}$$

$$\text{g) } \frac{2x-3}{7} - \frac{3x-1}{2} \leq x-11 \text{ or } (3x-1)^2 + (x-1)^2 > 10x^2 + 6$$

$$\text{h) } \frac{2x-3}{7} - \frac{3x-1}{2} \leq x-11 \text{ and } (3x-1)^2 + (x-1)^2 > 10x^2 + 6$$

22. Solve each of the given system of linear equations.

$$\text{a) } \begin{cases} 2x+3y=-4 \\ 4x-5y=-30 \end{cases} \quad \text{b) } \begin{cases} -x+2y=19 \\ 3x+y=-1 \end{cases} \quad \text{c) } \begin{cases} (x-2)^2 + (y-1)^2 = x^2 + y^2 - 1 \\ (x-3)^2 - x^2 = (y+3)^2 - y^2 \end{cases}$$

23. Prove that each of the following numbers is rational by re-writing them as a fraction of two integers.

$$\text{a) } 0.\overline{304} = 0.304040404\dots \quad \text{b) } 7.\overline{2673} = 7.2673673673673 \quad \text{c) } 0.\overline{9} = 0.99999\dots$$

24. The opposite of a number is thirty-five more than the sum of -5 and the number. Find this number.

25. The sum of three consecutive numbers is 63. Find these numbers.

26. The shortest side of a right triangle is 12 units long. Find the length of the other two sides if we know that the difference between them is 2 units.

27. Find the sides of the square if we know the following. If we increased the length of each side by 1 feet, the area of the square would increase by 17ft^2 .

28. We have some \$5 bills and some \$10 bills. All together, there are 74 bills. How many \$5 bills do we have if the value of all bills is \$430?

29. A number is 42 less than its own square. Find this number.

30. Find all numbers with the following property: if we multiply the number by seven, the result is the original number.

31. The first row in a theater has 15 seats in it. The second row has three more seats than the first row. The third row has three more seats than the second row. And so on, each row has three more seats than the row before. If the last row has 81 seats in it, how many seats are there in the entire theater?

32. Find all numbers with the following property: if we square the number, the result is the original number.

33. We have a jar full of coins, all dimes and quarters. How many of each coins do we have if all together, we have 154 coins in the total value of \$24.70?

34. An arch is in the shape of a semicircle. At a point along the base 2 foot from an end of the arch, the height of the arch is 8 feet. Find the maximum height of the arch.

35. We throw a small object upward from the top of a 432ft tall building. The vertical position (or height) h of the object, (measured in feet) t seconds after we threw it is

$$h = -16t^2 + 96t + 432$$

How long does it take for the object to hit the ground?

36. One number is twelve less than another. Find these numbers if their product is 160.

37. There are 120 people in a room. If everyone shakes hands with everyone else, how many handshakes took place?

38. Find the exact value of a square based pyramid if all of its edges are 20 meters long.

39. List all subsets of $A = \{1, 2, 3, 4\}$.

40. List all two-element subsets of $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Chapter 17

17.1 Graphing a Parabola - 2

Until now, we have been graphing quadratic equations of the form $y = x^2 + bx + c$, with leading coefficient 1. Of course, not all quadratic expressions are like this. In this section, we will learn how to graph parabolas with a leading coefficient different from 1.

Example 1. Graph the parabola $y = x^2 - 6x + 5$ and the equation $y = 2(x^2 - 6x + 5)$ in the same coordinate system.

Solution: We will start with $y = x^2 - 6x + 5$. We first factor it by completing the square to obtain the coordinates of vertex and x -intercepts.

$$\begin{aligned} y &= x^2 - 6x + 5 \\ y &= \underbrace{x^2 - 6x + 9}_{(x-3)^2} - 9 + 5 \\ y &= (x-3)^2 - 4 \implies \text{vertex } V(3, -4) \\ y &= (x-3)^2 - 2^2 \\ y &= (x-3+2)(x-3-2) \\ y &= (x-1)(x-5) \\ \implies x\text{-intercepts } (1, 0) \text{ and } (5, 0) \end{aligned}$$

The standard form, $y = (x-3)^2 - 4$ tells us that the vertex is $(3, -4)$. Recall that the vertex of a parabola occurs when the complete square part is zero.

The factored form tells us where the x -intercepts are: $(1, 0)$ and $(5, 0)$.

We can obtain additional points by substituting any value of x into the equation $y = x^2 - 6x + 5$.

x	-1	0	1	2	3	4	5	6	7
y	12	5	0	-3	-4	-3	0	5	12

To graph the second equation, $y = 2(x^2 - 6x + 5)$, we can simply use the same points we just collected in the table. All we have to do is to multiply the y -coordinate by 2. For example, if $x = 4$, then $y = x^2 - 6x + 5 = -3$, and also $y = 2(x^2 - 6x + 5) = 2(-3) = -6$.

x	-1	0	1	2	3	4	5	6	7
$y = x^2 - 6x + 5$	12	5	0	-3	-4	-3	0	5	12
$y = 2(x^2 - 6x + 5)$	24	10	0	-6	-8	-6	0	10	24

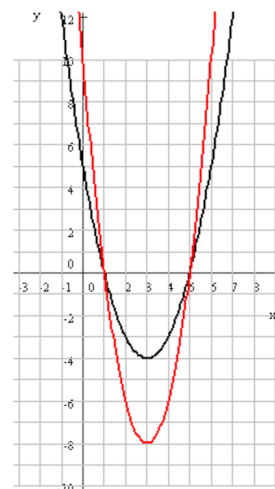
To obtain a point on the new graph, take any point on the parabola, keep the x -coordinate, and double the y -coordinate. We should repeat this process for every point on the original parabola.

The red graph is the new graph, it is also a parabola.

The x -intercepts remain the same. This makes sense: if we double zero, we do get zero again. Also, the equations $(x-1)(x-5) = 0$ and $2(x-1)(x-5) = 0$ have the same solutions.

The y -intercept was $(0, 5)$. The new y -intercept is $(0, 10)$.

The vertex is also affected by the multiplication. The x -coordinate is the same, but the y -coordinate is doubled. But if -4 is the lowest value of the expression $x^2 - 6x + 5$, then -8 will be the lowest value of $2(x^2 - 6x + 5)$.



Let us see how these properties show up in the algebra. Suppose we have to factor $y = 2x^2 - 12x + 10$.

$$y = 2x^2 - 12x + 10 \implies \text{the } y\text{-intercept is } (0, 10)$$

$$y = 2(x^2 - 6x + 5)$$

$$y = 2(x^2 - 6x + 9 - 9 + 5)$$

$$y = 2((x-3)^2 - 4) = \boxed{2(x-3)^2 - 8} \implies \text{the vertex is } (3, -8)$$

$$y = 2(x-3+2)(x-3-2)$$

$$y = 2(x-1)(x-5) \implies \text{the } x\text{-intercepts are } (1, 0) \text{ and } (5, 0)$$

The **standard form** of this parabola is $y = 2(x-3)^2 - 8$. This indicates that the vertex is at $(3, -8)$. Recall that the vertex is the point where the complete square part is zero. The smallest possible value of $(x-3)^2$ is zero, when $x = 3$. Then, if all the values are multiplied by 2, the smallest value is still zero. Then the smallest value of $2(x-3)^2 - 8$ is -8 .

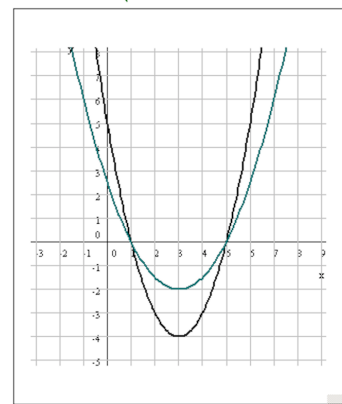
Example 2. Graph the given equations in the same coordinate system. $y = x^2 - 6x + 5$ and $y = \frac{1}{2}(x^2 - 6x + 5)$

Solution: We have already performed the computations and graphed the first equation. Let us then focus on the second equation. We can use the table of values to graph the equation, $y = \frac{1}{2}(x^2 - 6x + 5)$. This time we will multiply the y -coordinate of each point by $\frac{1}{2}$.

x	-1	0	1	2	3	4	5	6	7
$y = x^2 - 6x + 5$	12	5	0	-3	-4	-3	0	5	12
$y = \frac{1}{2}(x^2 - 6x + 5)$	6	$\frac{5}{2}$	0	$-\frac{3}{2}$	-2	$-\frac{3}{2}$	0	$\frac{5}{2}$	6

To obtain a point on the new graph, take any point on the parabola, keep the x -coordinate, and multiply the y -coordinate by $\frac{1}{2}$. We should repeat this process for every point on the original parabola.

The blue graph is the new graph, it is also a parabola. Again, the x -intercepts remain the same, and every other point is affected. The y -intercept, that used to be $(0, 5)$ is now $(0, \frac{5}{2})$. The vertex, that used to be $(3, -4)$ is now $(3, -2)$.



Example 3. Graph the given equations in the same coordinate system. $y = x^2 - 6x + 5$ and $y = -(x^2 - 6x + 5)$

Solution: We can use the table of values to graph the equation $y = -(x^2 - 6x + 5)$. This time we will multiply the y -coordinate of each point by -1 .

x	-1	0	1	2	3	4	5	6	7
$y = x^2 - 6x + 5$	12	5	0	-3	-4	-3	0	5	12
$y = -(x^2 - 6x + 5)$	-12	-5	0	3	4	3	0	-5	-12

$$y = -x^2 + 6x - 5 \implies \text{the } y\text{-intercept is } (0, -5)$$

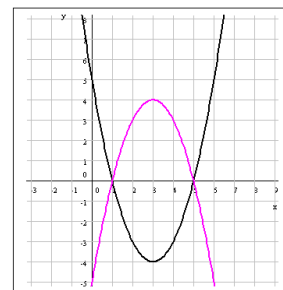
$$y = -(x^2 - 6x + 5)$$

$$y = -(x^2 - 6x + 9 - 9 + 5)$$

$$y = -((x-3)^2 - 4) = -(x-3)^2 + 4 \implies \text{the vertex is } (3, 4)$$

$$y = -(x-3+2)(x-3-2)$$

$$y = -(x-1)(x-5) \implies \text{the } x\text{-intercepts are } (1, 0) \text{ and } (5, 0)$$



Multiplication by -1 causes a reflection of the graph to the x -axis. We will call such a parabola a downward opening parabola. This time the vertex is the highest point on the graph. If we look at the standard form as $y = 4 - (x-3)^2$, we see that now we are *subtracting* a non-negative number from 4. We obtain the greatest value when we subtract the smallest value from 4.

Theorem: Suppose that b , and c are any real numbers and a is any non-zero real number.

Then the graph of $y = ax^2 + bx + c$ is a parabola.

If $a > 0$, the parabola opens upward and the vertex is the lowest point on the graph.

If $a < 0$, the parabola opens downward and the vertex is the highest point on the graph.

By completing the square, we can obtain the standard form of the parabola, which is $y = a(x-h)^2 + k$. The vertex of this parabola is (h, k) .

Example 4. Graph the parabola $y = 20x - 5x^2 + 60$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: We arrange the terms by degree to obtain the **polynomial form**, $y = -5x^2 + 20x + 60$. From the polynomial form, we obtain the y -intercept by substituting $x = 0$. If $x = 0$, then $y = 60$. And so the y -intercept is $(0, 60)$.

We factor out the leading coefficient and complete the square to obtain the standard form of the equation.

$$y = -5x^2 + 20x + 60$$

$$y = -5(x^2 - 4x - 12)$$

$$y = -5\left(\underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4 - 12\right)$$

$$y = -5((x-2)^2 - 16)$$

$$y = -5(x-2)^2 + 80$$

$$(x-2)^2 = x^2 - 4x + 4$$

distribute -5

The equation $y = -5(x-2)^2 + 80$ is the **standard form** of the equation of the parabola. It is the form we use to determine the vertex of the parabola. The vertex has to do with the complete square being zero. That happens when $x = 2$. And so the x -coordinate of the vertex is 2. And if the complete square is zero, then the y -coordinate of the vertex can be easily found: $y = -5(x-2)^2 + 80 = -5 \cdot 0 + 80 = 80$.

So the vertex is $(2, 80)$.

Since $(x-2)^2$ is always positive or zero, the expression $-5(x-2)^2$ is always negative or zero, and so now 80 is the greatest value that the expression $-5(x-2)^2 + 80$ achieves. We can look at the standard form as $80 - 5(x-2)^2$ where we *subtract* a non-negative quantity from 80. Indeed, **since the leading coefficient is negative, the parabola opens downward and so its vertex is where the greatest value of y is achieved.**

For the x -intercepts, we factor via the difference of squares theorem.

$$\begin{aligned} -5((x-2)^2 - 16) &= 0 \\ -5((x-2)^2 - 4^2) &= 0 \\ -5(x-2+4)(x-2-4) &= 0 \\ -5(x+2)(x-6) &= 0 \implies x_1 = -2 \quad x_2 = 6 \end{aligned}$$

Thus, there are two x -intercepts, $(-2, 0)$ and $(6, 0)$. We will compute a few more points before graphing the parabola. We will work with x -coordinates close to that of the vertex, and use the standard form of the parabola, $y = -5(x-2)^2 + 80$.

$$\text{if } x = -3, \text{ then } y = -5(-3-2)^2 + 80 = -45 \implies \text{found } (-3, -45)$$

$$\text{if } x = -1, \text{ then } y = -5(-1-2)^2 + 80 = 35 \implies \text{found } (-1, 35)$$

$$\text{if } x = 7, \text{ then } y = -5(7-2)^2 + 80 = -45 \implies \text{found } (7, -45)$$

$$\text{if } x = 1, \text{ then } y = -5(1-2)^2 + 80 = 75 \implies \text{found } (1, 75)$$

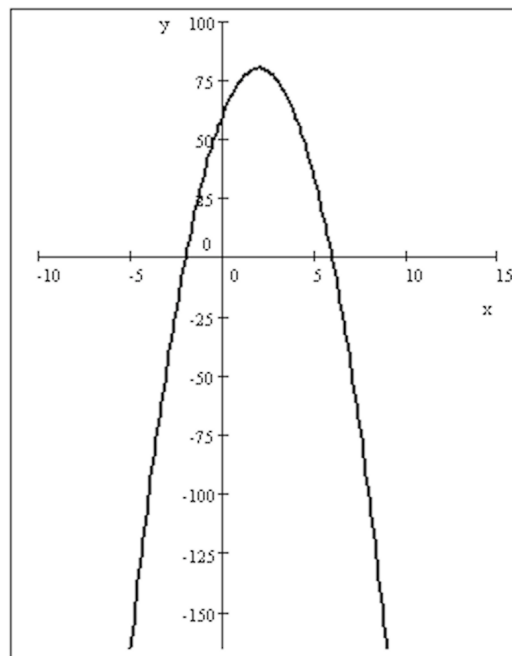
$$\text{if } x = 3, \text{ then } y = -5(3-2)^2 + 80 = 75 \implies \text{found } (3, 75)$$

$$\text{if } x = 4, \text{ then } y = -5(4-2)^2 + 80 = 60 \implies \text{found } (4, 60)$$

$$\text{if } x = 5, \text{ then } y = -5(5-2)^2 + 80 = 35 \implies \text{found } (5, 35)$$

We are ready to graph: we have the following points, listed left to right:

	$(-3, -45)$
x -intercept	$(-2, 0)$
	$(-1, 35)$
y -intercept	$(0, 60)$
	$(1, 75)$
vertex	$(2, 80)$
	$(3, 75)$
	$(4, 60)$
	$(5, 35)$
x -intercept	$(6, 0)$
	$(7, -45)$



Example 5. Graph the parabola $y = 6x^2 - 36x + 78$. Clearly label the coordinates of five points of the parabola, including vertex and intercepts.

Solution: Since the leading coefficient is positive, the graph will be an upward opening parabola. We first factor out the leading coefficient and then complete the square to obtain the standard form.

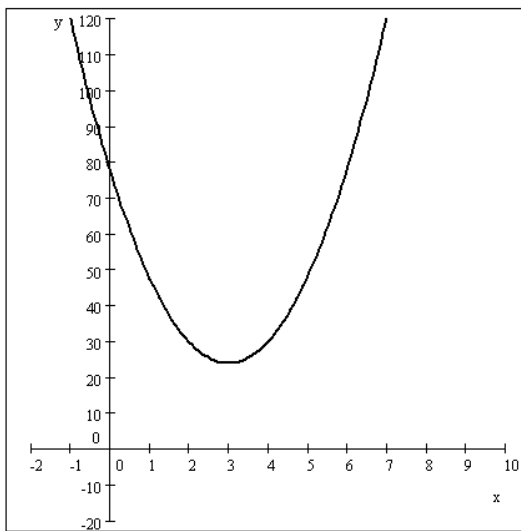
$$\begin{aligned}
 y &= 6x^2 - 36x + 78 && \implies y\text{-intercept: } (0, 78) \\
 y &= 6(x^2 - 6x + 13) && (x-3)^2 = x^2 - 6x + 9 \\
 y &= 6\left(\underbrace{x^2 - 6x + 9}_{(x-3)^2} - 9 + 13\right) \\
 y &= 6((x-3)^2 + 4) = 6(x-3)^2 + 24 && \implies \text{vertex: } (3, 24)
 \end{aligned}$$

This parabola has no x -intercepts. This is because it is an upward opening parabola with its vertex above the x -axis. Indeed, the expression $6(x-3)^2 + 24$ is always at least 24. Consequently, this expression is never zero.

Algebraically, this means that the factored form can not exist, because linear factors would certainly guarantee x -intercepts. Indeed, the expression $6(x-3)^2 + 24$ parabola does not have a factored form. This is because the sum of squares in the parentheses in $6((x-3)^2 + 4)$ can not be factored. Recall that **the sum of two squares can never be factored**.

We find a few more points close to the vertex and graph the parabola.

	$(-1, 120)$
y -intercept	$(0, 78)$
	$(1, 48)$
	$(2, 30)$
vertex	$(3, 24)$
	$(4, 30)$
	$(5, 48)$
	$(6, 120)$



With the introduction of quadratic expressions, we have discovered a new property: the existence of a smallest or greatest value of an expression. In real life, we can often describe situations using quadratic expressions. In such cases, it is natural to look for smallest or greatest values. For example, we want to find the smallest cost or smallest risk or the greatest profit. Problems in which we are looking for a greatest or smallest value (and not for a given number for value) are called **optimization** problems. Optimization will be a central topic in calculus, an important course in advanced mathematics. In case of quadratic expressions, completing the square enables us to find greatest and lowest values. In case of expressions of degree 3 or 4, or otherwise more complications, calculus develops methods to find these best values.

The following example is about a true to life formula about how objects move due to the gravitational force exerted on them by Earth.

Example 6. We are standing on the top of a 720ft tall building and throw a small object upward. The object's distance from the ground, measured in feet, after t seconds is

$$y = -16t^2 + 192t + 720$$

What is the highest point that the object reaches?

Solution: $y = -16t^2 + 192t + 720$ is quadratic, with a negative leading coefficient. Consequently, the graph is a downward opening parabola. The highest point is the vertex. We will find the vertex by completing the square.

$$\begin{aligned} y &= -16t^2 + 192t + 720 = -16(t^2 - 12t - 45) & (t-6)^2 &= t^2 - 12t + 36 \\ y &= -16(t^2 - 12t + 36 - 36 - 45) \\ y &= -16((t-6)^2 - 81) & \text{distribute } -16 \\ y &= -16(t-6)^2 + 1296 \end{aligned}$$

This indicates that the parabola has its vertex at $(6, 1296)$. Thus the highest point is reached 6 seconds after we threw the object, and it is at a height of 1296 feet.

Example 7. A company finds that if they price their product at \$60, they can sell 500 items of it. For every dollar increase in the price, the number of items sold will decrease by 5.

- Let x be the increase in price from \$60. Define the revenue function, R to be the sales revenue that results in such pricing. Find a formula for R .
- What price would guarantee an income of \$31 500?
- Find the price that guarantees the maximum revenue.
- Find the maximum revenue.

Solution: a) Let x be the increase in price from \$60. Define the revenue function, R to be the sales revenue (revenue is the same as income) that results in such pricing. Find a formula for R .

The revenue from sales of this product can be found by multiplying the price by the number of items sold. $R = (\text{Price}) \cdot (\text{Number of items sold})$

price	items sold	Revenue
60	500	30 000
61	495	30 195
62	490	30 380
$60 + x$	$500 - 5x$	$(60 + x)(500 - 5x)$

For example, if the price was \$60, then 500 items are sold and the revenue would be $\$60 \cdot 500 = \$30\,000$. If we increase the price to \$61, we can only sell 495 items. Then the revenue is $\$61 \cdot 495 = \$30\,195$. And so on, we should see a few examples to understand how the revenue works.

If we denote the change in the price from \$60, then the price is $60 + x$. With this price, we can sell $5x$ less items, that is, $500 - 5x$.

$$\begin{aligned} R &= (60 + x)(500 - 5x) & \text{rearrange} \\ &= (x + 60)(-5x + 500) & \text{factor out } -5 \\ &= -5(x + 60)(x - 100) \\ &= -5(x^2 - 100x + 60x - 6000) \\ &= -5(x^2 - 40x - 6000) \end{aligned}$$

So $\boxed{R = (x + 60)(-5x + 500) = -5x^2 + 200x + 30\,000}$.

- b) To determine the price that would guarantee an income of \$31 500, we need to solve the equation:
 $x = ?$ so that $R = 31\,500$.

$$\text{Solve } -5x^2 + 200x + 30\,000 = 31\,500 \text{ for } x$$

Since the equation is quadratic, we will reduce one side to zero. We will avoid a negative leading coefficient by reducing the left-hand side to zero.

$$\begin{aligned} -5x^2 + 200x + 30\,000 &= 31\,500 \\ 0 &= 5x^2 - 200x + 1500 \end{aligned}$$

We will factor by completing the square.

$$\begin{aligned} 5x^2 - 200x + 1500 &= 0 && \text{factor out 5} \\ 5(x^2 - 40x + 300) &= 0 && (x-20)^2 = x^2 - 40x + 400 \\ 5(\underbrace{x^2 - 40x + 400}_{(x-20)^2} - 400 + 300) &= 0 \\ 5((x-20)^2 - 100) &= 0 \\ 5((x-20)^2 - 10^2) &= 0 \\ 5(x-20+10)(x-20-10) &= 0 \\ 5(x-10)(x-30) &= 0 \implies x_1 = 10 \quad x_2 = 30 \end{aligned}$$

We check. If $x = 10$, the price is increased by \$10 and so it sells for \$70. We can then sell $500 - 5 \cdot 10 = 450$ items, and so the income is $R = 70 \cdot 450 = 31\,500$. On the other hand, if $x = 30$, the selling price is \$90. We then can sell $500 - 5 \cdot 30 = 350$ items, and so the revenue is $R = 90 \cdot 350 = 31\,500$. So, a price of both \$70 and \$90 would guarantee a revenue of \$31 500.

- c) To find the price that guarantees the maximum revenue, we need to find the greatest value of $-5x^2 + 200x + 30\,000$. Since $y = -5x^2 + 200x + 30\,000$ is clearly a downward turning parabola, its vertex is where the greatest value is taken. To find the vertex, we obtain the standard form by completing the square.

$$\begin{aligned} R &= -5x^2 + 200x + 30\,000 \\ &= -5(x^2 - 40x - 6000) && (x-20)^2 = x^2 - 40x + 400 \\ &= -5(\underbrace{x^2 - 40x + 400}_{(x-20)^2} - 400 - 6000) \\ &= -5((x-20)^2 - 6400) && \text{distribute } -5 \\ &= -5(x-20)^2 + 32\,000 \end{aligned}$$

The vertex of the parabola is $(20, 32\,000)$. Thus a \$20 increase in the price, implying a price of \$80, will guarantee a maximal income of \$32 000.

- d) The maximum revenue, (same as the greatest revenue), was found before, it is \$32 000.



Practice Problems

Graph each of the parabolas given below. In each case, clearly label the coordinates of five points of the parabola, including vertex and intercepts.

1. $y = 336 - 16x^2 - 64x$

4. $y = 6x - x^2$

7. $y = x(2 - x) - 1$

2. $y = 3x^2 - 30x + 75$

5. $y = 3x^2 - 12$

8. $y = -12x - 2x^2 - 10$

3. $y = 4x - 2x^2 - 20$

6. $y = 4x + \frac{1}{2}x^2 + 10$

9. Find the greatest or smallest value for each of the given expressions. In each case, also state whether you found a greatest or a smallest value.

a) $(x + 5)^2 - 49$

c) $2x^2 - 20x + 84$

e) $-2x^2 + 12$

g) $(20 - x)(x + 80)$

b) $-(2x - 3)^2 - 1$

d) $-x^2 - 6x$

f) $\frac{1}{3}x^2 - \frac{1}{2}x + 1$

10. We are standing on the top of a 720ft tall building and throw a small object upward. The object's distance from the ground, measured in feet, after t seconds is

$$y = -16t^2 + 64t + 720$$

What is the highest point that the object reaches?

11. A company finds that if they price their product at \$20, they can sell 3000 items of it. For every dollar increase in the price, the number of items sold will decrease by 10. Find the price that guarantees the maximum revenue. Find the maximum revenue possible.

17.2 Rational Exponents

Part 1 - The History Thus Far

Recall what we know about exponentiation thus far. Exponential notation was first defined to express repeated multiplication.

Definition: We define 2^7 to denote the factor 2 multiplied by itself repeatedly, such as

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{7 \text{ factors}} = 2^7$$

When mathematicians agreed to this definition, that was a free choice. Once this definition was accepted, certain properties are automatically true, and we had no other option but to recognize them as true. They just fell into our laps, or, as we say, they are consequence of the definition.

Theorem 1. If a is any number and m, n are any positive integers, then $a^n \cdot a^m = a^{n+m}$

Theorem 2. If a is any non-zero number and m, n are any positive integers, then $\frac{a^n}{a^m} = a^{n-m}$

Theorem 3. If a is any number and m, n are any positive integers, then $(a^n)^m = a^{nm}$

Theorem 4. If a, b are any numbers and n is any positive integer, then $(ab)^n = a^n b^n$

Theorem 5. If a, b are any numbers, $b \neq 0$, and n is any positive integer, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Consider the expression 2^x . The problem was that the definition of exponentiation only allows for a positive integer value of x . The expression 2^x was meaningful for $x = 2$ or 9 or 100, but not for values of x such as -3 or $\frac{3}{5}$ or 3.2. In short, the world of exponents was just the set of all natural numbers. Mathematicians usually don't like that. The best case scenario, the ultimate hope is that the definition of exponents could be extended to any number for x . That way, 2^x would be meaningful, no matter what the value of x is.

In the past, we had looked at how mathematicians extended exponential notation from the set of natural numbers ($\mathbb{N}$) to the set of all integers, ($\mathbb{Z}$). They defined zero- and negative exponent in such a way so that the rules listed above still worked. Recall that this principle is called the expansion principle.

Definition: Mathematicians often try to enlarge our world, in other words, to generalize definitions and rules to a larger set. The **expansion principle** is that when we enlarge our mathematics, we do so in such a way that the new definitions never create conflicts with the mathematics we already have.

As it turned out, the demand to preserve the rule of exponents allowed only one possible definition when we stepped out from $\mathbb{N}$ to $\mathbb{Z}$.

As we stepped out to integer exponents, we insisted on a definition that does not conflict with Rule 2, $\frac{a^n}{a^m} = a^{n-m}$.

Then there was only one possible choice for 2^0 .

$$2^0 = 2^{3-3} \stackrel{\text{rule 2}}{=} \frac{2^3}{2^3} = \frac{8}{8} = 1$$

The same idea gave us just one possible interpretation for 2^{-3} .

$$2^{-3} = 2^{1-4} \stackrel{\text{Rule 2}}{=} \frac{2^1}{2^4} = \frac{2}{16} = \frac{1}{8} = \frac{1}{2^3} \text{ or, more elegantly, } 2^{-3} = 2^{1-4} \stackrel{\text{Rule 2}}{=} \frac{2^1}{2^4} = \frac{-2}{-2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2^3}$$

So, here we are at this point: exponentiation is defined for all integer exponents, and

Theorem 1. If a is any number and m, n are any positive integers, then $a^n \cdot a^m = a^{n+m}$

Theorem 2. If a is any non-zero number and m, n are any positive integers, then $\frac{a^n}{a^m} = a^{n-m}$

Theorem 3. If a is any number and m, n are any positive integers, then $(a^n)^m = a^{nm}$

Theorem 4. If a, b are any numbers and n is any positive integer, then $(ab)^n = a^n b^n$

Theorem 5. If a, b are any numbers, $b \neq 0$, and n is any positive integer, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Theorem 6. If a is any non-zero number, then $a^0 = 1$.

0^0 is undefined.

Theorem 7. If a is any non-zero number, and n is any positive integer, then $a^{-n} = \frac{1}{a^n}$.

0^{-n} is undefined.

Next, mathematicians hoped to extend exponential notation for all rational numbers, and then perhaps for all real numbers. That way, the expression 2^x would be meaningful for every real number x .

Part 2 - Rational Exponents

Today we are going to see how exponential notation can be extended to rational numbers. The driving principle here was also the expansion principle, but this time with a focus on rule 3, $(a^n)^m = a^{nm}$. We also had to make some compromises in order to keep mathematical properties that are important for us.

Consider $5^{1/3}$. At this point, this expression has no meaning, as the definition, repeated multiplication can not be applied. We have the freedom to define $5^{1/3}$ any way we like. Let us denote $5^{1/3}$ by x for now.

$$5^{\frac{1}{3}} = x$$

Considering the rule $(a^n)^m = a^{nm}$, there is a good reason for us to want to raise $5^{1/3}$ to the third power.

$$\left(5^{\frac{1}{3}}\right)^3 = (5)^{\frac{1}{3} \cdot 3} = 5^1 = 5$$

This means that if we want to define $5^{1/3}$ in a way so that the rule $(a^n)^m = a^{nm}$ will still work, then $5^{1/3}$ must be a number that, when raised to third power, the result is 5.

$$\begin{aligned}
 5^{\frac{1}{3}} &= x && \text{raise to the 3rd power} \\
 \left(5^{\frac{1}{3}}\right)^3 &= x^3 \\
 5 &= x^3
 \end{aligned}$$

There is one number that does that, and we denote it by $\sqrt[3]{x}$. Recall the definition of $\sqrt[3]{x}$: it is the number, that, when raised to the third power, we get x .

Similarly, if we want to define $5^{1/2}$, the rule $(a^n)^m = a^{nm}$ demands that $5^{1/2}$ is a number that, when squared, the result is 5.

$$\left(5^{\frac{1}{2}}\right)^2 = (5)^{\frac{1}{2} \cdot 2} = 5^1 = 5$$

So, if we denote $5^{1/2}$ by y , we have that

$$\begin{aligned}
 5^{\frac{1}{2}} &= y \\
 \left(5^{\frac{1}{2}}\right)^2 &= y^2 \\
 5 &= y^2
 \end{aligned}$$

There are two such numbers, $\sqrt{5}$ and $-\sqrt{5}$. When we defined $\sqrt{5}$, we decided to define the positive candidate as $\sqrt{5}$. Similarly, it was not a difficult decision to go with the positive candidate, and so

$$5^{\frac{1}{2}} = \sqrt{5}$$

Both sides square to 5. The left-hand side by the rule $(a^n)^m = a^{nm}$, and the right-hand side by the definition of $\sqrt{5}$. So, we can extend exponential notation to rational numbers in the form of $a^{1/n}$ with the rule $(a^n)^m = a^{nm}$ in mind.

Theorem 8. If a is any number and n is any positive integer, then $a^{1/n} = \sqrt[n]{a}$.

There are some issues with this definition, but we already had them when we defined roots of numbers. Recall that $\sqrt{-9}$ and $\sqrt[4]{-16}$ and $\sqrt[6]{-1}$ are all expressions that can not be defined as real numbers, because any real number, positive or negative, will result in a positive number when raised to an *even* power. Therefore, $(-9)^{1/2}$ and $(-16)^{1/4}$ and $(-1)^{1/6}$ are all undefined.

Not all rational numbers are of the form $\frac{1}{n}$. How would we define something to the power of $\frac{2}{3}$? At first sight, this does not seem to be a difficult task. We could interpret $\frac{2}{3}$ as $\frac{1}{3} \cdot 2$, and then, using the same rule $(a^n)^m = a^{nm}$, we could interpret it as two different exponentiation. Consider, for example, $8^{2/3}$.

$$8^{2/3} = 8^{(1/3) \cdot 2} = \left(8^{1/3}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4 \quad \text{or} \quad 8^{2/3} = 8^{2 \cdot (1/3)} = (8^2)^{1/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

Based on the computation above, it appears that extending exponential notation to fractions is complete, with the definition $a^{n/m} = \left\{ \begin{array}{l} \sqrt[m]{a^n} \\ (\sqrt[m]{a})^n \end{array} \right.$. This is not the case. We are facing a new issue that will force us to make more

difficult choices than before.

Caution! Exponential notation was defined in *several different ways*. In advanced calculus, there will be a completely different approach to exponential notation, resulting in a different definition. The different definitions usually agree in the results, but not always. Occasionally, the different definitions lead to different results!

Here is the problem. Consider $(-8)^{4/6}$. The two computations shown in case of $8^{2/3}$ will not work the same way.

$$(-8)^{4/6} = \begin{cases} (-8)^{4 \cdot (1/6)} = ((-8)^4)^{1/6} = \sqrt[6]{(-8)^4} = \sqrt[6]{4096} = 4 \\ \text{or} \\ (-8)^{(1/6) \cdot 4} = (\sqrt[6]{-8})^4 = \text{undefined because } \sqrt[6]{-8} \text{ is undefined} \end{cases}$$

This is a troubling situation. If we interpret $\frac{4}{6}$ as $4 \cdot \frac{1}{6}$, we get a different result from when interpreting $\frac{4}{6}$ as $\frac{1}{6} \cdot 4$. What has happened here, that in case of a non-reduced fraction, we lose our grip on handling negative bases. There is no perfect way out of this mess.

We could define $a^{n/m}$ in such a way that we restrict the order between exponentiation $()^n$ and taking the m th root so that we only allow one order. We could also demand that fractions be always reduced when in exponents.

In intermediate algebra, we will follow a different route. We want to keep the order between exponentiation and taking the root flexible. We also demand that fractions behave the same way, whether they are in lowest terms or not. So, we simply gave up on defining fractions as exponents with negative bases. When the base is negative, the only fraction exponent allowed is of the form $\frac{1}{n}$.

Theorem 9. Suppose that m and n are integers with $m \neq 0$ and $n \neq 1$.

If a is negative, then $a^{n/m}$ is undefined. If a is non-negative, then $a^{n/m} = \begin{cases} \sqrt[m]{a^n} \\ (\sqrt[m]{a})^n \end{cases}$.

This is the first example we saw that expanding our world came with a price. As we extended exponential notation from the set of all integers ($\mathbb{Z}$) to the set of all rational numbers ($\mathbb{Q}$), we had to give up something. This definition results in giving up on negative bases (with the exception of exponent $\frac{1}{n}$). Other definitions of exponentiation are based on different decisions.

Part 3 - Applications

Example 1. Evaluate each of the given expressions. Present your answer using only positive integer exponents.

a) $16^{3/4}$ b) $-25^{1/2}$ c) $(-25)^{1/2}$ d) $(-8)^{1/3}$ e) $(-27)^{2/3}$ f) $27^{-2/3}$ g) $(\sqrt{8})^{2/3}$

Solution: a) $16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = \boxed{8}$

b) Recall the difference between -2^2 and $(-2)^2$. In the first case -2^2 , we are not squaring -2 . The base is 2. We can interpret -2^2 as $-1 \cdot 2^2$. So, in the case of $-25^{1/2}$, the base is 25.

$$-25^{1/2} = -1 \cdot 25^{1/2} = -1 \cdot \sqrt{25} = \boxed{-5}.$$

c) $(-25)^{1/2} = \sqrt{-25} = \boxed{\text{undefined}}$ because no real number has a negative square.

d) $(-8)^{1/3} = \sqrt[3]{-8} = \boxed{-2}$

e) $(-27)^{2/3} = \boxed{\text{undefined}}$ because when the base is negative, the only fraction allowed is of the form $\frac{1}{n}$.

f) $27^{-2/3} = \frac{1}{27^{2/3}} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{3^2} = \boxed{\frac{1}{9}}$

g) $(\sqrt{8})^{2/3} = \left((\sqrt{8})^2\right)^{1/3} = 8^{1/3} = \boxed{2}$

$$\text{or } (\sqrt{8})^{2/3} = (8^{1/2})^{2/3} = 8^{(1/2) \cdot (2/3)} = 8^{1/3} = \sqrt[3]{8} = 2$$

Notice that we have the free choice to decide the order in which we perform the exponentiation and taking the root. On a case by case basis, one method is preferred. For example, in the case of $27^{2/3}$, we took the 3rd root first, and then we squared. This is because the third root is nice (i.e. integer) and the other order, while also correct, would make the numbers unnecessarily large. In the last example, it was more advantageous to first exponentiate, and then take the root.

Whenever we deal with complicated exponents, we need to keep in mind that when rational exponents were defined, the goal was to preserve rules of exponents. So, we can apply all rules (1-9) to these exponents.

Example 2. Simplify each of the given expressions. Present your answer using only positive integer exponents.

a) $5^{1/3} \cdot 5^{1/2}$ b) $\frac{8^{7/6}}{8^{1/2}}$ c) $(64^{2/3})^{1/2}$ d) $2^{1/2} \cdot 18^{1/2}$ e) $\frac{96^{3/4}}{6^{3/4}}$

Solution: a) We will apply the first rule of exponents, $a^n \cdot a^m = a^{n+m}$.

$$5^{1/3} \cdot 5^{1/2} = 5^{1/3+1/2} = 5^{5/6} \quad \text{as } \frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6} \quad (\text{printed computations look terrible in the exponents})$$

b) We will apply the second rule of exponents, $\frac{a^n}{a^m} = a^{n-m}$.

$$\frac{8^{7/6}}{8^{1/2}} = 8^{7/6-1/2} = 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4 \quad \text{since } \frac{7}{6} - \frac{1}{2} = \frac{7-3}{6} = \frac{4}{6} = \frac{2}{3}$$

c) We will apply the third rule of exponents, $(a^n)^m = a^{nm}$.

$$(64^{2/3})^{1/2} = 64^{(2/3) \cdot (1/2)} = 64^{1/3} = \sqrt[3]{64} = 4 \quad \text{since } \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

d) We will apply the fourth rule of exponents, $(ab)^n = a^n b^n$.

$$2^{1/2} \cdot 18^{1/2} = (2 \cdot 18)^{1/2} = 36^{1/2} = \sqrt{36} = 6$$

e) We will apply the fifth rule of exponents, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

$$\frac{96^{3/4}}{6^{3/4}} = \left(\frac{96}{6}\right)^{3/4} = \left(\frac{48}{3}\right)^{3/4} = 16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$$

Example 3. Simplify each of the given expressions. Present your answer using only positive integer exponents. Assume that all variables represent positive numbers.

a) $\frac{x^{1/2}x^{2/3}}{x^{1/6}}$ b) $\left(\frac{p^{2/3}p^{5/3}}{(-p^{1/6})^2}\right)^{-1/2}$ c) $\left(\frac{x^{2/3}y^{-1/6}}{x^{-1/2}y^{-1/3}}\right)^{1/2}$

Solution: a) $\frac{x^{1/2}x^{2/3}}{x^{1/6}} = \frac{x^{1/2+2/3}}{x^{1/6}} = \frac{x^{7/6}}{x^{1/6}} = x^{7/6-1/6} = x^1 = \boxed{x}$ as $\frac{1}{2} + \frac{2}{3} = \frac{3+4}{6} = \frac{7}{6}$

b) We simplify the numerator and denominator first.

$$\left(\frac{p^{2/3}p^{5/3}}{(-p^{1/6})^2}\right)^{-1/2} = \left(\frac{p^{2/3+5/3}}{p^{2/6}}\right)^{-1/2} = \left(\frac{p^{7/3}}{p^{1/3}}\right)^{-1/2}$$

We now perform the division.

$$\left(\frac{p^{7/3}}{p^{1/3}}\right)^{-1/2} = (p^{7/3-1/3})^{-1/2} = (p^{6/3})^{-1/2} = (p^2)^{-1/2}$$

We now simplify the exponent.

$$(p^2)^{-1/2} = p^{2 \cdot (-1/2)} = p^{-1} = \boxed{\frac{1}{p}}$$

b) We first apply the rule $\frac{a^n}{a^m} = a^{n-m}$.

$$\left(\frac{x^{2/3}y^{-1/6}}{x^{-1/2}y^{-1/3}}\right)^{1/2} = (x^{2/3-(-1/2)}y^{-1/6-(-1/3)})^{1/2} = (x^{7/6}y^{1/6})^{1/2}$$

$$\text{because } \frac{2}{3} - \left(-\frac{1}{2}\right) = \frac{4+3}{6} = \frac{7}{6} \quad \text{and} \quad -\frac{1}{6} - \left(-\frac{1}{3}\right) = \frac{-1+2}{6} = \frac{1}{6}$$

We will further simplify the expression using the rules $(ab)^n = a^n b^n$ and $(a^n)^m = a^{nm}$.

$$(x^{7/6}y^{1/6})^{1/2} = (x^{7/6})^{1/2} (y^{1/6})^{1/2} = x^{(7/6) \cdot (1/2)} y^{(1/6) \cdot (1/2)} = x^{7/12} y^{1/12}$$

We need to present this expression using only positive integer exponents. This will take a few more steps.

$$x^{7/12} y^{1/12} = (x^7)^{1/12} (y)^{1/12} = (x^7 y)^{1/12} = \boxed{\sqrt[12]{x^7 y}}$$

Rational exponents are also useful in simplifying complicated radical expressions.

Example 4. Simplify each of the given expressions. Present your answer using only positive integer exponents. Assume that x represents a positive number.

a) $\sqrt{x}\sqrt[3]{x}\sqrt[4]{x^3}$ b) $\sqrt{x\sqrt[3]{x^2}\sqrt[4]{x^{-3}}}$ c) $\sqrt[18]{x^{12}}$

Solution: a) We first re-write the radicals as exponents. $\sqrt{x} = x^{1/2}$ and $\sqrt[3]{x} = x^{1/3}$ and $\sqrt[4]{x^3} = x^{3/4}$.

$$\sqrt{x}\sqrt[3]{x}\sqrt[4]{x^3} = x^{1/2}x^{1/3}x^{3/4} = x^{(1/2)+(1/3)+(3/4)} = x^{19/12} = \boxed{\sqrt[12]{x^{19}}} \text{ as } \frac{1}{2} + \frac{1}{3} + \frac{3}{4} = \frac{6+4+9}{12} = \frac{19}{12}$$

b) We re-write the radicals as exponents.

$$\begin{aligned} \sqrt{x\sqrt[3]{x^2}\sqrt[4]{x^{-3}}} &= \left(x(x^2x^{-3/4})^{1/3}\right)^{1/2} & 2 - \frac{3}{4} &= \frac{8-3}{4} = \frac{5}{4} \\ &= \left(x(x^{5/4})^{1/3}\right)^{1/2} = (x \cdot x^{(5/4) \cdot (1/3)})^{1/2} & \frac{5}{4} \cdot \frac{1}{3} &= \frac{5}{12} \text{ and } 1 + \frac{5}{12} = \frac{17}{12} \\ &= (x \cdot x^{5/12})^{1/2} = (x^{17/12})^{1/2} = x^{(17/12) \cdot (1/2)} = x^{17/24} = \boxed{\sqrt[24]{x^{17}}} \end{aligned}$$

c) $\sqrt[18]{x^{12}} = x^{12/18} = x^{2/3} = \boxed{\sqrt[3]{x^2}}$



Practice Problems

Simplify each of the following expressions.

- | | | | | | |
|-----------------|-----------------|-------------------|--------------------|-------------------|--------------------|
| 1. $9^{1/2}$ | 6. $9^{3/2}$ | 11. $5^{1/2}$ | 16. $36^{-1/2}$ | 21. $-4^{-3/2}$ | 26. $(-16)^{1/2}$ |
| 2. $9^{-1/2}$ | 7. $8^{1/3}$ | 12. $-4^{-5/2}$ | 17. $-36^{1/2}$ | 22. $(-4)^{-3/2}$ | 27. $8^{-2/3}$ |
| 3. $-9^{1/2}$ | 8. $8^{-1/3}$ | 13. $(-32)^{1/5}$ | 18. $(-36)^{1/2}$ | 23. $(-8)^{1/3}$ | 28. $(32)^{-4/5}$ |
| 4. $-9^{-1/2}$ | 9. $(-8)^{1/3}$ | 14. $32^{2/5}$ | 19. $(-36)^{-1/2}$ | 24. $(-8)^{2/3}$ | 29. $(-32)^{-4/5}$ |
| 5. $(-9)^{1/2}$ | 10. $16^{-3/4}$ | 15. $36^{1/2}$ | 20. $36^{-3/2}$ | 25. $-16^{1/2}$ | 30. $81^{-3/4}$ |

Simplify each of the following. Present your final answers using positive exponents only. For example,

$x^{-2/3} = \frac{1}{\sqrt[3]{x^2}}$. Assume that all variables represent positive numbers.

- | | | | |
|--|--|--|---|
| 31. $(x^{1/3})^6$ | 34. $\frac{(m^{1/2})(m^{3/2})}{m^{-3}}$ | 36. $(-8a^6b^{-24})^{1/3}$ | 39. $(A^{2/3})^{3/2}$ |
| 32. $\frac{x^{2/5}x^{-3/4}}{x^{-1/2}}$ | 35. $\sqrt[3]{\frac{x^2y^{-1}}{(-x^4y^{-2})^2}}$ | 37. $\left(\frac{16x^4y^{2/3}}{36x^2y^2}\right)^{\frac{3}{2}}$ | 40. $\frac{(y^{-3/4})^{-8}}{(y^{-1/2})^{10}}$ |
| 33. $(a^{-1/2})^{-4/5}$ | | 38. $(-3x^{-1/2}y^{3/8}x^{-7/2})^0$ | |

Use exponential notation to simplify each of the following. Present your answer using positive integers only.

41. $\sqrt[8]{x^6}$

43. $\left(\sqrt[6]{2}\right)^{18}$

45. $\frac{\sqrt[5]{x}\sqrt[3]{x^2}}{x\sqrt[6]{x}}$

42. $\left(\sqrt[6]{y}\right)^9$

44. $(\sqrt{x})(\sqrt[3]{x})$

46. $\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{x}}}}}$

Problem Set 17

Chapter 18

18.1 The Quadratic Formula

Part 1 - Deriving the Formula

Let $ax^2 + bx + c = 0$ be a quadratic equation, where $a \neq 0$. We will solve this equation by completing the square. We first factor out the leading coefficient, a .

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= 0 \end{aligned}$$

Half of the linear coefficient is $\frac{b}{2a}$ and so the complete square we need is

$$\left(x + \frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = x^2 + \frac{b}{2a}x + \frac{b}{2a}x + \frac{b^2}{4a^2} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

Now we "smuggle in" the last missing term of the complete square, $\frac{b^2}{4a^2}$

$$\begin{aligned} a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= 0 \\ a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) &= 0 \\ a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right) &= 0 \end{aligned}$$

We bring the fractions after the complete square to the common denominator, and factor out -1

$$\begin{aligned} a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} - \frac{-c \cdot 4a}{a \cdot 4a}\right) &= 0 \\ a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} - \frac{-4ac}{4a^2}\right) &= 0 \\ a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) &= 0 \end{aligned}$$

We prepare to factor via the difference of squares theorem. If $b^2 - 4ac$ is negative, then the expression $-\frac{b^2 - 4ac}{4a^2}$ is positive and the equation is

$$a \left(\left(x + \frac{b}{2a} \right)^2 + p \right) = 0 \quad \text{for some positive } p$$

Then this equation has no real solution.

If $b^2 - 4ac$ is positive or zero, then we can factor via the difference of squares theorem

$$a \left(\left(x + \frac{b}{2a} \right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}} \right)^2 \right) = 0$$

The second expression can be simplified since

$$\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{|2a|}$$

We will further simplify this and write $2a$ instead of $|2a|$. We factor via the difference of squares theorem

$$\begin{aligned} a \left(\left(x + \frac{b}{2a} \right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}} \right)^2 \right) &= 0 \\ a \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \right) \left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right) &= 0 \end{aligned}$$

We apply the zero product rule and solve for x in each of the two equations.

$$\begin{aligned} x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} &= 0 & \text{or} & & x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} &= 0 \\ x + \frac{\sqrt{b^2 - 4ac}}{2a} &= -\frac{b}{2a} & & & x - \frac{\sqrt{b^2 - 4ac}}{2a} &= -\frac{b}{2a} \\ x &= -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} & & & x &= -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} & & & x &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Theorem: (The quadratic formula) Suppose that a , b , and c are real numbers, where $a \neq 0$. Consider the quadratic equation

$$ax^2 + bx + c = 0$$

If $b^2 - 4ac$ is negative, the equation has no real solution. If $b^2 - 4ac$ is non-negative, then the solution(s) of the quadratic equation are x_1 and x_2 and they can be computed by the formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ is called the **discriminant**. When the discriminant is positive, the equation has two different real solutions. When the discriminant is zero, the equation has exactly one real solution. When the discriminant is negative, the equation has no real solution.

Example 1. Solve $2x^2 - 9x = 5$ over the real numbers.

Solution: We first reduce one side to zero: $2x^2 - 9x - 5 = 0$. Now we can apply the quadratic formula.

$a = 2$, $b = -9$, and $c = -5$

$$\begin{aligned} x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2} = \frac{9 \pm \sqrt{81 - (-40)}}{4} = \frac{9 \pm \sqrt{81 + 40}}{4} \\ &= \frac{9 \pm \sqrt{121}}{4} = \frac{9 \pm 11}{4} = \begin{cases} \frac{9+11}{4} = \frac{20}{4} = 5 \\ \frac{9-11}{4} = \frac{-2}{4} = -\frac{1}{2} \end{cases} \end{aligned}$$

So this equation has two real solutions, $\boxed{5 \text{ and } -\frac{1}{2}}$. We check: if $x = 5$, then

$$\text{LHS} = 2(5)^2 - 9(5) = 50 - 45 = 5 = \text{RHS}$$

and if $x = -\frac{1}{2}$, then $\text{LHS} = 2\left(-\frac{1}{2}\right)^2 - 9\left(-\frac{1}{2}\right) = 2 \cdot \frac{1}{4} + \frac{9}{2} = \frac{1}{2} + \frac{9}{2} = \frac{10}{2} = 5 = \text{RHS}$ and so both solutions are correct.

Example 2. Solve $x^2 - 7x = -10 - x$ over the real numbers.

Solution: We first reduce one side to zero.

$$\begin{aligned} x^2 - 7x &= -10 - x && \text{add } x \\ x^2 - 6x &= -10 && \text{add } 10 \\ x^2 - 6x + 10 &= 0 \end{aligned}$$

Now we can apply the quadratic formula with values $a = 1$, $b = -6$, and $c = 10$.

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm \sqrt{-4}}{2} = \text{undefined}$$

We now see that the discriminant is negative. This means that the equation has $\boxed{\text{no real solutions}}$.

Example 3. Solve $2x^2 - 10x + 26 = 6x$ over the real numbers.

Solution: Notice that each term is even, and so we can divide both sides by 2.

$$\begin{aligned} 2x^2 - 10x + 26 &= 6x && \text{subtract } 6x \\ 2x^2 - 16x + 26 &= 0 \\ 2(x^2 - 8x + 13) &= 0 \\ x^2 - 8x + 13 &= 0 \end{aligned}$$

Now we can apply the quadratic formula with values $a = 1$, $b = -8$, and $c = 13$.

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} = \frac{8 \pm \sqrt{64 - 52}}{2} = \frac{8 \pm \sqrt{12}}{2}$$

However, this is not the most simplified form of the solution. Recall that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ for all non-negative real numbers a and b . Using this rule, we obtain that $\sqrt{12} = 2\sqrt{3}$ as follows.

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$$

The solution can be further simplified:

$$x_{1,2} = \frac{8 \pm \sqrt{12}}{2} = \frac{8 \pm 2\sqrt{3}}{2} = \frac{2(4 \pm \sqrt{3})}{2} = 4 \pm \sqrt{3}$$

This of course means that $x_1 = 4 + \sqrt{3}$ and $x_2 = 4 - \sqrt{3}$.

We check: if $x = 4 + \sqrt{3}$, then

$$\begin{aligned} \text{LHS} &= 2(4 + \sqrt{3})^2 - 10(4 + \sqrt{3}) + 26 = 2(16 + 3 + 8\sqrt{3}) - 10(4 + \sqrt{3}) + 26 \\ &= 2(9 + 8\sqrt{3}) - 10(4 + \sqrt{3}) + 26 = 18 + 16\sqrt{3} - 40 - 10\sqrt{3} + 26 = 24 + 6\sqrt{3} \\ \text{RHS} &= 6(4 + \sqrt{3}) = 24 + 6\sqrt{3} \end{aligned}$$

and so $x_1 = 4 + \sqrt{3}$ works. And if $x = 4 - \sqrt{3}$, then

$$\begin{aligned} \text{LHS} &= 2(4 - \sqrt{3})^2 - 10(4 - \sqrt{3}) + 26 = 2(16 + 3 - 8\sqrt{3}) - 10(4 - \sqrt{3}) + 26 \\ &= 2(9 - 8\sqrt{3}) - 10(4 - \sqrt{3}) + 26 = 18 - 16\sqrt{3} - 40 + 10\sqrt{3} + 26 = 24 - 6\sqrt{3} \\ \text{RHS} &= 6(4 - \sqrt{3}) = 24 - 6\sqrt{3} \end{aligned}$$

and so $x_2 = 4 - \sqrt{3}$ also works.

Example 4. Solve $9x^2 + 1 = 6x$ over the real numbers.

Solution:

$$\begin{aligned} 9x^2 + 1 &= 6x \\ 9x^2 - 6x + 1 &= 0 \end{aligned}$$

We apply the quadratic formula with values $a = 9$, $b = -6$, and $c = 1$.

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 9 \cdot 1}}{2 \cdot 9} = \frac{6 \pm \sqrt{36 - 36}}{18} = \frac{6 \pm \sqrt{0}}{18} = \frac{6}{18} = \frac{1}{3}$$

So this equation has only one real solution: $x = \frac{1}{3}$. We check: if $x = \frac{1}{3}$, then

$$\text{LHS} = 9\left(\frac{1}{3}\right)^2 + 1 = 9 \cdot \frac{1}{9} + 1 = 2 \text{ and } \text{RHS} = 6 \cdot \frac{1}{3} = \frac{6}{3} = 2$$

And so our solution is correct.

Example 5. In each of the given equations, use the discriminant to determine the number of real solutions without solving the equation.

- a) $5x^2 - 3x + 1 = 0$ c) $\frac{1}{2}x^2 - x + 1 = 0$ d) $8x^2 + \frac{1}{2} = 4x$ f) $13x + 3x^2 = 56$
 b) $-2x^2 + 7x + 2 = 0$ e) $3x^2 - 12x + 87 = 0$ g) $9x^2 - 30x + 25 = 0$

Solution: We will compute the value of the discriminant, $D = b^2 - 4ac$. If the discriminant is positive, there are two positive solutions. If it is zero, there is exactly one real solution. If it is negative, there is no real solution.

- a) $5x^2 - 3x + 1 = 0$. In this case, $a = 5$, $b = -3$, and $c = 1$.

$$D = b^2 - 4ac = (-3)^2 - 4 \cdot 5 \cdot 1 = 9 - 20 = -11$$

Since the discriminant is negative, there is no real solution for this equation.

- b) $-2x^2 + 7x + 2 = 0$. In this case, $a = -2$, $b = 7$, and $c = 2$.

$$D = b^2 - 4ac = 7^2 - 4(-2)2 = 49 - (-16) = 65$$

Since the discriminant is positive, there are two real solutions for this equation.

- c) $\frac{1}{2}x^2 - x + 1 = 0$. In this case, $a = \frac{1}{2}$, $b = -1$, and $c = 1$.

$$D = b^2 - 4ac = \left(\frac{1}{2}\right)^2 - 4 \cdot \frac{1}{2} \cdot 1 = \frac{1}{4} - 2 = \frac{1}{4} - \frac{8}{4} = -\frac{7}{4}$$

Since the discriminant is negative, there is no real solution for this equation.

- d) $8x^2 + \frac{1}{2} = 4x$

Solution: Caution! We can not apply the quadratic formula if the equation is in this form. Remember, one side must be zero.

$$\begin{aligned} 8x^2 + \frac{1}{2} &= 4x && \text{subtract } 4x \\ 8x^2 - 4x + \frac{1}{2} &= 0 \end{aligned}$$

And so in this case, $a = 8$, $b = -4$, and $c = \frac{1}{2}$.

$$D = b^2 - 4ac = (-4)^2 - 4 \cdot 8 \cdot \frac{1}{2} = 16 - 16 = 0$$

Since the discriminant is zero, there is exactly one real solution for this equation.

e) $3x^2 - 12x + 87 = 0$

At first it appears that $a = 3$, $b = -12$, and $c = 87$. We might notice that all of a , b , and c are divisible by 3. It might make things easier if we divide both sides of the equation by 3 before applying the quadratic formula.

$$\begin{aligned} 3x^2 - 12x + 87 &= 0 \\ 3(x^2 - 4x + 29) &= 0 && \text{divide by 3} \\ x^2 - 4x + 29 &= 0 \end{aligned}$$

So now $a = 1$, $b = -4$, and $c = 29$.

$$D = b^2 - 4ac = (-4)^2 - 4 \cdot 1 \cdot 29 = 16 - 116 = -100$$

Since the discriminant is negative, there is no real solution for this equation.

f) $13x + 3x^2 = 56$

We first reduce one side to zero. $3x^2 + 13x - 56 = 0$. So $a = 3$, $b = 13$, and $c = -56$.

$$D = b^2 - 4ac = 13^2 - 4 \cdot 3 \cdot (-56) = 169 - (-672) = 169 + 672 = 841$$

Since the discriminant is positive, there are two real solutions for this equation.

g) $9x^2 - 30x + 25 = 0$

$a = 9$, $b = -30$, and $c = 25$.

$$D = b^2 - 4ac = 30^2 - 4 \cdot 9 \cdot 25 = 900 - 900 = 0$$

Since the discriminant is zero, there is exactly one real solution for this equation.



Practice Problems

1. Use the discriminant to determine how many real solutions are there for each of the given equation. You do not have to find the solutions.

a) $-3x^2 + 2x - 8 = 0$	c) $0.1x^2 + x + 1 = 0$	e) $x^2 - 6x = -7$	g) $-2x^2 - 5x + 10 = 0$
b) $5x^2 - 1 = 2x$	d) $\frac{4}{9}x^2 - \frac{4}{3}x + 1 = 0$	f) $x^2 - 8x + 20 = 0$	

2. Use the quadratic formula to solve each of the following equations over the real numbers.

a) $6x^2 - x + 9 = 0$	c) $9x^2 - 12x = 1$	e) $-2x^2 + 5x - 4 = 0$	g) $\frac{16}{25}x^2 - \frac{16}{5}x + 4 = 0$
b) $6x + 2x^2 + 1 = x + 4$	d) $x + 40x^2 = 15$	f) $x^2 + x = 1$	

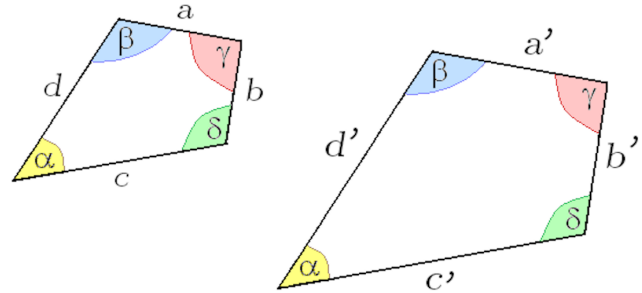
3. Find all positive numbers with the property that it is one greater than its own reciprocal.

18.2 Similar Triangles

Similarity is a very important relation used in geometry.

Definition: Two objects are **similar** to each other if they are either identical (also called congruent) or one is an enlargement of the other.

Similar objects always have the following properties: Their corresponding angles are the same, and their corresponding angles are proportional. The two quadrilaterals shown are similar. When labeling the sides, we often suggest which side corresponds to which side. The side corresponding to a is often denoted by a' .



The corresponding sides are proportional. This can be approached in two different ways as follows.

1. We can express how one quadrilateral was enlarged to obtain the other. In this case, every side was doubled. We can express this magnifying factor as the same for all pairs of corresponding sides between the two quadrilaterals.

$$\mu = \frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{d'}{d}$$

2. We can also express that from one quadrilateral to the other, the ratio between corresponding sides was preserved.

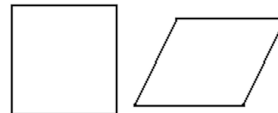
$$\frac{a}{b} = \frac{a'}{b'} \text{ and } \frac{a}{c} = \frac{a'}{c'} \text{ and } \frac{b}{c} = \frac{b'}{c'}$$

The two approaches express the same thing. Consider for example $\frac{a'}{a} = \frac{b'}{b}$. If we clear the denominators by multiplying both sides by ab , then we get $a'b = ab'$. If we start with $\frac{a}{b} = \frac{a'}{b'}$ and clear the denominators, we also get $ab' = a'b$. Consequently, we are free to express similarity in either way. We often select the form that will lead to easier computation. Most of the time all quantities involved are all positive. This means that we can take the reciprocal of both sides of an equation expressing these ratios without the need to worry about division by zero.

In case of quadrilaterals, they are similar to each other if their sides are proportional **and** have the same angles. Proportional sides alone or equal angles alone can not guarantee similarity.



The square and the rectangle clearly have the same angles and yet the two are not similar. Any enlargement of a square must be a square.



The picture shows a square and a parallelogram with all four of its sides of the same length. Such a quadrilateral is called a *rhombus*. The two are also not similar although they have the same side lengths.

In case of quadrilaterals, to establish similarity, we must prove that the angles are the same and that sides are proportional. Triangles are somehow simpler than quadrilaterals. If we know that two triangles have the same angles, that alone is enough for the two to be similar. If they are similar, then corresponding sides are proportional, therefore we can solve for sides using ratios. This is by far the most frequent use of similarity. We chase down angles to prove similarity so that we can solve for sides.

If two triangles have proportional sides, then that alone implies similarity. So now we can conclude that the angles are the same. There are three cases of similarity.

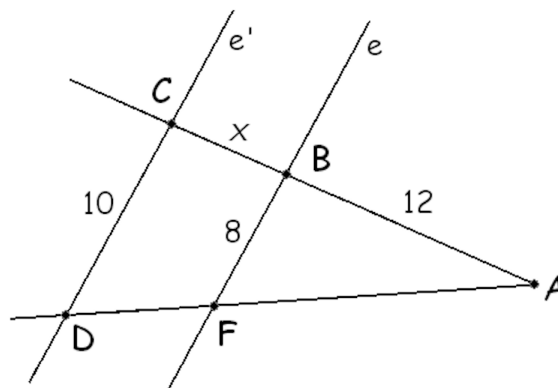
Theorem: Two triangles are similar to each other if

1. they have the same angles, or
2. if all three pairs of sides are proportional, or
3. if two pairs of sides are proportional and the angle enclosed by them are equal.

Most often we establish similarity using angles, and then solve for missing sides using proportionality.

Example 1. Find the value of x based on the figure, given that lines e and e' are parallel.

Solution: First we will use angles to establish that triangles ACD and ABF are similar. Then we use proportionality of sides to find the value of x . Angles ABF and ACD , (marked orange) have equal measures because lines e and e' are parallel. Similarly, angles AFB and ADC , (marked green) have equal measures because lines e and e' are parallel.



The third angle, at point A is shared by the two triangles. (Notice however, that we have similarity with only two pairs of equal angles.) So the two triangles are similar.

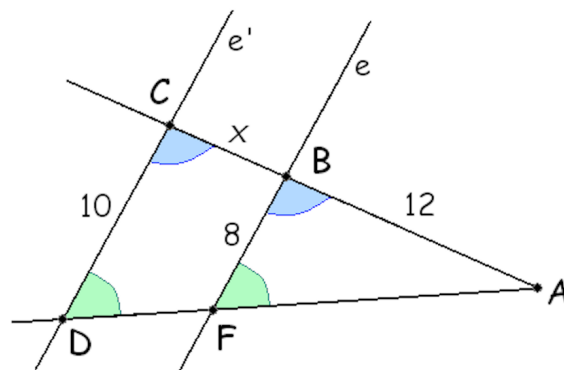
Because they are similar, corresponding sides of the two triangles are proportional. We can use this fact to find the value of x . Notice that x , or line segment BC is not a side of any triangles.

It is recommended that we keep track of sides by the angles opposite them. Our eyes seem to work much better with angles than with sides. So we will write the ratio

$$\frac{\text{side opposite the green angle}}{\text{side opposite the angle at point A}}$$

in both triangles. By similarity, this ratio must remain the same.

$$\text{In triangle } ACD, \text{ the ratio } \frac{\text{side opposite the green angle}}{\text{side opposite the angle at point A}} = \frac{x + 12}{10}.$$



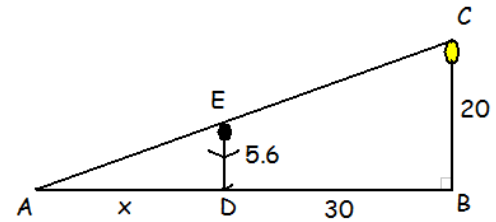
The same ratio in triangle ABF is $\frac{12}{8} = \frac{3}{2}$. Therefore, we have a linear equation for x .

$$\begin{array}{rcll} \frac{x+12}{10} & = & \frac{3}{2} & \text{multiply by 10} \\ x+12 & = & 15 & \text{subtract 12} \\ x & = & 3 & \end{array}$$

Now that we know the value of x , we see how the proportionality works: 15 to 12 is the same as 10 to 8. Therefore, our solution, $x = 3$ is correct.

Example 2. A 5.6ft tall person is standing 30ft away from a street light that is 20ft tall. How long is her shadow?

Solution: The shadow is created as light is absorbed by the person, so it can not hit the ground behind her. After we draw a picture, we notice that triangles ABC and ADE are similar. They share an angle at point A , and they both also have a right angle. Because the triangles are similar, corresponding



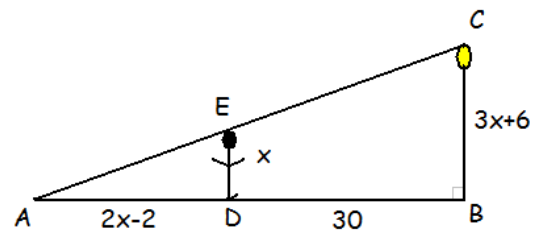
sides are proportional. We can write the ratio $\frac{\text{horizontal side}}{\text{vertical side}}$ for both triangles. We need to be careful with the horizontal side of the larger triangle: it is not 30 but rather $x + 30$.

$$\begin{array}{rcll} \frac{x}{5.6} & = & \frac{x+30}{20} & \text{multiply by } 20 \cdot 5.6 \\ 20x & = & 5.6(x+30) & \text{multiply by 10 (to get rid of the decimal)} \\ 200x & = & 56(x+30) & \text{distribute 56} \\ 200x & = & 56x + 1680 & \text{subtract } 56x \\ 144x & = & 1680 & \text{divide by 144} \\ x & = & \frac{1680}{144} = \frac{35}{3} = 11\frac{2}{3} & \end{array}$$

Thus her shadow is $11\frac{2}{3}$ ft long.

Example 3. A person is standing 30 feet away from a street light that is six feet taller than three times the height of the person. The length of his shadow is two feet less than twice his height. How tall is he?

Solution: The shadow is created as light is absorbed by the person, so it can not hit the ground behind her. After we draw a picture, we notice that triangles ABC and ADE are similar. They share an angle at point A , and they both also have a right angle.



If we label the person's height by x , then the height of the light is $3x + 6$ and the length of his shadow is $2x - 2$. Now we can use proportionality sides. We can write the ratio $\frac{\text{vertical side}}{\text{horizontal side}}$ for both triangles.

$$\frac{x}{2x-2} = \frac{3x+6}{30+(2x-2)} \quad \text{we solve for } x$$

$$\begin{aligned} \frac{x}{2x-2} &= \frac{3x+6}{2x+28} \\ \frac{x}{2x-2} &= \frac{3x+6}{2x+28} && \text{multiply by } (2x-2)(2x+28) \\ x(2x+28) &= (2x-2)(3x+6) && \text{expand products} \\ 2x^2+28x &= 6x^2+12x-6x-12 && \text{combine like terms} \\ 2x^2+28x &= 6x^2+6x-12 && \text{subtract } 2x^2, \text{ subtract } 28x \\ 0 &= 4x^2-22x-12 && \text{divide by 2} \\ 2x^2-11x-6 &= 0 \end{aligned}$$

Since this is a quadratic equation, we need to factor and apply the zero product rule. We will complete the square.

$$\begin{aligned} 2x^2-11x-6 &= 0 && \text{factor out 2} \\ 2\left(x^2-\frac{11}{2}x-3\right) &= 0 && \left(x-\frac{11}{4}\right)^2 = x^2-\frac{11}{2}x+\frac{121}{16} \\ 2\left(x^2-\frac{11}{2}x+\frac{121}{16}-\frac{121}{16}-3\right) &= 0 \\ 2\left(\left(x-\frac{11}{4}\right)^2-\frac{121}{16}-\frac{48}{16}\right) &= 0 \\ 2\left(\left(x-\frac{11}{4}\right)^2-\frac{169}{16}\right) &= 0 && \sqrt{169}=13 \\ 2\left(\left(x-\frac{11}{4}\right)^2-\left(\frac{13}{4}\right)^2\right) &= 0 \\ 2\left(x-\frac{11}{4}+\frac{13}{4}\right)\left(x-\frac{11}{4}-\frac{13}{4}\right) &= 0 \\ 2\left(x+\frac{2}{4}\right)\left(x-\frac{24}{4}\right) &= 0 \\ 2\left(x+\frac{1}{2}\right)(x-6) &= 0 \implies x_1 = -\frac{1}{2} \text{ and } x_2 = 6 \end{aligned}$$

Since distances are never negative, $-\frac{1}{2}$ is easily ruled out. Thus he is 6ft tall.

The following is an example of how similarity is applied in geometry. Also, it is an example for the third case of similarity: two sides proportional and the angle between them identical.

Example 4. Prove that the midpoints of any convex quadrilateral form a parallelogram.

Solution: Let $ABCD$ be any convex quadrilateral with midpoints M, N, P , and Q . Notice that AC and BD are the diagonals of quadrilateral $ABCD$.

We first claim that triangles MAN and DAB are similar to each other.

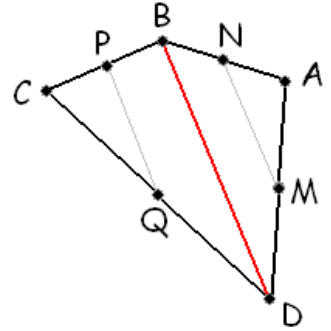
Because M and N are midpoints, side AB is twice side AN . Similarly, side AD is twice side AM . The angle between these sides is literally shared. Therefore, triangles MAN and DAB are similar, and all sides of triangle ABC are twice as long as the corresponding sides in triangle ANM . Because of similarity, these triangles have the same angles.

Also, we can see that the corresponding sides are parallel. Therefore, we can state that line segment BD is parallel to line segment NM , and is twice as long. We have the same situation on the other side, at point C .

Because of P and Q are midpoints, sides PC and CQ are half as long as sides BC and CD . The angle at point C is literally shared. Therefore, triangles BDC and PCQ are similar. In addition, corresponding sides are parallel. Therefore, line segment BD is parallel to line segment PQ and is twice as long.

We have therefore that both PQ and NM are line segments parallel to BD , therefore they are parallel to each other. (We also have that PQ and NM are equally long, but we don't need it).

Using an identical argument, we can see that both PN and QM are parallel to line segment AC . Therefore, they are parallel to each other. We have that PQ is parallel to NM and PN to QM , therefore $PQMN$ is a parallelogram.

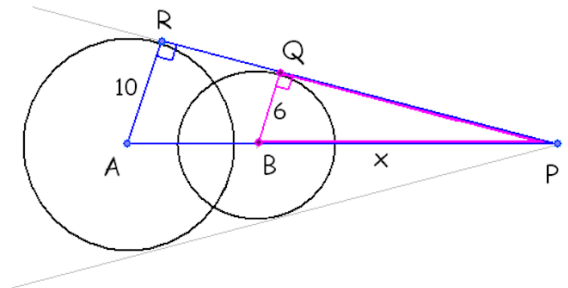


Recall that in circles, the radius drawn to the point of tangency is perpendicular to the tangent line.

Example 5. Suppose that A and B are the centers of two circles with radii 10 units and 6 units long. The distance between A and B is 12 units. The circles have two common tangent lines that intersect each other in point P . Find the distance between the points B and P .

Solution: Let Q and R be the points of tangency as shown on the picture. Triangles APR and BPQ are similar because they share an angle at point P and both have a right angle, at R and Q , correspondingly. If we denote distance PB by x , then

$$\frac{\text{side opposite the right angle}}{\text{side opposite } P} = \frac{x}{6} = \frac{x+12}{10}$$



We solve this equation for x .

$$\begin{aligned} \frac{x}{6} &= \frac{x+12}{10} \\ 10x &= 6(x+12) \end{aligned}$$

$$\begin{aligned} 10x &= 6x + 72 \\ 4x &= 72 \\ x &= 18 \end{aligned}$$

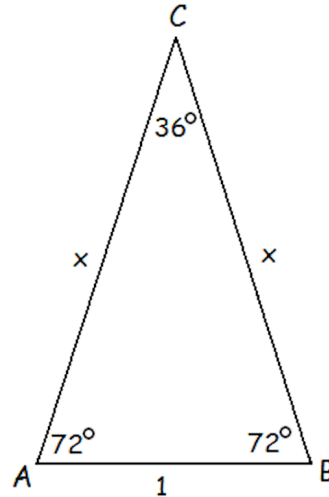
So $\overline{PB} = 18$ units.



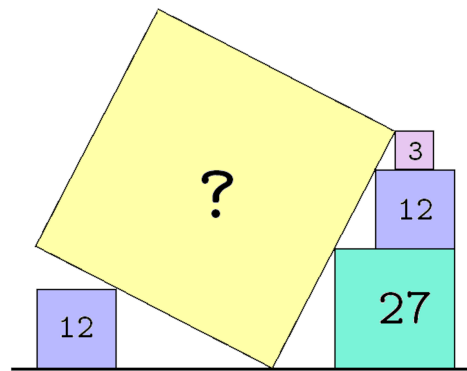
Enrichment

Consider triangle ABC shown on the picture.

1. Bisect the angle at point A . (To bisect an angle means to split it into two equal angles. Such a line is called the angle bisector) This angle bisector intersects side BC in point D . Prove that triangles ABC and ABD are similar.
2. Use the similarity established in the previous problem to find the exact value of x .

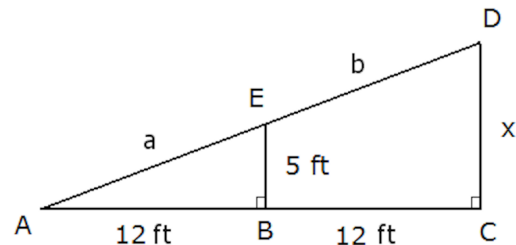
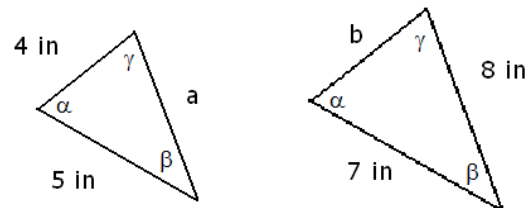


3. This problem is from the great Catriona Shearer. All rectangles on the picture are squares. The numbers show the area of each square. Find the exact value for the area of the yellow square.



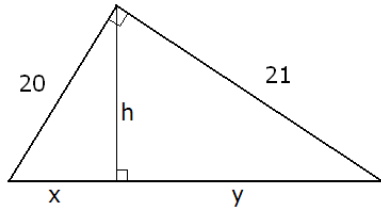
Sample Problems

1. The triangles shown are similar. Find the exact values of a and b shown on the picture below.
2. Consider the picture shown.
 - a) Use the Pythagorean Theorem to find the value of a .
 - b) Prove that the triangles ABE and ACD are similar.
 - c) Use similar triangles to find the value of x .
 - d) Find the value of b .



3. a) A person is standing 40ft away from a street light that is 30ft tall. How tall is he if his shadow is 10ft long?
 b) A 6ft tall person is standing 24ft away from a street light that is 15ft tall. How long is her shadow?
4. Prove the following statement. Let ABC be any right triangle, the right angle at point C . The altitude drawn from C to the hypotenuse splits the triangle into two right triangles that are similar to each other and to the original triangle.

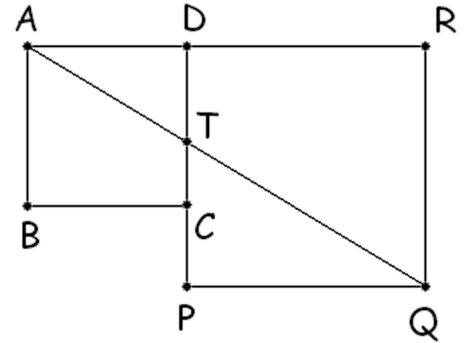
5. Find x , y , and h based on the picture.



6. The picture below shows a right triangle. Find the length of h , the height drawn to the hypotenuse.

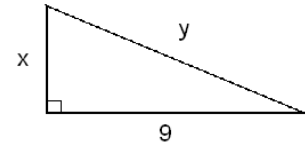
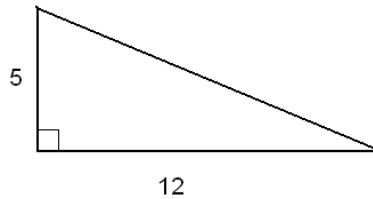


7. Quadrilaterals $ABCD$ and $PQRD$ are both squares. If $AB = 3$ and $PQ = 3$, then find the exact value of the length of TC .

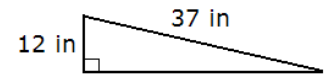
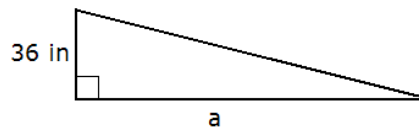


Practice Problems

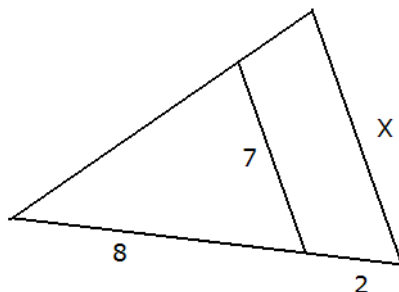
1. The picture shows two similar right triangles. Find the exact values of x and y .



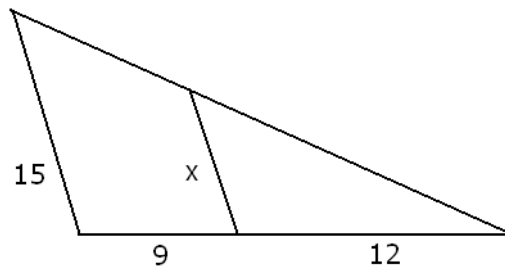
2. The picture shows two similar right triangles. Find the exact value of a .



3. Find the value of x based on the figures.

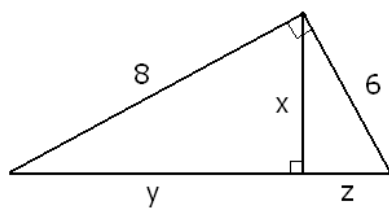


(a)

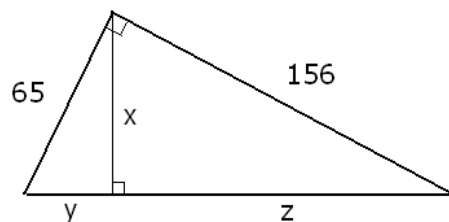


(b)

4. a) A person is standing 24ft away from a street light that is 25ft tall. How tall is he if his shadow is 6ft long?
 b) A 5.2ft tall person is standing 20ft away from a street light that is 15.6ft tall. How long is her shadow?
5. Find the exact value of x , y , and z , based on the figures shown.

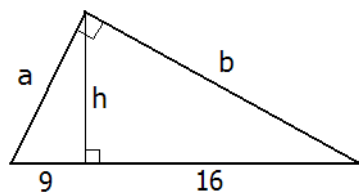


(a)

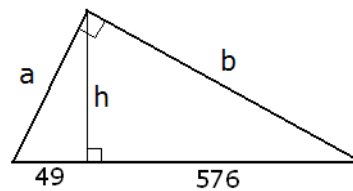


(b)

6. Find the exact value of a , b , and h , based on the picture shown.

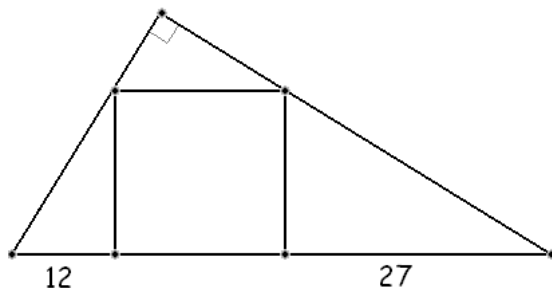


(a)



(b)

7. A square is written into a right triangle as shown on the picture. Find the exact value of the length of the side of the square.



Problem Set 18

Chapter 19

19.1 Writing Equations of Lines

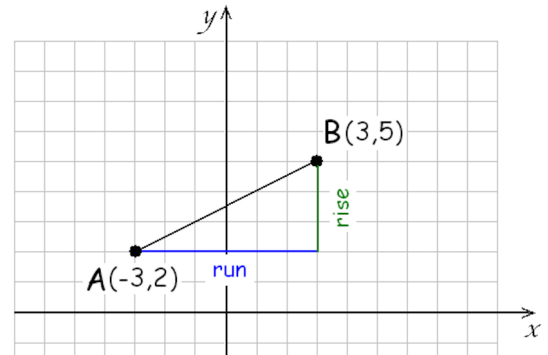
Until now, we focused on finding the graph of a given linear equation. In what follows, we will approach lines differently. Given some conditions of the line, *we* will come up with the equation. Let us first review a few things about straight lines. Let us first recall a definition.

Definition: Equations that are in x , or in y , or in x and y can be graphed. **The graph of such an equation is the set of all points $P(x,y)$ for which the coordinates x and y form a solution of the equation.**

If the equation is linear in x and y , the graph of the equation is a line. If one of the variables is missing, and the equation is linear in the other (such as $x = 2$ or $y = -3$), the graph of the equation is a line.

The slope of a line is a very important concept. It expresses the steepness of a line. Given two points on a line, the slope is a signed ratio expressing the vertical change per the horizontal change between the two points.

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in the } y - \text{coordinate}}{\text{change in the } x - \text{coordinate}} = \frac{y_2 - y_1}{x_2 - x_1}$$



Definition: The slope of a line (usually denoted by m) is a signed ratio expressing the steepness of the line. Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the slope of the line connecting these points is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

This formula is called the **slope formula**.

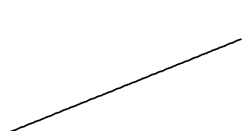
Example 1. Find the slope of the line determined by the points $A(-3, 2)$ and $B(3, 5)$.

Solution: a) We find the slope determined by the points. $(-3, 2) = (x_1, y_1)$ and $(3, 5) = (x_2, y_2)$ using the slope formula.

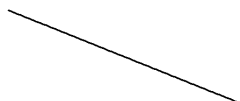
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 2}{3 - (-3)} = \frac{3}{6} = \frac{1}{2}$$

So the slope is $\boxed{\frac{1}{2}}$.

Note that **the slope of all horizontal lines is zero** and that **vertical lines have no slope**.



line with positive slope



line with negative slope



line with slope $m = 0$



line with no slope

When the equation of a line is given, we can find its slope by finding two points on the line and applying the slope formula. However, there are easier ways to find the slope of a line, given its equation.

Theorem: Suppose that the equation of a line is $y = mx + b$. Then the slope of this line is m , the coefficient of x in the equation, and the line's y -intercept is the point $(0, b)$. This form of an equation is called the **slope-intercept form**.

Proof: Suppose that $A(x_A, y_A)$ is a point on the line $y = mx + b$. Since this point is on the line, $y_A = mx_A + b$, and so the coordinates of this point is $(x_A, mx_A + b)$. Let us find another point B with an x -coordinate 1 greater than x_A . Then the new point, B has an x -coordinate $x_B = x_A + 1$, and the y -coordinate is $y_B = mx_B + b = m(x_A + 1) + b = mx_A + m + b$. Applying the slope formula for A and B

$$\text{slope} = \frac{y_B - y_A}{x_B - x_A} = \frac{(mx_A + m + b) - (mx_A + b)}{(x_A + 1) - x_A} = \frac{mx_A + m + b - mx_A - b}{x_A + 1 - x_A} = \frac{m}{1} = m$$

If we substitute $x = 0$ into the equation of the line $y = mx + b$, the result is $y = m \cdot 0 + b = b$. Therefore, the y -intercept of this line is $(0, b)$. ■

The intercept-slope form of a line tells us a great deal about the line. For example, the line $y = 2x - 3$ has slope $m = 2$ and y -intercept $(0, -3)$. The line $y = -\frac{2}{3}x + 1$ has slope $m = -\frac{2}{3}$, and y -intercept $(0, 1)$. The line $y = x - 5$ has slope $m = 1$ and y -intercept $(0, -5)$. We have seen before that graphing a line is a matter of seconds if its equation is given in the slope-intercept form.

We need to be careful. If the line is given in a form other than the slope-intercept form, the coefficient of x is NOT the slope. For example, the slope of the line $2x - 3y = -12$ is NOT 2.

Definition: The equation of a line can also be given in the form of $Ax + By = C$, where A , B , and C are real numbers. This form of the equation is called the **general form**.

Example 2. Find the slope of the line $2x - 3y = -12$.

Solution: The slope of this line is not 2. We have to first transform the equation into the slope-intercept form by

solving for y in terms of x .

$$2x - 3y = -12 \quad \text{add } 3y \text{ to both sides}$$

$$2x = 3y - 12 \quad \text{add } 12$$

$$2x + 12 = 3y \quad \text{divide by } 3 \text{ (also, swap the two sides)}$$

$$y = \frac{2x + 12}{3} = \frac{2}{3}x + \frac{12}{3} = \frac{2}{3}x + 4$$

So we have the equation $y = \frac{2}{3}x + 4$. This equation is in the slope-intercept form, so the slope is the coefficient of x . Thus $m = \boxed{\frac{2}{3}}$.

A natural question is: which equation is better to use, the general form or the slope-intercept form. One possible answer is that they are equally useful, with different advantages. The slope-intercept form is excellent when graphing the line or finding geometric properties of it. The general form better serves us when the focus is on algebraic computations. For example, the general form might be preferred when finding the intersection of two lines.

Example 3. Write an equation for the line with slope $m = -\frac{1}{4}$ and y -intercept $(0, 3)$.

Solution: There is not much we need to do. Since the slope and the y -intercept was given, we immediately have that $m = -\frac{1}{4}$ and $b = 3$ in the equation $y = mx + b$. Thus the equation is $y = -\frac{1}{4}x + 3$.

There is one more form of equation of a line that is extremely useful, it is called the point-slope form.

Definition: Suppose that a line has slope m and passes through the point $P(x_P, y_P)$. **The point-slope form** of the equation of this line is

$$y - y_P = m(x - x_P)$$

Example 4. Write the point-slope form equation for each of the lines passing through the given point and having the given slope.

a) $P(-5, 1)$ and $m = -2$ b) $P(3, -2)$ and $m = -\frac{2}{3}$ c) $P(4, 0)$ and $m = 1$

Solution: a) We write the point-slope form. $m = -2$, and $P(-5, 1)$ means that $x_P = -5$ and $y_P = 1$.

$$y - y_P = m(x - x_P) \text{ becomes } y - 1 = -2(x - (-5))$$

$$\text{We simplify the equation: } \boxed{y - 1 = -2(x + 5)}.$$

b) We write the point-slope form. $m = -\frac{2}{3}$, and $P(3, -2)$ means that $x_P = 3$ and $y_P = -2$.

$$y - y_P = m(x - x_P) \text{ becomes } y - (-2) = -\frac{2}{3}(x - 3)$$

$$\text{We simplify the equation: } \boxed{y + 2 = -\frac{2}{3}(x - 3)}.$$

c) We write the point-slope form. $m = 1$, and $P(4, 0)$ means that $x_P = 4$ and $y_P = 0$.

$$y - y_P = m(x - x_P) \text{ becomes } y - 0 = 1(x - 4)$$

We simplify the equation: $y = x - 4$.

Theorem: Two lines are parallel if and only if they have the same slope or both lines are vertical.

Example 5. Write an equation for the line that passes through the point $(6, -2)$ and is parallel to the line $3x + 4y = 10$.

Solution: Let us first figure out the slope of this other, parallel line. We transform the equation given to slope-intercept form and read the slope.

$$3x + 4y = 10 \quad \text{subtract } 3x \text{ from both sides}$$

$$4y = -3x + 10 \quad \text{divide by 4}$$

$$y = \frac{-3x + 10}{4} = -\frac{3}{4}x + \frac{10}{4} = -\frac{3}{4}x + \frac{5}{2}$$

If our line is to be parallel to this line, its slope must be the same. Now we can simply apply the point slope form with $m = -\frac{3}{4}$ and $P(6, -2)$.

$$y - y_P = m(x - x_P) \text{ becomes } y - (-2) = -\frac{3}{4}(x - 6)$$

We simplify the equation: $y + 2 = -\frac{3}{4}(x - 6)$

Recall that two lines are perpendicular if they intersect each other at an angle of 90° (also called right angle).

Theorem: Two lines are perpendicular if and only if their slopes m_1 and m_2 are negative reciprocals of each other, i.e. $m_1 m_2 = -1$, or one line is vertical and the other one is horizontal.

Example 6. Write an equation for the line that passes through the point $(6, -2)$ and is perpendicular to the line $3x + 4y = 10$.

Solution: This is the same line as in the previous example, so we know that its slope is $-\frac{3}{4}$. Our line is to be perpendicular to this line, so its slope needs to be the negative reciprocal of $-\frac{3}{4}$. That is $\frac{4}{3}$. We can check by multiplying the two slopes: the product must be -1 . Now we simply apply the point slope form with $m = \frac{4}{3}$ and $P(6, -2)$.

$$y - y_P = m(x - x_P) \text{ becomes } y - (-2) = \frac{4}{3}(x - 6) \text{ or } y + 2 = \frac{4}{3}(x - 6)$$

There are a number of interesting and important applications of our new skill of coming up with equations of lines based on certain geometric properties.

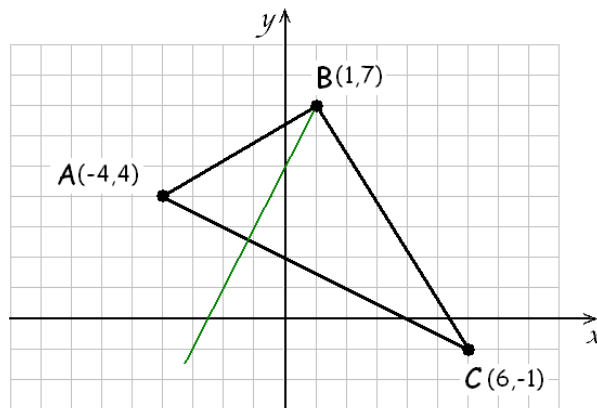
Example 7. Consider the triangle determined by $A(-4, 4)$, $B(1, 7)$, and $C(6, -1)$. Write an equation for the altitude drawn to side AC .

Solution: This task might seem intimidating at first, but it is not much different from our previous example. The height or altitude drawn to side AC is simply a line that is perpendicular to line AC and contains point B . We first compute the slope of line segment AC . We use the slope formula with points $A(-4, 4)$ and $C(6, -1)$.

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{-1 - 4}{6 - (-4)} = \frac{-5}{10} = -\frac{1}{2}$$

The height drawn to side AC is perpendicular to side AC and so its slope must be the negative reciprocal of m_{AC} . That is a slope of 2. Now we simply apply the point slope form with $m = 2$ and $B(1, 7)$.

$y - y_P = m(x - x_P)$ becomes $y - 7 = 2(x - 1)$.



If we are to present the equation in the slope-intercept form, we distribute the 2 on the right-hand side and then isolate y on the other side.

$$y - 7 = 2(x - 1)$$

$$y - 7 = 2x - 2$$

$$y = 2x + 5$$

Example 8. Find an equation for the line connecting the points $P(3, -8)$ and $Q(-2, 2)$.

Solution: We will use the slope formula to find the slope of the line connecting P and Q . Then we apply the point-slope form with either P or Q .

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{2 - (-8)}{-2 - 3} = \frac{10}{-5} = -2$$

Both $y + 8 = -2(x - 3)$ and $y - 2 = -2(x + 2)$ are correct. We can transform both equations to the slope intercept form.

$$y + 8 = -2(x - 3)$$

$$y - 2 = -2(x + 2)$$

$$y + 8 = -2x + 6$$

$$y - 2 = -2x - 4$$

$$y = -2x - 2$$

$$y = -2x - 2$$

So the two point-slope equations belong to the same line, whose slope-intercept form is $y = -2x - 2$.

Theorem: Suppose that $A(x_A, y_A)$ and $B(x_B, y_B)$ are two points given. Let M denote the midpoint of line segment AB . The coordinates of $M(x_M, y_M)$ are the average of the corresponding coordinates of A and B .

$$x_M = \frac{x_A + x_B}{2} \text{ and } y_M = \frac{y_A + y_B}{2}$$

Example 9. Find the midpoint of the line segment AB where points A and B are given as $A(-5, 4)$ and $B(3, -4)$.

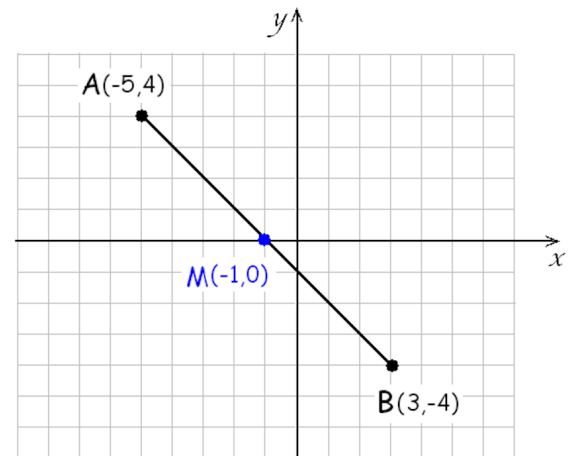
Solution: The x -coordinate of the midpoint is the average of the x -coordinates of A and B .

$$x_M = \frac{x_A + x_B}{2} = \frac{-5 + 3}{2} = \frac{-2}{2} = -1$$

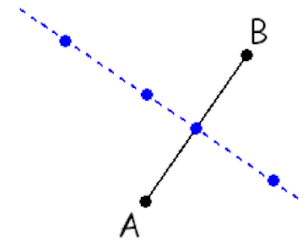
Similarly, the y -coordinate of the midpoint is the average of the y -coordinates of A and B .

$$y_M = \frac{y_A + y_B}{2} = \frac{4 + (-4)}{2} = \frac{0}{2} = 0$$

Thus the midpoint is $M(-1, 0)$.



Definition: Suppose that points A and B are given. The **perpendicular bisector** of line segment AB is the set of all points P in the plane that are equidistant to points A and B .



Although the definition is given in terms of distances, there are two properties of the perpendicular bisector that we use when finding their equation. **The perpendicular bisector of line segment AB is perpendicular to that line segment, and passes through the midpoint of the line segment.**

Example 10. Find an equation of the perpendicular bisector of the line segment AB where $A(4, -2)$ and $B(2, 4)$.

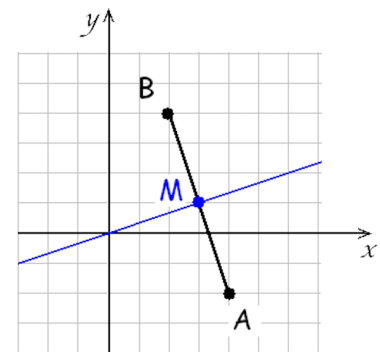
Solution: The perpendicular bisector of the line segment passes through its midpoint M . We compute the coordinates of M .

$$x_M = \frac{x_A + x_B}{2} = \frac{4 + 2}{2} = 3 \text{ and}$$

$$y_M = \frac{y_A + y_B}{2} = \frac{-2 + 4}{2} = 1 \quad \text{Thus } M(3, 1)$$

Next, we compute the slope of the line connecting A and B .

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{4 - (-2)}{2 - 4} = \frac{6}{-2} = -3$$



The slope of the perpendicular bisector is the negative reciprocal of m_{AB} , that is $\frac{1}{3}$. We are ready to write the equation in point-slope form, with $m = \frac{1}{3}$ and $M(3, 1)$. Thus an equation of the perpendicular bisector is $y - 1 = \frac{1}{3}(x - 3)$

An Application

When we convert units, we usually deal with a simple ratio problem. If 1 kilogram (kg) is 2.2 pounds (lb), then 2 kilograms are 4.4 pounds, 10 kilograms are 22 pounds, and so on. Conversion of temperature is an exception. In the USA, we use Fahrenheit, ($^{\circ}\text{F}$) and in most of Europe, Celsius ($^{\circ}\text{C}$) is used. But conversion between the two units is more complicated than a ratio problem.

Celsius was established as follows. The temperature at which water freezes was defined as 0°C . The temperature at which water boils was defined as 100°C . An increase of 1°C would be the same as one-hundredth of the difference between freezing and boiling points.

We can always come up with the conversion formula between Celsius and Fahrenheit if we remember the two temperatures listed before, and if we look at it as the equation of a line, given two points on it.

Suppose we want to convert from Fahrenheit to Celsius. Let us create a coordinate system in which the x -axis represents Fahrenheit, and the y -axis represents Celsius measurement. Water freezes at 32°F , this temperature was defined to be 0°C . Water boils at 212°F , this temperature was defined to be 100°C . Therefore the conversion formula would be the equation of the line connecting the points $(32, 0)$ and $(212, 100)$.

We compute the slope using the slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{100 - 0}{212 - 32} = \frac{100}{180} = \frac{5}{9}$.

We now use the point-slope form with $(32, 0)$ to write the equation of the line.

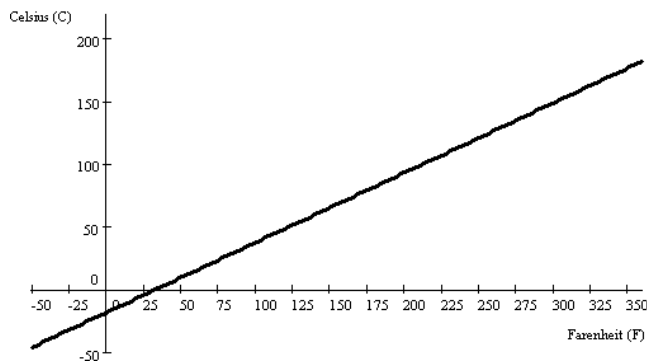
$$y - 0 = \frac{5}{9}(x - 32) \implies y = \frac{5}{9}(x - 32)$$

This means that the conversion formula from

Fahrenheit to Celsius is $C = \frac{5}{9}(F - 32)$.

For example, to convert 77°F to Celsius,

we write $C = \frac{5}{9}(77 - 32) = 25$. Therefore, $77^{\circ}\text{F} = 25^{\circ}\text{C}$.



Sample Problems

- In each case, find the slope of the line determined by the two points given.
 - $(5, -1)$ and $(1, 3)$
 - $(2, 1)$ and $(6, 3)$
 - $(8, -1)$ and $(-1, -1)$
 - $(7, 3)$ and $(7, -4)$
- For each straight line given, find its slope.
 - $y = -2x + 5$.
 - $2x - 5y = 8$
 - $y = 3$
 - $7(x - 3) + x - 2y = 2y + 1$
 - $x = -\frac{1}{2}$
 - a line parallel to $y = 3x - 1$
 - a line perpendicular to $y = 3x - 1$
- Find an equation for the straight line that has slope 4 and y -intercept $(0, -5)$.

4. Find an equation for the straight line that has slope $-\frac{1}{2}$ and passes through the point $(4, 1)$.
5. Find an equation of the straight line that is parallel to $3x - 2y = 12$ and passes through the point $(8, -1)$.
6. Find an equation of the straight line that is perpendicular to $x + 5y = -3$ and passes through the point $(-1, 4)$.
7. Find an equation of the straight line that passes through the points $(3, -1)$ and $(1, 5)$.
8. Find the midpoint of the line segment determined by $A(7, -4)$ and $B(3, -2)$.
9. Consider the triangle determined by $A(-5, -3)$, $B(-1, 3)$, and $C(7, 1)$.
 - a) Find an equation for the line that connects A and B .
 - b) Find the midpoint of line segment AB .
 - c) Find an equation for the perpendicular bisector of side AB .
 - d) Find an equation for the height (or altitude) belonging to side AC .



Practice Problems

1. In each case, find the slope of the line determined by the two points given.
 - a) $(-2, -3)$ and $(3, -1)$
 - b) $(12, -7)$ and $(6, -3)$
 - c) $(2, 5)$ and $(2, 10)$
 - d) $(12, 9)$ and $(-6, 9)$
2. For each straight line given, find its slope.
 - a) $y = \frac{3}{4}x$
 - b) $3x + 10y = 1$
 - c) $y = 0$
 - d) $3(y - 2) - 2(5 - x) = 3x - 4$
 - e) $x = 11$
 - f) a line parallel to $y = \frac{1}{2}x + 7$
 - g) a line perpendicular to $y = \frac{1}{2}x + 7$
3. Find an equation for the straight line that has slope -2 and y -intercept $(0, 7)$.
4. Find an equation for the straight line that has slope $\frac{2}{3}$ and passes through the point $(-6, 7)$.
5. Find an equation of the straight line that is parallel to $3x - 2y = 12$ and passes through the point $(8, -1)$.
6. Find an equation of the straight line that is perpendicular to $x + 5y = -3$ and passes through the point $(-1, 4)$.
7. Find an equation of the straight line that passes through the points $(2, 10)$ and $(5, -2)$.
8. Find an equation of the straight line that passes through the points $(-4, 5)$ and $(6, 0)$.
9. Consider the triangle determined by $A(6, 1)$, $B(0, 7)$, and $C(-4, -3)$.
 - a) Find an equation for the line that connects the given points.
 - i) A and B
 - ii) A and C
 - iii) B and C
 - b) Find the midpoint of each of the given line segments.
 - i) AB
 - ii) AC
 - iii) BC
 - c) Find an equation for the perpendicular bisector of each of the given line segments.
 - i) AB
 - ii) AC
 - iii) BC
 - d) Find an equation for the height (or altitude) belonging to each of the given sides.
 - i) AB
 - ii) AC
 - iii) BC

19.2 Radical Expressions 2

We already know how to perform operations with radical expressions. We can add, subtract, multiply them. The only thing missing is division. Dividing by a rational number is just a matter of the distributive law.

Example 1. Perform the division $\frac{2\sqrt{5}-4}{7}$.

Solution: To divide is to multiply by the reciprocal: $\frac{2\sqrt{5}-4}{7} = (2\sqrt{5}-4) \cdot \frac{1}{7} = \boxed{\frac{2}{7}\sqrt{5} - \frac{4}{7}}$

But how do we divide by an expression that contains radicals? Recall that if we multiply two expressions that are conjugates, the result is rational.

Example 2. Perform the given multiplications. a) $(3\sqrt{7}-5)(3\sqrt{7}+5)$ b) $(\sqrt{10}+3)(\sqrt{10}-3)$

Solution: a) We apply the distributive law. Because we are multiplying conjugates, the terms O and I from FOIL cancel out each other and we are left with the difference of two squares.

$$(3\sqrt{7}-5)(3\sqrt{7}+5) = (3\sqrt{7})^2 + 5(3\sqrt{7}) - 5(3\sqrt{7}) - 5^2 = 9 \cdot 7 - 25 = 63 - 25 = \boxed{38}$$

$$\text{b) } (\sqrt{10}+3)(\sqrt{10}-3) = (\sqrt{10})^2 - 3\sqrt{10} + 3\sqrt{10} - 3^2 = 10 - 9 = \boxed{1}$$

In this last example, the two numbers are not only conjugates, but they are also reciprocals of each other.

$$(\sqrt{10}+3)(\sqrt{10}-3) = 1 \implies \sqrt{10}+3 = \frac{1}{\sqrt{10}-3}$$

We should be able to compute the reciprocal of $\sqrt{10}-3$ directly, instead of just running into it by chance. Indeed, we can transform quotients so that their denominators become rational. This is also called rationalizing the denominator.

Example 3. Rationalize the denominator in each of the given expressions.

$$\text{a) } \frac{6}{2\sqrt{7}-5} \quad \text{b) } \frac{2}{\sqrt{5}-1} \quad \text{c) } \frac{\sqrt{3}+5}{\sqrt{3}-1}$$

Solution: a) We will multiply the expression by 1, where 1 is written as a fraction with the same numerator and denominator. That same expression is the conjugate of the part to be rationalized. In case of $2\sqrt{7}-5$, the conjugate is $2\sqrt{7}+5$.

$$\begin{aligned} \frac{6}{2\sqrt{7}-5} &= \frac{6}{2\sqrt{7}-5} \cdot 1 = \frac{6}{2\sqrt{7}-5} \cdot \frac{2\sqrt{7}+5}{2\sqrt{7}+5} = \frac{6(2\sqrt{7}+5)}{(2\sqrt{7}-5)(2\sqrt{7}+5)} = \frac{6(2\sqrt{7}+5)}{(2\sqrt{7})^2 - 5^2} \\ &= \frac{6(2\sqrt{7}+5)}{28-25} = \frac{6(2\sqrt{7}+5)}{28-25} = \frac{6(2\sqrt{7}+5)}{3} = \frac{2(2\sqrt{7}+5)}{1} = \boxed{4\sqrt{7}+10} \end{aligned}$$

$$\text{b) } \frac{2}{\sqrt{5}-1} = \frac{2}{\sqrt{5}-1} \cdot 1 = \frac{2}{\sqrt{5}-1} \cdot \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{2(\sqrt{5}+1)}{(\sqrt{5})^2 - 1^2} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{2(\sqrt{5}+1)}{4} = \boxed{\frac{\sqrt{5}+1}{2}}$$

$$\begin{aligned} \text{c) } \frac{\sqrt{3}+5}{\sqrt{3}-1} &= \frac{\sqrt{3}+5}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{(\sqrt{3}+5)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{(\sqrt{3})^2 + \sqrt{3} + 5\sqrt{3} + 5}{(\sqrt{3})^2 - 1^2} = \frac{3+6\sqrt{3}+5}{3-1} \\ &= \frac{8+6\sqrt{3}}{2} = \frac{2(4+3\sqrt{3})}{2} = \boxed{4+3\sqrt{3}} \end{aligned}$$



Enrichment

Simplify each of the following expressions.

1. $\frac{\sqrt{2} + \sqrt{6}}{\sqrt{2} + \sqrt{3}}$

2. $\frac{\sqrt{1008 + \sqrt{2015}} - \sqrt{1008 - \sqrt{2015}}}{\sqrt{2}}$



Sample Problems

1. Rationalize the denominator in each of the following expressions.

a) $\frac{3}{\sqrt{5}}$

b) $\frac{1}{\sqrt{10} - 3}$

c) $\frac{2}{\sqrt{7} + 1}$

d) $\frac{\sqrt{5} - 1}{\sqrt{5} - 2}$

e) $\frac{\sqrt{8} - \sqrt{5}}{\sqrt{8} + \sqrt{5}}$

2. Simplify each of the following expressions.

a) $\frac{3 - \sqrt{5}}{3 + \sqrt{5}} + \frac{3 + \sqrt{5}}{3 - \sqrt{5}}$

c) $\sqrt{\sqrt{41} + 4\sqrt{2}} \cdot \sqrt{\sqrt{41} - \sqrt{32}}$

d) $\sqrt{5\sqrt{3} + \sqrt{59}} \cdot \sqrt{\sqrt{75} - \sqrt{59}}$

b) $\left(\frac{8}{\sqrt{7} + \sqrt{3}} + \frac{12}{\sqrt{7} - \sqrt{3}} \right) (5\sqrt{7} - \sqrt{3})$

e) $\left(\sqrt{6 + \sqrt{11}} + \sqrt{6 - \sqrt{11}} \right)^2$

3. Find the exact value of $x^2 - 4x + 6$ if $x = 2 - \sqrt{3}$.

4. Solve each of the following quadratic equations by completing the square. Check your solution(s).

a) $x^2 = 4x + 1$

b) $x^2 + 13 = 8x$



Practice Problems

1. Rationalize the denominator in each of the following expressions.

a) $\frac{4}{\sqrt{7}}$

c) $\frac{1}{\sqrt{10} + 3}$

e) $\frac{\sqrt{5} - 1}{\sqrt{5} + 1}$

g) $\frac{-5x + 5}{\sqrt{x} + 1}$

b) $\frac{1}{\sqrt{7} - 3}$

d) $\frac{2}{\sqrt{17} + 4}$

f) $\frac{\sqrt{10} + \sqrt{2}}{\sqrt{10} - \sqrt{2}}$

h) $\frac{2}{\sqrt{x} + 4}$

2. Find the exact value of

a) $x^2 - 6x + 1$ if $x = 3 - \sqrt{10}$ b) $b^2 + 8b - 20$ if $b = \sqrt{5} - 2$ c) $x^2 - 10x + 16$ if $x = \sqrt{6} + 5$

3. Solve each of the following quadratic equations by completing the square. Check your solution(s).

a) $x^2 + 47 = 14x$

b) $x^2 + 12x + 31 = 0$

c) $x^2 = 2x + 1$

Problem Set 19

Chapter 20

20.1 Absolute Value Equations

Equations that have absolute values in them have interesting properties. Some will remind us to quadratic equations, some will be fundamentally different. Let us start with the most basic forms of such equations.

Example 1. Solve each of the given equations.

a) $|x| = 5$ b) $|x| = 0$ c) $|x| = -6$

Solution: a) How can the absolute value of a number be 5? There are only two ways: if the number is 5, or if the number is -5 . There are no other ways. Therefore, the solution of $|x| = 5$ is $\boxed{-5, \text{ and } 5}$.

b) How can the absolute value of a number be 0? There is only one way: if the number is 0. Therefore, the solution of $|x| = 0$ is $\boxed{x = 0}$.

c) How can the absolute value of a number be -5 ? Absolute values are never negative, so it is not possible. Therefore, this equation has $\boxed{\text{no solution}}$.

These basic equations with absolute values in them remind us to quadratic equations in the sense that there might be two, one, or no solution. More complicated equations can be solved by applying these ideas to expressions inside absolute value signs.

Example 2. Solve each of the given equations.

a) $|2x - 7| = 5$ b) $|x - 2| + 4 = 16$

Solution: a) We need to apply the same idea. This time, not on x , but on the expressions inside the absolute value sign. Let us think of the expression $2x - 7$ as a single quantity, say A . How can the absolute value of a number be 5? There are only two ways: if the number is 5, or if the number is -5 . But it is not x that needs to be 5 or -5 , but rather $A = 2x - 7$. This way, we will trade in one absolute value equation to two linear equations.

$$|2x - 7| = 5 \quad \text{if and only if} \quad 2x - 7 = 5 \quad \text{or} \quad 2x - 7 = -5$$

We solve both linear equations for x .

$$2x - 7 = 5 \quad \text{add 7}$$

$$2x = 12 \quad \text{divide by 2}$$

$$x = 6$$

$$2x - 7 = -5 \quad \text{add 7}$$

$$2x = 2 \quad \text{divide by 2}$$

$$x = 1$$

This equation seems to have two solutions. We check:

if $x = 6$, then $\text{LHS} = |2 \cdot 6 - 7| = |12 - 7| = |5| = 5 = \text{RHS} \checkmark$

And if $x = 1$, then $\text{LHS} = |2 \cdot 1 - 7| = |2 - 7| = |-5| = 5 = \text{RHS} \checkmark$. So the solutions are 1 and 6.



Discussion: Explain why the conclusion shown below is wrong.

$$|2x - 7| = 5 \implies 2x + 7 = 5$$

- b) In the previous example, the symmetry of 5 and -5 is no longer around x , but around the expression within the absolute value sign. We can exchange the one equation with absolute value to a pair of linear equations **only when the absolute value expression is isolated on one side of the equation**.

$$|x - 2| + 4 = 16 \quad \text{subtract 4}$$

$$|x - 2| = 12$$

$$|x - 2| = 12 \quad \text{if and only if} \quad x - 2 = 12 \text{ or } x - 2 = -12$$

We solve both linear equations for x .

$$\begin{array}{ll} x - 2 = 12 & \text{add 2} \\ x = 14 \end{array}$$

$$\begin{array}{ll} x - 2 = -12 & \text{add 2} \\ x = -10 \end{array}$$

We again have two solutions. We check:

if $x = 14$, then $\text{LHS} = |14 - 2| + 4 = |12| + 4 = 12 + 4 = 16 = \text{RHS} \checkmark$

And if $x = -10$, then $\text{LHS} = |-10 - 2| + 4 = |-12| + 4 = 12 + 4 = 16 = \text{RHS} \checkmark$

So the solutions are -10 and 14.



Discussion: Explain why the conclusion shown below is wrong.

$$|x - 12| + 4 = 16 \implies x - 2 + 4 = 16 \quad \text{or} \quad x - 2 + 4 = -16$$

Example 3. Solve each of the given equations.

a) $3|2x - 10| + 7 = 19$ b) $3 - \left| \frac{1}{2}x - 5 \right| = -10$ c) $8 + 3|5x - 4| = 2$

Solution: a) We can exchange the one equation with absolute values to a pair of linear equations only when the absolute value expression is isolated on one side of the equation. Let us introduce a new variable, say A for the absolute value expression. Let $A = |2x - 10|$. We first solve for A .

$3|2x - 10| + 7 = 19$ becomes $3A + 7 = 19$

$$3A + 7 = 19 \quad \text{subtract 7}$$

$$3A = 12 \quad \text{divide by 3}$$

$$A = 4 \quad \text{recall that } A = |2x - 10|$$

$$|2x - 10| = 4 \implies 2x - 10 = 4 \text{ or } 2x - 10 = -4$$

We solve both linear equations for x .

$$\begin{array}{rcl|lcl}
 2x - 10 = 4 & \text{add 10} & & 2x - 10 = -4 & \text{add 10} \\
 2x = 14 & \text{divide by 2} & & 2x = 6 & \text{divide by 2} \\
 x = 7 & & & x = 3 &
 \end{array}$$

This equation seems too has two solutions. We check: if $x = 7$, then

$$\text{LHS} = 3|2 \cdot 7 - 10| + 7 = 3|14 - 10| + 7 = 3|4| + 7 = 3 \cdot 4 + 7 = 12 + 7 = 19 = \text{RHS} \checkmark$$

And if $x = 3$, then

$$\text{LHS} = 3|2 \cdot 3 - 10| + 7 = 3|6 - 10| + 7 = 3|-4| + 7 = 3 \cdot 4 + 7 = 12 + 7 = 19 = \text{RHS} \checkmark. \text{ So the solutions are } \boxed{3 \text{ and } 7}.$$

- b) We first need to isolate the expression with the absolute value sign. This means that we first solve for $\left|\frac{1}{2}x - 5\right|$ in $3 - \left|\frac{1}{2}x - 5\right| = -10$. We can do that with or without introducing a new variable for $\left|\frac{1}{2}x - 5\right|$.

$$\begin{array}{rcl}
 3 - \left|\frac{1}{2}x - 5\right| & = & -10 \\
 3 & = & -10 + \left|\frac{1}{2}x - 5\right| \\
 13 & = & \left|\frac{1}{2}x - 5\right|
 \end{array}
 \begin{array}{l}
 \text{add } \left|\frac{1}{2}x - 5\right| \\
 \text{add 10}
 \end{array}$$

Now we can trade the single equation to a pair of linear equations and then solve for z in each.

$$\begin{array}{rcl|lcl}
 13 = \left|\frac{1}{2}x - 5\right| & \implies & \frac{1}{2}x - 5 = 13 & \text{or} & \frac{1}{2}x - 5 = -13 \\
 \frac{1}{2}x - 5 = 13 & \text{add 5} & & & \frac{1}{2}x - 5 = -13 & \text{add 5} \\
 \frac{1}{2}x = 18 & \text{multiply by 2} & & & \frac{1}{2}x = -8 & \text{multiply by 2} \\
 x = 36 & & & & x = -16 &
 \end{array}$$

We check both solutions:

$$\text{if } x = -16, \text{ then } \text{LHS} = 3 - \left|\frac{1}{2}(-16) - 5\right| = 3 - |-8 - 5| = 3 - |-13| = 3 - 13 = -10 = \text{RHS} \checkmark$$

$$\text{And if } x = 36, \text{ then } \text{LHS} = 3 - \left|\frac{1}{2} \cdot 36 - 5\right| = 3 - |18 - 5| = 3 - |13| = 3 - 13 = -10 = \text{RHS} \checkmark. \text{ So the solutions are } \boxed{-16 \text{ and } 36}.$$

c) Again, we first solve for the expression with the absolute value sign.

$$\begin{array}{rcl} 8 + 3|5x - 4| & = & 2 \quad \text{subtract 8} \\ 3|5x - 4| & = & -6 \quad \text{divide by 3} \\ |5x - 4| & = & -2 \end{array}$$

The absolute value of no number is negative. Therefore, we can conclude that this equation has no solution. Some students might solve for $5x - 4 = \pm 2$. In this case, it is essential that we check.

Some equations with absolute value are more complicated.

Example 4. Solve each of the given equations.

a) $|4x - 5| = |x + 1|$ b) $|2x - 3| = -x - 4$

Solution: a) How can the absolute value of two quantities be equal? There are only two ways: if the quantities are equal or if they are opposites of each other. This way we will, again, trade one equation to two linear.

$$|4x - 5| = |x + 1| \implies 4x - 5 = x + 1 \quad \text{or} \quad 4x - 5 = -(x + 1)$$

We solve both linear equations for x .

$$\begin{array}{rcl} 4x - 5 = x + 1 & & \text{subtract } x \\ 3x - 5 = 1 & & \text{add 5} \\ 3x = 6 & & \text{divide by 3} \\ x = 2 & & \end{array}$$

$$\begin{array}{rcl} 4x - 5 = -x - 1 & & \text{add } x \\ 5x - 5 = -1 & & \text{add 5} \\ 5x = 4 & & \text{divide by 5} \\ x = \frac{4}{5} & & \end{array}$$

This equation seems too has two solutions. We check:

if $x = 2$, then $\text{LHS} = |4 \cdot 2 - 5| = |8 - 5| = |3| = 3$ and

$\text{RHS} = |2 + 1| = |3| = 3$ ✓

If $x = \frac{4}{5}$, then $\text{LHS} = \left| 4 \cdot \frac{4}{5} - 5 \right| = \left| \frac{16}{5} - \frac{25}{5} \right| = \left| -\frac{9}{5} \right| = \frac{9}{5}$ and

$\text{RHS} = \left| \frac{4}{5} + 1 \right| = \left| \frac{4}{5} + \frac{5}{5} \right| = \left| \frac{9}{5} \right| = \frac{9}{5}$ ✓ So the solutions are 2 and $\frac{4}{5}$.

b) While this equation looks similar to the previous one, the slight difference will have significant consequences.

$$|2x - 3| = -x - 4 \implies 2x - 3 = -x - 4 \quad \text{or} \quad 2x - 3 = -(-x - 4)$$

We solve both linear equations for x .

$$\begin{array}{rcl} 2x - 3 = -x - 4 & & \text{add } x \\ 3x - 3 = -4 & & \text{add 3} \\ 3x = -1 & & \text{divide by 3} \\ x = -\frac{1}{3} & & \end{array}$$

$$\begin{array}{rcl} 2x - 3 = -(-x - 4) & & \text{simplify} \\ 2x - 3 = x + 4 & & \text{subtract } x \\ x - 3 = 4 & & \text{add 3} \\ x = 7 & & \end{array}$$

This equation seems too has two solutions. We check: $|2x - 3| = -x - 4$

if $x = -\frac{1}{3}$, then $\text{LHS} = \left| 2\left(-\frac{1}{3}\right) - 3 \right| = \left| -\frac{2}{3} - \frac{9}{3} \right| = \left| -\frac{11}{3} \right| = \frac{11}{3}$ and

$\text{RHS} = -\left(-\frac{1}{3}\right) - 4 = \frac{1}{3} - \frac{12}{3} = -\frac{11}{3}$ so $-\frac{1}{3}$ is NOT a solution!

And if $x = 7$, then $\text{LHS} = |2 \cdot 7 - 3| = |14 - 3| = |11| = 11$ and

$\text{RHS} = -(7 + 4) = -11$ Thus, 7 is also not a solution! So this equation has no solution.



Practice Problems

Solve each of the given equations.

1. $|2x + 7| = 1$

8. $|3x - 6| = 0$

14. $|4x - 7| - 7 = 0$

2. $\frac{1}{2}|x - 5| = 6$

9. $3 + \left| \frac{1}{2}x - 3 \right| = 7$

15. $|4x - 7| - 7 = 7$

3. $2 - |3x + 8| = 10$

10. $3 - \left| \frac{1}{2}x - 3 \right| = 7$

16. $|x + 1| = |2x - 7|$

4. $10 - |3x + 8| = 2$

17. $|5x - 2| = |-x + 8|$

5. $2 - \frac{1}{5}|2x + 17| = -3$

11. $2 - 3|2x - 5| = -10$

18. $|x - 5| = |x + 3|$

6. $|3x - 6| + 4 = 7$

12. $|3x - 1| = 5$

19. $x - 5 = |2x + 1|$

7. $|3x - 6| = 6$

13. $|3x - 5| = 1$

20. $|x - 5| = 2x + 1$

20.2 Complex Numbers

The set of all complex numbers, (denoted by $\mathbb{C}$) is a set containing the set of all real numbers, denoted by $\mathbb{R}$. It is best to think of complex numbers as an extension or enlargement of the set of real numbers. The definitions that established the set of all complex numbers can be explained by this fact. Let us recall the expansion principle.

Definition: The **Expansion Principle** is our desire to protect already true statements whenever new concepts are defined.

If a new definition would result in an already true statement becoming false, mathematicians do not accept the definition.

We will see that most definitions about complex numbers can be explained as our only choice for creating a structure around the real numbers that still has most properties and theorems of the real numbers.

Definition: A complex number is $z = a + bi$ where a and b are real numbers.

Complex numbers are often denoted by z so that we can use x and y to denote its real and imaginary components. For example, we will often see $z = x + yi$ where x and y are real numbers. If y happens to be zero, then $x + yi$ is a real number. Thus, every real number is also a complex number. The set of all real numbers (denoted by $\mathbb{R}$) is a subset of the set of all complex numbers (denoted by $\mathbb{C}$). In short, $\mathbb{R} \subseteq \mathbb{C}$. As a matter of fact, stepping out to the complex numbers is in various ways the last step in our story of expanding our number system. This story is

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

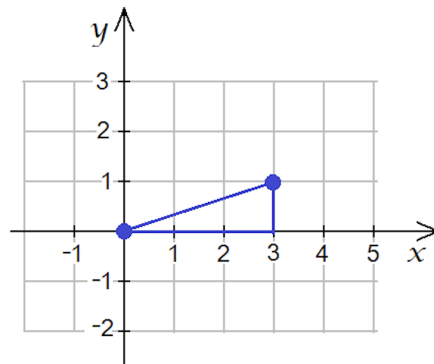
Real numbers are depicted as points on a line, also called the number line, complex numbers are often depicted as points on a plane, also called the complex plane. For example, the complex number $a + bi$ can be depicted as the point (a, b) on the complex plane. This means that the real number 5 can be depicted as $(5, 0)$ on the complex plane, and the number i can be depicted as $(0, 1)$. Notice that this correspondence between numbers and points is a one-to-one correspondence: to each complex number, there exists a unique point in the plane, and to each point in the plane, there exists a unique complex number. The origin represents the real number 0.

Recall the definition of absolute value. The absolute value of a real number is its distance from zero on the number line. As it turns out, the absolute value (or modulus) of a complex number can be defined very similarly.

Definition: The absolute value (or modulus) of a complex number is its distance from zero on the number plane.

For example, the absolute value of $3 + i$, denoted by $|3 + i|$ can be the distance between the points corresponding to $3 + i$ and 0. That is, the distance between $(0, 0)$ and $(3, 1)$. We can use the distance formula or the Pythagorean Theorem.

$$|3 + i| = \sqrt{3^2 + 1^2} = \sqrt{10}$$



Notice that the absolute value of a complex number is (just like that of a real number) a non-negative real number. Also, this definition clearly does not violate the expansion principle. If we take a real number and compute its absolute value as before, as a real number, we get the same result as we get when computing the absolute value as a complex number.

Theorem: The absolute value (or modulus) of a complex number $z = x + yi$ (where x and y are real numbers) is $|z| = \sqrt{x^2 + y^2}$.

Definition: Addition between complex numbers is defined as follows: if $z_1 = x_1 + y_1i$ and $z_2 = x_2 + y_2i$, then

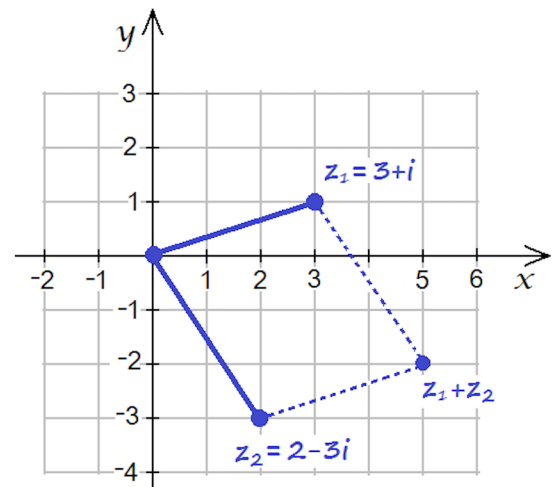
$$z_1 + z_2 = x_1 + x_2 + (y_1 + y_2)i$$

Suppose that $z_1 = 3 + i$ and $z_2 = 2 - 3i$. Then

$$z_1 + z_2 = (3 + i) + (2 - 3i) = 3 + i + 2 - 3i = (3 + 2) + (i - 3i) = 5 - 2i$$

This definition is not at all surprising: we would like to preserve the basic rules from real numbers that allow us to drop parentheses and combine like terms. With that in mind, this definition was pretty much the only way to go. Notice that this definition of addition guarantees that addition of complex numbers inherit the properties from real numbers: addition is commutative, associative, there is an identity element, zero. Once we define multiplication between a real and a every complex number, we will also establish that every complex number has an opposite.

Because of the definitions we just discussed, all of these properties have very nice geometric properties. If we depict z_1 , and z_2 in the same coordinate system, and connect them to the origin, we obtain a triangle. If we now draw a line through z_1 that is parallel to z_2 and another line through z_2 that is parallel to z_1 , we obtain a parallelogram. The fourth vertex of the parallelogram is $z_1 + z_2$. (Later we will see that this behavior is identical to addition of vectors. This is because addition and subtraction of vectors are defined very similarly.)

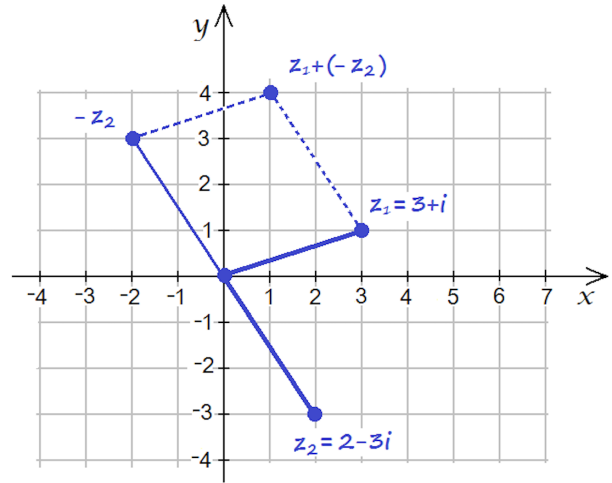


Definition: Multiplication between a real and a complex number: if $z = x + yi$ is a complex number and c is a real number, then

$$cz = c(x + yi) = cx + cyi$$

Now we can easily define subtraction between complex numbers. 'To subtract is to add the opposite' will remain true. If $z_1 = 3 + i$ and $z_2 = 2 - 3i$, then

$$\begin{aligned}
 z_1 - z_2 &= (3 + i) - (2 - 3i) \\
 &= 3 + i + (-1)(2 - 3i) \\
 &= 3 + i + (-2) + 3i \\
 &= (3 - 2) + (i + 3i) \\
 &= 1 + 4i
 \end{aligned}$$



The next logical step is to define multiplication on complex numbers. It is clear that we would like to preserve properties of multiplication from the real number system: we'd like multiplication to be commutative, associative, we would like to preserve the identity element being 1. And also, we would like to preserve the distributive law. So, using our previous example of z_1 and z_2 , we already have an idea about their product.

$$z_1 z_2 = (3 + i)(2 - 3i) = 3 \cdot 2 - 3 \cdot 3i + i \cdot 2 + i(-3i) = 6 - 9i + 2i - 3i^2 = 6 - 7i - 3i^2$$

We will know the product $z_1 z_2$ if we could just figure out what is i^2 . At this point of the construction of the complex numbers, we had the freedom to define i^2 any way we liked. Of course, different choices would result in different kind of rules we would observe. How i^2 was defined has a lot to do with the expansion theorem. We will see that if we want a simple rule from the real numbers, $|x| |y| = |xy|$ to be true on the complex numbers, then i^2 can only have the value -1 .

Theorem: If for all complex numbers z_1 and z_2 , the statement $|z_1| |z_2| = |z_1 z_2|$ is true, then i^2 can only be defined as -1 .

Proof: Assume that the complex number $i^2 = a + bi$. So, we will know the complex number i^2 if we know the real numbers a and b . In what follows, we will obtain two equations in a and b . First, let us look at the statement

$$|i| \cdot |i| = |i^2|$$

Since $|i| = 1$ and $|i^2| = |a + bi| = \sqrt{a^2 + b^2}$, this equation becomes

$$\begin{aligned}
 1 \cdot 1 &= \sqrt{a^2 + b^2} && \text{squaring both sides gives us} \\
 1 &= a^2 + b^2
 \end{aligned}$$

Next, let us look at the statement

$$|1 + i| \cdot |1 - i| = |(1 + i)(1 - i)|$$

On the left-hand side, we can determine the absolute value of each numbers:

$$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2} \quad \text{and} \quad |1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

On the right-hand side, we simplify $(1+i)(1-i)$ as much as we can: $(1+i)(1-i) = 1 - i^2$. So, our equation

$$\begin{aligned} |1+i| \cdot |1-i| &= |(1+i)(1-i)| \text{ becomes} \\ \sqrt{2} \cdot \sqrt{2} &= |1-i^2| \\ 2 &= |1-i^2| \end{aligned}$$

Recall that $i^2 = a + bi$, where a and b are real numbers.

$$\begin{aligned} 2 &= |1 - (a + bi)| \\ 2 &= |1 - a - bi| \\ 2 &= |(1 - a) - bi| \\ 2 &= \sqrt{(1-a)^2 + b^2} \quad \text{square both sides} \\ 4 &= (1-a)^2 + b^2 \end{aligned}$$

So, we have two equations:

$$\begin{aligned} a^2 + b^2 &= 1 \\ (1-a)^2 + b^2 &= 4 \end{aligned}$$

This should be enough for us to find a and b . Let us work on the second equation:

$$\begin{aligned} (1-a)^2 + b^2 &= 4 \\ 1 - 2a + \underbrace{a^2 + b^2} &= 4 \quad \text{according to the other equation, } a^2 + b^2 = 1 \\ 1 - 2a + 1 &= 4 \\ 2 - 2a &= 4 \\ 2 &= 4 + 2a \\ -2 &= 2a \\ -1 &= a \end{aligned}$$

Now let us go back to the first equation:

$$\begin{aligned} a^2 + b^2 &= 1 \quad \text{and now we know that } a = -1 \\ (-1)^2 + b^2 &= 1 \\ b^2 &= 0 \\ b &= 0 \end{aligned}$$

Thus $i^2 = a + bi = -1 + 0i = -1$. This concludes our proof. ■

What did we prove here? Did we prove that i^2 is -1 ? No, we have only proved that if we wanted to preserve the rule $|x||y| = |xy|$ from the real numbers, our only option for the value of i^2 is -1 . We didn't even prove that if we define i^2 to be -1 , some other things wouldn't go wrong. Or even that the rule $|x||y| = |xy|$ would *always* work. We just used this rule in two cases: $|i||i| = |i^2|$ and $|1+i||1-i| = |(1+i)(1-i)|$. Mathematicians still needed to verify that all the rules we desire to preserve are safe under the definition $i^2 = -1$. And, as it turned out, they all (or at least most) are safe.

Definition: $i^2 = -1$

Now we can easily define multiplication on complex numbers. Our unfinished example,

$$z_1 z_2 = (3+i)(2-3i) = 6-9i+2i-3i^2 = 6-7i-3(-1) = 6-7i+3 = 9-7i$$

In general, if $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then

$$\begin{aligned} z_1 z_2 &= (x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + x_2 y_1 i + y_1 y_2 (i^2) = x_1 x_2 + (x_1 y_2 + x_2 y_1) i + y_1 y_2 (-1) \\ &= x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1) i \end{aligned}$$

Theorem: If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then $z_1 z_2 = x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1) i$

The last operation that is left undefined is division. But before we do that, we will define the complex conjugate of a number. As we will see later, complex conjugates are extremely useful.

Definition: The complex conjugate of a number z , denoted by $\bar{z}$ is defined as follows. If $z = x + yi$, then $\bar{z} = x - yi$.

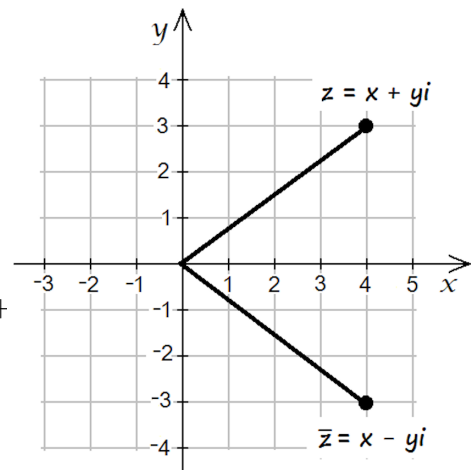
A complex number and its complex conjugate are symmetrical to the x -axis.

An interesting and important fact: if add or multiply a complex number and its conjugate, the result in both cases is a real number.

Consider for example $z = 1 + 7i$. Then

$$z + \bar{z} = (1 + 7i) + (1 - 7i) = 2 \quad \text{and} \quad z\bar{z} = (1 + 7i)(1 - 7i) = 1 - 49i^2 = 1 +$$

Actually, more is true: if we multiply a complex number z and its conjugate $\bar{z}$, the result is the square of the absolute value of z .



Theorem: $z\bar{z} = |z|^2$.

Proof: Suppose that $z = x + yi$. Then $\bar{z} = x - yi$ and recall that $|x + yi| = \sqrt{x^2 + y^2}$

$$z\bar{z} = (x + yi)(x - yi) = x^2 - y^2(i^2) = x^2 + y^2 = |z|^2$$

We are now ready to define division on complex numbers. First, we define division of a complex number by a real number. We actually already have this definition, remember, to divide is to multiply by the reciprocal. So, if c is a real number, not zero, and $z = x + yi$ is any complex number, then

$$\frac{z}{c} = \frac{x + yi}{c} = \frac{1}{c}(x + yi) = \frac{x}{c} + \frac{y}{c}i$$

When we are dividing by a complex number z , we will use the conjugate $\bar{z}$ to transform the division into another one where the denominator is real. Let us look at, for example, the division $\frac{3 - 4i}{2 + i}$. To make the denominator real, we multiply both numerator and denominator by the conjugate of $2 + i$. 1 is still the multiplicative identity so multiplication by 1 still does not change the value of any complex number.

$$\frac{3 - 4i}{2 + i} = \frac{3 - 4i}{2 + i} \cdot 1 = \frac{3 - 4i}{2 + i} \cdot \frac{2 - i}{2 - i} = \frac{(3 - 4i)(2 - i)}{(2 + i)(2 - i)} = \frac{6 - 3i - 8i + 4i^2}{4 - 2i + 2i - i^2} = \frac{6 - 11i + 4(-1)}{4 - (-1)} = \frac{2 - 11i}{5}$$

and now this is just a division by a real number:

$$\frac{2 - 11i}{5} = \frac{1}{5}(2 - 11i) = \frac{2}{5} - \frac{11}{5}i$$

Some questions are now different depending on whether we are working within $\mathbb{R}$ or $\mathbb{C}$. For example, $\sqrt{-9}$ is undefined over the real numbers but it is $3i$ over the complex numbers.

Another very important property of complex numbers is that over $\mathbb{C}$, every polynomial can be factored into a product of linear factors. Consider for example the polynomial $x^2 + 1$. While it is irreducible over $\mathbb{R}$, it is not over $\mathbb{C}$. We apply the difference of squares theorem:

$$x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x + i)(x - i)$$

We can factor more complicated quadratic expressions by completing the square.

$$\begin{aligned} x^2 - 6x + 13 &= \underbrace{x^2 - 6x + 9}_{(x-3)^2} - 9 + 13 = (x - 3)^2 + 4 = (x - 3)^2 - (-4) \\ &= (x - 3)^2 - (2i)^2 = (x - 3 + 2i)(x - 3 - 2i) \end{aligned}$$

So, every quadratic equation has solution(s) over the complex numbers.

While we preserved most of the mathematics we had over the real numbers, we did lose some parts. For example, we cannot order complex numbers. In other words, $<$ and $\leq$ are meaningless over the complex numbers, unless they happen to be real.

Another rule we have lost is that $\sqrt{ab} = \sqrt{a}\sqrt{b}$. While this was true among real numbers, it is no longer true over the complex numbers. Here is an easy example to consider:

$$\sqrt{-4}\sqrt{-4} = (2i)(2i) = 4i^2 = -4 \quad \text{but} \quad \sqrt{-4(-4)} = \sqrt{16} = 4$$

So we need to be careful not to use this rule over $\mathbb{C}$.



Sample Problems

Perform the indicated operations and simplify.

1. $|3 - 5i|$
2. $(2 - 5i) + (1 - i)$
3. $3 + 5i - (2 + 3i)$
4. $3(2 - i) - 2(3 + i)$
5. $-i(3 + i)$
6. $3(2 - i) - 2i(3 + i)$
7. $(2 - 3i)(5 + 2i)$
8. $(3 - 2i)^2$
9. $(3 - 2i)(3 + 2i)$
10. $(7 - 3i)(i + 1)$
11. $(3 + 5i)(-3 + 5i)$
12. $(1 - 3i)^2(1 + 3i)^2$
13. $(1 - i)^4$
14. $\frac{2i}{1 - i}$
15. $\frac{1 + 7i}{3i + 1}$
16. $\frac{10 + 5i}{3 + 4i}$
17. $\frac{5}{2 - i}$
18. $\frac{8i - 1}{2i + 3}$
19. $\frac{5}{1 + 2i} - \frac{10i}{1 - 2i}$
20. $\frac{(3 + 2i)(5 - 3i) - (7 - 2i)(3 - i)}{3 - 4i}$
21. $\frac{(3 - i)^2 - (1 + 3i)^2}{2i + 2}$
22. Simplify $\frac{\sqrt{2}i - 1}{\sqrt{2} + i} - \frac{\sqrt{2} - i}{1 + \sqrt{2}i}$

Completely factor each of the following over the complex numbers.

23. $x^2 + 9$
24. $x^2 - 10x + 29$
25. $x^4 - 1$

Solve each of the following equations over the complex numbers.

26. $x^2 = 4x - 29$
27. $x^2 + 3 = 2x$
- 28.* Find z such that $z^2 = -21 + 20i$.

Problem Set 20

Chapter 21

21.1 Complex Fractions

Complex fractions are fractions with more than one fraction bars. For example, $\frac{\frac{1}{2} - 3}{5 + \frac{1}{4}}$ and $\frac{1 - \frac{2}{x+1}}{\frac{5}{x-3} + \frac{x}{2}}$ are complex

fractions. A complex fraction can always be simplified, that is, brought to a form with only one fraction bar. We will present two methods. The first one might be convenient because it is familiar and consists of steps we know very well. The other method is new to us. However, it appears to be a very effective method.

Example 1. Simplify the given complex fractions.

$$\begin{array}{ll} \text{a) } \frac{\frac{2}{3} - \frac{1}{4}}{\frac{3}{8} + \frac{1}{2}} & \text{b) } \frac{\frac{2}{x} - 5}{1 + \frac{3}{x}} \end{array}$$

Solution 1: a) This method involves performing operations on fractions: a subtraction (in the numerator), an addition (in the denominator), and finally a division.

$$\text{The subtraction: } \frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12} = \frac{5}{12}$$

$$\text{The addition: } \frac{3}{8} + \frac{1}{2} = \frac{3}{8} + \frac{4}{8} = \frac{7}{8}$$

$$\text{The division: } \frac{\frac{5}{12}}{\frac{7}{8}} = \frac{5}{12} \cdot \frac{8}{7} = \frac{5 \cdot 2 \cdot \cancel{4}}{3 \cdot \cancel{4} \cdot 7} = \boxed{\frac{10}{21}}$$

We can perform these operations in a single computation:

$$\frac{\frac{2}{3} - \frac{1}{4}}{\frac{3}{8} + \frac{1}{2}} = \frac{\frac{8-3}{12}}{\frac{3+4}{8}} = \frac{\frac{5}{12}}{\frac{7}{8}} = \frac{5}{12} \cdot \frac{8}{7} = \frac{40}{84} = \boxed{\frac{10}{21}}$$

b) We can apply the same steps with the fraction containing variables.

$$\text{The subtraction in the numerator: } \frac{2}{x} - 5 = \frac{2}{x} - \frac{5}{1} = \frac{2}{x} - \frac{5x}{x} = \frac{2-5x}{x} = \frac{-5x+2}{x}$$

$$\text{The addition in the denominator: } 1 + \frac{3}{x} = \frac{x}{x} + \frac{3}{x} = \frac{x+3}{x}$$

$$\text{The division: } \frac{\frac{-5x+2}{x}}{\frac{x+3}{x}} = \frac{-5x+2}{x} \cdot \frac{x}{x+3} = \frac{-5x+2}{\cancel{x}} \cdot \frac{\cancel{x}}{x+3} = \boxed{\frac{-5x+2}{x+3}}$$

Solution 2: a) This method involves the fundamental property of fractions. Recall that we do not change the value of a fraction when multiplying both numerator and denominator by the same non-zero number. This technique involves 'blasting' the denominators.

$$\text{Consider the fraction } \frac{\frac{2}{3} - \frac{1}{4}}{\frac{3}{8} + \frac{1}{2}}. \text{ If we multiplied the numerator, } \frac{2}{3} - \frac{1}{4} \text{ by 12, that would clear the}$$

denominator of both $\frac{2}{3}$ and $\frac{1}{4}$. If we multiplied the denominator, $\frac{3}{8} + \frac{1}{2}$ by 8, that would clear both denominators. Since we must use the same number for both numerator and denominator, we will multiply both by 24. This technique is surprisingly powerful: we get the final result much faster than before.

$$\frac{\frac{2}{3} - \frac{1}{4}}{\frac{3}{8} + \frac{1}{2}} = \frac{24\left(\frac{2}{3} - \frac{1}{4}\right)}{24\left(\frac{3}{8} + \frac{1}{2}\right)} = \frac{24 \cdot \frac{2}{3} - 24 \cdot \frac{1}{4}}{24 \cdot \frac{3}{8} + 24 \cdot \frac{1}{2}} = \frac{8\cancel{24} \cdot \frac{2}{\cancel{3}} - 6\cancel{24} \cdot \frac{1}{\cancel{4}}}{3\cancel{24} \cdot \frac{3}{\cancel{8}} + 12\cancel{24} \cdot \frac{1}{\cancel{2}}} = \frac{8 \cdot 2 - 6 \cdot 1}{3 \cdot 3 + 12 \cdot 1} = \frac{16-6}{9+12} = \boxed{\frac{10}{21}}$$

b) We can apply the same steps with the fraction containing variables. In the case of $\frac{\frac{2}{x} - 5}{1 + \frac{3}{x}}$, both numerator and denominator would be cleared when multiplied by x .

$$\frac{\frac{2}{x} - 5}{1 + \frac{3}{x}} = \frac{x\left(\frac{2}{x} - 5\right)}{x\left(1 + \frac{3}{x}\right)} = \frac{x \cdot \frac{2}{x} - x \cdot 5}{x \cdot 1 + x \cdot \frac{3}{x}} = \frac{\cancel{x} \cdot \frac{2}{\cancel{x}} - x \cdot 5}{x \cdot 1 + \cancel{x} \cdot \frac{3}{\cancel{x}}} = \frac{2-5x}{x+3} = \boxed{\frac{-5x+2}{x+3}}$$

There might be cancelations after we simplify the complex fractions. Also, sometimes complex fractions are given using negative exponents.

Example 2. Simplify the complex fraction $\frac{25 - x^{-2}}{5 + x^{-1}}$

Solution 1: We first re-write the expression without negative exponents.

$$\begin{aligned} \frac{25 - x^{-2}}{5 + x^{-1}} &= \frac{25 - \frac{1}{x^2}}{5 + \frac{1}{x}} = \frac{\frac{25x^2}{x^2} - \frac{1}{x^2}}{\frac{5x}{x} + \frac{1}{x}} = \frac{\frac{25x^2 - 1}{x^2}}{\frac{5x+1}{x}} = \frac{(5x+1)(5x-1)}{x} \cdot \frac{x}{5x+1} \\ &= \frac{\cancel{(5x+1)}(5x-1)}{\cancel{x}} \cdot \frac{\cancel{x}}{\cancel{5x+1}} = \boxed{5x-1} \end{aligned}$$



Sample Problems

Simplify each of the following expressions.

$$1. \frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}}$$

$$3. \frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x}}$$

$$5. \frac{\frac{3}{x-1} - 1}{\frac{2}{x-1} + 1}$$

$$7. 1 - \frac{1}{1 - \frac{1}{x-3}}$$

$$8. \frac{a^{-1} - b^{-1}}{a^{-2} - b^{-2}}$$

$$2. \frac{5 - \frac{1}{a}}{\frac{1}{a^2} - 25}$$

$$4. \frac{\frac{4}{a^2} - 1}{1 - \frac{2}{a}}$$

$$6. \frac{2 - \frac{3}{x+1}}{3 - \frac{2x}{x+1}}$$

$$9. \frac{x^2 - 2y^{-3}}{x^{-1} + 3y^2}$$

$$10. (x - y^{-2})^{-3}$$



Practice Problems

Simplify each of the following expressions.

$$1. \frac{2a - \frac{1}{8a}}{4 + \frac{1}{a}}$$

$$6. \frac{\frac{1}{x+a} + \frac{1}{x-a}}{\frac{1}{x+a} - \frac{1}{x-a}}$$

$$10. \frac{1 - \frac{y}{y-1}}{\frac{y}{y+1} - 1}$$

$$14. \frac{1 - \frac{2}{3 - \frac{p}{2}}}{1 + \frac{2}{3 - \frac{4}{p}}}$$

$$2. \frac{1}{2 - \frac{1}{m-4}}$$

$$7. 1 - \frac{1}{1 - \frac{1}{1 - \frac{1}{q-1}}}$$

$$11. \frac{\frac{4}{z-6} - \frac{7}{z}}{\frac{5}{z} + \frac{7}{z-6}}$$

$$15. \frac{9a^{-2} - b^{-2}}{3a^{-1} + b^{-1}}$$

$$3. \frac{1}{1 - \frac{1}{x-1}}$$

$$8. \frac{6 + \frac{2}{x}}{3x + 1}$$

$$12. \frac{\frac{4}{x^2} - \frac{3}{x}}{\frac{1}{x^2} + \frac{2}{3x}}$$

$$16. \frac{x^{-3}y - 2x^2y^{-1}}{x^{-1} + y^{-1}}$$

$$4. \frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$$

$$9. \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

$$13. \frac{\frac{5}{x-3} - \frac{1}{x}}{\frac{2}{x} + \frac{3}{x-3}}$$

$$17. (3 - a^{-2})^{-2}$$

$$5. \frac{5 - \frac{x}{x-1}}{2 - \frac{x}{x+1}}$$

21.2 More on Linear Systems

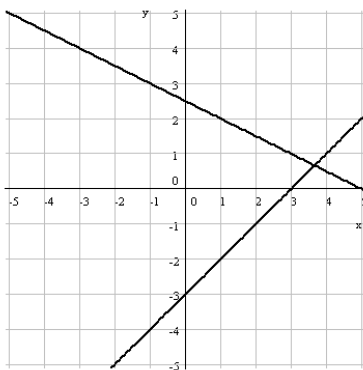
Recall that linear equations in x and y has many, many solutions, and one way to meaningfully represent these solutions is to graph them. Every linear equation can be graphed, and every point on the graph has coordinates that form a solution of the equation.

Therefore, a system of two linear equations can also be graphed, and the intersection point is a point whose coordinates form a simultaneous solution of both equations.

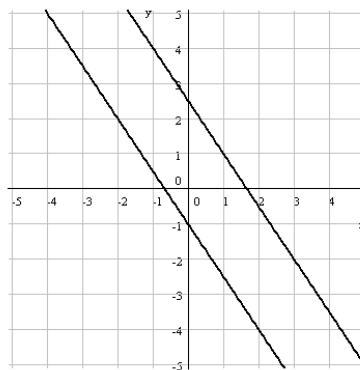
Most systems of linear equations have a unique solution (x, y) . However, this is not always the case.

If we think of a system of linear equations as two lines, it is clear that geometrically, there are three distinct ways two lines in a plane can behave.

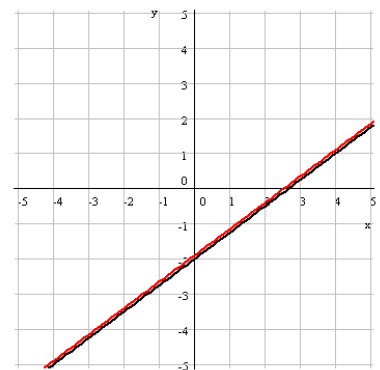
One intersection point



No intersection point



All points on the line are intersection points



Until now, we have only seen linear systems with a unique solution, corresponding to two lines intersecting each other in a unique point. However, there are linear systems that end up with different results.

Example 1. Solve the given system of linear equations.
$$\begin{cases} x - 2y = 6 \\ y = \frac{1}{2}x + 1 \end{cases}$$

Solution: Since the second equation is already solved for y , we will use substitution, but elimination would also work fine. We substitute $y = \frac{1}{2}x + 1$ into the first equation and solve the linear equation for x .

$$\begin{aligned} x - 2\left(\frac{1}{2}x + 1\right) &= 6 && \text{distribute 2} \\ x - x - 2 &= 6 && \text{combine like terms} \\ -2 &= 6 \end{aligned}$$

As we solve the equation for x , it disappears from the equation and we are left with an unconditionally false equation. We have seen this before when we solved linear equations. An equation such as $-2 = 6$ is called a contradiction, and it has no solution. What does this result mean for a system?

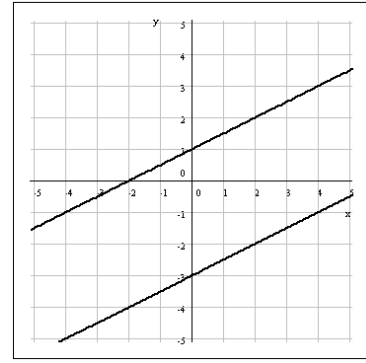
Let us solve the first equation for y .

$$\begin{aligned} x - 2y &= 6 && \text{add } 2y \\ x &= 2y + 6 && \text{subtract 6} \\ x - 6 &= 2y && \text{divide by 2} \\ \frac{x - 6}{2} &= y \implies y = \frac{x}{2} - \frac{6}{2} = \frac{1}{2}x - 3 \end{aligned}$$

Let us look again at the system, but this time from a geometric point of view. Both equations represent a straight line.

$$\begin{cases} y = \frac{1}{2}x - 3 \\ y = \frac{1}{2}x + 1 \end{cases} \quad \begin{array}{l} \text{These lines are parallel because they have the} \\ \text{same slope, } m = \frac{1}{2}, \text{ and parallel lines have no} \\ \text{intersection points.} \end{array}$$

In case of such a system, the last line is an unconditionally false equation, a contradiction. Such a system is called an **inconsistent system**, and there is no solution of it.



Example 2. Solve the given system of linear equations.
$$\begin{cases} 2x + 6y = -12 \\ y = -\frac{1}{3}x - 2 \end{cases}$$

Solution: Since the second equation is already solved for y , we will use substitution, but elimination would also work fine. We substitute $y = -\frac{1}{3}x - 2$ into the first equation and solve the linear equation for x .

$$\begin{aligned} 2x + 6\left(-\frac{1}{3}x - 2\right) &= -12 && \text{distribute 6} \\ 2x - 2x - 12 &= -12 && \text{combine like terms} \\ -12 &= -12 \end{aligned}$$

As we solve the equation for x , it disappears from the equation again, but this time we are left with an unconditionally true equation. We have seen this before when we solved linear equations. Such an equation is called an identity, and all real numbers are solutions of it.

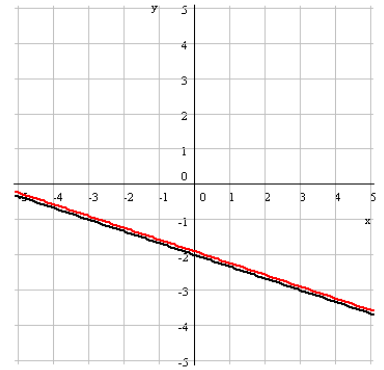
Just as before, we solve the first equation for y .

$$\begin{aligned} 2x + 6y &= -12 && \text{subtract } 2x \\ 6y &= -2x - 12 && \text{divide by 6} \\ y &= \frac{-2x - 12}{6} && \text{divide by 2} \\ y &= \frac{-2x - 12}{6} = -\frac{2x}{6} - \frac{12}{6} = -\frac{1}{3}x - 2 \end{aligned}$$

So, the linear system now look like this:
$$\begin{cases} y = -\frac{1}{3}x - 2 \\ y = -\frac{1}{3}x - 2 \end{cases}$$

$$\begin{cases} y = -\frac{1}{3}x - 2 \\ y = -\frac{1}{3}x - 2 \end{cases}$$

These lines are identical and so every point on the line is a solution. We can express this as the set of points $\left(x, -\frac{1}{3}x - 2\right)$.



In case of such a system, the last line is an unconditionally true equation, an identity. Such a system is called a **dependent system**, and all points on the line are solutions.



Practice Problems

1. Solve each of the following systems of linear equations.

a) $\begin{cases} x + 3y = 11 \\ 12y = -4x + 7 \end{cases}$

c) $\begin{cases} (x-2)^2 + y = (x+1)^2 - y \\ 2y = 6x - 3 \end{cases}$

b) $\begin{cases} 4y = 6x + 10 \\ 3x - 2y = -5 \end{cases}$

d) $\begin{cases} 5x - 6y = -10 \\ \frac{1}{2}x - \frac{3}{5}y = 1 \end{cases}$

2. Consider the linear system $\begin{cases} 2x + 3y = A \\ x = -\frac{3}{2}y - 12 \end{cases}$. Find the value of A so that the system

- a) has no solution b) has infinitely many solutions

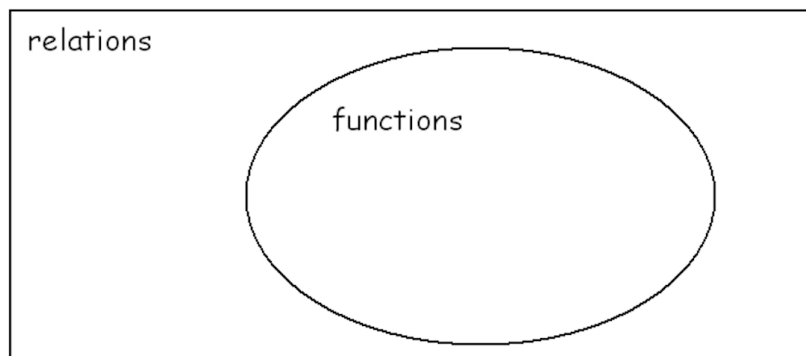
21.3 Functions

One of the most fundamental and important concepts in mathematics is that of a function. Functions also play a key role in sciences such as physics. What is a function?

Definition: A **relation** is an assignment between elements of a non-empty set called domain and another set called the range. We usually denote relations by lower case letters such as f or g .

Definition: A **function** is an assignment between elements of a non-empty set called domain and another set called the range, with one restriction: that to each element of the domain, only one thing can be assigned. We usually denote functions by lower case letters such as f or g .

We should notice the connection between relations and functions. The definitions are identical, with one exception: the restriction on the assignment in case of functions. So, functions are special relations. Every function is a relation, but not every relation is a function. Another way to say this is that the set of all functions is a subset of the set of all relations.



A relation or function can be given in numerous ways: by a description, by an equation, or a diagram, a table, or a graph. As long as the reader can determine the elements of the domain and range and the assignment, the relation is well defined.

Example 1. Functions given by description.

a) Let the domain be the set of all people currently living in the United States. Let us assign to each person their phone number. Is this a function?

The answer is no. This assignment fails the restriction that the assignment must be unique. Since many people have more than one phone number (cell phone, home phone, office, etc.) this assignment is not a function. This is an example of a relation that is not a function.

b) Let the domain be the set of all people currently living in the United States. Let us assign to each person their social security number. Is this a function?

Unless there is some fraud involved, every person has only one social security number. Because of that, this assignment is unique, and so it indeed defines a function.

What is the range in this example? The range is not the set of all 9-digit numbers, only the ones currently assigned to people living in the United States. Once the domain and the assignment are specified, those two determine the range. For this reason, it is usually customary to describe a function by specifying only its domain and assignment.

When we identify a function, we must specify its domain and its assignment.

c) Let the domain be the set of all people currently enrolled in this mathematics class. Let us assign to each person a grade of A or B or C or D or F, based on their current average. If the class has more than 5 students enrolled in it, we are certain that some people will be assigned the same grade. Is this a function?

It is a common reaction to think that this is not a function because some people will be assigned the same grade. However, this is a function as long as every student gets only one grade assigned to them. Because the assignment is unique, this is indeed a function. It would still be a function if every student got an A. If this is the case, we say that the function is constant. Constant functions may not be too interesting but they are well defined and even important functions.

d) Another very typical example for a function is that of location as a function of time. Consider an elevator. Suppose we monitor the location of this elevator at every minute for an hour. If the elevator is moving between two floors, we agree to wait until it stops for the first time and record that floor as its location. This is an excellent example for what a function is. It is perfectly normal for the elevator to sit at the same location for several minutes - so it also normal for several time-values in the domain to have the same floor-value assigned. An elevator sitting on the same floor for an hour would give us a constant function.

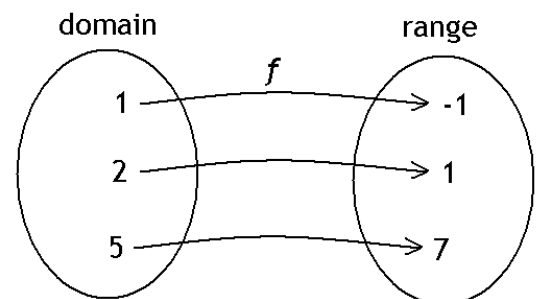
What would cause this assignment to fail as a function? The assignment would no longer be a function if there existed a time for which we would have two different floors assigned. For example, at the end of the 5th minute, the elevator is both on the 3rd floor and on the 10th floor. Our physical laws rule out that the same object could be at two different locations at the same time. So, location as a function of time is a very natural application of the concept of functions.

Example 2. Functions given by diagrams.

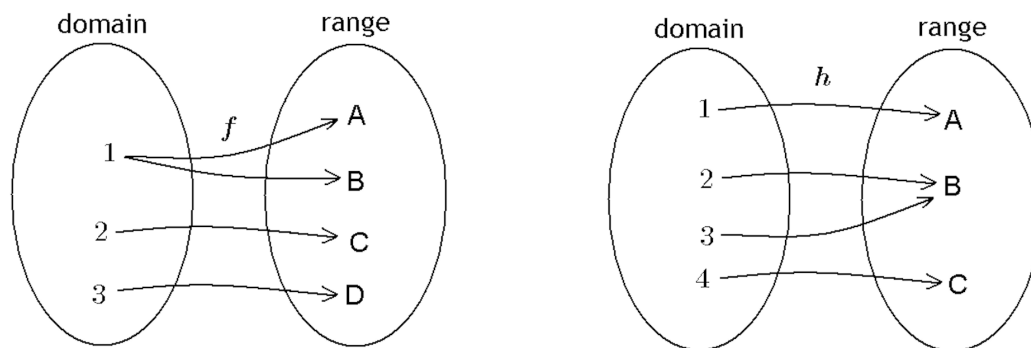
a) The diagram specifies a function f .

We can see that the domain is a set containing only three elements: $D = \{1, 2, 5\}$ the range is also a three-element set, $R = \{-1, 1, 7\}$ and f assigns -1 to 1 , 1 to 2 , and 7 to 5 . We introduce function notation to describe the assignment with less writing:

$$f(1) = -1, \quad f(2) = 1, \quad \text{and} \quad f(5) = 7$$



b) Let f and h be given as described by the diagrams shown below. Which of them is also a function? Why?



Solution: While h is a function, f is not. f is not a function because it assigns both A and B to 1. On the other hand, h is a function because to each element of the domain, only one thing is being assigned. Note the fact that h is a function even though $h(2) = h(3)$. This kind of "branching" is allowed in case of functions.

Example 3. Functions given as sets of ordered pairs. This involves simply listing all elements of the domain, together with their assigned values.

a) Suppose that f is given as $f = \{(2, 5), (3, 5), (5, 1)\}$. Is this a function?

The set $\{(2, 5), (3, 5), (5, 1)\}$ means that to 2, the value 5 was assigned; to 3, the value 5 was assigned, and to 5, the value 1 was assigned. Using function notation,

$$f(2) = 5, f(3) = 5, \text{ and } f(5) = 1$$

Thus the domain is $\{2, 3, 5\}$ and the range is $\{1, 5\}$. This is indeed a function.

b) Suppose that g is given as $g = \{(1, 4), (4, 9), (9, 1), (1, 5)\}$. Is this a function?

g is not a function because $(1, 4)$ and $(1, 5)$ means $g(1) = 4$ and $g(1) = 5$, so g assigns two values to the number 1.

Example 4. Functions given by tables or equations.

a) A function can also be described using a table. Consider the function f given by the table below.

element of domain	assigned element by f
1	-2
3	6
5	22
-4	13
10	97

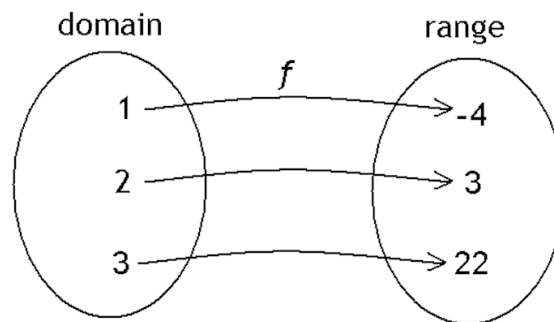
b) Functions can also be described by an equation. Of course, we still have to specify the domain.

Define g as follows: domain: $D = \{-7, -2, 0, 1, 4\}$ and $g(x) = 3x + 1$.

When a function is given in such a way, we can determine the assigned values using the equation of the function. For example, $g(-7)$ is the value assigned to -7 . We can compute $g(-7)$ using the definition of g , by replacing x with -7 .

$$\begin{aligned} g(x) &= 3x + 1 \\ g(-7) &= 3(-7) + 1 = -20 \implies g(-7) = -20 \end{aligned}$$

When are two functions equal to each other? Consider now the functions f and g as given below. f is given by a diagram and g is given by an equation.



Define g by domain $\{1, 2, 3, 4\}$ and the assignment $g(a) = a^3 - 5$. We can find the range by computing the values assigned to elements of the domain.

$$g(1) = 1^3 - 5 = -4 \qquad g(2) = 2^3 - 5 = 3 \qquad g(3) = 3^3 - 5 = 22 \qquad g(4) = 4^3 - 5 = 59$$

f and g have the same assignment, but have different domains. Can we say that these two functions are the same? Based on the current definition, the answer is no.

Definition: Two functions are equal to each other if they have the same domain and the same assignment.

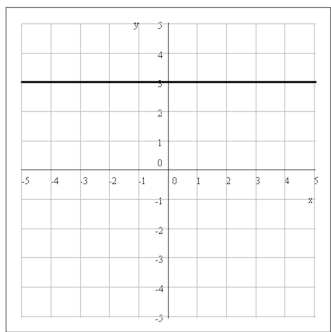
Basic Functions and Their Properties

1. Constant functions

$f(x) = c$ where c is a real number.

The graph is a horizontal line.

domain: $\mathbb{R}$ range: $\{c\}$



2. Linear functions

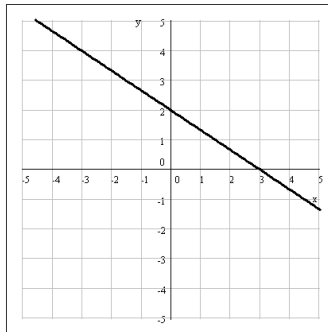
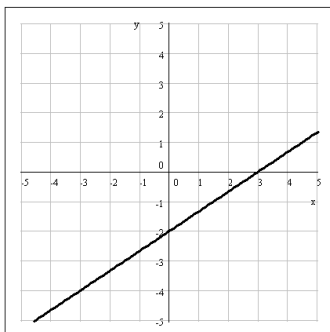
$f(x) = mx + b$ where $m \neq 0$. The graph is a straight line.

positive slope

$m > 0$

domain: $\mathbb{R}$

range: $\mathbb{R}$



negative slope

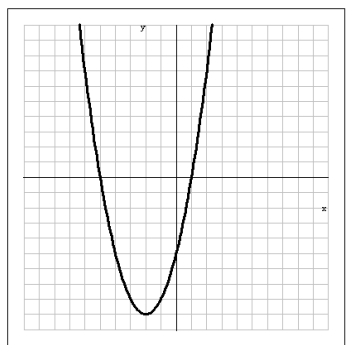
$m < 0$

domain: $\mathbb{R}$

range: $\mathbb{R}$

3. Quadratic functions $f(x) = ax^2 + bx + c$ where $a \neq 0$.

The graph is a parabola. It opens upward if $a > 0$ and opens downward if $a < 0$.



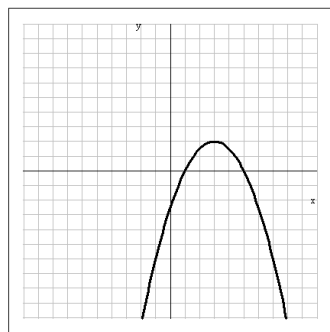
Case 1. If $a > 0$

Example: $f(x) = x^2 + 4x - 5$

standard form: $f(x) = (x + 2)^2 - 9$

factored form: $f(x) = (x + 5)(x - 1)$

domain: $\mathbb{R}$ range: $[-9, \infty)$



Case 2. If $a < 0$

Example: $f(x) = -\frac{1}{2}x^2 + 3x - \frac{5}{2}$

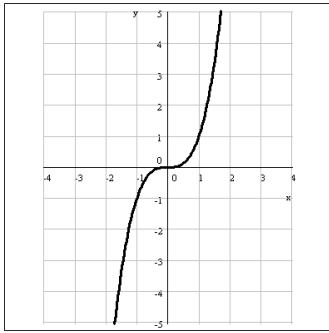
standard form: $f(x) = -\frac{1}{2}(x - 3)^2 + 2$

factored form: $f(x) = -\frac{1}{2}(x - 1)(x - 5)$

domain: $\mathbb{R}$ range: $(-\infty, 2]$

4. $f(x) = x^3$

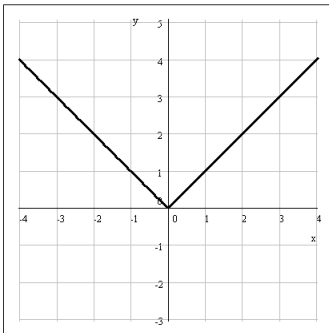
domain: $\mathbb{R}$ range: $\mathbb{R}$



5. **Absolute value function**

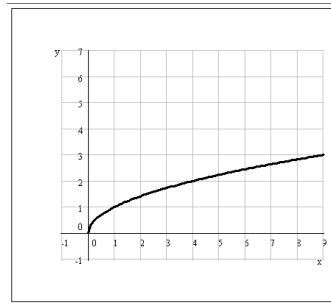
$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

domain: $\mathbb{R}$ range: $[0, \infty)$



6. $f(x) = \sqrt{x}$

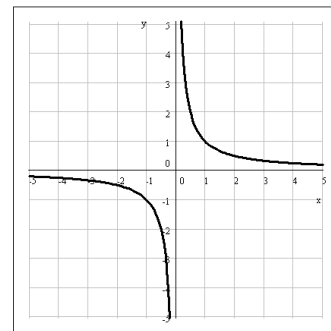
domain: $[0, \infty)$ range: $[0, \infty)$



7. $f(x) = \frac{1}{x}$

domain: (all real numbers except for 0) $\mathbb{R} \setminus \{0\}$

range: (all real numbers except for 0) $\mathbb{R} \setminus \{0\}$



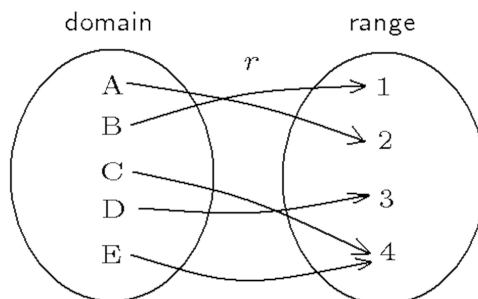
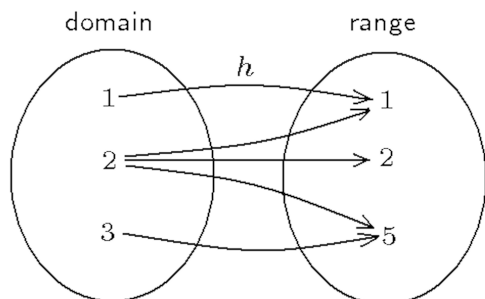
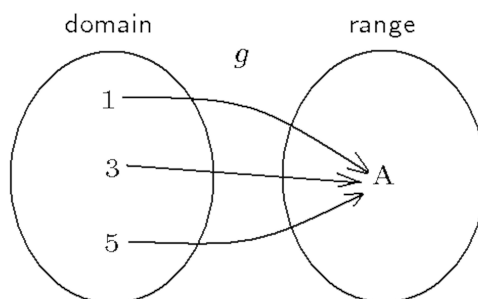
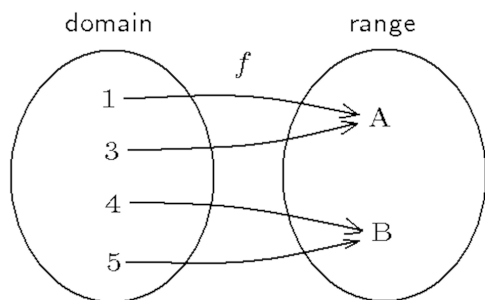


Sample Problems

1. Which of the following are functions?

- Let the domain be the set of students enrolled in this course. To each student, let us assign the number of minutes they spent studying yesterday for this class.
- Let the domain be the set of students enrolled in this course. To each student, let us assign their uncle.
- Let the domain be the set $D = \{0, 1, 4, 9\}$. To each number d in the domain, let us assign a number whose square is d . For example, $f(9) = 3$ because $3^2 = 9$.

2. Which of the following are functions?



3. The following relations are given in form of ordered pairs. Which of the following are functions?

- $f = \{(1, 4), (2, 7), (4, 1), (5, 7), (7, 7)\}$
- $g = \{(1, 1), (-2, 4), (2, 4), (3, 9), (-3, 9)\}$
- $h = \{(1, 1), (3, 5), (5, 3), (5, 5)\}$

4. Define f by its domain being $\mathbb{R}$, the set of all real numbers, and its assignment being $f(x) = -x^2 + 7$. Compute or simplify each of the following expressions.

- $f(3)$
- $f(\sqrt{6})$
- $f(5)$
- $f(2x)$
- $f(a-1)$

5. Define f as follows. The domain is $\mathbb{R}$, and the assignment is $f(t) = t^2 - t + 1$. Compute or simplify each of the following expressions.

- $f(0)$
- $f(-1)$
- $f(5)$
- $f(-5)$
- $f(2t)$

6. Define f as follows. The domain is $\mathbb{R}$, and the assignment is $f(x) = 2x^2 - 5$. Compute or simplify each of the following expressions.

- $f(2)$
- $f(3)$
- $f(5)$
- $f(2) + f(3)$
- $f(2+3)$

7. In each case, a pair of functions is given. Are they equal or not?

a) $f = \{(-2, 5), (0, 1), (2, 5), (3, 10)\}$ and g is defined by domain $\{-2, 0, 2, 3\}$ and assignment $g(a) = a^2 + 1$.

b) f is given by the table below

element of domain	assigned element by f
1	0
2	-3
3	-10
5	-24

and g is defined by domain $\{1, 2, 3, 5\}$ and assignment $g(x) = -x^2 + 1$.

c) $f = \{(0, -3), (1, 2), (2, 7), (3, 12)\}$ and g is defined by domain $\mathbb{R}$ and assignment $g(m) = 5m - 3$.

Problem Set 21

Chapter 22

22.1 Rational Expressions 2

We have been studying fractions formed from polynomials. Such abstract fractions are called rational expressions. In case of the usual fractions multiplication and division are simpler to perform than are addition and subtraction. This is because when we add or subtract fractions, we must bring them to the common denominator. This will also be the case in rational expressions. One difference is that when we subtract algebraic expressions, we need to make sure not to violate the distributive law.

Example 1. Perform the indicated operations.

$$\text{a) } \frac{5}{6} - \frac{1}{4} \qquad \text{b) } \frac{2x-1}{10} - \frac{x-3}{8}$$

Solution: a) We need to bring the fractions to the common denominator. Recall that we may multiply both numerator and denominator of fractions without changing their values. The least common multiple of 4 and 6 is 12.

To re-write $\frac{5}{6}$ with a denominator of 12, we multiply both numerator and denominator by 2. To re-write $\frac{1}{4}$ with a denominator of 12, we multiply both numerator and denominator by 3. Once they have the same denominator, we subtract the numerators.

$$\frac{5}{6} - \frac{1}{4} = \frac{5 \cdot 2}{6 \cdot 2} - \frac{1 \cdot 3}{4 \cdot 3} = \frac{10}{12} - \frac{3}{12} = \frac{10-3}{12} = \boxed{\frac{7}{12}}$$

b) The computation with rational expressions is similar. The only difference is between subtracting numbers and subtracting algebraic expressions. We need to watch out for the distributive law.

The least common multiple of 10 and 8 is 40. To re-write $\frac{2x-1}{10}$ with a denominator of 40, we multiply both numerator and denominator by 4. To re-write $\frac{x-3}{8}$ with a denominator of 40, we multiply both numerator and denominator by 5. Once they have the same denominator, we subtract the numerators. This time however, the numerators are algebraic expressions.

$$\begin{aligned} \frac{2x-1}{10} - \frac{x-3}{8} &= \frac{(2x-1) \cdot 4}{10 \cdot 4} - \frac{(x-3) \cdot 5}{8 \cdot 5} = \frac{4(2x-1)}{40} - \frac{5(x-3)}{40} = \frac{4(2x-1) - 5(x-3)}{40} \\ &= \frac{8x-4-5x+15}{40} = \boxed{\frac{3x+11}{40}} \end{aligned}$$

Note: It is good strategy to wait with distributing 4 and 5 in the numerators. Consider the following computations:

$$\frac{4(2x-1)}{40} - \frac{5(x-3)}{40} = \frac{8x-4}{40} - \frac{5x-15}{40} = \frac{8x-4-5x-15}{40} \text{ is incorrect}$$

$$\frac{4(2x-1)}{40} - \frac{5(x-3)}{40} = \frac{8x-4}{40} - \frac{5x-15}{40} = \frac{8x-4-(5x-15)}{40} \text{ is correct}$$

$$\frac{4(2x-1)}{40} - \frac{5(x-3)}{40} = \frac{4(2x-1)-5(x-3)}{40} = \frac{8x-4-5x+15}{40} \text{ is safer}$$

Example 2. Perform the indicated operations.

$$\text{a) } \frac{7x-1}{x^2-1} - \frac{4}{x+1} \quad \text{b) } \frac{6}{2x+x^2} + \frac{1-x}{x+2}$$

Solution: a) In case of numbers, we do not have to find the least common denominator, any common multiple will get us to the correct answer. However, in case of rational expressions we need to find the least common multiple. Otherwise, we might end up with polynomials of degree too high for us to factor. To find the least common multiple, we need to factor first the denominators.

$$\frac{7x-1}{x^2-1} - \frac{4}{x+1} = \frac{7x-1}{(x+1)(x-1)} - \frac{4}{x+1}$$

The common denominator is $(x+1)(x-1)$. To bring $\frac{4}{x+1}$ to the common denominator, we multiply both numerator and denominator by $x-1$.

$$\begin{aligned} \frac{7x-1}{x^2-1} - \frac{4}{x+1} &= \frac{7x-1}{(x+1)(x-1)} - \frac{4}{x+1} = \frac{7x-1}{(x+1)(x-1)} - \frac{4(x-1)}{(x+1)(x-1)} = \frac{7x-1-4(x-1)}{(x+1)(x-1)} \\ &= \frac{7x-1-4x+4}{(x+1)(x-1)} = \frac{3x+3}{(x+1)(x-1)} \end{aligned}$$

Just as with fractions, we need to reduce the result to lowest terms. To do that, we factor the numerator and check for common factors to cancel out.

$$\frac{3x+3}{(x+1)(x-1)} = \frac{3(x+1)}{(x+1)(x-1)} = \frac{\cancel{3(x+1)}}{\cancel{(x+1)}(x-1)} = \boxed{\frac{3}{x-1}}$$

b) The first denominator is $x^2+2x = x(x+2)$. The other denominator is $x+2$. Therefore, the common denominator is $x(x+2)$.

$$\begin{aligned} \frac{6}{2x+x^2} + \frac{1-x}{x+2} &= \frac{6}{x(x+2)} + \frac{(1-x) \cdot x}{(x+2) \cdot x} = \frac{6}{x(x+2)} + \frac{x(1-x)}{x(x+2)} = \frac{6+x(1-x)}{x(x+2)} \\ &= \frac{6+x(1-x)}{x(x+2)} = \frac{6+x-x^2}{x(x+2)} = \frac{-x^2+x+6}{x(x+2)} \end{aligned}$$

Before we present the final answer, we need to make sure it is reduced to lowest terms. So we need to factor the numerator, $-x^2+x+6$.

$$-x^2+x+6 = -(x^2-x-6) = -(x-3)(x+2)$$

$$\text{So our final answer is } \frac{-x^2+x+6}{x(x+2)} = \frac{-(x-3)(x+2)}{x(x+2)} = \frac{-(x-3)\cancel{(x+2)}}{x\cancel{(x+2)}} = \boxed{\frac{-x+3}{x}}$$



Sample Problems

Perform the indicated operations and simplify.

$$1. \frac{3x}{x-2} - \frac{x+4}{x-2}$$

$$4. \frac{2}{p-5} - \frac{p+11}{p^2-2p-15}$$

$$6. \left[\left(\frac{1}{b} - b \right) \div \left(1 - \frac{1}{b} \right) \right] (1+b)$$

$$2. \frac{10}{x-y} - \frac{5}{y-x}$$

$$5. \frac{x^2-5x+78}{18x+x^2-208} - \frac{x}{x+26}$$

$$3. \frac{1}{x-y} - \frac{1}{x+y}$$



Practice Problems

Perform the indicated operations and simplify.

$$1. \frac{x+y}{2} - \frac{x-y}{2}$$

$$4. \frac{5y+1}{y+5} + \frac{y^2-1}{y+5}$$

$$7. \frac{3}{x+1} - \frac{2}{x-1} + \frac{x+3}{x^2-1}$$

$$2. \frac{3a}{a-1} - \frac{a+2}{a-1}$$

$$5. \frac{1}{p-5} - \frac{1}{p+5}$$

$$8. \frac{2(m+2)}{4m+m^2-12} - \frac{1}{m-2}$$

$$3. \frac{a+1}{a-b} - \frac{b-1}{b-a}$$

$$6. \frac{x-5}{x+2} - \frac{3}{2-x} - \frac{14-x}{x^2-4}$$

$$9. \frac{1}{x-2} - \frac{3}{x+6}$$

22.2 Rational Equations

In what follows, we will study rational equations. As we will see, rational equations are quite interesting. They often require nonequivalent steps. Recall first a few definitions.

Definition: When solving equations, an **equivalent step** is one that preserves the solution set from one line to the next. If a step is not like that, we call it a **non-equivalent step**.

Most steps we have been using in solving equations are equivalent steps. However, solving rational equations will sometimes require nonequivalent steps.

Definition: A **rational expression** is a quotient of two polynomials.

For example, $\frac{x-2}{x+7}$, $\frac{x+8}{x^2+4}$, and $\frac{a^2-2a+1}{a^2-9}$ are all rational expressions. Rational expressions are slightly different from the algebraic expressions we have seen so far. If we consider algebraic expressions such as x^2 , $|2x-3|+1$ and x^3-x^2+5x-7 can be evaluated for every value of x . Every real number can be squared. On the other hand, -7 will not work well with $\frac{x-2}{x+7}$. This is because of division by zero: if $x = -7$, then

$$\frac{x-2}{x+7} = \frac{-7-2}{-7+7} = \frac{-9}{0} = \text{undefined}$$

Definition: The **domain of an expression** is the set of all real numbers that can be meaningfully substituted into the expression.

This is to say that the domain of an expression is the set of real numbers for which we get a number and not undefined. For example, the domain of $\frac{x-2}{x+7}$ is the set of all real numbers except $x = -7$. In set notation, it is $\{x \in \mathbb{R} : x \neq -7\}$. Or the domain of the expression $\sqrt{x}$ is the set of all non-negative real numbers, $\{x \in \mathbb{R} : x \geq 0\}$ or $[0, \infty)$ in interval notation. This is because the square root of a negative number is undefined.

At this point, there are only two ways we can get 'undefined' when evaluating expressions are: division by zero and a negative number under an even root such as $\sqrt{\quad}$, $\sqrt[4]{\quad}$, $\sqrt[6]{\quad}$, etc.

Example 1. Find the domain for each of the following expressions.

a) $\frac{x+6}{x^2-25}$ b) $\frac{2x}{x^2-4x-12}$ c) $\frac{x^2-1}{x^2+1}$

Solution: a) The only way this expression can be undefined is by division by zero. We know this because there are no radicals in the expression. Every number can be squared or added 6 to it. The only way this expression can be undefined because of division by zero. Therefore, the domain of $\frac{x+6}{x^2-25}$ is

the set of all real numbers, for which the denominator is not zero. We solve the quadratic equation $x^2-25=0$ and get that $x = -5$ and 5 are the solutions. These are the only two numbers for which we get undefined. So the domain of this expression is $\boxed{\{x \in \mathbb{R} : x \neq \pm 5\}}$.

- b) Since there are no radicals in this expression, the only worry is division by zero. The misbehaving numbers will only be those for which the denominator is zero. So we solve the equation $x^2 - 4x - 12 = 0$.

$$\begin{aligned}x^2 - 4x - 12 &= 0 \\(x+2)(x-6) &= 0 \implies x_1 = -2, x_2 = 6\end{aligned}$$

Therefore, the domain of $\frac{2x}{x^2 - 4x - 12}$ can't have the absolute value of a number be 0? There is only one way: if the number is 0. Therefore, the domain of $\frac{2x}{x^2 - 4x - 12}$ is the set of all real numbers except for -2 and 6 .

6. Using set notation, this is $\{x \in \mathbb{R} : x \neq -2, 6\}$. The same set can be also expressed using interval notation: $(-\infty, -2) \cup (2, 6) \cup (6, \infty)$.

- c) Since there are no radicals in this expression, the only worry is division by zero. The only problematic numbers will only be those for which the denominator is zero. So we solve the equation $x^2 + 1 = 0$. The expression $x^2 + 1$ cannot be factored over the real numbers. The equation $x^2 = -1$ has no real solution. Therefore, there is no danger of division by zero. We will always get a real number, no

matter what number we substitute for x . So the domain of $\frac{x^2 - 1}{x^2 + 1}$ is the set of all real numbers, $\mathbb{R}$.

The same set can be also expressed in interval notation as $(-\infty, \infty)$.

When solving rational equations, the domain of the expressions plays an interesting and sometimes significant role.

Example 2. Solve the given rational equation. $\frac{3x-1}{x-8} = \frac{2x+7}{x-8}$

Solution: Both sides of the equation are rational expressions with the same denominator. We could clear the denominator by multiplying both sides by $x-8$. This way the equation will become a simple linear one.

$$\begin{aligned}\frac{3x-1}{x-8} &= \frac{2x+7}{x-8} && \text{multiply by } x-8 \\3x-1 &= 2x+7 && \text{subtract } 2x \\x-1 &= 7 && \text{add } 1 \\x &= 8\end{aligned}$$

We check: if $x = 8$, then $\text{LHS} = \frac{3 \cdot 8 - 1}{8 - 8} = \frac{23}{0}$ = undefined. We do not even need to consider the other side. Since 8 makes one side undefined, it cannot be a solution of this equation. Therefore, this equation has no solution.

What happened here? Why did we get a solution that proved to be not a solution? Such a solution is called an **extraneous solution**. The number 8 is a solution of the second line, $3x - 1 = 2x + 7$, but not of the first line, $\frac{3x-1}{x-8} = \frac{2x+7}{x-8}$. Therefore, multiplying by $x-8$ proved to be a non-equivalent step. This is because multiplying by $x-8$ changed the domain of the expressions making up the equation. 8 is in the domain of the second line, but not of the first line.

There are several lessons from this: first, that we HAVE to check the solutions of a rational equation. In case of equivalent steps, checking is a matter of making sure we didn't make a mistake. In this case, we could have an extraneous solution even if we did everything correctly.

Example 3. Solve the given rational equation. $x - \frac{6x}{x+3} = \frac{x+21}{x+3}$

Solution: In order to clear the denominator, we will multiply both sides by $x+3$. Because this is a nonequivalent step, we must check our solution.

$$\begin{aligned}
 x - \frac{6x}{x+3} &= \frac{x+21}{x+3} && \text{multiply by } x+3 \\
 (x+3) \left(x - \frac{6x}{x+3} \right) &= (x+3) \left(\frac{x+21}{x+3} \right) \\
 (x+3)x - \cancel{(x+3)} \cdot \frac{6x}{\cancel{x+3}} &= \cancel{(x+3)} \cdot \left(\frac{x+21}{\cancel{x+3}} \right) \\
 x(x+3) - 6x &= x+21 \\
 x^2 + 3x - 6x &= x+21 && \text{combine like terms} \\
 x^2 - 3x &= x+21 && \text{subtract } x \\
 x^2 - 4x &= 21 && \text{subtract 21} \\
 x^2 - 4x - 21 &= 0 && \text{factor} \\
 (x-7)(x+3) &= 0 \implies x_1 = 7 \quad x_2 = -3
 \end{aligned}$$

We check: if $x = 7$, then

$$\begin{aligned}
 \text{LHS} &= 7 - \frac{6 \cdot 7}{7+3} = 7 - \frac{42}{10} = 7 - \frac{21}{5} = \frac{35}{5} - \frac{21}{5} = \frac{14}{5} \quad \text{and} \\
 \text{RHS} &= \frac{7+21}{7+3} = \frac{28}{10} = \frac{14}{5} \quad \checkmark
 \end{aligned}$$

and so 7 is a solution of the equation.

If $x = -3$, then

$$\text{LHS} = -3 - \frac{6(-3)}{-3+3} = -3 - \frac{-18}{0} = \text{undefined}$$

Therefore, $x = -3$ is not a solution of the equation. The only solution is $\boxed{7}$.

Naturally, not all rational equations have extraneous solutions. It is also worth noting that after we cleared the denominator, we can end up with either a linear or a quadratic equation.



Sample Problems

Solve each of the given equations.

1. $\frac{3x}{2x-1} = 6$

3. $\frac{x}{x+1} + \frac{4}{x+7} = \frac{5x-1}{x^2+8x+7}$

5. $\frac{3}{x-5} - \frac{1}{x+3} = \frac{10}{x^2-2x-15}$

2. $\frac{5}{x+7} - \frac{1}{6} = \frac{8}{4x+28}$

4. $\frac{1}{4y} - \frac{1}{2} = \frac{1}{y}$

6. $\frac{x}{x-7} - \frac{3}{x+3} = \frac{4x+42}{x^2-4x-21}$



Practice Problems

Solve each of the given equations.

$$1. \frac{3x}{7x+6} = 1$$

$$2. \frac{6}{3x-1} = 2$$

$$3. \frac{3}{x-2} - \frac{8}{x+2} = \frac{1}{6}$$

$$4. \frac{x}{x-3} + \frac{2}{x+9} = \frac{19x-21}{x^2+6x-27}$$

$$5. \frac{x}{x+6} + \frac{5}{x+2} = \frac{-8x-24}{x^2+8x+12}$$

$$6. \frac{12}{x-5} - \frac{6}{x-2} = 3$$

$$7. \frac{x}{x+3} - \frac{1}{x-7} = \frac{-4x+18}{x^2-4x-21}$$

$$8. \frac{1}{x-6} - 1 = \frac{5}{x-6}$$

$$9. \frac{3}{x-7} + \frac{x+7}{x} = \frac{7x-28}{x^2-7x}$$

$$10. \frac{10}{x} - \frac{4}{x-1} = 1$$

$$11. \frac{x}{x-2} = \frac{x-2}{x}$$

$$12. \frac{12}{x-1} + \frac{14}{x} = 4$$

$$13. \frac{x}{x+1} + \frac{7}{x-3} = \frac{-x+3}{x^2-2x-3}$$

$$14. \frac{2x-1}{x+3} - \frac{1}{x-7} = 2 + \frac{38-6x}{x^2-4x-21}$$

Problem Set 22

Chapter 23

23.1 Radical Equations

We will now study radical equations. As the name suggests, these are equations with square roots or cubic roots, etc. Let us first recall a few facts.

Example 1. Simplify each of the given expressions.

$$\text{a) } (\sqrt{x-2})^2 \quad \text{b) } (\sqrt{x}-2)^2 \quad \text{c) } (-3\sqrt{x})^2$$

Solution: a) Recall that $\sqrt{5}$ is the non-negative number whose square is 5. Similarly, $\sqrt{x-2}$ is the non-negative quantity that, when squared, the result is $x-2$. That is exactly what happens here and so

$$(\sqrt{x-2})^2 = \boxed{x-2}$$

b) This is a different situation. $\sqrt{x}$ is the non-negative number, that, when squared, the result is x . To square $\sqrt{x}-2$, we need to apply the distributive property.

$$(\sqrt{x}-2)^2 = (\sqrt{x}-2)(\sqrt{x}-2) = \sqrt{x}\sqrt{x} - 2\sqrt{x} - 2\sqrt{x} + 4 = (\sqrt{x})^2 - 4\sqrt{x} + 4 = \boxed{x-4\sqrt{x}+4}$$

This computation shows that the radical is not always eliminated just because we square the expression.

$$\text{b) } (-3\sqrt{x})^2 = (-3\sqrt{x})(-3\sqrt{x}) = 9(\sqrt{x})^2 = \boxed{9x}$$

When we are dealing with radical expressions and want to get rid of radicals, squaring is useful for expressions such as $\sqrt{x-2}$ or $-3\sqrt{x}$, but not for expressions such as $\sqrt{x}-2$. This will be important to keep in mind.

Also recall that when solving an equations, an **equivalent step** is one that does not change the solution set.

Example 2. Find all real solutions of the equation $\sqrt{x-1} = -x+3$

Solution: If we square both sides of the equation, the radical on the left will disappear, and the expression on the right becomes quadratic, resulting in a quadratic equation. So it looks like it is a good idea to square both sides first and then solve the quadratic equation we just created.

$$\sqrt{x-1} = -x+3 \quad \text{square both sides}$$

$$x-1 = (-x+3)^2 \quad \text{expand complete square}$$

$$x-1 = x^2-6x+9 \quad \text{subtract } x$$

$$-1 = x^2-7x+9 \quad \text{add 1}$$

$$0 = x^2 - 7x + 10$$

$$0 = (x - 2)(x - 5)$$

$$x_1 = 2 \quad x_2 = 5$$

It appears that this equation has two solutions, 2 and 5. Let us check.

If $x = 2$, then $\text{LHS} = \sqrt{2-1} = \sqrt{1} = 1$ and $\text{RHS} = -2 + 3 = 1$ and so $x = 2$ works.

If $x = 5$, then $\text{LHS} = \sqrt{5-1} = \sqrt{4} = 2$ and $\text{RHS} = -5 + 3 = -2$, and $2 \neq -2$, so $x = 5$ is not a solution of this equation. Thus this equation has one solution, $\boxed{x = 2}$.

Until now, checking a solution was just a matter of making certain that we did not make a mistake. This is a different situation: our computation was correct, and yet we have a number that is a solution of the last equation, but not of the first. This is because squaring both sides of an equation is a **non-equivalent step** that increases the solution set.

When we substituted $x = 5$ into the original equation, we had $2 = -2$, which is a false statement. But next we squared both sides, and the false statement $2 = -2$ became $4 = 4$, which is true. This is how $x = 5$ works after we squared both sides, but not before.

If we square both sides of an equation $L = R$, and solve the equation $L^2 = R^2$, then some of the solutions of $L^2 = R^2$ might be numbers for which $L = -R$. Such a number is called an **extreaneous solution**. When we square both sides of an equation, we *must* check our solution(s) because squaring both sides of an equation is a non-equivalent step. In our previous example, $x = 5$ was an extreaneous solution.

Example 3. Find all real solutions of the equation $11 = \sqrt{4x+1} + x$

Solution: If we squared both sides as they are, we would not eliminate the radical. Recall that sums involving radicals do not respond well to squaring both sides. Before we square, we need to isolate the radical expression on one side. Therefore, we will start by subtracting x .

$$\begin{aligned} 11 &= \sqrt{4x+1} + x && \text{subtract } x \\ 11 - x &= \sqrt{4x+1} && \text{square both sides} \\ (11 - x)^2 &= 4x + 1 \\ x^2 - 22x + 121 &= 4x + 1 && \text{subtract } x, \text{ subtract } 1 \\ x^2 - 26x + 120 &= 0 \\ (x - 6)(x - 20) &= 0 \implies x_1 = 6, \quad x_2 = 20 \end{aligned}$$

We check: If $x = 6$, then $\text{RHS} = \sqrt{4 \cdot 6 + 1} + 6 = \sqrt{25} + 6 = 5 + 6 = 11 = \text{LHS} \checkmark$

If $x = 20$, then $\text{RHS} = \sqrt{4 \cdot 20 + 1} + 20 = \sqrt{81} + 20 = 9 + 20 = 29 \neq 11$. Thus $x = 20$ is not a solution.

The only solution is $\boxed{x = 6}$.

Why would $x = 20$ show up as a solution? Let us substitute 20 into the equation we squared. That is the second line, $11 - x = \sqrt{4x+1}$. If we substitute 20 into x , we get $-9 = 9$. That is a false statement, but we see that it will become true after squaring both sides.

We have only seen equations so far that had one solution and one extreaneous solution. This is not the only possibility: some radical equations have two solution, some have none.

Example 4. Find all real solutions of the equation $\sqrt{3x+1} - \sqrt{x-1} = 2$

Solution: If we squared both sides as they are, we would be left with expressions such as $2\sqrt{3x+1}\sqrt{x-1}$. To avoid that, we should isolate the radical expression. In this equation, however, there are two radical expressions, and we cannot isolate both. We should select the more complicated one, isolate that and square. Then we will only left with one radical expression, so we repeat the process of isolating it and then squaring both sides.

$$\begin{aligned}\sqrt{3x+1} - \sqrt{x-1} &= 2 && \text{add } \sqrt{x-1} \\ \sqrt{3x+1} &= \sqrt{x-1} + 2 && \text{square} \\ 3x+1 &= (\sqrt{x-1} + 2)^2\end{aligned}$$

To expand $(\sqrt{x-1} + 2)^2$, we apply the distributive law:

$$\begin{aligned}(\sqrt{x-1} + 2)^2 &= (\sqrt{x-1} + 2)(\sqrt{x-1} + 2) = \sqrt{x-1}\sqrt{x-1} + 2\sqrt{x-1} + 2\sqrt{x-1} + 4 \\ &= (\sqrt{x-1})^2 + 4\sqrt{x-1} + 4 = x-1 + 4\sqrt{x-1} + 4 = x + 4\sqrt{x-1} + 3\end{aligned}$$

and so our equation is

$$\begin{aligned}3x+1 &= (\sqrt{x-1} + 2)^2 \\ 3x+1 &= x + 4\sqrt{x-1} + 3 && \text{subtract } x \\ 2x+1 &= 4\sqrt{x-1} + 3 && \text{subtract } 3 \\ 2x-2 &= 4\sqrt{x-1}\end{aligned}$$

Notice that all coefficients are even. Therefore, we may divide both sides by 2. This is a step that is not necessary, but it saves us work as the numbers don't get as large.

$$\begin{aligned}2x-2 &= 4\sqrt{x-1} && \text{divide by } 2 \\ x-1 &= 2\sqrt{x-1} && \text{square both sides} \\ (x-1)^2 &= (2\sqrt{x-1})^2 \\ x^2 - 2x + 1 &= 4(x-1) \\ x^2 - 2x + 1 &= 4x - 4 && \text{subtract } 4x \\ x^2 - 6x + 1 &= -4 && \text{add } 4 \\ x^2 - 6x + 5 &= 0 \\ (x-1)(x-5) &= 0 && \implies x_1 = 1, x_2 = 5\end{aligned}$$

We check both candidates. If $x = 1$, then

$$\text{LHS} = \sqrt{3 \cdot 1 + 1} - \sqrt{1 - 1} = \sqrt{4} - \sqrt{0} = 2 - 0 = 2 = \text{RHS} \checkmark$$

and if $x = 5$, then

$$\text{LHS} = \sqrt{3 \cdot 5 + 1} - \sqrt{5 - 1} = \sqrt{16} - \sqrt{4} = 4 - 2 = 2 = \text{RHS} \checkmark$$

In case of this equation, both 1 and 5 are solutions.



Enrichment

1. Consider the equation $x^2 - x - 10 = 8 + \sqrt{x^2 - x - 16}$

The problem here is that if we isolate the radical expression and square, we will end up with an equation of degree 4. It is worth a try, as we can solve some degree 4 equations, but chances are that this one would be tougher. Here is another method.

Let us introduce a new variable, $a = \sqrt{x^2 - x - 16}$. Then $x^2 - x - 10$ on the left-hand side can be written as $x^2 - x - 10 = x^2 - x - 16 + 6 = a^2 + 6$.

Substituting a on both sides, our equation becomes $a^2 + 6 = 8 + a$. Solve for a . Once you have a , solve for x .



Sample Problems

1. $\sqrt{3x-2} = x$

5. $2\sqrt{x-1} = x-4$

9. $2\sqrt{2x+29} = x+12$

2. $x + 2\sqrt{2x+16} = -2$

6. $x = 2\sqrt{x+8}$

10. $\sqrt{x+10} + 10 = x$

3. $10 + \sqrt{4x-7} = 7$

7. $5\sqrt{x} + 1 = 3\sqrt{x} + 17$

4. $5 + \sqrt{x+15} = x$

8. $\sqrt{2x+5} + 5 = x$



Practice Problems

1. $\sqrt{3x-5} = 4$

8. $\sqrt{18+x} = x-2$

15. $2\sqrt{x+1} = x+2$

2. $2\sqrt{a-1} + 7 = 1$

9. $w + 13 = 2\sqrt{3w+30}$

16. $m - 4 = 2\sqrt{2m-8}$

3. $3\sqrt{7x+1} + 2 = 20$

10. $3\sqrt{4x-39} = x-3$

17. $-k - 1 = -2\sqrt{k+16}$

4. $\sqrt{x+3} = x-9$

11. $-4a = 3\sqrt{2a+22}$

18. $\sqrt{1-x} + 1 = x+12$

5. $2\sqrt{x+5} = x-3$

12. $2y - 3 = \sqrt{2y+3}$

19. $2 = \sqrt{x^2+1} - x$

6. $p + 1 = 3\sqrt{p+1}$

13. $\sqrt{b-2} + b = 8$

7. $2\sqrt{x+5} = x+6$

14. $\sqrt{3x+1} - x = -1$

20. $3\sqrt{4t-35} = t-6$

23.2 Work Word Problems

Work word problems are quite different from the word problems we have seen so far. As we will soon see, the labeling of unknowns and the subsequent equation is unusual at first.

Example 1. It takes Randy four hours to paint a room. It takes Cecile six hours to paint the same room. How long does it take for them to paint the room together?

Solution: Let us denote the job of painting the room by 1. If Randy paints the room in four hours, then he paints $\frac{1}{4}$ of the room in an hour, $\frac{2}{4}$ of the room in two hours, $\frac{3}{4}$ of the room in three hours. In four hours, he paints $\frac{4}{4}$ of the room, which is what we denoted by 1.

Similarly, Cecile paints $\frac{1}{6}$ of the room in an hour, $\frac{2}{6}$ in two hours, $\frac{3}{6}$ in three hours, etc.

Let x denote the time it takes for the two of them to paint the room, measured in hours. In x hours, Randy paints $\frac{x}{4}$ of the room and Cecile paints $\frac{x}{6}$ of the room. Together, they paint the entire room, denoted by 1.

$$\begin{aligned} \frac{x}{4} + \frac{x}{6} &= 1 && \text{bring to common} \\ &&& \text{denominator} \\ \frac{3x + 2x}{12} &= 1 \end{aligned}$$

$$\begin{aligned} \frac{5x}{12} &= 1 && \text{multiply by 12} \\ 5x &= 12 && \text{divide by 5} \end{aligned}$$

$$x = \frac{12}{5} = 2.4$$

So it takes them together 2.4 hours to paint the room. If we convert 0.4 hours to minutes, we get

$$0.4\text{h} = \frac{0.4\cancel{\text{h}}}{1} \cdot \frac{60\text{min}}{1\cancel{\text{h}}} = \frac{24\text{min}}{1} = 24\text{min}$$

Our answer is that they will need 2.4 hours or 2 hours and 24 minutes to paint the room together.

Example 2. A swimming pool can be filled by three water pipes, A, B and C. Pipe A alone can fill the pool in 8 hours. Pipe B alone can fill the pool in 10 hours, and pipe C can fill it in 12 hours. How long does it take to fill the swimming pool using all three pipes?

Solution: Although the situation sounds different, mathematically this problem is similar to the previous example.

If pipe A alone can fill the pool in 8 hours, then it fills up $\frac{1}{8}$ of the pool in an hour, $\frac{2}{8}$ in two hours, and $\frac{x}{8}$ in x hours. Pipe B fills up the pool in 10 hours, so it fills up $\frac{1}{10}$ in one hour and $\frac{x}{10}$ in x hours. Similarly, pipe C fills up $\frac{1}{12}$ in an hour and $\frac{x}{12}$ in x hours. We set up the equation as before.

$$\begin{aligned} \frac{x}{8} + \frac{x}{10} + \frac{x}{12} &= 1 && \text{factor out } x \\ x \left(\frac{1}{8} + \frac{1}{10} + \frac{1}{12} \right) &= 1 \\ x \cdot \frac{15 + 12 + 10}{120} &= 1 \end{aligned}$$

$$\begin{aligned} x \cdot \frac{37}{120} &= 1 && \text{divide by } \frac{37}{120} \\ x &= 1 \div \frac{37}{120} = 1 \cdot \frac{120}{37} = \frac{120}{37} \end{aligned}$$

Therefore, it will take $\frac{120}{37} \approx 3.2432$ hours to fill the pool using all three pipes. For a more precise answer, we can convert the answer to minutes.

$$\frac{120}{37}\text{h} = \frac{120\cancel{\text{h}}}{37} \cdot \frac{60\text{min}}{1\cancel{\text{h}}} = \frac{7200\text{min}}{37} \approx 194.6\text{min}$$

It will take 194.6 minutes or 3 hours and 14.6 minutes for the three pipes to fill the pool together.

Example 3. Ann can paint her room in 10 hours. If Ann and Britney work together, it takes them 6 hours to paint the room. How long would it take for Britney to paint the room alone?

Solution: Let x denote the time it takes for Britney to paint the room. The equation will express that together it will take 6 hours to paint the room.

$6\left(\frac{1}{10} + \frac{1}{x}\right) = 1 \quad \text{distribute 6}$ $\frac{6}{10} + \frac{6}{x} = 1 \quad \text{subtract } \frac{6}{10} = \frac{3}{5}$ $\frac{6}{x} = 1 - \frac{3}{5}$	$\frac{6}{x} = \frac{2}{5} \quad \text{take reciprocal of both sides}$ $\frac{x}{6} = \frac{5}{2} \quad \text{multiply by 6}$ $x = \frac{30}{2} = 15$
---	---

Thus it will take 15 hours for Britney to paint the room alone.

Example 4. A swimming pool can be filled by 3 water pipes, A, B and C. Pipe C alone can fill the pool in 8 hours. If pipes B and C are used together, it can be filled in 6 hours. If pipes A and B are used together, it takes 10 hours. How long does it take to fill the pool if all three pipes are used?

Solution: Let x denote the time it takes for pipe C to fill the pool alone. Then pipe C pumps $\frac{1}{x}$ of the pool in one hour. Pipes B and C fill up the pool in 6 hours.

$6\left(\frac{1}{8} + \frac{1}{x}\right) = 1 \quad \text{distribute 6}$ $\frac{6}{8} + \frac{6}{x} = 1 \quad \text{subtract } \frac{6}{8} = \frac{3}{4}$ $\frac{3}{4} + \frac{6}{x} = 1$	$\frac{6}{x} = 1 - \frac{3}{4}$ $\frac{6}{x} = \frac{1}{4} \quad \text{take reciprocal of both sides}$ $\frac{x}{6} = 4 \quad \text{multiply by 6}$ $x = 24$
--	--

Therefore, it takes 24 hours for pipe B to fill up the pool. Let y denote the time it takes for pipe A to fill up the pool. Pipes A and B take 10 hours to fill up the pool.

$$10 \left(\frac{1}{24} + \frac{1}{y} \right) = 1 \quad \text{distribute 10}$$

$$\frac{10}{24} + \frac{10}{y} = 1 \quad \text{subtract } \frac{10}{24} = \frac{5}{12}$$

$$\frac{10}{y} = 1 - \frac{5}{12}$$

$$\frac{10}{y} = \frac{7}{12} \quad \text{take reciprocal of both sides}$$

$$\frac{y}{10} = \frac{12}{7} \quad \text{multiply by 10}$$

$$y = \frac{120}{7}$$

Now we are ready to find the time it takes to fill the pool using all three pipes. Pipe A takes $\frac{120}{7} \approx 17.143$ hours to fill the pool, pipe B takes 24 hours, and pipe C takes 8 hours. Together they fill the pool in t hours.

$$t \left(\frac{1}{8} + \frac{1}{24} + \frac{1}{\frac{120}{7}} \right) = 1 \quad \text{distribute 10}$$

$$t \left(\frac{1}{8} + \frac{1}{24} + \frac{7}{120} \right) = 1 \quad \text{bring to common denominator}$$

$$t \cdot \frac{15 + 5 + 7}{120} = 1$$

$$t \cdot \frac{27}{120} = 1 \quad \text{divide by } \frac{27}{120} = \frac{9}{40}$$

$$t = 1 \div \frac{9}{40} = 1 \cdot \frac{40}{9} = \frac{40}{9}$$

Therefore, it will take $\frac{40}{9} = 4.\bar{4}$ hours to fill up the pool using all three pipes. We can convert our result to minutes. $\frac{40}{9} = 4\frac{4}{9}$ and converting $\frac{4}{9}$ hours to minutes, we get

$$\frac{4}{9}\text{h} = \frac{4\cancel{\text{h}}}{9} \cdot \frac{60\text{min}}{1\cancel{\text{h}}} = \frac{240\text{min}}{9} = 26.\bar{6}\text{min}$$

It will take about 5 hours and 27 minutes to fill up the pool using all three pipes.

Example 5. A swimming pool can be filled by a water pipe in 8 hours and can be emptied by another pipe in 10 hours. One night, when the pool is empty, we accidentally leave both pipes open. How long will it take for the pool to be filled with water?

Solution: This problem is similar, the unknown will end up in the denominator like before. Only, in this case, the two pipes work against each other. One pipe adds water to the pool, the other removes water from the pool.

$$x \left(\frac{1}{8} - \frac{1}{10} \right) = 1 \quad \text{subtract fractions}$$

$$x \cdot \frac{5-4}{40} = 1$$

$$x \cdot \frac{1}{40} = 1 \quad \text{multiply by 40}$$

$$x = 40$$

So it will take 40 hours to fill up the pool.

Example 6. Andy and Lilian together clean their house in 6 hours. Working alone, it takes Andy 5 hours longer to clean the house than for Lilian. How long does it take for Andy to clean the house?

Solution: Let x denote the time it takes for Lilian to clean the house. Then Andy needs $x + 5$ hours. Together, they clean $\frac{1}{x}$ and $\frac{1}{x+5}$ of the house in each hour.

$$6\left(\frac{1}{x} + \frac{1}{x+5}\right) = 1 \quad \text{distribute 6}$$

$$\frac{6}{x} + \frac{6}{x+5} = 1 \quad \text{multiply by } x(x+5)$$

$$6(x+5) + 6x = x(x+5)$$

$$6x + 30 + 6x = x^2 + 5x$$

$$12x + 30 = x^2 + 5x$$

$$0 = x^2 - 7x - 30$$

$$0 = (x - 10)(x + 3)$$

$$x = 10 \text{ or } x = -3$$

The negative number can be obviously eliminated. Therefore, $x = 10$. This means that Lilian cleans the house in 10 hours, and Andy in 15 hours.



Practice Problems

1. Ayesha paints her room in 12 hours. Bella can paint the same room in 15 hours. Courtney can paint the same room in 10 hours. How long would it take for the three of them to paint the room together?
2. Jimmy can clean the house in 5 hours. If Susan helps to clean the house the job can be done in 2 hours. How long should it take Susan to clean the house alone?
3. Emily can mow the lawn in 5 hours. If Sammie helps her, the lawn is mowed in 3 hours. How long would it take for Sammie to mow the lawn alone?
4. A swimming pool can be filled by a water pipe in 12 hours and can be emptied by another pipe in 20 hours. One night, when the pool is empty, we accidentally leave both pipes open. How long will it take for the pool to be filled with water?
5. A swimming pool can be filled by three water pipes, A, B and C. Pipe A alone can fill the pool in 30 hours, pipe B alone is 6 hours, and pipe C in 5 hours. How long does it take to fill the pool if all three pipes are used?
6. *A swimming pool can be filled by three water pipes, A, B and C. Pipe A and B together fill the pool in 1.2 hours. Pipes A and C fill the pool in 1.5 hours, and pipes B and C fill the pool in 2 hours. How long does it take to fill the pool if all three pipes are used?
7. *A swimming pool can be filled by three water pipes, A, B and C. Pipe A alone can fill the pool in 12 hours. Pipe C alone fills the pool four times as fast as pipe B alone. When all three pipes are used, the pool is filled in 3 hours. How long would pipe B alone fill up the pool? Pipe C?

Problem Set 23

Chapter 24

24.1 Factoring by the AC-Method

Factoring by the AC-method is extremely useful because it addresses the most difficult situation, factoring a general quadratic expression with a leading coefficient, such as $6x^2 - 5x - 4$. Trial and error still works, but it is more difficult because there are more possibilities to be considered. The AC-method is a neat and powerful method to quickly factor a trinomial. The main steps are: we cleverly take apart the linear term into two parts and then factor by grouping.

In case of a general quadratic equation, we often denote the coefficients by a , b , and c , where the trinomial is $ax^2 + bx + c$. Notice the addition. This means that coefficients also carry the negative signs if there are any. In the case of $6x^2 - 5x - 4$, $a = 6$, $b = -5$, and $c = -4$.

Example 1. Completely factor the expression $6x^2 - 5x - 4$.

Solution: We will use the AC-method. The first step is to re-write the middle term, $-5x$ into a sum of two terms in such a way that grouping would work after that. To do that, we need to find two numbers p and q with a sum of -5 and a product that is the same as the product ac (hence the name, AC-method.) In this particular case, $ac = 6(-4) = -24$. So we need to find two numbers, p and q , such that $p + q = -5$ and $pq = -24$

This sounds similar to the trial and error method. Indeed, we will apply the same method to find p and q . A negative product pq indicates that one of p and q is positive and the other is negative. The negative sum -5 indicates that between the two numbers p and q , the negative one has a greater absolute value. These observations will make our task much easier. There are infinitely many integer pair solutions for $p + q$. There are just a few ways to solve the second equation for integers, and so we start there.

We list all the pairs of positive numbers with a product of 24, and take the opposite of the greater one in each pair. We are looking for the pair with sum -5 .

	-24	
1	-24	Now we consider these pairs as candidates for p and q . We are looking for the pair with sum -5 . Clearly that is -8 and 3 . Once we found p and q with product -24 and sum -5 , we can rewrite the middle term, $-5x$ as $-8x + 3x$ or $3x - 8x$ and factor by grouping.
2	-12	
3	-8	
4	-6	

$$\begin{aligned}
 6x^2 - 5x - 4 &= 6x^2 - 8x + 3x - 4 \\
 &= 2x(3x - 4) + 1(3x - 4) \\
 &= (2x + 1)(3x - 4)
 \end{aligned}$$

and so the factored form is $\boxed{(2x + 1)(3x - 4)}$. We can check our solution by multiplying back:

$$(2x + 1)(3x - 4) = 6x^2 - 8x + 3x - 4 = 6x^2 - 5x - 4 \text{ and so our solution is correct.}$$

Example 2. Completely factor the expression $10x^2 - 19x + 6$.

Solution: In this case, $a = 10$, $b = -19$, and $c = 6$. The product ac is 60. We are looking for two numbers p and q with a product of 60 and a sum of -19 . A positive product indicates that both p and q are positive, or both of them negative. The negative sum indicates that both p and q are negative.

As always, we start with $pq = 60$. We list all the pairs of negative numbers with a product of 60.

60	The only pair with a sum -19 is -15 and -4 . Now we know how to re-write the linear term, $-19x$ and then factor by grouping.	
-1	-60	
-2	-30	
-3	-20	$10x^2 - 19x + 6 = 10x^2 - 15x - 4x + 6$
-4	-15	$= 5x(2x - 3) - 2(2x - 3)$
-5	-12	$= (5x - 2)(2x - 3)$
-6	-10	

factor by grouping

Therefore, $10x^2 - 19x + 6 = \boxed{(5x - 2)(2x - 3)}$. We check by multiplying back:

$$(5x - 2)(2x - 3) = 10x^2 - 15x - 4x + 6 = 10x^2 - 19x + 6 \text{ and so our solution is correct.}$$

What happens if we cannot find a suitable pair of numbers p and q ? If that is the case, the trinomial cannot be factored using integer coefficients. At this point, we will just say that the expression is **prime** or **irreducible**. We are still responsible however, to factor out the GCF if there is any.

Example 3. Completely factor the expression $3x^2 - 4x + 2$.

Solution: In this case, $ac = 6$. Therefore, we are looking for two numbers p and q with a product of 6 and a sum -4 . A positive product indicates that both p and q are negative or both are positive. The negative sum -4 indicates that both p and q are negative.

We start with the equation $pq = 6$. We list all the pairs of negative numbers with a product of 6.

6	The sum of the first pair is -7 and the sum of the second pair is -5 . There are no pairs with sum -4 .	
-1	-6	
-2	-3	

This indicates that the trinomial cannot be factored. Therefore, our answer is $\boxed{3x^2 - 4x + 2}$ for the factored form.

Sometimes the product ac is too rich in divisors to list all pairs. If the second term has a small coefficient, there is a neat shortcut to find the right p and q . The following example illustrates this trick.

Example 4. Completely factor the expression $24x^2 + x - 10$.

Solution: In this case, $ac = -240$. We are looking for p and q with $pq = -240$ and $p + q = 1$. The number -240 has quite a number of divisors, so we will find the right pair without listing all the pairs. This trick is not possible for all cases, but it will work here. The negative product pq indicates that one of p and q is positive and the other is negative. The positive sum 1 indicates that between the two numbers p and q , the positive one has a greater absolute value.

When we add a positive and a negative number, we subtract the absolute values. Considering the absolute values $|p|$ and $|q|$, their product is 240 and their difference is 1. That means that the right pair of factors are very close to each other. Consequently, both numbers must be fairly close to the square root of 240. We enter $\sqrt{240}$ into our calculator. This is not an integer, the calculator shows $\sqrt{240} = 15.491933\dots$. If two numbers are close to each other and their product is 240, then they both are fairly close to 15. So we round up to 16 and start looking for divisors counting backward: 16, 15, 14, etc. Except, we won't even reach 14 because we immediately bump into 15 and 16. Our p and q is 16 and -15 . We can now easily factor by grouping.

$$\begin{aligned} 24x^2 + x - 10 &= 24x^2 + 16x - 15x - 10 \\ &= 8x(3x + 2) - 5(2x + 2) = (8x - 5)(3x + 2) \end{aligned}$$

Example 5. One side of a rectangle is 2 feet shorter than three times another side. Find the sides of the rectangle if we also know that its area is 176ft^2 .

Solution: If we label one side as x , then the other side can be written as $3x - 2$. The equation will express the area of the rectangle.

$$\begin{aligned} x(3x - 2) &= 176 \\ 3x^2 - 2x &= 176 \\ 3x^2 - 2x - 176 &= 0 \end{aligned}$$

We will factor this trinomial by the AC-method. In this case, $ac = 3(-176) = -528$. To factor $3x^2 - 2x - 176$, we need to find two integers p and q with product -528 and sum -2 . The negative product indicates that one of p and q is positive, the other one is negative. Therefore, the difference between their absolute values is 2. That is fairly small for a large product such as -528 . If the two numbers multiplying each other to 528 are close to each other, they also must be close to $\sqrt{528} = 22.978251\dots$. We roll up to 23 and start looking for pairs of integers multiplying to 528. We will try 23, 22, 21, and so on. We almost immediately find 22 and 24. Therefore p and q are -24 and 22. We can now easily factor the trinomial.

$$\begin{aligned} 3x^2 - 2x - 176 &= 3x^2 - 24x + 22x - 176 \\ &= 3x(x - 8) + 11(2x - 8) \\ &= (3x + 11)(x - 8) \end{aligned}$$

Applying the zero product rule, we solve $3x + 11 = 0$ and $x - 8 = 0$ and obtain $x = -\frac{11}{3}$ or $x = 8$. Since x represents a distance and distances cannot be negative, we rule out $-\frac{11}{3}$ and are left with $x = 8$. If x is 8, then $3x - 2 = 3 \cdot 8 - 2 = 22$. Thus the sides of the rectangle are 8ft and 22ft long.

We check: 22 is indeed two less than three times 8, and the area of the rectangle is $8\text{ft}(22\text{ft}) = 176\text{ft}^2$, and so our solution is correct.

Quadratic equations often have two solutions. In the previous example, one solution of the equation was easily ruled out, but that is not always the case. Often times both solutions of the equation result in a meaningful solution. The next example illustrates this.

Example 6. Twice the square of a number is 35 greater than three times the number. Find all such numbers.

Solution: If we label this number by x , then the square of this number is x^2 and twice the square is $2x^2$. The equation will compare twice the square of the number and three times the number.

$$\begin{aligned} 2x^2 &= 3x + 35 \\ 2x^2 - 3x - 35 &= 0 \end{aligned}$$

We will factor $2x^2 - 3x - 35$. The product ac is -70 . Therefore, we are looking for two integers p and q with product -70 and sum -3 . These are easily found: -10 and 7 . We now know how to take apart the middle term before grouping.

$$\begin{aligned} 2x^2 - 3x - 35 &= 0 \\ 2x^2 - 10x + 7x - 35 &= 0 \\ 2x(x - 5) + 7(x - 5) &= 0 \\ (2x + 7)(x - 5) &= 0 \end{aligned}$$

We solve the linear equations $2x + 7 = 0$ and $x - 5 = 0$ and obtain $x = -\frac{7}{2}$ and $x = 5$. We check both:

If the number is 5, then twice its square is $2 \cdot 5^2 = 50$, and three times the number is 15. Indeed, 50 is 35 greater than 15. So 5 works.

If the number is $-\frac{7}{2}$, then twice its square is $2 \left(-\frac{7}{2}\right)^2 = 2 \cdot \frac{49}{4} = \frac{49}{2}$. Three times the number is $3 \left(-\frac{7}{2}\right) = -\frac{21}{2}$. The difference between $\frac{49}{2}$ and $-\frac{21}{2}$ is $\frac{49}{2} - \left(-\frac{21}{2}\right) = \frac{49 + 21}{2} = \frac{70}{2} = 35$.

We found that both $\boxed{-\frac{7}{2}}$ and $\boxed{5}$ works.



Practice Problems

1. Completely factor each of the following.

- | | | |
|------------------------------------|--|---------------------------|
| a) $30x - 15y + 6ax - 3ay$ | f) $b^2 - a + ab^2 - 1$ | k) $14x - 12x^2 + 10$ |
| b) $xy - y - x + 1$ | g) $2m^2 - 18n^4 + 2m^2p^2 - 18n^4p^2$ | l) $5m^2 - 7mn + 2n^2$ |
| c) $6a^2b^2 - 4a^2bc - 10a^2c^2$ | h) $a^2x^2 - a^2y^2 + b^2x^2 - b^2y^2$ | m) $29px - 21p^2 + 10x^2$ |
| d) $a^2m + 2a^2n - b^2m - 2b^2n$ | i) $3x^2 - 2x - 1$ | n) $2x^4 - 3y^4 + x^2y^2$ |
| e) $x^2 - 4y^2 + m^2x^2 - 4m^2y^2$ | j) $6x^2 - 5x + 1$ | |

2. Solve each of the following equations.

- | | | |
|------------------------|---------------------|----------------------|
| a) $6x^4 - x^3 = 2x^2$ | b) $5a^2 + 5 = 26a$ | c) $11p + 35p^2 = 6$ |
|------------------------|---------------------|----------------------|

3. One side of a rectangle is 4 in shorter than 3 times the other side. Find the sides of the rectangle if its area is 319 in^2 .

24.2 Systems of Linear Equations in Three Variables

Systems become more complicated in several different ways. One way is that the equations are of higher degree than one. Quadratic systems will be studied later. Another way is that the equations remain linear, but the number of variables and equations grow. We will now learn how to solve systems in three variables. However, this method can be used with systems of equations with any number of unknowns.

If we have just one equation with two unknowns, we cannot solve for unique values of the unknowns. We can only solve for one unknown in terms of the other. Similarly, we can only completely solve a system in three variables if we have three equations.

The method we will learn is a very general method, a generalization of elimination. It is useful to keep in mind however, that symmetries in systems often enable us to solve it faster. It is worth to inspect a system for such shortcuts.

Example 1. Solve the given system of linear equations. $x = 4$ $y = -1$ $z = -9$

$$\begin{cases} x - y + z = -4 \\ -x + 3y - 2z = 11 \\ -x - 5y - z = 10 \end{cases}$$

Solution: We will not need new techniques if we just notice that we can solve for y after we add the first and second equation.

$$\begin{array}{r} \begin{cases} x - y + z = -4 \\ + \quad -x - 5y - z = 10 \end{cases} \\ \hline -6y = 6 \quad \text{divide by } -6 \\ y = -1 \end{array}$$

Now that we have the value of y , the problem is reduced to a system of linear equations in two variables that we already know how to handle.

$$\begin{cases} x - (-1) + z = -4 \\ -x + 3(-1) - 2z = 11 \\ -x - 5(-1) - z = 10 \end{cases} \implies \begin{cases} x + 1 + z = -4 \\ -x - 3 - 2z = 11 \\ -x + 5 - z = 10 \end{cases} \implies \begin{cases} x + z = -5 \\ -x - 2z = 14 \\ -x - z = 5 \end{cases}$$

The first equation and the third are dependent: if we multiply the first equation by -1 , we get the third one. However, given that we have two unknown left, we only need two equations. So we discard the last equation and solve the remaining system.

$$\begin{array}{r} \begin{cases} x + z = -5 \\ + \quad -x - 2z = 14 \end{cases} \\ \hline -z = 9 \\ z = -9 \end{array}$$

Now that we know the values of y and z , we can easily find that of x using either one of the equations.

$$x + z = -5 \implies x + (-9) = -5 \implies x - 9 = -5 \implies x = 4$$

So our solution is $\boxed{x = 4, y = -1, \text{ and } z = -9}$. We check:

$$\begin{cases} x - y + z = -4 \\ -x + 3y - 2z = 11 \\ -x - 5y - z = 10 \end{cases} \implies \begin{cases} \text{LHS} = 4 - (-1) + (-9) = 4 + 1 - 9 = 5 - 9 = -4 = \text{RHS} \checkmark \\ \text{LHS} = -4 + 3(-1) - 2(-9) = -4 - 3 + 18 = -7 + 18 = 11 = \text{RHS} \checkmark \\ \text{LHS} = -4 - 5(-1) - (-9) = -4 + 5 + 9 = 1 + 9 = 10 = \text{RHS} \checkmark \end{cases}$$

So our solution is correct.

The following example does not offer such shortcuts, so we will use a general method which is a generalization of elimination. The computations are all easy, the difficulty lies in keeping track of the equations. We will number the equations to keep track of them.

Example 2. Solve the given system of linear equations.

$$\begin{cases} 1) & 3x - y + 5z = 9 \\ 2) & 2x + 3y + z = 29 \\ 3) & -4x - 2y - 3z = -36 \end{cases}$$

Solution: First, we inspect the three equations and look for an unknown with the simplest coefficient, 1 and -1 highly preferred. In this case, y in the first equation and z in the second equation are the best candidates. We will choose y in the first equation.

We will use 'up' the first equation to eliminate y from the second and third equations.

To eliminate y from the second equation, we multiply both sides of the first equation by three and then add the two equations. The resulting equation will be our new second equation.

$$\begin{array}{rcl} 1) & 3x - y + 5z = 9 & / \cdot 3 \\ 2) & 2x + 3y + z = 29 & \\ \hline & & \implies \end{array} \quad \begin{array}{rcl} 1) & 9x - 3y + 15z = 27 & \\ 2) & 2x + 3y + z = 29 & \end{array}$$

We add the two equations:

$$\begin{array}{rcl} 1) & 9x - 3y + 15z = 27 & \\ 2) & + \quad 2x + 3y + z = 29 & \\ \hline & 11x + 16z = 56 & \text{This is our new second equation.} \end{array}$$

We repeat the process. This time we use the first equation to eliminate y from the third equation. The resulting equation will be our new third equation. To eliminate y , we multiply the first equation by -2 and then add the two equations.

$$\begin{array}{rcl} 1) & 3x - y + 5z = 9 & / \cdot (-2) \\ 3) & -4x - 2y - 3z = -36 & \\ \hline & & \implies \end{array} \quad \begin{array}{rcl} 1) & -6x + 2y - 10z = -18 & \\ 3) & -4x - 2y - 3z = -36 & \end{array}$$

We add the two equations:

$$\begin{array}{rcl} 1) & -6x + 2y - 10z = -18 & \\ 3) & + \quad -4x - 2y - 3z = -36 & \\ \hline & -10x - 13z = -54 & \text{This is our new third equation.} \end{array}$$

We now obtained a system of equations with two variables:

$$\begin{array}{rcl} 2) & -10x - 13z = -54 & \\ 3) & 11x + 16z = 56 & \end{array}$$

We already know how to solve those. We will use elimination. We will multiply the first equation by 11 and the second equation by 10 and then we add the two equations.

$$\begin{array}{rcl} 2) & -10x - 13z = -54 & / \cdot 11 \\ 3) & 11x + 16z = 56 & / \cdot 10 \\ \hline & & \implies \end{array} \quad \begin{array}{rcl} 2) & -110x - 143z = -594 & \\ 3) & 110x + 160z = 560 & \end{array}$$

We add the two equations:

$$\begin{array}{rcl} 2) & -110x - 143z & = -594 \\ 3) & +110x + 160z & = 560 \\ \hline & & 17z = -34 \quad \text{divide by 17} \\ & & z = -2 \end{array}$$

Now we can solve for x in the second equation:

$$\begin{array}{rcl} 2) & -10x - 13z = -54 & \text{becomes} \quad -10x - 13(-2) = -54 \quad \text{Solve for } x \\ & & -10x + 26 = -54 \quad \text{subtract 26} \\ & & -10x = -80 \quad \text{divide by } -10 \\ & & x = 8 \end{array}$$

Now that we know the values of x and z , we can easily solve for y in either of the equations. We will use the first equation.

$$\begin{array}{rcl} 1) & 3x - y + 5z = 9 & \text{becomes} \quad 3 \cdot 8 - y + 5(-2) = 9 \quad \text{Solve for } y \\ & & 24 - y - 10 = 9 \quad \text{combine like terms} \\ & & 14 - y = 9 \quad \text{subtract 14} \\ & & -y = -5 \quad \text{multiply by } -1 \\ & & y = 5 \end{array}$$

Our solution is: $x = 8$, $y = 5$, and $z = -2$. We check:

$$\begin{array}{l} 1) \text{ LHS} = 3x - y + 5z = 3 \cdot 8 - 5 + 5(-2) = 24 - 5 - 10 = 9 = \text{RHS} \quad \checkmark \\ 2) \text{ LHS} = 2x + 3y + z = 2 \cdot 8 + 3 \cdot 5 + (-2) = 16 + 15 - 2 = 29 = \text{RHS} \quad \checkmark \\ 3) \text{ LHS} = -4x - 2y - 3z = -4 \cdot 8 - 2 \cdot 5 - 3(-2) = -32 - 10 + 6 = -36 = \text{RHS} \quad \checkmark \end{array}$$

Therefore, our solution is correct.

The following example is an application of systems with three equations.

Example 3. Find the equation of a parabola if we know that its graph contains the points $(-1, 15)$, $(2, 6)$, and $(4, 10)$.

Solution: We start by setting up the coefficients in the formula as the unknown. Suppose that $y = ax^2 + bx + c$ is the equation. Finding the equation of the parabola is equivalent to finding the values of a , b , and c . To find these three unknowns, we will need three equations. For that, we will use the three points given on the graph of the parabola.

$$(-1, 15) \text{ is on the graph} \implies a(-1)^2 + b(-1) + c = 15$$

$$(2, 6) \text{ is on the graph} \implies a \cdot 2^2 + b \cdot 2 + c = 6$$

$$(4, 10) \text{ is on the graph} \implies a \cdot 4^2 + b \cdot 4 + c = 10$$

We simplify each of the equations:

$$\left\{ \begin{array}{l} 1) \quad a - b + c = 15 \\ 2) \quad 4a + 2b + c = 6 \\ 3) \quad 16a + 4b + c = 10 \end{array} \right.$$

We will first use the first equation to eliminate c from the second and third equations.

$$\begin{array}{rcl} 1) & a - b + c = 15 & / \cdot (-1) \\ 2) & 4a + 2b + c = 6 & \\ \hline & & \implies \end{array} \quad \begin{array}{l} 1) \quad -a + b - c = -15 \\ 2) \quad 4a + 2b + c = 6 \end{array}$$

We add the two equations:

$$\begin{array}{rcl}
 1) & -a + b - c & = -15 \\
 2) & + \quad 4a + 2b + c & = 6 \\
 \hline
 & 3a + 3b & = -9 \quad \text{divide by 3} \\
 & a + b & = -3
 \end{array}$$

Similarly, we will use the first equation to eliminate c from the third equation.

$$\begin{array}{rcl}
 1) & a - b + c = 15 & / \cdot (-1) \\
 3) & 16a + 4b + c = 10 & \\
 \hline
 & & \Rightarrow \\
 1) & -a + b - c = -15 & \\
 3) & 16a + 4b + c = 10 &
 \end{array}$$

We add the two equations:

$$\begin{array}{rcl}
 1) & -a + b - c & = -15 \\
 3) & + \quad 16a + 4b + c & = 10 \\
 \hline
 & 15a + 5b & = -5 \quad \text{divide by 5} \\
 & 3a + b & = -1
 \end{array}$$

We have reduced the problem to a system in a and b only. We solve the system.

$$\begin{array}{rcl}
 2) & a + b = -3 & / \cdot (-1) \\
 3) & 3a + b = -1 & \\
 \hline
 & & \Rightarrow \\
 2) & -a - b = 3 & \\
 3) & 3a + b = -1 &
 \end{array}$$

We add the two equations:

$$\begin{array}{rcl}
 2) & -a - b & = 3 \\
 3) & + \quad 3a + b & = -1 \\
 \hline
 & 2a & = 2 \quad \text{divide by 2} \\
 & a & = 1
 \end{array}$$

Then the second equation, $-a - b = 3$ becomes $-1 - b = 3 \Rightarrow b = -4$

Then the first equation, $a - b + c = 15$ becomes $1 - (-4) + c = 15 \Rightarrow c = 10$

So $a = 1$, $b = -4$, and $c = 10$, and this means that the parabola is $y = x^2 - 4x + 10$.

We check:

If $x = -1$, then $y = (-1)^2 - 4(-1) + 10 = 1 + 5 + 10 = 15$, so $(-1, 15)$ is on the graph. ✓

If $x = 2$, then $y = 2^2 - 4 \cdot 2 + 10 = 4 - 8 + 10 = 6$, so $(2, 6)$ is on the graph. ✓

If $x = 4$, then $y = 4^2 - 4 \cdot 4 + 10 = 16 - 16 + 10 = 10$, so $(4, 10)$ is also on the graph. ✓



Practice Problems

1. Solve each of the following system of equations.

$$\text{a) } \begin{cases} 2x - y + z = -1 \\ x + 2y + 2z = 5 \\ -3x - 4y - 3z = 2 \end{cases}$$

$$\text{c) } \begin{cases} -5x + 3y - 2z = 44 \\ 2x - y + 3z = -31 \\ -x + 4y + 7z = -9 \end{cases}$$

$$\text{e) } \begin{cases} 2x + 3y + 7z = 3 \\ 4x - 5y - 3z = 19 \\ 5x - 3y + z = 21 \end{cases}$$

$$\text{b) } \begin{cases} -2a - 3b + c = 5 \\ 3a - 2b + c = -6 \\ 6a + 4b + 3c = 31 \end{cases}$$

$$\text{d) } \begin{cases} 4a - 3b + 5c = -3 \\ -2a + 5b - 3c = 11 \\ -a + 2b - 4c = -1 \end{cases}$$

$$\text{f) } \begin{cases} 2a - 5b - c = 7 \\ -3a + 10b + c = -7 \\ a - 8b + 3c = 15 \end{cases}$$

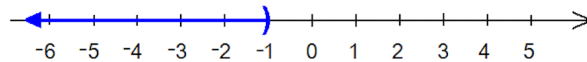
2. Find the equation of the parabola if we know that the points $(1, 4)$, $(-2, 10)$, and $(3, -10)$ are all points on the graph of the parabola.
3. Find the equation of the parabola if we know that the points $(2, 1)$, $(-8, 11)$, and $(4, -13)$ are all points on the graph of the parabola.
4. We went to the market to buy some fruit. We decided to buy some apples, strawberries, and oranges. If we buy 2 apples, 3 boxes of strawberry, and 4 oranges, the fruit would cost \$15.30. If we buy 1 box of strawberry, 4 apples, and 2 oranges, the fruit would cost \$10.90. If we buy 1 orange, 5 apples and 2 boxes of strawberry, the fruit would cost \$13.70. What is the price of each type of fruit?

Problem Set 24

- Label each of the following statements as true or false.
 - 3 is an odd number or 10 is a prime number.
 - 3 is an odd number and 10 is a prime number.
 - -2 is a natural number or 5 is an even number.
 - $-\frac{2}{3}$ is an even number or an odd number.
- Suppose that $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 4, 9\}$, and $B = \{2, 4, 6, 7\}$. Find each of the following.
 - $A \cap B$
 - $A \cup B$
 - $P = \{x \in U : x > 4 \text{ or } x \leq 7\}$
 - $Q = \{x \in U : x > 4 \text{ and } x \leq 7\}$
- Suppose that S is the set of all squares and R is the set of all rectangles. Label each of the following statements as true or false.
 - $S \subseteq R$
 - $R \subseteq S$
 - $R \cup S = S$
 - $R \cap S = S$
 - $R \cup \emptyset = R$
 - $R \subseteq S$
 - $\emptyset \subseteq S$
 - $R \cup S = R$
 - $R \cap S = R$
- Suppose that F is the set of all integers divisible by four, S is the set of all integers divisible by six, and T is the set of all integers divisible by three. Label each of the following statements as true or false.
 - $S \subseteq T$
 - $F \subseteq S$
 - $F \cap T = S$
 - $F \cap T \subseteq S$
 - $F \subseteq S \cup T$
- Suppose that A and B are sets such that $A \cup B = \{1, 3, 4, 5, 6, 7, 8\}$ and $A \cap B = \{1, 3, 5, 6, 8\}$. How many different sets are possible for A ?
- Perform the division with remainder. $2020 \div 17$
- List all factors of 84.
- Consider the following numbers. 2011, 11060904, 321, 3106
 Select all the numbers from the list that are divisible
 - by 2
 - by 3
 - by 6
- Which of the following numbers is a prime? 2007, 143, 151, 91
- Find the prime factorization of 720.
- Find the prime factorization for x if
 - $x = 12^{100}$
 - $x = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
 Note that there is a shorter notation for the product above: $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$ (read as ten factorial)
- Find the smallest positive integer that is divisible by 2, 3, 4, 5, and 6.
- Find the greatest common factor and least common multiple of 420 and 2400.
- Label each of the following statements as true or false.
 - Every integer is a rational number.
 - Every prime number is odd.
 - If a is divisible by 2 and b is divisible by 3, then the product ab is divisible by 6.
 - The sum of two consecutive integers is always an odd number.
 - The product of two consecutive integers is always an odd number.
 - Suppose that a and b are two positive integers. Let us denote the greatest common factor of a and b by F and the least common multiple of a and b by M . Then $ab = FM$.
 - If x and y are positive integers such that x and y are both divisible by 7, then $x - y$ is also divisible by 7.

h) If x and y are positive integers such that $x - y$ is divisible by 7, then both x and y are divisible by 7.

15. Re-write the set depicted on the picture in interval notation.



16. Find the multiplicative inverse of -0.75 .

17. If $A = \left\{ -\sqrt{9}, \pi, -\frac{3}{16}, 0.67, \sqrt{3} \right\}$, then A contains how many irrational numbers?

18. Perform the indicated operations and simplify.

a) $-3^2 - 2(5 - 3(2 \cdot 7 - 4^2))$

e) $12 - 2(5 - 2 + 1)$

h) $2^{-1} - 3^{-1}$

b) $-6 - 7(-3)$

f) 3^{-2}

i) $\frac{(-2)^{-1} - (-3)^{-1}}{(-2)^{-2} + (-3)^{-2}}$

c) $\frac{5^2 - 4^2}{11 - 3 \cdot 2^2 + 1}$

g) $\frac{1}{(-2)^{-3}}$

j) $\frac{5}{8} - \frac{2}{3} + \frac{1}{6} \left(-\frac{1}{2} \right)$

d) $\frac{48 \div 8 \cdot 6}{-2^2 - 5}$

19. Simplify each of the following.

a) $|2 - 4| - 3 - 7||$

c) $|2| - 4 - 3 - 7||$

e) $||2 - 4| - 3| - 7$

b) $|2 - 4 - |3 - 7||$

d) $|2 - 4| | -3 - 7|$

f) $||2 - 4 - 3| - 7|$

20. Prove that the given decimals represent rational numbers by re-writing each of them as a quotient of integers. You do not need to reduce the fraction.

a) $0.49\overline{3}$ b) $0.228\overline{5}$ c) $0.06\overline{52}$

21. Suppose that $A = 640000000000$ and $B = 0.000000000025$. Perform the indicated operations. Present your answer using scientific notation.

a) A b) B c) B^2 d) AB e) $\sqrt{A}$ f) $\frac{A}{B}$ g) A^2B^3

22. Evaluate each of the given expressions.

a) $b^2 - 4ac$ if $a = -1$, $b = -2$, and $c = -3$ b) $\frac{|2x-8|}{2x+1}$ if $x = -5$ c) $\frac{3a+b}{2a-b}$ if $a = -3$ and $b = -6$

23. Compute the value of the expression $-12p + 3$ when $p = \frac{5}{6}$.

24. Simplify each of the given expressions.

a) $8t - (-6t + 2)$ b) $2x - 4(y - x) - 3y$ c) $\frac{-10x + 24}{2}$ d) $2(x - 3y) - 5(2x - 4y)$

25. Simplify each of the given expressions.

a) $\frac{5x^5y^4z}{30x^3yz^2}$ c) $\left(\frac{-2ab^{-3}}{b^{-2}} \right)^{-4}$ e) $-2x^3(-x^2)^4$
 b) $(x^4)^2(x^{-2})^3$ d) $\frac{-2ab^5(3a^2b^{-3})^4}{(-3a^{-1}b^0a^3)^5(3a)^{-2}}$ f) $(-2x^3(-x^2))^4$

26. Consider the rule of exponents $(a^n)^m = a^{nm}$. Based on that, evaluate each of the given expressions.

a) $\sqrt{x^4}$ b) $\sqrt{x^{12}}$ c) $\sqrt{A^{100}}$

27. Suppose that $x = 2^{100}$. Express each of the given expression in terms of x .

a) 2^{101} b) $2^{103} - 2^{101}$ c) 2^{98} d) $3 \cdot 2^{102} + 5 \cdot 2^{101} - 2^{100}$ e) 8^{100} f) 2^{50}

28. Solve each of the given linear equations.

a) $5x - (x + 4) = -8$ d) $\frac{2}{3}(x - 6) - \frac{3}{4}(x + 8) = x - 10$ f) $x - \frac{2x - 1}{3} = \frac{x + 5}{2} - \frac{1}{6}$
 b) $-2(x - 5) + 3x = 4x - 2$ e) $x + \frac{4}{3} = \frac{5}{6}$ g) $\frac{2x - 3}{8} + \frac{x}{4} = \frac{10}{16}$
 c) $3(x - 7) - 2(x - 5) = x + 11$

29. Solve each of the given linear inequalities. Present your answer using interval notation.

a) $\frac{3x - 5}{-2} > 4$ b) $3(x - 2) - 5(2x - 1) \leq 4(3x - 5)$ c) $\frac{x - 6}{5} - \frac{2x + 3}{3} \leq x + 11$

30. Solve each of the formulas for the indicated variable.

a) $A = \frac{(a + c)h}{2}$ for c b) $y = mx + b$ for x c) $V = \frac{1}{3}bh$ for h

31. Graph each of the following.

a) $y = 2x + 4$ c) $x = -2$ e) $y = 3$
 b) $y = -\frac{2}{3}x + 1$ d) $2x - 3y = -12$ f) $x + 3y = -6$

32. a) Find the x -intercept of $x - 2y = 4$.

b) Find the y -intercept of $4x - 3y = 12$.

33. Find the slope of the straight line passing through the indicated points.

a) $(1, -2)$ and $(3, -4)$ b) $(-1, -3)$ and $(-4, 5)$.

34. Solve each of the given system of linear equations.

a) $\begin{cases} \frac{x}{3} + 6y = 4 \\ y = -5 - x \end{cases}$ b) $\begin{cases} 12x - 2y = 10 \\ y = 6x - 5 \end{cases}$ c) $\begin{cases} 2x + 3y = 11 \\ x - 4y = 0 \end{cases}$ d) $\begin{cases} 3x + 5y = 20 \\ 2x - 10y = 0 \end{cases}$

35. Expand or simplify each of the following.

a) $(2x - 1)^2$ b) $(2x - 1)^3$ c) $(x - 3)^2 - (x - 2)(3x + 1)$ d) $(3y - 2)(y - 1) - (2y - 1)^2$

36. Completely factor each of the following.

a) $p^2 - 4p - 32$ b) $3t^2 - 5t - 2$ c) $9x^2 - 25$ d) $2x^2 - 2x - 12$ e) $2x^4 - 32$

37. Completely factor $x^2 + x - 2$ and $x^2 - 4$. What is the factor they have in common?

38. Solve each of the given quadratic equations.

a) $5x + 2x^2 = 3$ b) $x^2 - 18 = -7x$ c) $6x^2 - 11x = 10$ d) $(x - 5)(x + 2) = x - 10$

39. Simplify each of the given expressions.

a) $\frac{2x - 3}{3 - 2x}$ c) $\frac{(x + 1)^2}{x^2 - 1}$ e) $\frac{x^2 - 2x - 3}{x^2 - 1} \cdot \frac{x^2 - x}{x^2 + 2x - 15}$
 b) $\frac{x^2 - 36}{x^2 - 4x - 12}$ d) $\frac{3ax - 6ay - bx + 2by}{3a - b}$ f) $\frac{x}{x^2 - 2x} \div \frac{3x + 6}{4x - 8}$

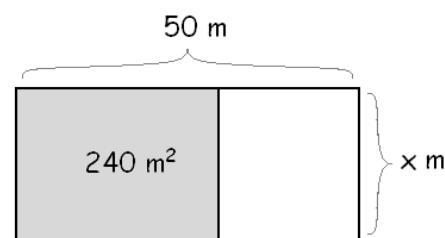
40. 28 students in our class worked during the summer. 20 students in our class took a trip. Some students might have done both. 6 students did neither.

a) What is the smallest number possible for the size of our class?

- b) What is the greatest number possible for the size of our class?
- c) If 15 students traveled and worked during the summer, what is the size of the class?
41. There is an animal farm where chickens and cows live. All together, there are 73 heads and 204 legs. How many chickens and how many cows are there on the farm?
42. What is the area of the rectangle whose longer side is twice the length of the shorter side, and whose perimeter is 36in?
43. If Jupiter is 483,800,000 miles from the sun, what is its distance from the sun in inches? Round to the nearest hundredths and answer in scientific notation. (1 mile = 5,280 feet and 1 foot = 12 inches)
44. Most countries measure temperature in either Celsius or in Farenheit. The connection between these two measurements is $C = \frac{5}{9}(F - 32)$. Solve this formula for F .
45. You thought that dinner and a movie would cost \$35, but your estimate was \$17 less than the actual cost. If the movie was \$10, how much was dinner?
46. A coat is on a special sale at a 20% discount. If the sale price is \$96, what was the price of the coat before the discount?
47. A sweater originally cost \$65 and is now \$39. What is the percent change of the price of the shirt? Round your answers to the nearest percentage.
48. Find the x -coordinate of the point where the lines $x + 3y = -8$ and $4x - 3y = 23$ intersect.
49. We throw a small object upward from the top of a 1024ft tall building. The height of the object, (measured in feet) t seconds after we threw it is
- $$h = -16t^2 + 192t + 1024$$
- a) Where is the object 2 seconds after we threw it?
- b) How long does it take for the object to hit the ground?

50. A school purchases tickets to a show. A child ticket costs \$8 and an adult ticket costs \$14. The school has paid a total of \$610 for 65 tickets. How many of the 65 tickets were for adults?

51. The area of the shaded region of the rectangle shown on the picture is 240 square meters. Express the area of the unshaded region in terms of x .



52. One side of a rectangle is 2 inches shorter than three times another side. How long is the longer side in the rectangle if its area is 96 square inches?
53. Find an expression that describes the area in square meters of a rectangle that has width $4x^2y^2$ meters and length $3x^3y^3$ meters. Simplify your answer.
54. The formula for the volume of a rectangular box $V = lwh$; where l is the length, w is the width, and h is the height. If $V = 64$, $l = 8$, and $w = 4$, find the value of h .
55. Find the value of k so that the line connecting the points $(2, 3)$ and $(5, k)$ has a slope of $\frac{1}{3}$.
56. Monico invests a total of \$12500 in two accounts paying 4% and 3% annual interest, respectively. How much was invested in each account if, after one year, the total interest was \$455?

Appendix A – Answers

13.1 – Radical Expressions 1

Sample Problems

1. $4\sqrt{2}$ 2. $3\sqrt{5}$ 3. $4x^2y\sqrt{3xy}$ 4. $-4\sqrt{5}$ 5. $\frac{2}{3}$ 6. $13a^5\sqrt{5a}$ 7. 3 8. $11 - 4\sqrt{7}$
 9. $-10 + 6\sqrt{3}$ 10. $5x + \sqrt{5x} - 6$ 11. $6 + \sqrt{x} - 2x$ 12. $x - 2\sqrt{2x} + 2$ 13. 5

Practice Problems

1. $2\sqrt{5}$ 2. $2\sqrt{3}$ 3. $8\sqrt{5}$ 4. $7a^2\sqrt{2a}$ 5. 1 6. $9 - 4\sqrt{5}$ 7. $17\sqrt{5} - 38$ 8. $103 - \sqrt{5}$
 9. a) 2 b) $-27 + 4\sqrt{5}$ c) -3

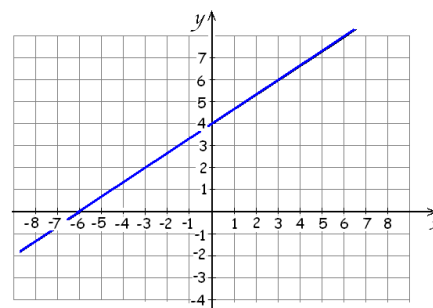
13.2 – Smallest Value of a Quadratic Expression

Practice Problems

1. 3 2. -11 3. -1 4. 33 5. -2440 6. 4 7. -16 8. -60 9. 0 10. 0

Problem Set 13

1. a) true b) true c) false 2. a) m^5 b) m^6 c) a^8b^2 d) 3 e) $-p^4q$ f) $-\frac{1}{16y^6}$
 3. a) $\frac{1}{36}$ b) $\frac{9}{25}$ c) $-\frac{a^4}{b^9}$ d) 1 4. 408430 5. $\frac{173}{990}$
 6. a) $25a^2 - 10a + 1$ b) $9x^{10} - 16y^2$ c) $-4\sqrt{5}$ d) $9\sqrt{3} - 13$ e) $11 - 4\sqrt{7}$ f) $-50 + 19\sqrt{7}$ g) $7 - 2\sqrt{10}$
 7. a) $6\sqrt{2} - 1$ b) 7 c) $-12\sqrt{3} - 11$
 8. a) $(3x - 4y^5)(3x + 4y^5)$ b) $5x(x + 2)$ c) $-3x(x + 12)x - 20$ d) $2(4x^2 + 1)(2x + 1)(2x - 1)$
 9. a) 35, 0, and -25 b) identity, all real numbers are solution c) -41 12.
 d) -4 e) -7, 3 f) $\frac{25}{4}, 0$ g) $-\frac{5}{2}, 0, \frac{5}{2}$ h) 7 i) 0, 1
 10. a) $(14, \infty)$ b) $[0, \infty)$ c) $[0, \infty)$ d) $(14, \infty)$
 11. a) $(-4, -12)$ b) $(-5, -1)$ c) $(4, 0)$ 13. 51% decrease
 14. 52 chickens and 21 cows



15. -100, when the numbers are 10 and -10 16. a) 9ft and 23ft b) 15ft and 21ft 17. 11 miles
 18. \$12400 in bonds and 7600 in stocks 19. a) 16 seconds b) 1600, when $t = 6$ 20. \$12

14.1 – Completing the Square – Part 3

Practice Problems

1. $(x+4)(x-3)$ 2. $(x+9)(x-10)$ 3. $(x+6)(x+5)$ 4. $(x-8)(x-9)$ 5. $3(x+5)(x-4)$
6. can not be factored 7. $2(x+1)(x-\frac{1}{2}) = (x+1)(2x-1)$ 8. $-2x(x+7)(x-8)$
9. $5(a+\frac{1}{5})(a-3) = (5a+1)(a-3)$ 10. $-c^3(c-15)(c-18)$ 11. $6(m+\frac{2}{3})(m-\frac{1}{2}) = (3m+2)(2m-1)$
12. $6(x+\frac{1}{3})(x-\frac{3}{2}) = (3x+1)(2x-3)$ 13. $10(p+\frac{2}{5})(p-\frac{3}{2}) = (5p+2)(2p-3)$
14. $15(x-\frac{5}{3})(x-\frac{5}{3}) = (5x-3)(3x-5)$

14.2 – Completing the Square – Part 4

Sample Problems

1. $\frac{5-\sqrt{109}}{6}$ and $\frac{5+\sqrt{109}}{6}$ 2. $\frac{1+\sqrt{13}}{2}$ and $\frac{1-\sqrt{13}}{2}$

Practice Problems

1. $3 \pm \sqrt{22}$ 2. $5 \pm 2\sqrt{7}$ 3. $\frac{-1 \pm \sqrt{5}}{2}$ 4. $\frac{11 \pm \sqrt{61}}{6}$ 5. no real solution 6. $\frac{2 \pm 2\sqrt{26}}{5}$
7. $-4 \pm \sqrt{26}$ 8. $\frac{-1 \pm \sqrt{13}}{6}$ 9. no real solution 10. $1 + \sqrt{3}$ and $1 - \sqrt{3}$ 11. $\frac{1 + \sqrt{17}}{2}$ and $\frac{1 - \sqrt{17}}{2}$

14.3 – Rational Expressions 1

Practice Problems

1. a) $-\frac{1}{2}$ b) $x-1$ c) $\frac{2t+3}{2}$ d) $\frac{p}{p+1}$ e) $-\frac{m+18}{m+6}$ f) $\frac{x+6}{x-8}$ g) $-\frac{5}{4}$
2. a) $\frac{x^2}{2}$ b) $-a^2(a-4)$ c) 1 d) $\frac{x-5}{x+5}$ e) $\frac{3}{y-1}$ f) $4x-4$

14.4 – The Pythagorean Theorem

Sample Problems

1. a) There is a right angle opposite the 10cm long side. b) not a right triangle 2. 17m 3. 35in
4. 13 units 5. 20 units 6. 32cm and 60cm 7. $\sqrt{45}$ ft 8. 24m 9. 25ft

Practice Problems

1. a) not a right triangle b) There is a right angle opposite the 37ft long side. c) not a right triangle
2. 13mi 3. 16cm 4. 29ft 5. 24m 6. 15cm or $\sqrt{353}$ cm 7. a) 10 units b) 25 units
8. 9cm, 40cm, and 41cm 9. 14in, 48in, and 50in 10. 16, 30, and 34 units
11. a) 20 b) 7 c) 10 12. $\sqrt{2}$ units 13. a) 15m b) $\sqrt{75}$ ft
14. a) $x = \sqrt{105}$ ft $y = \sqrt{89}$ ft b) $q = 25$ cm, $p = \sqrt{850}$ cm = $5\sqrt{34}$ cm

Problem Set 14

1. a) i) $180 = 2^2 \cdot 3^2 \cdot 5$ ii) $1575 = 3^2 \cdot 5^2 \cdot 7$ iii) $80^{100} = 2^{400} \cdot 5^{100}$ b) 45 and 6300

c) Only 90 is possible d) Yes, but only if the two numbers are equal.

2. a) $(x-3+\sqrt{2})(x-3-\sqrt{2})$ b) $6\left(x-\frac{5}{3}\right)\left(x+\frac{1}{2}\right) = (3x-5)(2x+1)$ c) $9\left(x-\frac{2-\sqrt{5}}{3}\right)\left(x-\frac{2+\sqrt{5}}{3}\right)$

3. a) $\frac{5 \pm \sqrt{7}}{3}$ b) When $x = \frac{5-\sqrt{7}}{3}$, then

$$\begin{aligned} \text{LHS} &= 3\left(\frac{5-\sqrt{7}}{3}\right)^2 - 10\left(\frac{5-\sqrt{7}}{3}\right) = 3 \cdot \frac{(5-\sqrt{7})^2}{3^2} - 10 \cdot \frac{5-\sqrt{7}}{3} = -3 \cdot \frac{25+7-10\sqrt{7}}{-3 \cdot 3} - \frac{10(5-\sqrt{7})}{3} \\ &= \frac{32-10\sqrt{7}-10(5-\sqrt{7})}{3} = \frac{32-10\sqrt{7}-50+10\sqrt{7}}{3} = \frac{-18}{3} = -6 = \text{RHS} \end{aligned}$$

and when $x = \frac{5+\sqrt{7}}{3}$, then

$$\begin{aligned} \text{LHS} &= 3\left(\frac{5+\sqrt{7}}{3}\right)^2 - 10\left(\frac{5+\sqrt{7}}{3}\right) = 3 \cdot \frac{(5+\sqrt{7})^2}{3^2} - 10 \cdot \frac{5+\sqrt{7}}{3} = -3 \cdot \frac{25+7+10\sqrt{7}}{-3 \cdot 3} - \frac{10(5+\sqrt{7})}{3} \\ &= \frac{32+10\sqrt{7}-10(5+\sqrt{7})}{3} = \frac{32+10\sqrt{7}-50-10\sqrt{7}}{3} = \frac{-18}{3} = -6 = \text{RHS} \end{aligned}$$

4. a) -4, (when $x = 5$) b) 6 (when $x = -1$) c) $\frac{3}{4}$ (when $x = -\frac{1}{2}$) 5. $\frac{369}{990}$

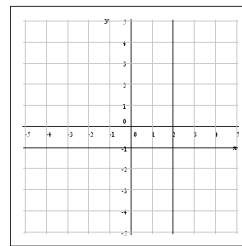
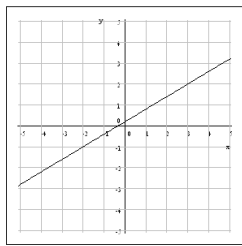
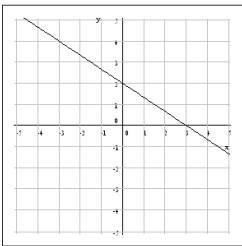
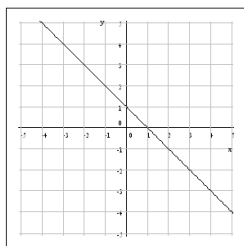
6. a) $8M$ b) $\frac{1}{9}M^2$ c) 24 e) $\sqrt{M}$ 7. a) $\sqrt{10}$ b) $2\sqrt{2}$ c) $6-\sqrt{3}$ d) $9+7\sqrt{2}$ e) $9-4\sqrt{5}$ f) 11
g) $-6+\sqrt{5}$ h) $3-2\sqrt{2}$ i) $-7+5\sqrt{2}$ j) $17-12\sqrt{2}$ k) 3

8. a) $2\sqrt{10}+5$ b) $4\sqrt{2}$ c) 6 9. a) $\frac{3x-1}{2}$ b) $\frac{x-1}{x-5}$ c) -1 e) x^2+1

10. a) 0, 2 a) $1+\sqrt{6}, 1-\sqrt{6}$ c) no real solution d) $-5, \frac{1}{2}$ e) $\frac{-9+\sqrt{17}}{4}, \frac{-9-\sqrt{17}}{4}$ f) no real solution

11. a) $(-2, -5)$ b) $(5, -8)$

12. a) $y = -x+1$ b) $y = -\frac{2}{3}x+2$ c) $3x-5y = -1$ d*) $(x-2)(y+1) = 0$



13. $P = 84$ 14. a) 5 unit b) $\sqrt{50} = 5\sqrt{2}$ unit c) -7, 9 15. a) 35cm and 37cm b) 18cm and 80cm

16. a) -30 with -26 and 26 and 30 b) Allie is 26 and Bethany is 30.

c) The equation $x(x+4) = 780$ or $x(x-4) = 780$ has two solutions. Therefore, there are two pairs of numbers that satisfy the equation. In the case when we are looking for the age of a person, the negative number for x is ruled out. After all, -30 and -26 does satisfy the equation, but is meaningless within the context of the word problem.

17. $\sqrt{213}\text{m}$ 18. $h = \sqrt{224}\text{ unit} = 4\sqrt{14}\text{ unit}$, $H = \sqrt{124}\text{ unit} = 2\sqrt{31}\text{ unit}$

19. a) $\sqrt{24}\text{m} = 2\sqrt{6}\text{m}$ b) $10\sqrt{6}\text{m}^2$ 20. a) $P = 60\text{cm}$ $A = 120\text{cm}^2$ b) $P = 15 + 5\sqrt{5}\text{ft}$ $A = 25\text{ft}$

21. $P = 168\text{in}$, $A = 1224\text{in}^2$ 22. 41 ten-dollar and 34 twenty-dollar bills 23. 24m, 70m, 74m

24. -256 25. 4960 26. a) 2 and -1 b) $\frac{1+\sqrt{13}}{2}$ and $\frac{1-\sqrt{13}}{2}$ 27. 120ft^2
 28. 24m , 32m , and 40m 29. 1008.5 30. 6054 31. 76
 32. a) If two numbers are rational, then they can be written as $\frac{a}{b}$ and $\frac{c}{d}$, where a, b, c, d are all integers, $b, d \neq 0$. Then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$$

Since both $ad+bc$ and bd are integers, the sum $\frac{a}{b} + \frac{c}{d}$ is rational.

b) Suppose, for a contradiction, that $\sqrt{2}-1$ is rational. Clearly, 1 is rational because it can be written as $\frac{1}{1}$. If both $\sqrt{2}-1$ and 1 are rational, then so is their sum, $\sqrt{2}-1+1=\sqrt{2}$. Therefore, $\sqrt{2}$ is rational. That's impossible, so our assumption that $\sqrt{2}-1$ is rational must be false. Thus $\sqrt{2}-1$ is irrational.

15.1 – Graphing a Parabola 1

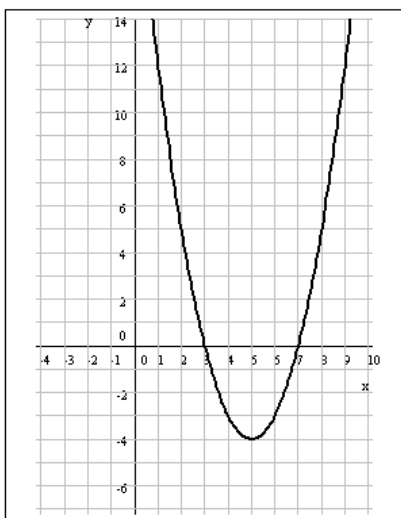
Practice Problems

1. $y = x^2 - 10x + 21$

$$y = x^2 - 10x + 21 \implies y\text{-intercept: } (0, 21)$$

$$y = (x-5)^2 - 4 \implies \text{vertex: } (5, -4)$$

$$y = (x-3)(x-7) \implies x\text{-intercepts: } (3, 0) \text{ and } (7, 0)$$

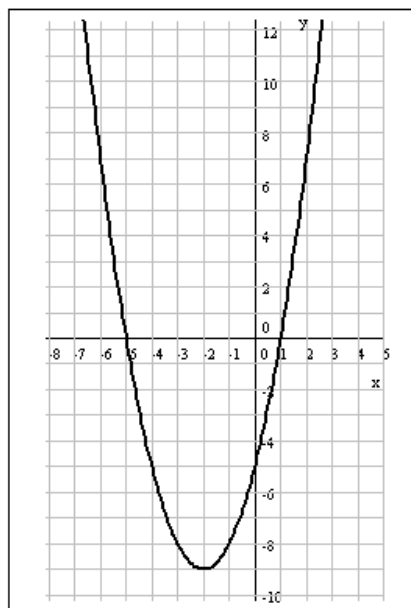


2. $y = x^2 + 4x - 5$

$$y = x^2 + 4x - 5 \implies y\text{-intercept: } (0, -5)$$

$$y = (x+2)^2 - 9 \implies \text{vertex: } (-2, -9)$$

$$y = (x+5)(x-1) \implies x\text{-intercepts: } (-5, 0) \text{ and } (1, 0)$$

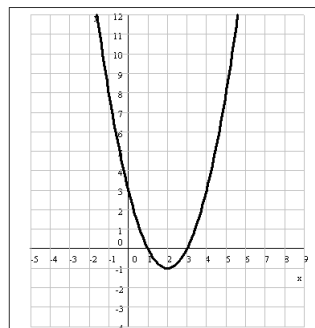


3. $y = x^2 - 4x + 3$

$$y = x^2 - 4x + 3 \implies y\text{-intercept: } (0, 3)$$

$$y = (x-2)^2 - 1 \implies \text{vertex: } (2, -1)$$

$$y = (x-1)(x-3) \implies x\text{-intercepts: } (1, 0) \text{ and } (3, 0)$$



15.2 – Compound Inequalities

Practice Problems

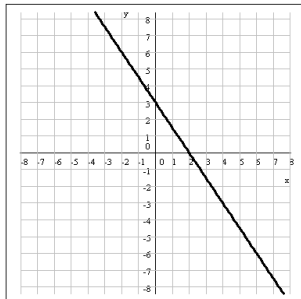
1. a) $(-3, 5]$ b) $\mathbb{R} = (-\infty, \infty)$ 2. a) $(-\infty, 0]$ b) $(-\infty, -4]$ 3. a) $\emptyset$ b) $(-\infty, 5) \cup (10, \infty)$
 4. a) $(-\infty, 8)$ b) $(-\infty, 4)$ 5. a) $\emptyset$ b) $(-\infty, -1] \cup (1, \infty)$

16.1 – Slope of a Line

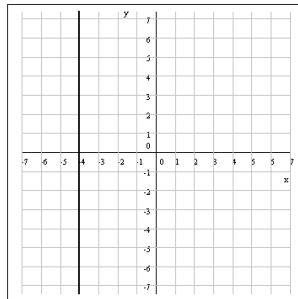
Practice Problems

1. a) -2 b) $\frac{3}{4}$ c) undefined d) 0

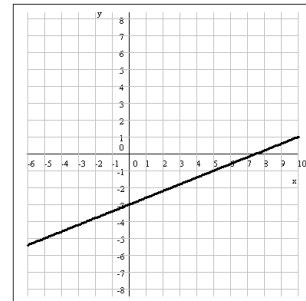
1. a) $3x + 2y = 6$



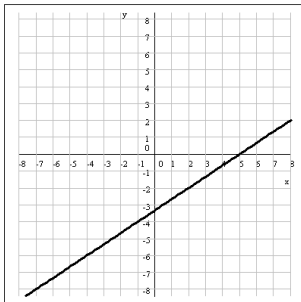
b) $x = -4$



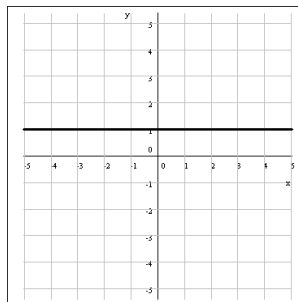
c) $y = \frac{2}{5}x - 3$



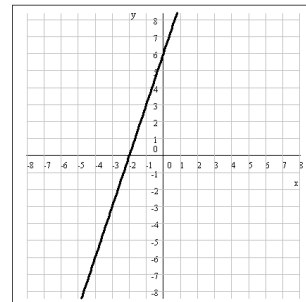
d) $2x - 3y = 10$



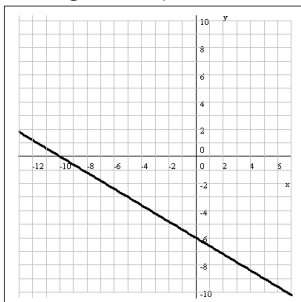
e) $y = 1$



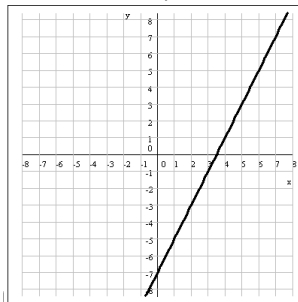
f) $y = 3x + 6$



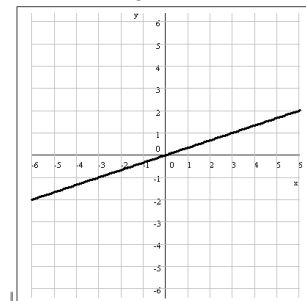
g) $3x + 5y = -30$



h) $2x - y = 7$



i) $y = \frac{1}{3}x$



2. a) $m = \frac{2}{3}$ b) $m = -\frac{3}{5}$ c) $m = 1$ d) $m = -2$ e) $m = \frac{1}{2}$ f) $m = 0$

16.2 – Combinatorics 1

Practice Problems

1. 4900 2. 2450

- 3.
- | | | | |
|------------------|------------------|------------------|----------------------------------|
| $\{1, 2, 3, 4\}$ | | | |
| $\{1, 2, 3, 5\}$ | $\{1, 2, 4, 5\}$ | | |
| $\{1, 2, 3, 6\}$ | $\{1, 2, 4, 6\}$ | $\{1, 2, 5, 6\}$ | smallest two numbers are 1 and 2 |
| $\{1, 3, 4, 5\}$ | $\{1, 3, 5, 6\}$ | | smallest two numbers are 1 and 3 |
| $\{1, 3, 4, 6\}$ | | | |
| $\{1, 4, 5, 6\}$ | | | smallest two numbers are 1 and 4 |
| $\{2, 3, 4, 5\}$ | | | |
| $\{2, 3, 4, 6\}$ | $\{2, 3, 5, 6\}$ | | smallest numbers are 2 and 3 |
| $\{2, 4, 5, 6\}$ | | | smallest numbers are 2 and 4 |
| $\{3, 4, 5, 6\}$ | | | smallest numbers are 3 and 4 |

4. There are 32 subsets, listed below.

0-element subsets:	$\emptyset$	1 subset
1-element subsets:	$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$	5 subsets
2-element subsets:	$\{1, 2\}$ $\{1, 3\} \quad \{2, 3\}$ $\{1, 4\} \quad \{2, 4\} \quad \{3, 4\}$ $\{1, 5\} \quad \{2, 5\} \quad \{3, 5\} \quad \{4, 5\}$ $\{1, 2, 3\}$	10 subsets
3-element subsets:	$\{1, 2, 4\} \quad \{1, 3, 4\} \quad \{2, 3, 4\}$ $\{1, 2, 5\} \quad \{1, 3, 5\} \quad \{1, 4, 5\} \quad \{2, 3, 5\} \quad \{2, 4, 5\} \quad \{3, 4, 5\}$	10 subsets
4-element subsets:	$\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}$	5 subsets
5-element subsets:	$\{1, 2, 3, 4, 5\}$	1 subsets

5. rectangles: 90 squares: 26

Problem Set 16

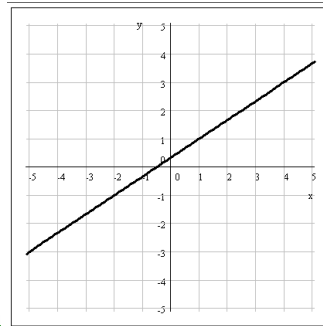
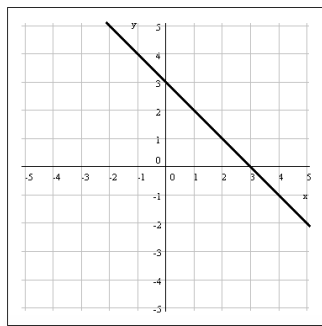
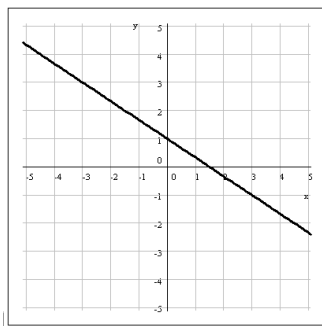
Practice Problems

1. 36 (24 has 8 divisors, 36 has 9) 2. a) 2A b) $-9A$ c) A^3 3. a) -5 b) -18 c) -9
 4. a) $1200 = 2^4 \cdot 3 \cdot 5^2$ b) $10^{100} = 2^{100} \cdot 5^{100}$ c) $75 \cdot 80 = 2^4 \cdot 3 \cdot 5^3$ 5. $\text{lcm} = 270$ $\text{gcd} = 45$
 6. a) $-\frac{1}{2}$ b) $\frac{10}{3}$ c) 10 d) $\frac{x^8}{4y^{21}}$ e) $-\frac{1}{2x^3}$ 7. a) 14672 b) 28800 c) 67980
 8. a) $11 + 5\sqrt{7}$ b) $19 - 6\sqrt{2}$ c) $17 - 12\sqrt{2}$ d) -1 e) $14\sqrt{2}$ f) 64 g) $6 - \sqrt{5}$ h) $\frac{4}{3}$ i) 16 j) $-16\sqrt{2}$
 9. a) $(1, 7)$ b) $(2, 4)$ c) $(-\infty, 5]$ d) $(-\infty, 3)$ e) $\mathbb{R}$, same as $(-\infty, \infty)$ f) $(7, 10]$ g) can not be simplified
 h)
 $\emptyset$
 10. a) $76 - 42\sqrt{3}$ b) $3\sqrt{3}$ c) 0

11. a) $y = -\frac{2}{3}x + 1$

b) $x + y = 3$

c) $2x - 3y = -1$



12. a) $\frac{1}{2}$ b) -1 c) $\frac{3}{5}$ d) $-\frac{3}{2}$

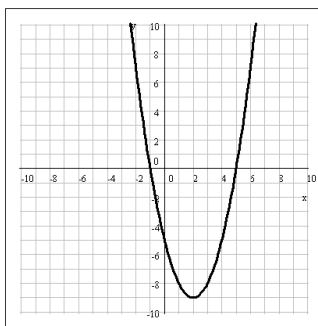
13. $y = x^2 - 4x - 5 = (x - 2)^2 - 9$

vertex: $(2, -9)$

x -intercepts: $(-1, 0)$ and $(5, 0)$

y -intercept: $(0, -5)$

14. $P = 24$ unit, $A = 35$ unit²



15. a) -14 b) -1 c) -40 d) $\frac{1}{4}$

e) $\frac{19}{4}$ f) 5 g) $\frac{9}{4}$ h) $\frac{23}{8}$ i) 4

16. a) $3(3a + 8)(a - 2)$ b) $2x(x^2 + 100)$

c) $5x^2(x^2 + 4)(x + 2)(x - 2)$

d) $(x^5 + 7)(x^5 - 7)$

e) $-5(x^2 + 9)(x + 3)(x - 3)$

f) $2x(x + 11)(x - 5)$

g) $-4(2x^2 - 1)(x - 3)$

17. the set of all integers divisible by 15 18. a) 4 b) 0 c) undefined

19. a) $-10x + 20$ b) $6x^2 - 4$ c) $-x - 7$ d) $5a - 55$ e) $x^2 - 7x + 10$ f) $4x^2 - 9$ g) $-11m + 33$

h) $a^2 - a - 13$ i) $27x^3 - 27x^2 + 9x - 1$ j) $\frac{x+5}{x-7}$ k) -1 l) $\frac{3}{a-6}$

20. a) $\frac{3}{4}$ b) 8 c) 2 d) There is no solution. e) $0, \frac{1}{4}$ f) $-\frac{1}{2}, 0, \frac{1}{2}$ g) 0 h) 2, 0, -6 i) 0

j) $\mathbb{R}$ (all real numbers are solution) k) 2, -2

21. a) $[5, \infty)$ b) $(-5, \infty)$ c) $\mathbb{R}$ d) $\left[-1, \frac{1}{2}\right)$ e) $(4, \infty)$ f) $[0, \infty)$ g) $\left(-\infty, -\frac{1}{2}\right) \cup [5, \infty)$ h) $\emptyset$

22. a) $(-5, 2)$ b) $(-3, 8)$ c) $(3, -3)$ 23. a) $\frac{301}{990}$ b) $\frac{72601}{9990}$ c) 1 24. -15 25. 20, 21, 22

26. 35 units and 37 units 27. 8ft 28. 12 ten-dollar bills and 62 five-dollar bills 29. -6 and 7

30. 0 31. 1104 32. 0, 1 33. 62 quarters and 92 dimes 34. 17 feet 35. 9 seconds

36. -8 with -20 or 8 with 20 37. 7140 38. $10\sqrt{2}m$

39. All subsets of $A = \{1, 2, 3, 4\}$

0-element subsets: $\emptyset$

1-elt subsets: $\{1\}, \{2\}, \{3\}, \{4\}$

2-elt subsets: $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$

3-elt subsets: $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$

4-elt subsets: $\{1, 2, 3, 4\}$

40. List all two-element subsets of $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

$\{1, 2\}$

$\{1, 3\}$ $\{2, 3\}$

$\{1, 4\}$ $\{2, 4\}$ $\{3, 4\}$

$\{1, 5\}$ $\{2, 5\}$ $\{3, 5\}$ $\{4, 5\}$

$\{1, 6\}$ $\{2, 6\}$ $\{3, 6\}$ $\{4, 6\}$ $\{5, 6\}$

$\{1, 7\}$ $\{2, 7\}$ $\{3, 7\}$ $\{4, 7\}$ $\{5, 7\}$ $\{6, 7\}$

$\{1, 8\}$ $\{2, 8\}$ $\{3, 8\}$ $\{4, 8\}$ $\{5, 8\}$ $\{6, 8\}$ $\{7, 8\}$

$\{1, 9\}$ $\{2, 9\}$ $\{3, 9\}$ $\{4, 9\}$ $\{5, 9\}$ $\{6, 9\}$ $\{7, 9\}$ $\{8, 9\}$

$\{1, 10\}$ $\{2, 10\}$ $\{3, 10\}$ $\{4, 10\}$ $\{5, 10\}$ $\{6, 10\}$ $\{7, 10\}$ $\{8, 10\}$ $\{9, 10\}$

17.1 Graphing a Parabola –2

Practice Problems

1. $y = 336 - 16x^2 - 64x$

polynomial form: $y = -16x^2 - 64x + 336$

$\Rightarrow$ y-intercept: $(0, 336)$

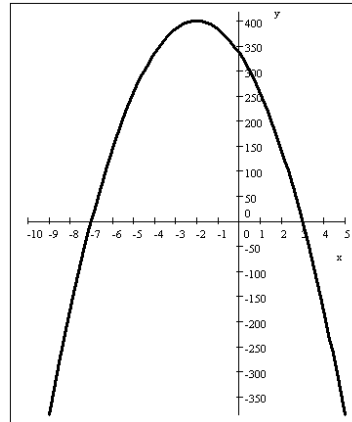
standard form: $y = -16(x+2)^2 + 400$

$\Rightarrow$ vertex: $(-2, 400)$

factored form: $y = -16(x+7)(x-3)$

$\Rightarrow$ x-intercepts: $(-7, 0)$ and $(3, 0)$

additional points: $(-4, 336), (-3, 384), (-1, 384),$
 $(1, 256), (2, 144), (4, -176)$



2. $y = 3x^2 - 30x + 75$

polynomial form: $y = 3x^2 - 30x + 75$

$\Rightarrow$ y-intercept: $(0, 75)$

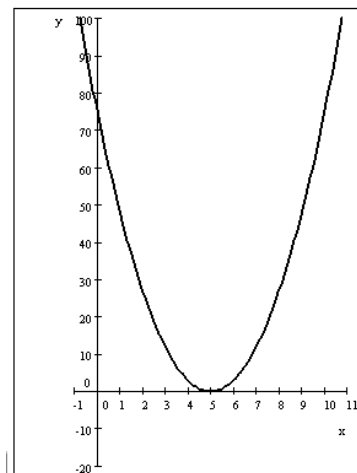
standard form: $y = 3(x-5)^2$

$\Rightarrow$ vertex: $(5, 0)$

factored form: $y = 3(x-5)^2$

$\Rightarrow$ x-intercept: $(5, 0)$

additional points: $(1, 48), (2, 27), (3, 12), (4, 3), (6, 3)$
 $(7, 12), (8, 27)$



3. $y = 4x - 2x^2 - 20$

polynomial form: $y = -2x^2 + 4x - 20$

$\Rightarrow$ y-intercept: $(0, -20)$

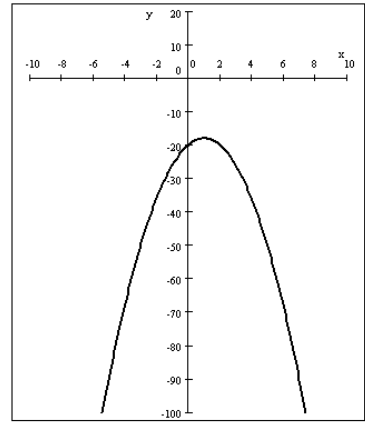
standard form: $y = -2(x - 1)^2 - 18$

$\Rightarrow$ vertex: $(1, -18)$

factored form: doesn't exist

$\Rightarrow$ no x-intercepts

additional points: $(-3, -50), (-2, -36), (2, -20),$
 $(3, -26), (4, -36), (5, -50)$



4. $y = 6x - x^2$

polynomial form: $y = -x^2 + 6x$

$\Rightarrow$ y-intercept: $(0, 0)$

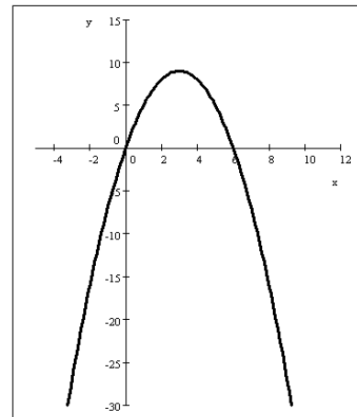
standard form: $y = -(x - 3)^2 + 9$

$\Rightarrow$ vertex: $(3, 9)$

factored form: $y = -x(x - 6)$

$\Rightarrow$ x-intercepts: $(0, 0)$ and $(6, 0)$

additional points: $(-1, -7), (1, 5), (2, 8), (4, 8)$
 $(5, 5), (7, -7), (8, -16)$



5. $y = 3x^2 - 12$

polynomial form: $y = 3x^2 - 12$

$\Rightarrow$ y-intercept: $(0, -12)$

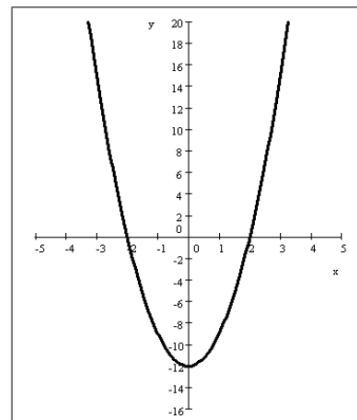
standard form: $y = 3x^2 - 12$

$\Rightarrow$ vertex: $(0, -12)$

factored form: $y = 3(x + 2)(x - 2)$

$\Rightarrow$ x-intercepts: $(-2, 0)$ and $(2, 0)$

additional points: $(-4, 36), (-3, 15), (-1, -9), (1, -9),$
 $(3, 15), (4, 36), (5, 63)$



6. $y = 4x + \frac{1}{2}x^2 + 10$

polynomial form: $y = \frac{1}{2}x^2 + 4x + 10$

$\Rightarrow$ y-intercept: (0, 10)

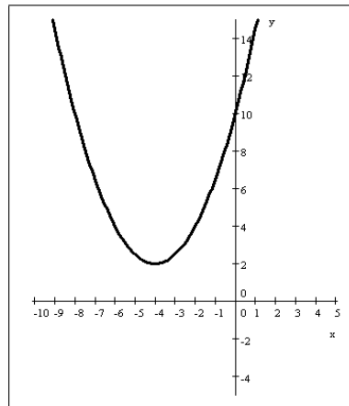
standard form: $y = \frac{1}{2}(x+4)^2 + 2$

$\Rightarrow$ vertex: (-4, 2)

factored form: doesn't exist

$\Rightarrow$ no x-intercepts

additional points: $(-7, \frac{13}{2})$, $(-6, 4)$, $(-5, \frac{5}{2})$, $(-3, \frac{5}{2})$,
 $(-1, \frac{13}{2})$, $(1, \frac{29}{2})$, $(2, 20)$



7. $y = x(2-x) - 1$

polynomial form: $y = -x^2 + 2x - 1$

$\Rightarrow$ y-intercept: (0, -1)

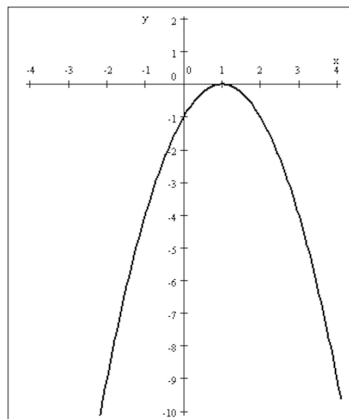
standard form: $y = -(x-1)^2$

$\Rightarrow$ vertex: (1, 0)

factored form: $y = -(x-1)^2$

$\Rightarrow$ x-intercept: (1, 0)

additional points: $(-2, -9)$, $(-1, -4)$, $(2, -1)$,
 $(3, -4)$, $(4, -9)$, $(5, -16)$



8. $y = -12x - 2x^2 - 10$

polynomial form: $y = -2x^2 - 12x - 10$

$\Rightarrow$ y-intercept: (0, -10)

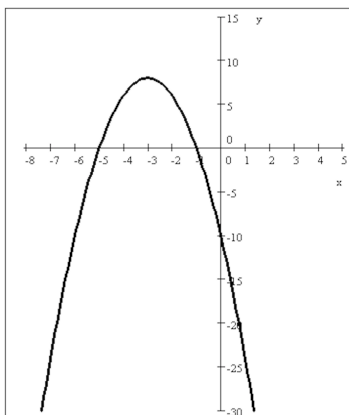
standard form: $y = -2(x+3)^2 + 8$

$\Rightarrow$ vertex: (-3, 8)

factored form: $y = -2(x+5)(x+1)$

$\Rightarrow$ x-intercepts: (-5, 0) and (-1, 0)

additional points: $(-7, -24)$, $(-6, -10)$, $(-4, 6)$,
 $(-2, 6)$, $(1, -24)$, $(2, -42)$



9. a) -49 is the smallest value b) -1 is the greatest value c) 34 is the smallest value d) 9 is the greatest value
 e) 12 is the greatest value f) $\frac{13}{16}$ is the greatest value g) 2500 is the greatest value

10. 784 feet 11. max income: \$256 000 when the price is \$160

17.2 – Rational Exponents

Practice Problems

1. 3 2. $\frac{1}{3}$ 3. -3 4. $-\frac{1}{3}$ 5. undefined 6. 27 7. 2 8. $\frac{1}{2}$ 9. -2 10. $\frac{1}{8}$ 11. $\sqrt{5}$ 12. $-\frac{1}{32}$
13. -2 14. 4 15. 6 16. $\frac{1}{6}$ 17. -6 18. undefined 19. undefined 20. $\frac{1}{216}$ 21. $-\frac{1}{8}$
22. undefined 23. -2 24. undefined 25. -4 26. undefined 27. $\frac{1}{4}$ 28. $\frac{1}{16}$ 29. undefined
30. $\frac{1}{27}$ 31. x^2 32. $\sqrt[20]{x^3}$ 33. $\sqrt[5]{a^2}$ 34. m^5 35. $\frac{y}{x^2}$ 36. $\frac{-2a^2}{b^8}$ 37. $\frac{8x^3}{27y^2}$ 38. 1 39. A
40. y^{11} 41. $\sqrt[4]{x^3}$ 42. $(\sqrt{y})^3$ 43. 8 44. $\sqrt[6]{x^5}$ 45. $\frac{\sqrt[5]{x}\sqrt[3]{x^2}}{x\sqrt[6]{x}} = x^{\left(\frac{1}{5}+\frac{2}{3}-\left(1+\frac{1}{6}\right)\right)} = x^{-\frac{3}{10}} = \frac{1}{\sqrt[10]{x^3}}$
46. $\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{x}}}}} = \left(\left(\left(\left(x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = x^{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}} = x^{\frac{1}{32}} = \sqrt[32]{x}$

18.1 – The Quadratic Formula

Practice Problems

1. a) no real solutions b) two real solutions c) two real solutions d) one real solution
e) two real solutions f) no real solutions g) two real solutions
2. a) no real solutions b) $-3, \frac{1}{2}$ c) $\frac{2 \pm \sqrt{5}}{3}$ d) $-\frac{5}{8}, \frac{3}{5}$ e) no real solutions f) $\frac{-1 \pm \sqrt{5}}{2}$ g) $\frac{5}{2}$
3. $\frac{1 + \sqrt{5}}{2}$ (the other solution is negative)

18.2 – Similar Triangles

Sample Problems

1. $a = \frac{40}{7}$ in, $b = \frac{28}{5}$ in 2. a) 13ft b) see solutions c) 10ft d) 13ft
3. a) 6ft b) 16ft 4. see solutions 5. $h = \frac{420}{29}$, $x = \frac{400}{29}$, $y = \frac{441}{29}$ 6. 30 units

Practice Problems

1. $x = \frac{15}{4} = 3.75$, $y = \frac{39}{4} = 9.75$ 2. 105in 3. a) $\frac{35}{4} = 8.75$ b) $\frac{60}{7}$ 4. a) 5ft b) 10ft
5. a) $x = \frac{24}{5} = 4.8$, $y = \frac{32}{5} = 6.4$, $z = \frac{18}{5} = 3.6$ b) $x = 60$, $y = 25$, $z = 144$
6. a) $h = 12$, $a = 15$, $b = 20$ b) $h = 168$, $a = 175$, $b = 600$ 7. 18 units

19.1 – Writing Equations of Lines

Sample Problems

1. a) -1 b) $\frac{1}{2}$ c) 0 d) This line has no slope
2. a) -2 b) $\frac{2}{5}$ c) 0 d) 2 e) This line has no slope f) 3 g) $-\frac{1}{3}$
3. $y = 4x - 5$ 4. $y = -\frac{1}{2}x + 3$ 5. $y = \frac{3}{2}x - 13$ 6. $y = 5x + 9$ 7. $y = -3x + 8$ 8. $(5, -3)$
9. a) $y + 3 = \frac{3}{2}(x + 5)$ or $y - 3 = \frac{3}{2}(x + 1)$ b) $M(-2, 0)$ c) $y = -\frac{2}{3}x - 2$ d) $y = -3x$

Practice Problems

1. a) $\frac{2}{5}$ b) $-\frac{2}{3}$ c) This line has no slope d) 0
2. a) $\frac{3}{4}$ b) $-\frac{3}{10}$ c) 0 d) $\frac{1}{3}$ e) This line has no slope f) $\frac{1}{2}$ g) -2 3. $y = -2x + 7$
4. $y = \frac{2}{3}x + 11$ 5. $y = \frac{3}{2}x - 13$ 6. $y = 5x + 9$ 7. $y = -4x + 18$ 8. $y = -\frac{1}{2}x + 3$
9. a) i) $y = -x + 7$ ii) $y - 1 = \frac{2}{5}(x - 6)$ or $y = \frac{2}{5}x - \frac{7}{5}$ iii) $y = \frac{5}{2}x + 7$
 b) i) $(3, 4)$ ii) $(1, -1)$ iii) $(-2, 2)$
 c) i) $y = x + 1$ ii) $y + 1 = -\frac{5}{2}(x - 1)$ or $y = -\frac{5}{2}x + \frac{3}{2}$ iii) $y - 2 = -\frac{2}{5}(x + 2)$ or $y = -\frac{2}{5}x + \frac{6}{5}$
 d) i) $y = x + 1$ ii) $y = -\frac{5}{2}x + 7$ iii) $y - 1 = -\frac{2}{5}(x - 6)$ or $y = -\frac{2}{5}x + \frac{17}{5}$

19.2 – Radical Expressions 2

Sample Problems

1. a) $4\sqrt{2}$ b) $3\sqrt{5}$ c) $4x^2y\sqrt{3xy}$ d) $-4\sqrt{5}$ e) $\frac{2}{3}$ f) $13a^5\sqrt{5a}$ g) 3 h) $11 - 4\sqrt{7}$
 i) $-10 + 6\sqrt{3}$ j) $5x + \sqrt{5x} - 6$ k) $6 + \sqrt{x} - 2x$ l) $x - 2\sqrt{2x} + 2$
2. a) $\frac{3\sqrt{5}}{5}$ b) $3 + \sqrt{10}$ c) $\frac{-1 + \sqrt{7}}{3}$ d) $3 + \sqrt{5}$ e) $\frac{13 - 4\sqrt{10}}{3}$
3. a) 7 b) 172 c) 3 d) 4 e) 22 4. 5 5. a) $2 - \sqrt{5}, 2 + \sqrt{5}$ b) $4 - \sqrt{3}, 4 + \sqrt{3}$

Practice Problems

1. a) $2\sqrt{5}$ b) $2\sqrt{3}$ c) $8\sqrt{5}$ d) $7a^2\sqrt{2a}$ e) 1 f) $9 - 4\sqrt{5}$ g) $17\sqrt{5} - 38$ h) $103 - \sqrt{5}$
2. a) $\frac{4\sqrt{7}}{7}$ b) $-\frac{3 + \sqrt{7}}{2}$ c) $-3 + \sqrt{10}$ d) $-8 + 2\sqrt{17}$ or $2(-4 + \sqrt{17})$ e) $\frac{3 - \sqrt{5}}{2}$ f) $\frac{\sqrt{5} + 3}{2}$
 g) $-5\sqrt{x} + 5$ h) $\frac{2\sqrt{x} - 8}{x - 16}$ 3. a) 2 b) $-27 + 4\sqrt{5}$ c) -3
4. a) $7 - \sqrt{2}, 7 + \sqrt{2}$ b) $-6 - \sqrt{5}, -6 + \sqrt{5}$ c) $1 - \sqrt{2}, 1 + \sqrt{2}$

20.1 – Absolute Value Equations

Practice Problems

1. $-4, -3$
2. $-7, 17$
3. no solution
4. $-\frac{16}{3}, 0$
5. $-21, 4$
6. $1, 3$
7. $0, 4$
8. 2
9. $-2, 14$
10. no solution
11. $\frac{1}{2}, \frac{9}{2}$
12. $-\frac{4}{3}, 2$
13. $\frac{4}{3}, 2$
14. $0, \frac{7}{2}$
15. $-\frac{7}{4}, \frac{21}{4}$
16. $2, 8$
17. $-\frac{3}{2}, \frac{5}{3}$
18. 1
19. no solution
20. $\frac{4}{3}$

20.2 – Complex Numbers

Sample Problems

1. $\sqrt{34}$
2. $3 - 6i$
3. $1 + 2i$
4. $-5i$
5. $1 - 3i$
6. $8 - 9i$
7. $16 - 11i$
8. $5 - 12i$
9. 13
10. $10 + 4i$
11. -34
12. 100
13. -4
14. $-1 + i$
15. $\frac{11}{5} + \frac{2}{5}i$
16. $2 - i$
17. $2 + i$
18. $1 + 2i$
19. $5 - 4i$
20. $-2 + 2i$
21. $1 - 7i$
22. $2i$
23. $(x + 3i)(x - 3i)$
24. $(x - 5 + 2i)(x - 5 - 2i)$
25. $(x + 1)(x - 1)(x + i)(x - i)$
26. $2 + 5i$ and $-2 - 5i$
27. $1 + \sqrt{2}i$ and $1 - \sqrt{2}i$
28. $2 + 5i$ and $-2 - 5i$

Appendix B – Solutions

Solutions for 13.1 – Radical Expressions 1

Simplify each of the following expressions. Assume that a represents a positive number.

1. $\sqrt{32}$

Solution: We first factor the number under the square root sign into two factors, where the first factor is the largest square we can find in the number. Then this first part can come out from under the square root sign and become a coefficient.

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16}\sqrt{2} = \boxed{4\sqrt{2}}$$

2. $\sqrt{45}$

Solution: We first factor the number under the square root sign into two factors, where the first factor is the largest square we can find in the number. Then this first part can come out from under the square root sign and become a coefficient.

$$\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9}\sqrt{5} = \boxed{3\sqrt{5}}$$

3. $\sqrt{48x^5y^3}$

Solution: We first factor the expressions under the square root sign into two factors, where the first factor is the largest square we can find in the expression. Then this first part can come out from under the square root sign.

$$\sqrt{48x^5y^3} = \sqrt{16x^4y^2 \cdot 3xy} = \sqrt{16x^4y^2}\sqrt{3xy} = \boxed{4x^2y\sqrt{3xy}}$$

4. $\sqrt{125} - 3\sqrt{80} + \sqrt{45}$

Solution: We simplify each expression as in the previous problem. Then we combine like radicals.

$$\begin{aligned}\sqrt{125} - 3\sqrt{80} + \sqrt{45} &= \sqrt{25 \cdot 5} - 3\sqrt{16 \cdot 5} + \sqrt{9 \cdot 5} \\ &= \sqrt{25}\sqrt{5} - 3\sqrt{16}\sqrt{5} + \sqrt{9}\sqrt{5} \\ &= 5\sqrt{5} - 3 \cdot 4 \cdot \sqrt{5} + 3\sqrt{5} \\ &= 5\sqrt{5} - 12\sqrt{5} + 3\sqrt{5} \\ &= (5 - 12 + 3)\sqrt{5} = \boxed{-4\sqrt{5}}\end{aligned}$$

5. $\frac{\sqrt{24}}{\sqrt{54}}$

Solution: We simplify both radical expressions and simplify.

$$\frac{\sqrt{24}}{\sqrt{54}} = \frac{\sqrt{4 \cdot 6}}{\sqrt{9 \cdot 6}} = \frac{\sqrt{4}\sqrt{6}}{\sqrt{9}\sqrt{6}} = \frac{2\sqrt{6}}{3\sqrt{6}} = \boxed{\frac{2}{3}}$$

6. $\sqrt{80a^{11}} - 2\sqrt{180a^{11}} + 3\sqrt{245a^{11}}$

Solution:

$$\begin{aligned} \sqrt{80a^{11}} - 2\sqrt{180a^{11}} + 3\sqrt{245a^{11}} &= \\ \sqrt{16a^{10} \cdot 5a} - 2\sqrt{36a^{10} \cdot 5a} + 3\sqrt{49a^{10} \cdot 5a} &= \\ \sqrt{16a^{10}}\sqrt{5a} - 2\sqrt{36a^{10}}\sqrt{5a} + 3\sqrt{49a^{10}}\sqrt{5a} &= \\ 4a^5\sqrt{5a} - 2(6a^5)\sqrt{5a} + 3(7a^5)\sqrt{5a} &= \\ 4a^5\sqrt{5a} - 12a^5\sqrt{5a} + 21a^5\sqrt{5a} &= \\ (4 - 12 + 21)a^5\sqrt{5a} &= \boxed{13a^5\sqrt{5a}} \end{aligned}$$

Note: We can rearrange the final answer as $13a^5\sqrt{5a} = 13\sqrt{5a^5}\sqrt{a}$. This other form is just as correct and might even be preferable in some cases.

7. $(\sqrt{7} + 2)(\sqrt{7} - 2)$

Solution:

$$(\sqrt{7} + 2)(\sqrt{7} - 2) = \sqrt{7}\sqrt{7} - 2\sqrt{7} + 2\sqrt{7} - 4 = 7 - 4 = \boxed{3}$$

8. $(\sqrt{7} - 2)^2$

Solution:

$$\begin{aligned} (\sqrt{7} - 2)^2 &= (\sqrt{7} - 2)(\sqrt{7} - 2) \\ &= \sqrt{7}\sqrt{7} - 2\sqrt{7} - 2\sqrt{7} + 4 \\ &= 7 - 4\sqrt{7} + 4 = \boxed{11 - 4\sqrt{7}} \end{aligned}$$

9. $(\sqrt{3} - 1)^3$

Solution: We will first work out $(\sqrt{3} - 1)^2$ and then multiply the result by $(\sqrt{3} - 1)$.

$$\begin{aligned} (\sqrt{3} - 1)^3 &= (\sqrt{3} - 1)(\sqrt{3} - 1)(\sqrt{3} - 1) \\ &= (\sqrt{3}\sqrt{3} - 1\sqrt{3} - 1\sqrt{3} + 1)(\sqrt{3} - 1) \\ &= (3 - 2\sqrt{3} + 1)(\sqrt{3} - 1) \\ &= (4 - 2\sqrt{3})(\sqrt{3} - 1) \\ &= 4\sqrt{3} - 4 - 2\sqrt{3}\sqrt{3} + 2\sqrt{3} \\ &= 4\sqrt{3} - 4 - 2 \cdot 3 + 2\sqrt{3} \\ &= 4\sqrt{3} - 4 - 6 + 2\sqrt{3} \\ &= \boxed{-10 + 6\sqrt{3}} \end{aligned}$$

10. $(\sqrt{5x}-2)(\sqrt{5x}+3)$

Solution:

$$\begin{aligned}(\sqrt{5x}-2)(\sqrt{5x}+3) &= \sqrt{5x}\sqrt{5x}+3\sqrt{5x}-2\sqrt{5x}-2\cdot 3=5x+3\sqrt{5x}-2\sqrt{5x}-6 \\ &= \boxed{5x+\sqrt{5x}-6}\end{aligned}$$

11. $(2-\sqrt{x})(3+2\sqrt{x})$

Solution:

$$(2-\sqrt{x})(3+2\sqrt{x})=6+4\sqrt{x}-3\sqrt{x}-2\sqrt{x}\sqrt{x}=\boxed{6+\sqrt{x}-2x}$$

12. $(\sqrt{x}-\sqrt{2})^2$

Solution:

$$\begin{aligned}(\sqrt{x}-\sqrt{2})^2 &= (\sqrt{x}-\sqrt{2})(\sqrt{x}-\sqrt{2}) && \text{we FOIL} \\ &= \sqrt{x}\sqrt{x}-\sqrt{x}\sqrt{2}-\sqrt{2}\sqrt{x}+\sqrt{2}\sqrt{2} \\ &= x-\sqrt{2x}-\sqrt{2x}+2=\boxed{x-2\sqrt{2x}+2}\end{aligned}$$

13. Find the exact value of x^2-4x+6 if $x=2-\sqrt{3}$.

Solution: We work out x^2 first.

$$\begin{aligned}x^2 &= (2-\sqrt{3})^2 = (2-\sqrt{3})(2-\sqrt{3}) \\ &= 4-2\sqrt{3}-2\sqrt{3}+\sqrt{3}\sqrt{3} \\ &= 4-4\sqrt{3}+3=7-4\sqrt{3}\end{aligned}$$

Now we substitute $x=2-\sqrt{3}$ into x^2-4x+6 .

$$\begin{aligned}x^2-4x+6 &= (2-\sqrt{3})^2-4(2-\sqrt{3})+6= \\ &= 7-4\sqrt{3}-8+4\sqrt{3}+6=7-8+6=\boxed{5}\end{aligned}$$

Solutions for 14.2 – Completing the Square – Part 4

1. Solve the equation $3x^2 - 5x - 7 = 0$ over the real numbers. Make sure to check your solutions.

Solution: We factor out the leading coefficient and factor by completing the square.

$$\begin{aligned} 3x^2 - 5x - 7 &= 0 \\ 3\left(x^2 - \frac{5}{3}x - \frac{7}{3}\right) &= 0 \end{aligned}$$

Half of the linear coefficient is $\frac{5}{3} \div 2 = \frac{5}{3} \cdot \frac{1}{2} = \frac{5}{6}$. Thus the complete square is $\left(x - \frac{5}{6}\right)^2$

$$\left(x - \frac{5}{6}\right)^2 = \left(x - \frac{5}{6}\right)\left(x - \frac{5}{6}\right) = x^2 - \frac{5}{6}x - \frac{5}{6}x + \frac{25}{36} = x^2 - \frac{5}{3}x + \frac{25}{36} \quad \text{So we smuggle in } \frac{25}{36}$$

$$\begin{aligned} 3\left(\underbrace{x^2 - \frac{5}{3}x + \frac{25}{36}}_{\left(x - \frac{5}{6}\right)^2} - \frac{25}{36} - \frac{7}{3}\right) &= 0 \\ 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{25}{36} - \frac{7 \cdot 12}{3 \cdot 12}\right) &= 0 \\ 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{25}{36} - \frac{84}{36}\right) &= 0 \\ 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{109}{36}\right) &= 0 \end{aligned}$$

We next apply the difference of squares theorem. $\frac{109}{36}$ is the square of a number - its own square root.

$$\frac{109}{36} = \left(\sqrt{\frac{109}{36}}\right)^2 = \left(\frac{\sqrt{109}}{\sqrt{36}}\right)^2 = \left(\frac{\sqrt{109}}{6}\right)^2$$

$$\begin{aligned} 3\left(\left(x - \frac{5}{6}\right)^2 - \left(\frac{\sqrt{109}}{6}\right)^2\right) &= 0 \\ 3\left(x - \frac{5}{6} + \frac{\sqrt{109}}{6}\right)\left(x - \frac{5}{6} - \frac{\sqrt{109}}{6}\right) &= 0 \end{aligned}$$

We solve for the zeroes of both linear factors:

$$x - \frac{5}{6} + \frac{\sqrt{109}}{6} = 0$$

$$x + \frac{\sqrt{109}}{6} = \frac{5}{6}$$

$$x_1 = \frac{5}{6} - \frac{\sqrt{109}}{6} = \boxed{\frac{5 - \sqrt{109}}{6}}$$

or

$$x - \frac{5}{6} - \frac{\sqrt{109}}{6} = 0$$

$$x - \frac{\sqrt{109}}{6} = \frac{5}{6}$$

$$x_2 = \frac{5}{6} + \frac{\sqrt{109}}{6} = \boxed{\frac{5 + \sqrt{109}}{6}}$$

We check: if $x = \frac{5 - \sqrt{109}}{6}$, then

$$\begin{aligned}
 \text{LHS} &= 3x^2 - 5x - 7 = 3 \left(\frac{5 - \sqrt{109}}{6} \right)^2 - 5 \left(\frac{5 - \sqrt{109}}{6} \right) - 7 \\
 &= \frac{3}{1} \cdot \frac{(5 - \sqrt{109})^2}{36} - \frac{5}{1} \left(\frac{5 - \sqrt{109}}{6} \right) - 7 = \\
 &= \frac{3(25 - 5\sqrt{109} - 5\sqrt{109} + 109)}{36} - \frac{5(5 - \sqrt{109})}{6} - 7 = \frac{134 - 10\sqrt{109}}{12} - \frac{25 - 5\sqrt{109}}{6} - 7 \\
 &= \frac{134 - 10\sqrt{109}}{12} - \frac{2(25 - 5\sqrt{109})}{12} - 7 = \frac{134 - 10\sqrt{109} - 2(25 - 5\sqrt{109})}{12} - 7 \\
 &= \frac{134 - 10\sqrt{109} - 50 + 10\sqrt{109}}{12} - 7 = \frac{84}{12} - 7 = 7 - 7 = 0 = \text{RHS}
 \end{aligned}$$

and if $x = \frac{5 + \sqrt{109}}{6}$, then

$$\begin{aligned}
 \text{LHS} &= 3x^2 - 5x - 7 = 3 \left(\frac{5 + \sqrt{109}}{6} \right)^2 - 5 \left(\frac{5 + \sqrt{109}}{6} \right) - 7 \\
 &= \frac{3}{1} \cdot \frac{(5 + \sqrt{109})^2}{36} - \frac{5}{1} \left(\frac{5 + \sqrt{109}}{6} \right) - 7 = \\
 &= \frac{3(25 + 5\sqrt{109} + 5\sqrt{109} + 109)}{36} - \frac{5(5 + \sqrt{109})}{6} - 7 = \frac{134 + 10\sqrt{109}}{12} - \frac{25 + 5\sqrt{109}}{6} - 7 \\
 &= \frac{134 + 10\sqrt{109}}{12} - \frac{2(25 + 5\sqrt{109})}{12} - 7 = \frac{134 + 10\sqrt{109} - 2(25 + 5\sqrt{109})}{12} - 7 \\
 &= \frac{134 + 10\sqrt{109} - 50 - 10\sqrt{109}}{12} - 7 = \frac{84}{12} - 7 = 7 - 7 = 0 = \text{RHS}
 \end{aligned}$$

Thus our solution, $\frac{5 + \sqrt{109}}{6}$ and $\frac{5 - \sqrt{109}}{6}$ is correct.

2. Find all numbers that are exactly three less than their own square.

Solution: Let us denote this number by x . Then its square is x^2 and

$$\begin{aligned}
 x &= x^2 - 3 && \text{reduce one side to zero} \\
 0 &= x^2 - x - 3 \\
 x^2 - x - 3 &= 0 && \left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4} \\
 \underbrace{x^2 - x + \frac{1}{4}}_{\left(x - \frac{1}{2}\right)^2} - \frac{1}{4} - 3 &= 0 && 3 = \frac{12}{4} \\
 \left(x - \frac{1}{2}\right)^2 - \frac{13}{4} &= 0 && \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{\sqrt{4}} = \frac{\sqrt{13}}{2} \\
 \left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{13}}{2}\right)^2 &= 0 \\
 \left(x - \frac{1}{2} + \frac{\sqrt{13}}{2}\right) \left(x - \frac{1}{2} - \frac{\sqrt{13}}{2}\right) &= 0
 \end{aligned}$$

$$\begin{array}{ll}
 \text{Either } x - \frac{1}{2} + \frac{\sqrt{13}}{2} = 0 & \text{or} \quad x - \frac{1}{2} - \frac{\sqrt{13}}{2} = 0 \\
 x + \frac{\sqrt{13}}{2} = \frac{1}{2} & x - \frac{\sqrt{13}}{2} = \frac{1}{2} \\
 x_1 = \frac{1}{2} - \frac{\sqrt{13}}{2} = \frac{1 - \sqrt{13}}{2} & x_2 = \frac{1}{2} + \frac{\sqrt{13}}{2} = \frac{1 + \sqrt{13}}{2}
 \end{array}$$

We check: If $x_1 = \frac{1 - \sqrt{13}}{2}$, then

$$\begin{aligned}
 x_1^2 - 3 &= \left(\frac{1 - \sqrt{13}}{2} \right)^2 - 3 = \frac{(1 - \sqrt{13})^2}{2^2} - 3 = \frac{1 + 13 - 2\sqrt{13}}{4} - 3 = \frac{14 - 2\sqrt{13}}{4} - 3 = \frac{2(7 - \sqrt{13})}{4} - 3 \\
 &= \frac{7 - \sqrt{13}}{2} - \frac{6}{2} = \frac{1 - \sqrt{13}}{2} = x_1 \checkmark \quad \text{Thus } x_1 = \frac{1 - \sqrt{13}}{2} \text{ works.}
 \end{aligned}$$

Let us see about the other number. If $x_2 = \frac{1 + \sqrt{13}}{2}$, then

$$\begin{aligned}
 x_2^2 - 3 &= \left(\frac{1 + \sqrt{13}}{2} \right)^2 - 3 = \frac{(1 + \sqrt{13})^2}{2^2} - 3 = \frac{1 + 13 + 2\sqrt{13}}{4} - 3 = \frac{14 + 2\sqrt{13}}{4} - 3 = \frac{2(7 + \sqrt{13})}{4} - 3 \\
 &= \frac{7 + \sqrt{13}}{2} - \frac{6}{2} = \frac{1 + \sqrt{13}}{2} = x_2 \checkmark \quad \text{Thus } x_2 = \frac{1 + \sqrt{13}}{2} \text{ also works.}
 \end{aligned}$$

We found two numbers; they both work.

Solutions for 14.4 – The Pythagorean Theorem

1. Could the three line segments given be the three sides of a right triangle? Explain your answer.

a) 6 cm, 10 cm, and 8 cm

Solution: The longest side is 10 cm long. Thus, only this side can be the hypotenuse. We use the Pythagorean theorem to check for a right angle:

$$6^2 + 8^2 \stackrel{?}{=} 10^2$$

We get that the two quantities are equal, thus this triangle has a right angle.

b) 4 m, 5 m, and 6 m

Solution: The longest side is 6 m long. Thus, only this side can be the hypotenuse. We use the Pythagorean theorem to check for a right angle:

$$4^2 + 5^2 \stackrel{?}{=} 6^2$$

$$\text{LHS} = 4^2 + 5^2 = 16 + 25 = 41$$

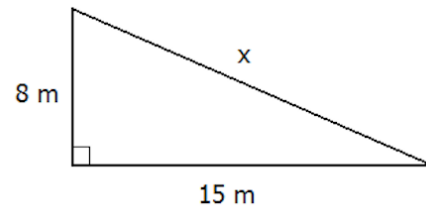
$$\text{RHS} = 6^2 = 36$$

$$\text{LHS} \neq \text{RHS}$$

We get that the two quantities are not equal, thus this triangle does not have a right angle.

Find the hypotenuse of the triangle shown on the figure.

2. Solution: We apply the Pythagorean theorem. The longest side is always the one opposite the right angle.



$$8^2 + 15^2 = x^2$$

$$289 = x^2$$

$$\pm 17 = x$$

Since distance can not be negative, -17 is ruled out. The answer is 17m.

Please note that the step taking us from $x^2 = 289$ to $x = \pm 17$ is a very nice shortcut. The traditional way of solving quadratic equations is to reduce one side to zero, factor, and apply the zero product rule.

$$x^2 = 289$$

$$x^2 - 289 = 0$$

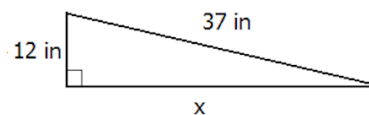
$$x^2 - 17^2 = 0$$

$$(x + 17)(x - 17) = 0 \implies x = -17 \text{ or } x = 17$$

Students are encouraged to use the shorter version, **as long as they don't make the serious algebraic error** of concluding from $x^2 = 289$ that $x = 17$. While in the context of the geometry the negative solution is not possible, the equation $x^2 = 289$ has two solutions, 17 and -17 .

3. Find the missing leg of the right triangle shown on the picture.

Solution: We apply the Pythagorean theorem. The longest side is always the one opposite the right angle.



$$\begin{aligned} (12\text{in})^2 + x^2 &= (37\text{in})^2 \\ x^2 + 144\text{in}^2 &= 1369\text{in}^2 && \text{subtract } 144\text{in}^2 \\ x^2 &= 1225\text{in}^2 && \sqrt{1225} = 35 \\ x &= \pm 35\text{in} \end{aligned}$$

Since distance can not be negative, -35in is ruled out. The answer is 35 inches.

4. Find the distance between the points $(3, 8)$ and $(8, -4)$.

Solution: We graph the points, and draw a horizontal and vertical line connecting the points as shown on the picture. We can compute the distance as the hypotenuse of the right triangle we created. How long are the shorter sides?

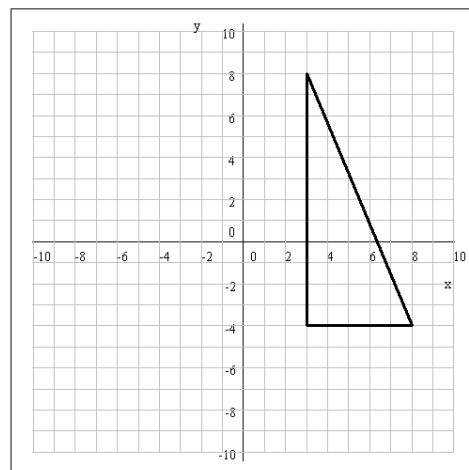
Algebraic approach: Subtract the coordinates. The length of the horizontal side is the difference between the x -coordinates: $8 - 3 = 5$ and the length of the vertical side is the difference between the y -coordinates:

$8 - (-4) = 12$. The difference will always work. Even if we get -5 instead of 5, it will not matter since we will square it in the Pythagorean theorem.

Geometric approach: From 3 to 8 we have to step 5 units up. From -4 to 8: first we step 4 to get from -4 to 0. Then another 8 steps to 8, and so $4 + 8 = 12$ steps. The message here is that the algebra and geometry will always agree.

Now we know that the shorter sides are 5 and 12 units long, and we need to find the hypotenuse.

$$\begin{aligned} 5^2 + 12^2 &= x^2 \\ 25 + 144 &= x^2 \\ 169 &= x^2 \\ \pm 13 &= x \end{aligned}$$

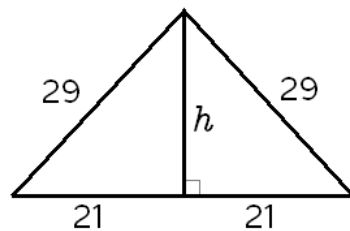


Since distances are never negative, -13 is ruled out and so the answer is 13 units.

5. The sides of an isosceles triangle are 42 units, 29 units, and 29 units long. Find the length of the height drawn to the 42 units long side.

Solution: In case of isosceles triangles, the height drawn to the base splits the triangle into two identical right triangles as shown on the picture. The height now can be easily computed via the Pythagorean theorem.

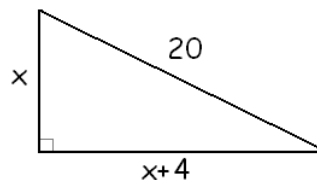
$$\begin{aligned} 21^2 + h^2 &= 29^2 \\ 441 + h^2 &= 841 \\ h^2 &= 400 \\ h &= \pm 20 \implies h = 20 \end{aligned}$$



Again, the negative solution of the equation is ruled out because distances cannot be negative. The height belonging to the base is 20 units long.

6. The hypotenuse of a right triangle is 20cm. The difference between the other two sides is 4cm. Find the sides of the triangle.

Solution: Let x denote the shortest side. Then the other missing side is $x + 4$ cm long. We state the Pythagorean theorem for the triangle and solve the quadratic equation for x .



$$\begin{aligned}
 x^2 + (x+4)^2 &= 20^2 && \text{expand } (x+4)^2 \\
 x^2 + x^2 + 8x + 16 &= 400 && \text{combine like terms} \\
 2x^2 + 8x + 16 &= 400 && \text{subtract 400} \\
 2x^2 + 8x - 384 &= 0 && \text{factor out 2} \\
 2(x^2 + 4x - 192) &= 0 && (x+2)^2 = x^2 + 4x + 4 \\
 2(\underbrace{x^2 + 4x + 4}_{(x+2)^2} - 192) &= 0 \\
 2((x+2)^2 - 196) &= 0 \\
 2((x+2)^2 - 14^2) &= 0 \\
 2(x+2+14)(x+2-14) &= 0 \\
 2(x+16)(x-12) &= 0 \\
 x_1 = -16, x_2 = 12
 \end{aligned}$$

Since distances are never negative, -16 is ruled out. If the shortest side is 12cm, the other side is $12\text{cm} + 4\text{cm} = 16\text{cm}$. Thus the solution is 12cm and 16cm. We check:

$$16 - 12 = 4 \checkmark \text{ and } 16^2 + 12^2 = 256 + 144 = 400 = 20^2 \checkmark$$

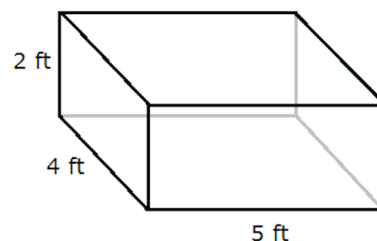
7. Find the length of the longest line segment inside the rectangular prism shown on the picture. (This line segment is called the **main diagonal** of the prism.)

Solution: We will apply the Pythagorean theorem twice.

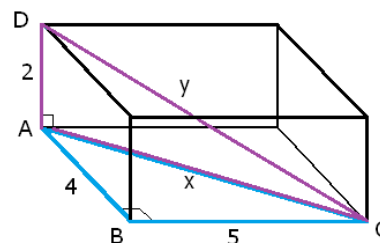
Let us label the points and sides we will use on the picture first.

We will find x using the Pythagorean theorem in triangle ABC .

Then we can find y using the Pythagorean theorem in triangle ACD .



$$\begin{aligned}
 4^2 + 5^2 &= x^2 && 2^2 + x^2 = y^2 \\
 41 &= x^2 && 45 = y^2 \\
 x &= \sqrt{41} && y = \sqrt{45}
 \end{aligned}$$

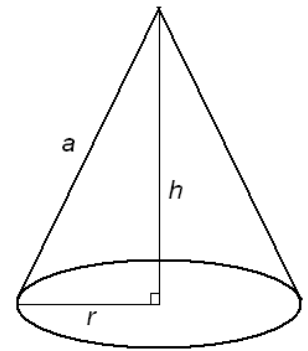


Note: Our result is actually $\sqrt{2^2 + 4^2 + 5^2}$. Indeed, we can see that the length of the main diagonal in a rectangular prism with sides x , y , and z is $L = \sqrt{x^2 + y^2 + z^2}$. This is sometimes called the 3-dimensional Pythagorean theorem.

8. Find the height h of the cone shown on the picture, if the base has a radius of 10m and $a = 26$ m.

Solution: There is a right triangle formed by a , h , and r as the picture shows. We state the Pythagorean theorem for this triangle and solve for h .

$$\begin{aligned} r^2 + h^2 &= a^2 \\ 10^2 + h^2 &= 26^2 \\ h^2 + 100 &= 676 \\ h^2 &= 576 \\ h &= \pm 24 \end{aligned}$$



The negative value is ruled out, and so the height is 24m.

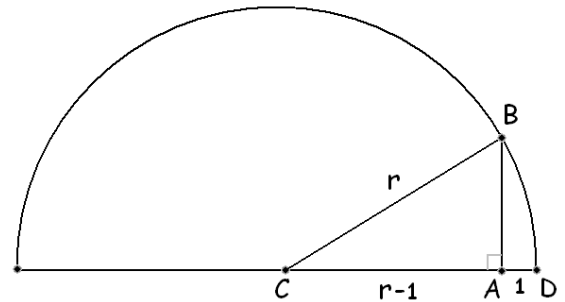
9. An arch is in the shape of a semicircle. At a point along the base 1 foot from an end of the arch, the height of the arch is 7 feet. Find the maximum height of the arch.

Solution: Consider the picture shown.

Let r denote the radius of the semi-circle. We are asked to find the value of r . $AD = 1$ and $AB = 7$ were given. Clearly $BC = r$, and this problem becomes easy once we realize that $AC = r - 1$ since the entire line segment CD is the radius r .

Now we state the Pythagorean theorem on the right triangle ABC and solve for r .

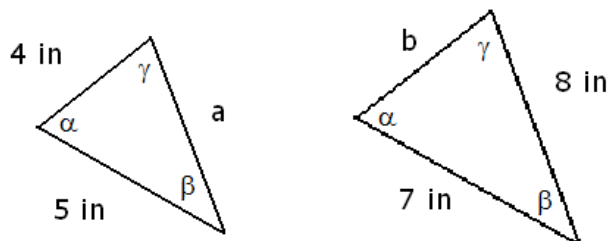
$$\begin{aligned} (r-1)^2 + 7^2 &= r^2 \\ r^2 - 2r + 1 + 49 &= r^2 && \text{subtract } r^2 \\ -2r + 50 &= 0 && \text{add } 2r \\ 50 &= 2r && \text{divide by 2} \\ 25 &= r \end{aligned}$$



Thus the maximum height of the arch is 25 feet.

Solutions for 18.2 – Similar Triangles

1. The triangles shown are similar. Find the exact values of a and b shown on the picture.



Solution: In similar triangles, the ratios of corresponding sides are preserved. To find a , we write the ratio $\frac{\text{side opposite angle } \alpha}{\text{side opposite angle } \gamma}$ for both triangles.

$$\frac{\text{side opposite angle } \alpha}{\text{side opposite angle } \gamma} = \frac{a}{5} = \frac{8}{7}$$

We now solve the equation for a : $\frac{a}{5} = \frac{8}{7}$ multiply both sides by 35

$$7a = 40 \quad \text{divide by 7}$$

$$a = \frac{40}{7}$$

Similarly, we can find b by writing the ratio $\frac{\text{side opposite angle } \beta}{\text{side opposite angle } \gamma}$ for both triangles.

$$\frac{\text{side opposite angle } \beta}{\text{side opposite angle } \gamma} = \frac{4}{5} = \frac{b}{7}$$

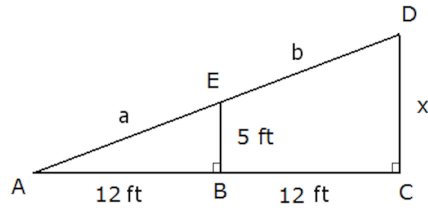
We now solve the equation for b . $\frac{4}{5} = \frac{b}{7}$ multiply both sides by 35

$$28 = 5b \quad \text{divide by 5}$$

$$b = \frac{28}{5}$$

Thus $a = \frac{40}{7}$ in and $b = \frac{28}{5}$ in.

2. Consider the picture shown.



a) Use the Pythagorean Theorem to find the value of a .

Solution: The shorter sides are 5ft and 12ft long. The hypotenuse is a . We state the Pythagorean Theorem for this triangle and solve the equation for a .

$$\begin{aligned} 5^2 + 12^2 &= a^2 \\ 25 + 144 &= a^2 \\ 169 &= a^2 \\ \pm 13 &= a \end{aligned}$$

Since distances can never be negative, $a = -13$ is ruled out. Thus $a = 13\text{ft}$.

b) Prove that the triangles ABE and ACD are similar.

Solution: First, angles ABE and ACD are both right angles. Second, the two triangles literally share angle EAB (or angle DAC). Finally, if two triangles agree in the measure of two of their angles, the third angles must be equal since in every triangle, the three angles add up to 180° . The two triangles are similar because they have identical angles.

c) Use similar triangles to find the value of x .

Solution: The triangles $\triangle ABE$ and $\triangle ACD$ are similar. To find x , we write the ratio $\frac{\text{side opposite point A}}{\text{horizontal side}}$ for both triangles.

$$\frac{\text{side opposite point A}}{\text{horizontal side}} = \frac{5}{12} = \frac{x}{24}$$

and solve the equation for x .

$$\begin{aligned} \frac{5}{12} &= \frac{x}{24} && \text{multiply both sides by 24} \\ 10 &= x \end{aligned}$$

Thus x is 10ft . Indeed, once we established that the triangles are similar, and noticed that the horizontal side was doubled from 12ft to 24ft, we could easily predict this answer.

d) Find the value of b .

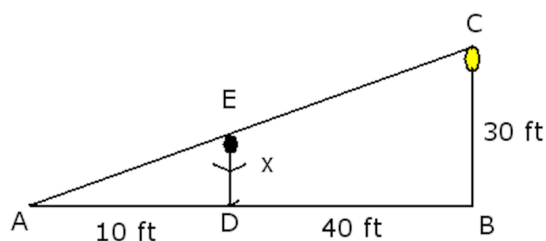
Solution: We can either use similar triangles or the Pythagorean Theorem to find the side AD . Either way, we easily get that 26ft. However, the length of side AD is not b , but $a + b$. From part a), we know that $a = 13\text{ft}$.

$$\begin{aligned} 13 + b &= 26 \\ b &= 13 \end{aligned}$$

Thus $b = 13\text{ft}$.

3. a) A person is standing 40ft away from a street light that is 30ft tall. How tall is he if his shadow is 10ft long?

Solution: After we draw a picture, we see that this problem is very similar to the previous one. Triangles $\triangle ADE$ and $\triangle ABC$ are similar.



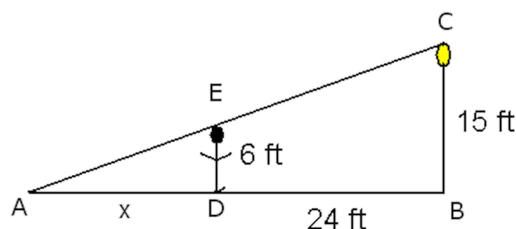
We use the ratio $\frac{DE}{AD} = \frac{BC}{AB}$ and solve for x .

$$\begin{aligned} \frac{x}{10} &= \frac{30}{50} && \text{multiply both sides by 50} \\ 5x &= 30 && \text{divide by 5} \\ x &= 6 \end{aligned}$$

Thus the person is 6ft tall. Notice that the number 40 did not occur in the equation. It is a common error to use 40 instead of 50.

- b) A 6ft tall person is standing 24ft away from a street light that is 15ft tall. How long is her shadow?

Solution: After we draw a picture, write an equation expressing that triangles ADE and ABC are similar.



We can use the same ratio as before, $\frac{DE}{AD} = \frac{BC}{AB}$ and solve for x .

$$\begin{aligned} \frac{6}{x} &= \frac{15}{x+24} && \text{multiply both sides by } x(x+24) \\ 6(x+24) &= 15x && \text{distribute} \\ 6x+144 &= 15x && \text{subtract } 6x \\ 144 &= 9x && \text{divide by 9} \\ 16 &= x \end{aligned}$$

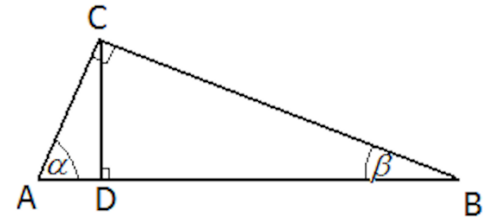
Thus her shadow is 16ft long.

Note: If the first step, multiplying by $x(x+24)$ (same as cross-multiplying) is confusing, here is the break-down:

$$\begin{aligned} \frac{6}{x} &= \frac{15}{x+24} && \text{multiply by } x(x+24) \\ x(x+24) \cdot \frac{6}{x} &= \frac{15}{x+24} \cdot x(x+24) && \text{expressing everything as a fraction} \\ \frac{x(x+24)}{1} \cdot \frac{6}{x} &= \frac{15}{x+24} \cdot \frac{x(x+24)}{1} \\ \frac{x(x+24)6}{x} &= \frac{15x(x+24)}{x+24} && \text{cancel} \\ \frac{(x+24)6}{1} &= \frac{15x}{1} && \text{simplify} \\ (x+24)6 &= 15x \end{aligned}$$

4. Prove the following statement. Let ABC be any right triangle, the right angle at point C . The altitude drawn from C to the hypotenuse splits the triangle into two right triangles that are similar to each other and to the original triangle.

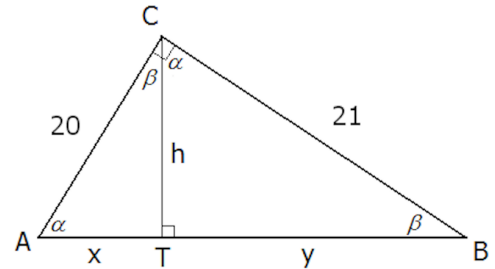
Solution: Let us draw a picture and use standard labeling of points.



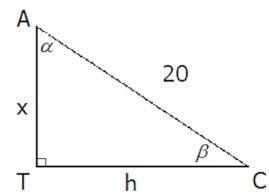
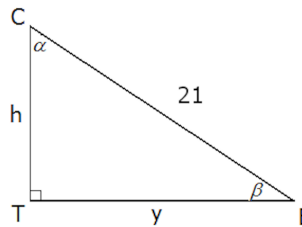
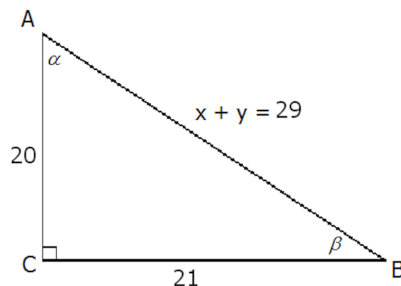
The two triangles created, $\triangle ADC$ and $\triangle DBC$ are both right triangles. $\triangle ADC$ is similar to the original triangle, because they agree in two angles: the right angle and α . $\triangle DBC$ is similar to the original triangle, because they agree in two angles: the right angle and β . Thus all three triangles are similar. Also, this will be very useful later: $\angle ACD = \beta$ and $\angle BCD = \alpha$.

5. Find x , y , and h based on the picture.

Solution: We can easily find the hypotenuse of this triangle via the Pythagorean Theorem. The hypotenuse, $x + y$ is $\sqrt{20^2 + 21^2} = 29$ units long. Next, let us first label the points, angles and sides in the triangle.



We now re-draw the three similar triangles in a separate figure, all three of them rotated and reflected into the same direction. This way, it is easy to realize what sides correspond to each other. (Hint: start with the angles, they are in the same location. Then identify the points, and finally the sides.)



We can find y using the following ratio in the first two triangles: $\frac{\text{side opposite } \alpha}{\text{hypotenuse}} = \frac{21}{29} = \frac{y}{21}$

$$\begin{aligned} \frac{21}{29} &= \frac{y}{21} && \text{multiply both sides by } 21 \cdot 29 \\ 441 &= 29y && \text{divide by } 29 \\ \frac{441}{29} &= y \end{aligned}$$

A different ratio in the same triangles can be used to obtain

$$\frac{\text{side opposite } \beta}{\text{hypotenuse}} = \frac{20}{29} = \frac{h}{21}$$

$$\begin{aligned} \frac{20}{29} &= \frac{h}{21} && \text{multiply both sides by } 21 \cdot 29 \\ 420 &= 29h && \text{divide by } 29 \\ \frac{420}{29} &= h \end{aligned}$$

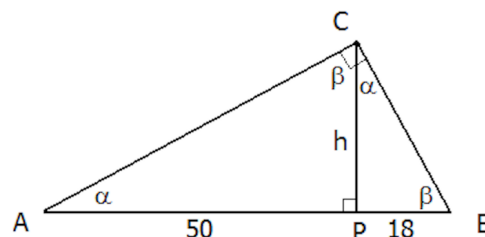
For x , we can simply use the fact that $x + y = 29$ and we already computed $y = \frac{441}{29}$.

$$\begin{aligned}x + \frac{441}{29} &= 29 \\x &= 29 - \frac{441}{29} = \frac{400}{29}\end{aligned}$$

Therefore, $x = \frac{400}{29}$, $y = \frac{441}{29}$, and $h = \frac{420}{29}$.

6. The picture shows a right triangle. Find the length of h , the height drawn to the hypotenuse.

Solution: Let us first label the points, angles and sides in the triangle. As we proved it in the previous problem, the two new triangles are similar to the original triangle.



Consider now the ratio $\frac{\text{side opposite } \beta}{\text{side opposite } \alpha}$ in triangles $\triangle APC$ and $\triangle PBC$. Since these triangles are similar, this ratio is preserved.

$$\frac{\text{side opposite } \beta}{\text{side opposite } \alpha} = \frac{50}{h} = \frac{h}{18}$$

We solve this equation for h .

$$\begin{aligned}\frac{50}{h} &= \frac{h}{18} \\50 \cdot 18 &= h^2 \\900 &= h^2 \\h &= \pm 30\end{aligned}$$

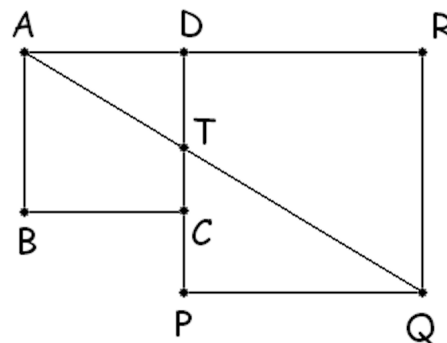
$h = -30$ is ruled out since distances can not be negative. Thus $h = 30$.

7. Quadrilaterals $ABCD$ and $PQRD$ are both squares. If $AB = 3$ and $PQ = 3$, then find the exact value of the length of TC .

Solution: Triangles ADT and ARQ are similar, because they share an angle at A and they both have a right angle. $AD = 2$, $AR = 5$, and $RQ = 3$.

$$\frac{DT}{AD} = \frac{RQ}{AR} \quad \text{becomes} \quad \frac{x}{2} = \frac{3}{5} \implies x = \frac{6}{5} = 1.2$$

So $DT = 1.2$ units



Solutions for 19.1 – Writing Equations of Lines

1. In each case, find the slope of the line determined by the two points given.

- a) $(5, -1)$ and $(1, 3)$

Solution: We will use the slope formula to find the slope. Let $(x_1, y_1) = (5, -1)$ and $(x_2, y_2) = (1, 3)$. Then the slope formula gives us

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-1)}{1 - 5} = \frac{4}{-4} = \boxed{-1}$$

- b) $(2, 1)$ and $(6, 3)$

Solution: We will use the slope formula. Let $(x_1, y_1) = (2, 1)$ and $(x_2, y_2) = (6, 3)$. Then the slope formula gives us

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{6 - 2} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

- c) $(8, -1)$ and $(-1, -1)$

Solution: We will use the slope formula. $(x_1, y_1) = (8, -1)$ and $(x_2, y_2) = (-1, -1)$

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - (-1)}{-1 - 8} = \frac{0}{-9} = \boxed{0}$$

Indeed, **horizontal lines all have a slope of zero**. Since the line is horizontal, all of its points have the same y -coordinate and so the rise, $y_2 - y_1$ is zero.

- d) $(7, 3)$ and $(7, -4)$

Solution: We will use the slope formula. $(x_1, y_1) = (7, 3)$ and $(x_2, y_2) = (7, -4)$.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 3}{7 - 7} = \frac{-7}{0} = \text{undefined}$$

Vertical lines do not have slopes. Since the line is vertical, all of its points have the same x -coordinate and so the run, $x_2 - x_1$ is zero. This line has no slope.

2. For each straight line given, find its slope.

- a) the line $y = -2x + 5$

Solution: The equation of the line is given in slope-intercept form. We simply read the coefficient of x . The slope is -2.

- b) the line $2x - 5y = 8$

Solution: We first bring the equation of the line to its slope-intercept form by solving for y . Then the coefficient of x is the slope.

$$\begin{array}{lll} 2x - 5y & = & 8 \quad \text{add } 5y \\ 2x & = & 5y + 8 \quad \text{subtract } 8 \\ 2x - 8 & = & 5y \quad \text{divide by } 5 \\ \frac{2x - 8}{5} & = & y \\ y & = & \frac{2}{5}x - \frac{8}{5} \quad \text{The slope is } \boxed{\frac{2}{5}}. \end{array}$$

- c) the line $y = 3$

Solution: The equation of the line is given in slope-intercept form. We simply read the coefficient of x . In this case, we don't see x because its coefficient is zero. We can re-write it as $y = 0x + 3$. If we are not sure, we can graph the line and see from its graph that it is horizontal. Therefore, the slope is 3.

- d) the line
- $7(x-3) + x - 2y = 2y + 1$

Solution: We first bring the equation of the line to its slope-intercept form by solving for y . Then the coefficient of x is the slope.

$$\begin{aligned}
 7(x-3) + x - 2y &= 2y + 1 && \text{distribute} \\
 7x - 21 + x - 2y &= 2y + 1 && \text{combine like terms} \\
 8x - 2y - 21 &= 2y + 1 && \text{add } 2y \\
 8x - 21 &= 4y + 1 && \text{subtract } 1 \\
 8x - 22 &= 4y && \text{divide by } 4 \\
 \frac{8x - 22}{4} &= y \\
 y &= \frac{8}{4}x - \frac{22}{4} = 2x - \frac{11}{2} && \text{Thus the slope is } \boxed{2}.
 \end{aligned}$$

- e) the line
- $x = -\frac{1}{2}$
- .

Solution: We can not bring this equation to its slope-intercept form by solving for y because y does not appear in it. This means that there is no slope. If this is confusing, select any two points from the line, say $\left(-\frac{1}{2}, 1\right)$ and $\left(-\frac{1}{2}, 5\right)$ and apply the slope formula. It will be clear that m is undefined since we will end up dividing by zero.

This line has no slope.

- f) a line parallel to
- $y = 3x - 1$
- .

Solution: The line $y = 3x - 1$ has slope 3. Parallel lines have the same slope. So our line also has a slope 3.

- g) a line perpendicular to
- $y = 3x - 1$
- .

Solution: The line $y = 3x - 1$ has slope 3. Perpendicular lines have slopes that are negative reciprocals of each other. The negative reciprocal of 3 is $-\frac{1}{3}$.

3. Find an equation for the straight line that has slope 4 and
- y
- intercept
- $(0, -5)$
- .

Solution: The equation is very easy to find with the data given. The slope-intercept form of the line is $y = mx + b$ and we know that $m = 4$ and $b = -5$. Thus the answer is $y = 4x - 5$.

4. Find an equation for the straight line that has slope
- $-\frac{1}{2}$
- and passes through the point
- $(4, 1)$
- .

Solution: Since the point given is not the y -intercept, we will use the point-slope form. We know that the slope is $m = -\frac{1}{2}$ and a point on the line is $(4, 1)$. We can write the line's equation in one easy step: $y - 1 = -\frac{1}{2}(x - 4)$.

5. Find an equation of the straight line that is parallel to
- $3x - 2y = 12$
- and passes through the point
- $(8, -1)$
- .

Solution: We start with the slope of $3x - 2y = 12$. We bring the equation to its slope-intercept form by solving for y .

$$\begin{aligned}
 3x - 2y &= 12 && \text{add } 2y \\
 3x &= 2y + 12 && \text{subtract } 12 \\
 3x - 12 &= 2y && \text{divide by } 2 \\
 y &= \frac{3}{2}x - 6
 \end{aligned}$$

Thus the line $3x - 2y = 12$ has slope $\frac{3}{2}$. Since parallel, our line must have the same slope. Now the problem is like the previous one: the slope and a point is given. Using the point-slope form, we write

$$y - (-1) = \frac{3}{2}(x - 8) \quad \text{and simplify: } \boxed{y + 1 = \frac{3}{2}(x - 8)}$$

If we are to present our answer in the slope-intercept form, that takes just a few additional steps from here.

$$\begin{aligned}
 y + 1 &= \frac{3}{2}(x - 8) && \text{distribute } \frac{3}{2} \\
 y + 1 &= \frac{3}{2}x - \frac{3}{2} \cdot 8 && \text{subtract 1} \\
 y &= \frac{3}{2}x - 12 - 1 \\
 &\boxed{y = \frac{3}{2}x - 13}
 \end{aligned}$$

6. Find an equation of the straight line that is perpendicular to $x + 5y = -3$ and passes through the point $(-1, 4)$.

Solution: We start with the slope of $x + 5y = -3$. We bring the equation $x + 5y = -3$ to its slope-intercept form by solving for y .

$$\begin{aligned}
 x + 5y &= -3 && \text{subtract } x \\
 5y &= -x - 3 && \text{divide by 5} \\
 y &= -\frac{1}{5}x - \frac{3}{5}
 \end{aligned}$$

Thus the line $x + 5y = -3$ has slope $-\frac{1}{5}$. Since they are perpendicular, the slope of our line is the negative reciprocal of $-\frac{1}{5}$, which is 5. Now we have the slope the point given. Using the point-slope form, we write

$$y - 4 = 5(x - (-1)) \quad \text{and simplify: } \boxed{y - 4 = 5(x + 1)}$$

$$\begin{aligned}
 y - 4 &= 5x + 5 \\
 y &= 5x + 9
 \end{aligned}$$

The slope-intercept form can be easily found from the point-slope form:

$$\begin{aligned}
 y - 4 &= 5(x + 1) \\
 y - 4 &= 5x + 5 \\
 &\boxed{y = 5x + 9}
 \end{aligned}$$

7. Find an equation of the straight line that passes through the points $(3, -1)$ and $(1, 5)$.

Solution: We will present two different methods.

Method 1. Point-slope form.

We first find the slope determined by the points. If the point $(x_1, y_1) = (3, -1)$ and $(x_2, y_2) = (1, 5)$, then the slope formula gives us

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-1)}{1 - 3} = \frac{6}{-2} = -3$$

We know that the slope is $m = -3$ and a point on the line is $(1, 5)$. We can write the line's equation in the point-slope form in one easy step: $\boxed{y - 5 = -3(x - 1)}$.

Method 2. This is an application of systems of equations.

We are looking for the slope-intercept form of the straight line, $y = mx + b$. We will know the answer once we have found the value of m and b . Both points $(1, 5)$ and $(3, -1)$ are on the line, and so their coordinates are solution of the equation $y = mx + b$. We obtain two equations by stating that each point is on the line.

$$5 = m(1) + b \quad \text{since } (1, 5) \text{ is on the line, and } -1 = m(3) + b \quad \text{since } (3, -1) \text{ is on the line}$$

This gives us a system of linear equations in m and b :

$$\begin{cases} m + b = 5 \\ 3m + b = -1 \end{cases}$$

This system can easily be solved by elimination. We multiply the first equation by -1 and then add the two equations

$$\begin{array}{r} -m - b = -5 \\ + \quad 3m + b = -1 \\ \hline 2m = -6 \\ m = -3 \end{array}$$

Now that we know the value of m , we find b by substituting $m = -3$ into the first equation (any of the two can be used) and solve for b .

$$\begin{array}{r} -3 + b = 5 \\ b = 8 \end{array}$$

Thus $m = -3$ and $b = 8$ and so the equation of the line is $y = -3x + 8$.

8. Find the midpoint of the line segment determined by $A(7, -4)$ and $B(3, -2)$.

Solution: The x -coordinate of the midpoint is the average of the x -coordinates of A and B .

$$x_M = \frac{x_A + x_B}{2} = \frac{7 + 3}{2} = \frac{10}{2} = 5$$

Similarly, the y -coordinate of the midpoint is the average of the y -coordinates of A and B .

$$y_M = \frac{y_A + y_B}{2} = \frac{-4 + (-2)}{2} = \frac{-6}{2} = -3 \quad \text{Thus the midpoint is } M(5, -3).$$

9. Consider the triangle determined by $A(-5, -3)$, $B(-1, 3)$, and $C(7, 1)$.

- a) Find an equation for the line that connects A and B .

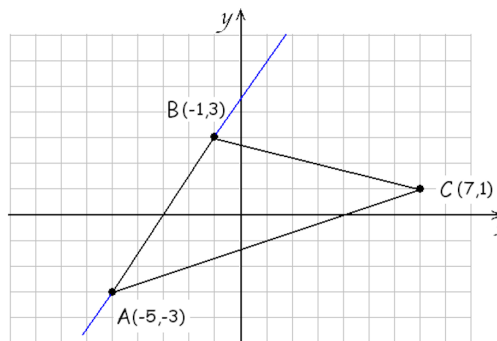
Solution: We find the slope of line AB using the slope formula.

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{3 - (-3)}{-1 - (-5)} = \frac{3 + 3}{-1 + 5} = \frac{6}{4} = \frac{3}{2}$$

We can now use the slope and point A to write the point-slope form

of the equation: $y + 3 = \frac{3}{2}(x + 5)$,

or the slope with point B : $y - 3 = \frac{3}{2}(x + 1)$.



Both solutions are correct. We can bring the equation to the slope-intercept form:

$$\begin{aligned} y + 3 &= \frac{3}{2}(x + 5) \\ y + 3 &= \frac{3}{2}x + \frac{15}{2} \\ y &= \frac{3}{2}x + \frac{15}{2} - 3 = \frac{3}{2}x + \frac{15}{2} - \frac{6}{2} = \frac{3}{2}x + \frac{9}{2} = \frac{3}{2}x + 4\frac{1}{2} \end{aligned}$$

We can see on the picture that this equation is correct.

- b) Find the midpoint of line segment AB .

Solution: The x -coordinate of the midpoint is the average of the x -coordinates of A and B .

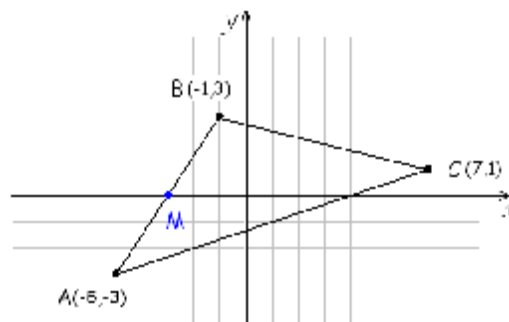
$$x_M = \frac{x_A + x_B}{2} = \frac{-5 + (-1)}{2} = \frac{-6}{2} = -3$$

Similarly, the y -coordinate of the midpoint is the average of the y -coordinates of A and B .

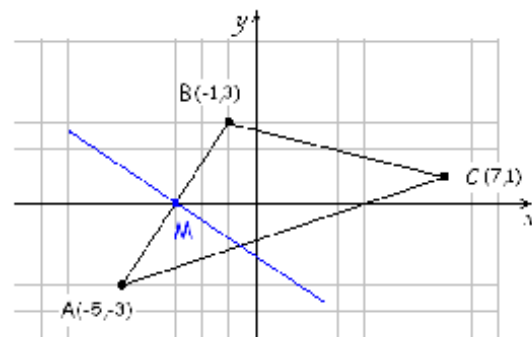
$$y_M = \frac{y_A + y_B}{2} = \frac{-3 + 3}{2} = \frac{0}{2} = 0$$

Thus the midpoint is $M(-3, 0)$.

- c) Find an equation for the perpendicular bisector of side AB .



Solution: In part a), we found the slope of line AB to be $\frac{3}{2}$. The perpendicular bisector is perpendicular (duh) to this line segment and so its slope must be the negative reciprocal of $\frac{3}{2}$, which is $-\frac{2}{3}$. The perpendicular bisector contains the midpoint M of line segment AB . We found the midpoint in part b) to be $M(-2, 0)$.



We can write the point-slope form of the equation: $y - (-2) = -\frac{2}{3}(x - 0)$ and simplify this to

$$y + 2 = -\frac{2}{3}x \quad \text{or} \quad \boxed{y = -\frac{2}{3}x - 2}. \quad \text{We can see from the picture that our equation is correct.}$$

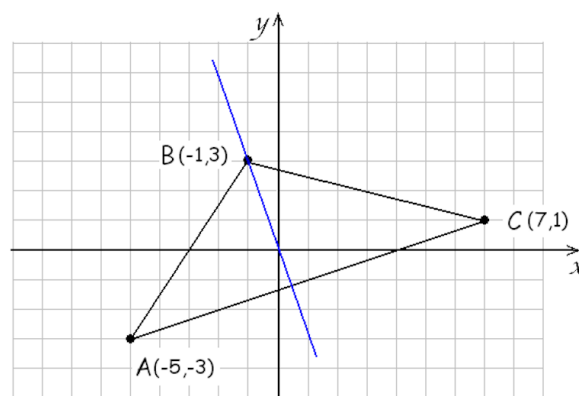
d) Find an equation for the height (or altitude) belonging to side AC .

Solution: The height belonging to side AC is a line that is perpendicular to AC and passes through the opposite vertex, which is point B .

We compute the slope of line segment AC .

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{1 - (-3)}{7 - (-5)} = \frac{1 + 3}{7 + 5} = \frac{4}{12} = \frac{1}{3}$$

The slope of the height, since perpendicular, is the negative reciprocal of $\frac{1}{3}$, which is -3 . We can now write the point-slope form with slope -3 and point $B(-1, 3)$. The answer is $\boxed{y - 3 = -3(x + 1)}$.



If we are to present the slope-intercept form of the equation, we can get it with just a few steps from the slope-intercept form.

$$y - 3 = -3(x + 1)$$

$$y - 3 = -3x - 3$$

$$y = -3x$$

Thus the slope-intercept form of this line is $\boxed{y = -3x}$. We can see from the picture that our equation is correct.

Solutions for 19.2 – Radical Expressions 2

1. Rationalize the denominator in each of the following expressions.

a) $\frac{3}{\sqrt{5}}$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by $\sqrt{5}$.

$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \boxed{\frac{3\sqrt{5}}{5}}$$

b) $\frac{1}{\sqrt{10}-3}$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{10}+3$.

$$\frac{1}{\sqrt{10}-3} = \frac{1}{\sqrt{10}-3} \cdot \frac{\sqrt{10}+3}{\sqrt{10}+3} = \frac{\sqrt{10}+3}{1} = \boxed{3+\sqrt{10}}$$

The denominator is 1 since

$$(\sqrt{10}-3)(\sqrt{10}+3) = \sqrt{10}\sqrt{10} + 3\sqrt{10} - 3\sqrt{10} - 9 = 10 - 9 = 1$$

c) $\frac{2}{\sqrt{7}+1}$

Solution: To rationalize the denominator, we will multiply both the numerator and denominator by the conjugate of the denominator, which is $\sqrt{7}-1$

$$\frac{2}{\sqrt{7}+1} = \frac{2}{\sqrt{7}+1} \cdot \frac{\sqrt{7}-1}{\sqrt{7}-1} = \frac{2(\sqrt{7}-1)}{7-1} = \frac{2(\sqrt{7}-1)}{6} = \boxed{\frac{\sqrt{7}-1}{3}}$$

d) $\frac{\sqrt{5}-1}{\sqrt{5}-2}$

Solution:

$$\frac{\sqrt{5}-1}{\sqrt{5}-2} \cdot \frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{(\sqrt{5}-1)(\sqrt{5}+2)}{(\sqrt{5}-2)(\sqrt{5}+2)} = \frac{5+2\sqrt{5}-\sqrt{5}-2}{5+2\sqrt{5}-2\sqrt{5}-4} = \frac{3+\sqrt{5}}{1} = \boxed{3+\sqrt{5}}$$

e) $\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}}$

Solution: We multiply the fraction by 1 as a fraction of whose both numerator and denominator are the conjugate of the denominator.

$$\frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}} = \frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}} \cdot 1 = \frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}+\sqrt{5}} \cdot \frac{\sqrt{8}-\sqrt{5}}{\sqrt{8}-\sqrt{5}} = \frac{(\sqrt{8}-\sqrt{5})(\sqrt{8}-\sqrt{5})}{(\sqrt{8}+\sqrt{5})(\sqrt{8}-\sqrt{5})}$$

We FOIL out both numerator and denominator

$$\frac{\sqrt{8}\sqrt{8}-\sqrt{8}\sqrt{5}-\sqrt{5}\sqrt{8}+\sqrt{5}\sqrt{5}}{\sqrt{8}\sqrt{8}-\sqrt{8}\sqrt{5}+\sqrt{5}\sqrt{8}-\sqrt{5}\sqrt{5}} = \frac{8-\sqrt{40}-\sqrt{40}+5}{8-5} = \frac{13-2\sqrt{40}}{3}$$

Note: although this answer is acceptable, the expression can be further simplified.

$$\frac{13-2\sqrt{40}}{3} = \frac{13-2\sqrt{4 \cdot 10}}{3} = \frac{13-2\sqrt{4} \cdot \sqrt{10}}{3} = \frac{13-2 \cdot 2 \cdot \sqrt{10}}{3} = \boxed{\frac{13-4\sqrt{10}}{3}}$$

2. Simplify each of the following expressions.

a) $\frac{3-\sqrt{5}}{3+\sqrt{5}} + \frac{3+\sqrt{5}}{3-\sqrt{5}}$

Solution:

$$\frac{3-\sqrt{5}}{3+\sqrt{5}} + \frac{3+\sqrt{5}}{3-\sqrt{5}} = \frac{(3-\sqrt{5})^2 + (3+\sqrt{5})^2}{(3+\sqrt{5})(3-\sqrt{5})} = \frac{14-6\sqrt{5}+14+6\sqrt{5}}{9-5} = \frac{28}{4} = \boxed{7}$$

b) $\left(\frac{8}{\sqrt{7}+\sqrt{3}} + \frac{12}{\sqrt{7}-\sqrt{3}}\right)(5\sqrt{7}-\sqrt{3})$

Solution: We first bring the fractions to the common denominator - they are conjugates, that helps, because the common denominator is then rational. We then multiply the factors.

$$\begin{aligned} &\left(\frac{8}{\sqrt{7}+\sqrt{3}} + \frac{12}{\sqrt{7}-\sqrt{3}}\right)(5\sqrt{7}-\sqrt{3}) = \\ &= \frac{8(\sqrt{7}-\sqrt{3}) + 12(\sqrt{7}+\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})}(5\sqrt{7}-\sqrt{3}) \\ &= \frac{8\sqrt{7}-8\sqrt{3}+12\sqrt{7}+12\sqrt{3}}{7-3}(5\sqrt{7}-\sqrt{3}) \\ &= \frac{20\sqrt{7}+4\sqrt{3}}{4}(5\sqrt{7}-\sqrt{3}) = (5\sqrt{7}+\sqrt{3})(5\sqrt{7}-\sqrt{3}) \\ &= 25(7)-3 = 175-3 = \boxed{172} \end{aligned}$$

c) $\sqrt{\sqrt{41}+4\sqrt{2}} \cdot \sqrt{\sqrt{41}-\sqrt{32}}$

Solution:

$$\begin{aligned} \sqrt{\sqrt{41}+4\sqrt{2}} \cdot \sqrt{\sqrt{41}-\sqrt{32}} &= \sqrt{\sqrt{41}+\sqrt{32}} \cdot \sqrt{\sqrt{41}-\sqrt{32}} = \sqrt{(\sqrt{41}+\sqrt{32})(\sqrt{41}-\sqrt{32})} \\ &= \sqrt{\left((\sqrt{41})^2 - (\sqrt{32})^2\right)} = \sqrt{41-32} = \sqrt{9} = \boxed{3} \end{aligned}$$

d) $\sqrt{5\sqrt{3}+\sqrt{59}} \cdot \sqrt{\sqrt{75}-\sqrt{59}}$

Solution:

$$\begin{aligned} \sqrt{5\sqrt{3}+\sqrt{59}} \cdot \sqrt{\sqrt{75}-\sqrt{59}} &= \sqrt{\sqrt{75}+\sqrt{59}} \cdot \sqrt{\sqrt{75}-\sqrt{59}} = \sqrt{(\sqrt{75}+\sqrt{59})(\sqrt{75}-\sqrt{59})} \\ &= \sqrt{\left((\sqrt{75})^2 - (\sqrt{59})^2\right)} = \sqrt{75-59} = \sqrt{16} = \boxed{4} \end{aligned}$$

e) $\left(\sqrt{6+\sqrt{11}} + \sqrt{6-\sqrt{11}}\right)^2$

Solution:

$$\begin{aligned} \left(\sqrt{6+\sqrt{11}} + \sqrt{6-\sqrt{11}}\right)^2 &= \left(\sqrt{6+\sqrt{11}}\right)^2 + \left(\sqrt{6-\sqrt{11}}\right)^2 + 2\sqrt{6+\sqrt{11}}\sqrt{6-\sqrt{11}} \\ &= 6+\sqrt{11}+6-\sqrt{11}+2\sqrt{(6+\sqrt{11})(6-\sqrt{11})} \\ &= 12+2\sqrt{36-11} = 12+2\sqrt{25} = 12+2\cdot 5 = \boxed{22} \end{aligned}$$

3. Find the exact value of $x^2 - 4x + 6$ if $x = 2 - \sqrt{3}$.

Solution: We work out x^2 first.

$$\begin{aligned} x^2 &= (2 - \sqrt{3})^2 = (2 - \sqrt{3})(2 - \sqrt{3}) \\ &= 4 - 2\sqrt{3} - 2\sqrt{3} + \sqrt{3}\sqrt{3} = 4 - 4\sqrt{3} + 3 = 7 - 4\sqrt{3} \end{aligned}$$

Now we substitute $x = 2 - \sqrt{3}$ into $x^2 - 4x + 6$.

$$\begin{aligned} x^2 - 4x + 6 &= (2 - \sqrt{3})^2 - 4(2 - \sqrt{3}) + 6 = \\ &= 7 - 4\sqrt{3} - 8 + 4\sqrt{3} + 6 = 7 - 8 + 6 = \boxed{5} \end{aligned}$$

4. Solve each of the following quadratic equations by completing the square. Check your solution(s).

a) $x^2 = 4x + 1$

Solution: We factor by completing the square.

$$\begin{aligned} x^2 &= 4x + 1 && \text{reduce one side to zero} \\ x^2 - 4x - 1 &= 0 && (x-2)^2 = x^2 - 4x + 4 \\ \underbrace{x^2 - 4x + 4} - 4 - 1 &= 0 \\ (x-2)^2 - 5 &= 0 \\ (x-2)^2 - (\sqrt{5})^2 &= 0 \\ (x-2+\sqrt{5})(x-2-\sqrt{5}) &= 0 \end{aligned}$$

$$\boxed{x_1 = 2 - \sqrt{5} \text{ and } x_2 = 2 + \sqrt{5}}$$

We check: if $x = 2 - \sqrt{5}$, then

$$\begin{aligned} \text{LHS} &= (2 - \sqrt{5})^2 = (2 - \sqrt{5})(2 - \sqrt{5}) = 4 - 2\sqrt{5} - 2\sqrt{5} + \sqrt{5}\sqrt{5} = 4 - 4\sqrt{5} + 5 = 9 - 4\sqrt{5} \\ \text{RHS} &= 4(2 - \sqrt{5}) + 1 = 8 - 4\sqrt{5} + 1 = 9 - 4\sqrt{5} \end{aligned}$$

and if $x = 2 + \sqrt{5}$, then

$$\begin{aligned} \text{LHS} &= (2 + \sqrt{5})^2 = (2 + \sqrt{5})(2 + \sqrt{5}) = 4 + 2\sqrt{5} + 2\sqrt{5} + \sqrt{5}\sqrt{5} = 4 + 4\sqrt{5} + 5 = 9 + 4\sqrt{5} \\ \text{RHS} &= 4(2 + \sqrt{5}) + 1 = 8 + 4\sqrt{5} + 1 = 9 + 4\sqrt{5} \end{aligned}$$

So, both solutions are correct.

b) $x^2 + 13 = 8x$

Solution: We factor by completing the square.

$$\begin{aligned} x^2 - 8x + 13 &= 0 && (x-4)^2 = x^2 - 8x + 16 \\ \underbrace{x^2 - 8x + 16} - 16 + 13 &= 0 \\ (x-4)^2 - 3 &= 0 \\ (x-4)^2 - (\sqrt{3})^2 &= 0 \\ (x-4+\sqrt{3})(x-4-\sqrt{3}) &= 0 \end{aligned}$$

$$\boxed{x_1 = 4 - \sqrt{3} \text{ and } x_2 = 4 + \sqrt{3}}$$

We check: if $x = 4 - \sqrt{3}$, then

$$\text{LHS} = (4 - \sqrt{3})^2 + 13 = (4 - \sqrt{3})(4 - \sqrt{3}) + 13 = 16 - 4\sqrt{3} - 4\sqrt{3} + 3 + 13 = 32 - 8\sqrt{3}$$

$$\text{RHS} = 8(4 - \sqrt{3}) = 32 - 8\sqrt{3}$$

and if $x = 4 + \sqrt{3}$, then

$$\text{LHS} = (4 + \sqrt{3})^2 + 13 = (4 + \sqrt{3})(4 + \sqrt{3}) + 13 = 16 + 4\sqrt{3} + 4\sqrt{3} + 3 + 13 = 32 + 8\sqrt{3}$$

$$\text{RHS} = 8(4 + \sqrt{3}) = 32 + 8\sqrt{3}$$

So, both solutions are correct.

Solutions for 20.2 – Complex Numbers

Perform the indicated operations and simplify.

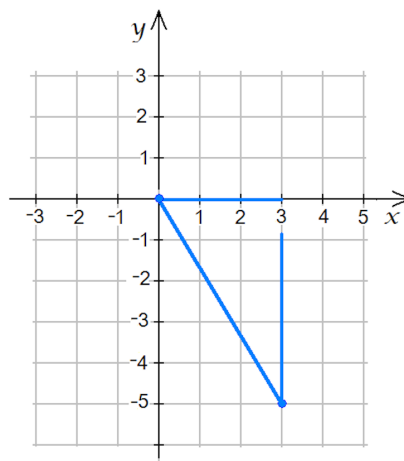
1. $|3 - 5i|$

Solution: The absolute value of complex number is its distance from zero on the number plane. In the case of $3 - 5i$ this means the distance between the points $(3, -5)$ and $(0, 0)$.

We state the Pythagorean theorem on the right triangle and obtain the equation

$$\begin{aligned} 3^2 + 5^2 &= x^2 \\ 34 &= x^2 \\ \pm\sqrt{34} &= x \end{aligned}$$

Since distances can never be negative, we rule out the negative solution and so the answer is $\boxed{\sqrt{34}}$.



2. $(2 - 5i) + (1 - i)$

Solution: We simply drop the parentheses and combine like terms.

$$(2 - 5i) + (1 - i) = 2 - 5i + 1 - i = 2 + 1 + (-5i - i) = \boxed{3 - 6i}$$

3. $3 + 5i - (2 + 3i)$

Solution: To subtract is to add the opposite. Then we drop the parentheses and combine like terms.

$$3 + 5i - (2 + 3i) = 3 + 5i + (-1)(2 + 3i) = 3 + 5i + (-2 - 3i) = 3 - 2 + 5i - 3i = \boxed{1 + 2i}$$

4. $3(2 - i) - 2(3 + i)$

Solution: We apply the distributive law and combine like terms.

$$3(2 - i) - 2(3 + i) = 6 - 3i - 6 - 2i = \boxed{-5i}$$

5. $-i(3 + i)$

Solution: We apply the distributive law and combine like terms.

$$-i(3 + i) = -3i - i^2 = -3i - (-1) = \boxed{1 - 3i}$$

6. $3(2 - i) - 2i(3 + i)$

Solution: We apply the distributive law and combine like terms.

$$3(2 - i) - 2i(3 + i) = 6 - 3i - 6i - 2i^2 = 6 - 9i - 2(-1) = 6 - 9i + 2 = \boxed{8 - 9i}$$

7. $(2 - 3i)(5 + 2i)$

Solution: We apply the distributive law (in this case, FOIL) and combine like terms.

$$(2 - 3i)(5 + 2i) = 10 + 4i - 15i - 6i^2 = 10 + 4i - 15i - 6(-1) = \boxed{16 - 11i}$$

8. $(3 - 2i)^2$

Solution: We apply the distributive law (in this case, FOIL) and combine like terms.

$$(3 - 2i)^2 = (3 - 2i)(3 - 2i) = 9 - 6i - 6i + 4i^2 = 9 - 6i - 6i + 4(-1) = \boxed{5 - 12i}$$

9. $(3 - 2i)(3 + 2i)$

Solution: We apply the distributive law (in this case, FOIL) and combine like terms.

$$(3 - 2i)(3 + 2i) = (3 - 2i)(3 + 2i) = 9 - 6i + 6i - 4i^2 = 9 - 4(-1) = \boxed{13}$$

Since we multiplied conjugates, we have the difference of squares theorem.

10. $(7 - 3i)(i + 1)$

Solution: We apply the distributive law (in this case, FOIL) and combine like terms.

$$(7 - 3i)(i + 1) = 7i + 7 - 3i^2 - 3i = 7i + 7 - 3(-1) - 3i = \boxed{10 + 4i}$$

11. $(3 + 5i)(-3 + 5i)$

Solution: We apply the distributive law (in this case, FOIL) and combine like terms. After FOIL, there will be cancellations because we are multiplying a number and its conjugate, so the product is the difference of the two squares.

$$(3 + 5i)(-3 + 5i) = -9 + 15i - 15i + 25i^2 = -9 + 25(-1) = -9 - 25 = \boxed{-34}$$

12. $(1 - 3i)^2(1 + 3i)^2$

Solution 1: We apply the distributive law for each square and multiply the two numbers we obtained.

$$\begin{aligned} (1 - 3i)^2(1 + 3i)^2 &= (1 - 3i - 3i + 9i^2)(1 + 3i + 3i + 9i^2) = (1 - 6i + 9(-1))(1 + 6i + 9(-1)) \\ &= (-8 - 6i)(-8 + 6i) = -2(4 + 3i)2(-4 + 3i) = -4(4 + 3i)(-4 + 3i) \\ &= -4(-16 + 12i - 12i + 9i^2) = -4(-16 + 9(-1)) = -4(-25) = \boxed{100} \end{aligned}$$

Solution 2: We "outsmart" the problem by computing the product in a different order. We use the exponent rule $a^2b^2 = (ab)^2$.

$$(1 - 3i)^2(1 + 3i)^2 = ((1 - 3i)(1 + 3i))^2 = (1 + 3i - 3i - 9i^2)^2 = (1 - 9(-1))^2 = 10^2 = \boxed{100}$$

13. $(1 - i)^4$

Solution: We will compute $((1 - i)^2)^2$.

$$(1 - i)^4 = ((1 - i)^2)^2 = ((1 - i)(1 - i))^2 = (1 - i - i + i^2)^2 = (1 - 2i + (-1))^2 = (-2i)^2 = 4i^2 = 4(-1) = \boxed{-4}$$

14. $\frac{2i}{1 - i}$

Solution: We will use the same technique as with rationalizing quotients with irrational denominator: we multiply the fraction by 1, where both numerator and denominator are the conjugate of the denominator.

$$\frac{2i}{1 - i} = \frac{2i}{1 - i} \cdot 1 = \frac{2i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{2i(1 + i)}{1^2 - i^2} = \frac{2i + 2i^2}{1 - (-1)} = \frac{2i + 2(-1)}{2} = \frac{-2 + 2i}{2} = \frac{2(-1 + i)}{2} = \boxed{-1 + i}$$

15. $\frac{1 + 7i}{3i + 1}$

Solution: We first multiply the fraction by 1, where both numerator and denominator are the conjugate of the denominator.

$$\begin{aligned} \frac{1 + 7i}{3i + 1} &= \frac{1 + 7i}{1 + 3i} \cdot 1 = \frac{1 + 7i}{1 + 3i} \cdot \frac{1 - 3i}{1 - 3i} = \frac{(1 + 7i)(1 - 3i)}{1^2 - (3i)^2} = \frac{1 - 3i + 7i - 21i^2}{1 - 9i^2} = \frac{1 + 4i - 21(-1)}{1 - 9(-1)} \\ &= \frac{1 + 4i + 21}{10} = \frac{22 + 4i}{10} = \frac{22}{10} + \frac{4}{10}i = \boxed{\frac{11}{5} + \frac{2}{5}i} \end{aligned}$$

16. $\frac{10 + 5i}{3 + 4i}$

Solution: We first multiply the fraction by 1, where both numerator and denominator are the conjugate of the denominator.

$$\begin{aligned} \frac{10 + 5i}{3 + 4i} &= \frac{10 + 5i}{3 + 4i} \cdot 1 = \frac{10 + 5i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{(10 + 5i)(3 - 4i)}{(3 + 4i)(3 - 4i)} = \frac{30 - 40i + 15i - 20i^2}{9 - 12i + 12i - 16i^2} \\ &= \frac{30 - 25i - 20(-1)}{9 - 16(-1)} = \frac{30 - 25i + 20}{9 + 16} = \frac{50 - 25i}{25} = \frac{25(2 - i)}{25} = \boxed{2 - i} \end{aligned}$$

17. $\frac{5}{2-i}$

Solution: We first multiply the fraction by 1, where both numerator and denominator are the conjugate of the denominator.

$$\frac{5}{2-i} = \frac{5}{2-i} \cdot 1 = \frac{5}{2-i} \cdot \frac{2+i}{2+i} = \frac{5(2+i)}{(2+i)(2-i)} = \frac{10+5i}{2^2-i^2} = \frac{10+5i}{4-(-1)} = \frac{10+5i}{4+1} = \frac{10+5i}{5} = \frac{5(2+i)}{5} = \boxed{2+i}$$

18. $\frac{8i-1}{2i+3}$

Solution: We will use the same technique as with rationalizing quotients with irrational denominator: we multiply the fraction by 1, where both numerator and denominator are the conjugate of the denominator.

$$\begin{aligned} \frac{8i-1}{2i+3} &= \frac{-1+8i}{3+2i} \cdot 1 = \frac{-1+8i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{(-1+8i)(3-2i)}{3^2-(2i)^2} = \frac{-3+2i+24i-16i^2}{9-4i^2} = \frac{-3+26i-16(-1)}{9-4(-1)} \\ &= \frac{13+26i}{13} = \frac{13(1+2i)}{13} = \boxed{1+2i} \end{aligned}$$

19. $\frac{5}{1+2i} - \frac{10i}{1-2i}$

Solution: We perform each division and then subtract. As you see, it is sometimes wise to wait with the distribution on the top because we save work as terms might cancel out before the multiplication.

$$\frac{5}{1+2i} = \frac{5}{1+2i} \cdot 1 = \frac{5}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{5(1-2i)}{1^2-(2i)^2} = \frac{5(1-2i)}{1-4i^2} = \frac{5(1-2i)}{1-4(-1)} = \frac{5(1-2i)}{5} = 1-2i$$

$$\begin{aligned} \frac{10i}{1-2i} &= \frac{10i}{1-2i} \cdot 1 = \frac{10i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{10i(1+2i)}{1^2-(2i)^2} = \frac{10i(1+2i)}{1-4i^2} = \frac{10i(1+2i)}{1-4(-1)} = \frac{10i(1+2i)}{5} = 2i(1+2i) \\ &= 2i+4i^2 = -4+2i \end{aligned}$$

We are ready to subtract:

$$\frac{5}{1+2i} - \frac{10i}{1-2i} = 1-2i - (-4+2i) = 1-2i+4-2i = \boxed{5-4i}$$

20. $\frac{(3+2i)(5-3i) - (7-2i)(3-i)}{3-4i}$

Solution: We apply the order of operations agreement and perform the multiplications first, then the subtraction, and finally the division.

$$(3+2i)(5-3i) = 15-9i+10i-6i^2 = 15+i-6(-1) = 21+i$$

$$(7-2i)(3-i) = 21-7i-6i+2i^2 = 21-13i+2(-1) = 19-13i$$

$$(3+2i)(5-3i) - (7-2i)(3-i) = (21+i) - (19-13i) = 21+i-19+13i = 2+14i$$

$$\frac{2+14i}{3-4i} = \frac{2+14i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{6+8i+42i+56i^2}{3^2-(4i)^2} = \frac{6+50i+56(-1)}{9-(-16)} = \frac{-50+50i}{25} = \frac{25(-2+2i)}{25} = \boxed{-2+2i}$$

21. $\frac{(3-i)^2 - (1+3i)^2}{2i+2}$

Solution:

$$\begin{aligned} \frac{(3-i)^2 - (1+3i)^2}{2i+2} &= \frac{9-6i+i^2 - (1+6i+9i^2)}{2i+2} = \frac{9-6i+(-1) - (1+6i-9)}{2i+2} \\ &= \frac{8-6i-(-8+6i)}{2i+2} = \frac{8-6i+8-6i}{2i+2} = \frac{16-12i}{2i+2} = \frac{2(8-6i)}{2(1+i)} = \frac{8-6i}{1+i} \cdot 1 \\ &= \frac{8-6i}{1+i} \cdot \frac{1-i}{1-i} = \frac{8-8i-6i+6i^2}{1^2-i^2} = \frac{2-14i}{2} = \frac{2(1-7i)}{2} = \boxed{1-7i} \end{aligned}$$

22. Simplify $\frac{\sqrt{2}i - 1}{\sqrt{2} + i} - \frac{\sqrt{2} - i}{1 + \sqrt{2}i}$

Solution: We will use the complex conjugates to turn the denominators to real numbers before bringing the fractions to the common denominator.

$$\begin{aligned}\frac{\sqrt{2}i - 1}{\sqrt{2} + i} - \frac{\sqrt{2} - i}{1 + \sqrt{2}i} &= \frac{(\sqrt{2}i - 1)(\sqrt{2} - i)}{(\sqrt{2} + i)(\sqrt{2} - i)} - \frac{(\sqrt{2} - i)(1 - \sqrt{2}i)}{(1 + \sqrt{2}i)(1 - \sqrt{2}i)} = \frac{2i - \sqrt{2}i^2 - \sqrt{2} + i}{(\sqrt{2})^2 - i^2} - \frac{\sqrt{2} - 2i - i + \sqrt{2}i^2}{1^2 - (\sqrt{2}i)^2} \\ &= \frac{2i + \sqrt{2} - \sqrt{2} + i}{2 - (-1)} - \frac{\sqrt{2} - 2i - i - \sqrt{2}}{1 - (-2)} = \frac{3i}{3} - \frac{-3i}{3} = i - (-i) = \boxed{2i}\end{aligned}$$

Completely factor each of the following over the complex numbers.

23. $x^2 + 9$

Solution: $x^2 + 9 = x^2 - (-9) = x^2 - (3i)^2 = \boxed{(x + 3i)(x - 3i)}$

We can check via multiplication: $(x + 3i)(x - 3i) = x^2 - 3ix + 3ix - 9i^2 = x^2 - (-9) = x^2 + 9$.

So our solution is correct.

24. $x^2 - 10x + 29$

Solution:
$$\begin{aligned}x^2 - 10x + 29 &= \underbrace{x^2 - 10x + 25}_{(x-5)^2} - 25 + 29 = (x-5)^2 + 4 = (x-5)^2 - (-4) = (x-5)^2 - (2i)^2 \\ &= \boxed{(x-5+2i)(x-5-2i)}\end{aligned}$$

We can check via multiplication, but first we will make a slight modification that will make the computation easier. We will re-write $x - 5 + 2i$ as $x - (5 - 2i)$ and $x - 5 - 2i$ as $x - (5 + 2i)$.

$$\begin{aligned}(x - 5 + 2i)(x - 5 - 2i) &= (x - (5 - 2i))(x - (5 + 2i)) \\ &= x^2 - (5 - 2i)x - (5 + 2i)x + (5 - 2i)(5 + 2i) \\ &= x^2 - x(5 - 2i + 5 + 2i) + 25 - (2i)^2 = x^2 - 10x + 29\end{aligned}$$

and so our solution is correct.

25. $x^4 - 1$

Solution:

$$x^4 - 1 = (x^2)^2 - 1^2 = (x^2 + 1)(x^2 - 1) = (x^2 - (-1))(x^2 - 1) = \boxed{(x + i)(x - i)(x + 1)(x - 1)}$$

Solve each of the following equations over the complex numbers.

26. $x^2 = 4x - 29$

Solution 1 (Completing the square).

$$\begin{aligned}x^2 &= 4x - 29 \\ x^2 - 4x + 29 &= 0 && \text{complete the square} \\ \underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4 + 29 &= 0 \\ (x - 2)^2 + 25 &= 0 \\ (x - 2)^2 - (-25) &= 0 \\ (x - 2)^2 - (5i)^2 &= 0 && \text{factor via the difference of squares theorem} \\ (x - 2 + 5i)(x - 2 - 5i) &= 0\end{aligned}$$

$$\begin{aligned}x - 2 + 5i &= 0 && \text{or} && x - 2 - 5i = 0 \\ x &= 2 - 5i && \text{or} && x = 2 + 5i\end{aligned}$$

Solution 2 (The quadratic formula)

$$\begin{aligned}x^2 &= 4x - 29 \\x^2 - 4x + 29 &= 0 & a = 1, b = -4, \text{ and } c = 29 \\x_{1,2} &= \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 29}}{2 \cdot 1} = \frac{4 \pm \sqrt{16 - 116}}{2} = \frac{4 \pm \sqrt{-100}}{2} = \frac{4 \pm 10i}{2} = 2 \pm 5i\end{aligned}$$

We check $x = 2 - 5i$.

$$\begin{aligned}\text{LHS} &= x^2 = (2 - 5i)^2 = 4 - 20i + 25i^2 = 4 - 20i - 25 = -21 - 20i \\ \text{RHS} &= 4x - 29 = 4(2 - 5i) - 29 = 8 - 20i - 29 = -21 - 20i\end{aligned}$$

So $x = 2 - 5i$ is a solution. We also check $x = 2 + 5i$.

$$\begin{aligned}\text{LHS} &= x^2 = (2 + 5i)^2 = 4 + 20i + 25i^2 = 4 + 20i - 25 = -21 + 20i \\ \text{RHS} &= 4x - 29 = 4(2 + 5i) - 29 = 8 + 20i - 29 = -21 + 20i\end{aligned}$$

So $\boxed{2 - 5i \text{ and } 2 + 5i}$ are both solutions.

27. $x^2 + 3 = 2x$

Solution 1 (Completing the square).

$$\begin{aligned}x^2 + 3 &= 2x \\x^2 - 2x + 3 &= 0 & \text{complete the square} \\ \underbrace{x^2 - 2x + 1}_{(x-1)^2} - 1 + 3 &= 0 \\(x-1)^2 + 2 &= 0 \\(x-1)^2 - (-2) &= 0 \\(x-1)^2 - (\sqrt{2}i)^2 &= 0 & \text{factor via the difference of squares theorem} \\(x-1+\sqrt{2}i)(x-1-\sqrt{2}i) &= 0 \\x-1+\sqrt{2}i &= 0 & \text{or } x-1-\sqrt{2}i=0 \\x &= 1-\sqrt{2}i & \text{or } x=1+\sqrt{2}i\end{aligned}$$

Solution 2 (The quadratic formula)

$$\begin{aligned}x^2 + 3 &= 2x \\x^2 - 2x + 3 &= 0 & a = 1, b = -2, \text{ and } c = 3 \\x_{1,2} &= \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{2 \pm \sqrt{4 - 12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm \sqrt{8}i}{2} = \frac{2 \pm 2\sqrt{2}i}{2} = 1 \pm \sqrt{2}i\end{aligned}$$

We check $x = 1 + \sqrt{2}i$.

$$\begin{aligned}\text{LHS} &= x^2 + 3 = (1 + \sqrt{2}i)^2 + 3 = 1 + 2i^2 + 2\sqrt{2}i + 3 = 2 + 2\sqrt{2}i \\ \text{RHS} &= 2x = 2(1 + \sqrt{2}i) = 2 + 2\sqrt{2}i\end{aligned}$$

So $x = 1 + \sqrt{2}i$ is a solution. We also check $x = 1 - \sqrt{2}i$.

$$\begin{aligned}\text{LHS} &= x^2 + 3 = (1 - \sqrt{2}i)^2 + 3 = 1 + 2i^2 - 2\sqrt{2}i + 3 = 2 - 2\sqrt{2}i \\ \text{RHS} &= 2x = 2(1 - \sqrt{2}i) = 2 - 2\sqrt{2}i\end{aligned}$$

So $\boxed{1 - \sqrt{2}i \text{ and } 1 + \sqrt{2}i}$ are both solutions.

28. Find z such that $z^2 = -21 + 20i$.

Solution: Let $z = x + yi$. (Remember, x and y are real numbers.) Then

$$z^2 = (x + yi)^2 = x^2 + 2x(yi) + (yi)^2 = x^2 + 2xyi + y^2 i^2 = \underbrace{x^2 - y^2}_{\text{real part}} + \underbrace{2xyi}_{\text{imaginary part}}$$

$$\begin{aligned} z^2 &= -21 + 10i \\ (x^2 - y^2) + (2xy)i &= -21 + 20i \end{aligned}$$

The equation of complex numbers gives us a system of equations on real numbers.

$$\begin{aligned} x^2 - y^2 &= -21 \\ 2xy &= 20 \implies xy = 10 \end{aligned}$$

We square the second equation

$$\begin{aligned} x^2 + 21 &= y^2 \\ x^2 y^2 &= 100 \end{aligned}$$

We solve the system by substitution:

$$\begin{aligned} x^2(x^2 + 21) &= 100 \\ x^4 + 21x^2 - 100 &= 0 \\ (x^2 + 25)(x^2 - 4) &= 0 \\ x_1^2 = -25 \quad x_2^2 = 4 \end{aligned}$$

Since the square of real numbers can not be negative, only $x^2 = 4$ can apply to this problem. If $x = 2$, we get $y = 5$ and so $z = 2 + 5i$. If $x = -2$, then $y = -5$ and so $z = -2 - 5i$. So the solutions are $\boxed{2 + 5i \text{ and } -2 - 5i}$. This does make sense: a number and its opposite will square to the same thing even among complex numbers: $(-z)^2 = (-z)(-z) = z^2$.

Solutions for 21.1 – Complex Fractions

Simplify each of the following expressions.

$$1. \frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}}$$

Solution: We will first perform the addition indicated in the numerator and denominator. After the addition and subtraction, we will divide by multiplying by the reciprocal.

$$\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}} = \frac{\frac{1 \cdot 3}{2 \cdot 3} + \frac{1 \cdot 2}{3 \cdot 2}}{\frac{1 \cdot 3}{2 \cdot 3} - \frac{1 \cdot 2}{3 \cdot 2}} = \frac{\frac{3}{6} + \frac{2}{6}}{\frac{3}{6} - \frac{2}{6}} = \frac{\frac{2+3}{6}}{\frac{3-2}{6}} = \frac{\frac{5}{6}}{\frac{1}{6}} = \frac{5}{\cancel{6}} \cdot \frac{6}{1} = \frac{5}{1} = \boxed{5}$$

$$2. \frac{5 - \frac{1}{a}}{\frac{1}{a^2} - 25}$$

Solution: Method 1. Start with the subtractions. We need to work with common denominators. The numerator:

$$5 - \frac{1}{a} = \frac{5}{1} - \frac{1}{a} = \frac{5a}{a} - \frac{1}{a} = \frac{5a-1}{a}$$

The denominator:

$$\frac{1}{a^2} - 25 = \frac{1}{a^2} - \frac{25}{1} = \frac{1}{a^2} - \frac{25a^2}{a^2} = \frac{1-25a^2}{a^2}$$

Notice that the numerator here factors via the difference of squares theorem.

$$\frac{1-25a^2}{a^2} = \frac{1^2 - (5a)^2}{a^2} = \frac{(1-5a)(1+5a)}{a^2}$$

Now we are ready to perform the division. To divide is to multiply by the reciprocal.

$$\begin{aligned} \frac{5 - \frac{1}{a}}{\frac{1}{a^2} - 25} &= \frac{\frac{5a-1}{a}}{\frac{(1-5a)(1+5a)}{a^2}} = \frac{5a-1}{a} \cdot \frac{a^2}{(1-5a)(1+5a)} \quad \text{cancel out } a \\ &= \frac{5a-1}{\cancel{a}} \cdot \frac{a^{\cancel{2}}}{(1-5a)(1+5a)} = \frac{a(5a-1)}{(1-5a)(1+5a)} \end{aligned}$$

There is one more cancellation: $1-5a$ and $5a-1$ are opposites. We re-write $5a-1$ as $-(1-5a)$. Then

$$\frac{a(5a-1)}{(1-5a)(1+5a)} = \frac{-(1-5a)a}{(1-5a)(1+5a)} = \frac{\cancel{-(1-5a)}a}{\cancel{(1-5a)}(1+5a)} = \boxed{\frac{-a}{1+5a} \quad \text{or} \quad -\frac{a}{5a+1}}$$

Method 2. This method is very efficient. The main idea is to clear denominators by multiplying both numerator and denominator by the same quantity. We usually do not apply this method with usual fractions but it does work there too. The first step is to multiply both numerator and denominator by a^2 , the quantity that clears the denominator in the fractions in both numerator and denominator.

$$\frac{5 - \frac{1}{a}}{\frac{1}{a^2} - 25} = \frac{5 - \frac{1}{a}}{\frac{1}{a^2} - 25} \cdot \frac{a^2}{a^2} = \frac{a^2 \left(5 - \frac{1}{a}\right)}{a^2 \left(\frac{1}{a^2} - 25\right)} = \frac{5a^2 - a^2 \cdot \frac{1}{a}}{a^2 \cdot \frac{1}{a^2} - 25a^2} = \frac{5a^2 - a}{1 - 25a^2}$$

We now factor the expressions in the numerator and denominator so that we could see if further simplification is possible. We factor out the greatest common factor from the numerator and factor the denominator via the difference of squares theorem.

$$\frac{5a^2 - a}{1 - 25a^2} = \frac{a(5a - 1)}{(1 - 5a)(1 + 5a)}$$

Notice that $5a - 1$ and $1 - 5a$ are opposites. We factor out -1 from the numerator, so we re-write $5a - 1$ as $-(1 - 5a)$. Then we cancel out the common factor.

$$\frac{a(5a - 1)}{(1 - 5a)(1 + 5a)} = \frac{a(-1)(1 - 5a)}{(1 - 5a)(1 + 5a)} = \frac{a(-1)\cancel{(1 - 5a)}}{\cancel{(1 - 5a)}(1 + 5a)} = \frac{a(-1)}{1 + 5a} = \boxed{\frac{-a}{5a + 1}}$$

3. $\frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x}}$

Solution: Method 1. We bring fractions to the common denominator and add and subtract. Then we perform the division as multiplication by the reciprocal. Finally, we factor and cancel out common factors.

$$\frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x}} = \frac{\frac{1 \cdot x^2}{1 \cdot x^2} - \frac{1}{x^2}}{\frac{1 \cdot x}{1 \cdot x} + \frac{1}{x}} = \frac{\frac{x^2 - 1}{x^2}}{\frac{x + 1}{x}} = \frac{x^2 - 1}{x^2} \cdot \frac{x}{x + 1} = \frac{(x + 1)(x - 1)}{x^2} \cdot \frac{x}{x + 1} = \frac{\cancel{(x + 1)}(x - 1)}{x^{\cancel{2}} \cdot \cancel{x + 1}} \cdot \frac{x}{\cancel{x + 1}} = \boxed{\frac{x - 1}{x}}$$

Method 2. First we multiply both numerator and denominator by x^2 .

$$\frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x}} = \frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x}} \cdot \frac{x^2}{x^2} = \frac{x^2 \left(1 - \frac{1}{x^2}\right)}{x^2 \left(1 + \frac{1}{x}\right)} = \frac{x^2 \cdot 1 - x^2 \cdot \frac{1}{x^2}}{x^2 \cdot 1 + x^2 \cdot \frac{1}{x}} = \frac{x^2 - 1}{x^2 + x}$$

We now factor and cancel out common factors.

$$\frac{x^2 - 1}{x^2 + x} = \frac{(x + 1)(x - 1)}{x(x + 1)} = \frac{\cancel{(x + 1)}(x - 1)}{x\cancel{(x + 1)}} = \boxed{\frac{x - 1}{x}}$$

$$4. \frac{\frac{4}{a^2} - 1}{1 - \frac{2}{a}}$$

Solution: Method 1

$$\begin{aligned} \frac{\frac{4}{a^2} - 1}{1 - \frac{2}{a}} &= \frac{\frac{4}{a^2} - \frac{1}{1}}{\frac{1}{1} - \frac{2}{a}} = \frac{\frac{4}{a^2} - \frac{1 \cdot a^2}{1 \cdot a^2}}{\frac{1 \cdot a}{1 \cdot a} - \frac{2}{a}} = \frac{\frac{4 - a^2}{a^2}}{\frac{a - 2}{a}} = \frac{4 - a^2}{a^2} \cdot \frac{a}{a - 2} = \frac{(2 + a)(2 - a)}{a} \cdot \frac{1}{a - 2} \\ &= \frac{(2 + a)(-1)(a - 2)}{a(a - 2)} = \frac{(2 + a)(-1)\cancel{(a - 2)}}{a\cancel{(a - 2)}} = \boxed{-\frac{a + 2}{a}} \end{aligned}$$

Method 2. We first multiply both numerator and denominator by a^2 . Then we factor all expressions and simplify the fraction.

$$\frac{\frac{4}{a^2} - 1}{1 - \frac{2}{a}} = \frac{a^2 \left(\frac{4}{a^2} - 1 \right)}{a^2 \left(1 - \frac{2}{a} \right)} = \frac{4 - a^2}{a^2 - 2a} = \frac{(2 + a)(2 - a)}{a(a - 2)}$$

Notice that $a - 2$ and $2 - a$ are opposites. We will factor out -1 from $2 - a$ in the numerator.

$$\frac{(2 + a)(2 - a)}{a(a - 2)} = \frac{(2 + a)(-1)(a - 2)}{a(a - 2)} = \frac{-(2 + a)\cancel{(a - 2)}}{a\cancel{(a - 2)}} = \frac{-(2 + a)}{a} = \boxed{-\frac{a + 2}{a}}$$

$$5. \frac{\frac{3}{x-1} - 1}{\frac{2}{x-1} + 1}$$

Solution: Method 1.

$$\begin{aligned} \frac{\frac{3}{x-1} - 1}{\frac{2}{x-1} + 1} &= \frac{\frac{3}{x-1} - \frac{1}{1}}{\frac{2}{x-1} + \frac{1}{1}} = \frac{\frac{3}{x-1} - \frac{1 \cdot (x-1)}{1 \cdot (x-1)}}{\frac{2}{x-1} + \frac{1 \cdot (x-1)}{1 \cdot (x-1)}} = \frac{\frac{3 - (x-1)}{x-1}}{\frac{2 + (x-1)}{x-1}} = \frac{3 - x + 1}{2 + x - 1} = \frac{3 - x + 1}{x + 1} \\ &= \frac{\frac{-x + 4}{x-1}}{\frac{x+1}{x-1}} = \frac{-x + 4}{x-1} \cdot \frac{x-1}{x+1} = \frac{-x + 4}{\cancel{x-1}} \cdot \frac{\cancel{x-1}}{x+1} = \boxed{\frac{-x + 4}{x + 1}} \end{aligned}$$

Method 2. We multiply both numerator and denominator by $x - 1$. We need to be careful with subtractions as we are subtracting entire expressions.

$$\frac{\frac{3}{x-1} - 1}{\frac{2}{x-1} + 1} = \frac{(x-1) \left(\frac{3}{x-1} - 1 \right)}{(x-1) \left(\frac{2}{x-1} + 1 \right)} = \frac{3 - 1(x-1)}{2 + 1(x-1)} = \frac{3 - x + 1}{2 + x - 1} = \frac{4 - x}{x + 1} = \boxed{\frac{-x + 4}{x + 1}}$$

$$6. \frac{2 - \frac{3}{x+1}}{3 - \frac{2x}{x+1}}$$

Solution: Method 1. We first work out the numerator

$$\begin{aligned} 2 - \frac{3}{x+1} &= \frac{2}{1} - \frac{3}{x+1} \quad \text{common denominator is } x+1 \\ &= \frac{2(x+1)}{x+1} - \frac{3}{x+1} = \frac{2(x+1)-3}{x+1} = \frac{2x+2-3}{x+1} = \frac{2x-1}{x+1} \end{aligned}$$

Now for the denominator

$$\begin{aligned} 3 - \frac{2x}{x+1} &= \frac{3}{1} - \frac{2x}{x+1} \quad \text{common denominator is } x+1 \\ &= \frac{3(x+1)}{x+1} - \frac{2x}{x+1} = \frac{3(x+1)-2x}{x+1} = \frac{3x+3-2x}{x+1} = \frac{x+3}{x+1} \end{aligned}$$

Now the division:

$$\begin{aligned} \frac{2 - \frac{3}{x+1}}{3 - \frac{2x}{x+1}} &= \frac{\frac{2x-1}{x+1}}{\frac{x+3}{x+1}} = \quad \text{to divide is to multiply by the reciprocal} \\ &= \frac{2x-1}{x+1} \cdot \frac{x+1}{x+3} = \frac{2x-1}{x+3} \cdot \frac{\cancel{x+1}}{\cancel{x+1}} = \boxed{\frac{2x-1}{x+3}} \end{aligned}$$

Method 2.

$$\frac{2 - \frac{3}{x+1}}{3 - \frac{2x}{x+1}} \cdot \frac{x+1}{x+1} = \frac{\left(2 - \frac{3}{x+1}\right)(x+1)}{\left(3 - \frac{2x}{x+1}\right)(x+1)} = \frac{2(x+1) - \frac{3}{x+1}(x+1)}{3(x+1) - \frac{2x}{x+1}(x+1)} = \frac{2(x+1)-3}{3(x+1)-2x} = \frac{2x+2-3}{3x+3-2x} = \boxed{\frac{2x-1}{x+3}}$$

$$7. 1 - \frac{1}{1 - \frac{1}{x-3}}$$

Solution:

$$\begin{aligned} 1 - \frac{1}{1 - \frac{1}{x-3}} &= 1 - \frac{1}{\frac{1}{1} - \frac{1}{x-3}} = 1 - \frac{1}{\frac{1 \cdot (x-3)}{1 \cdot (x-3)} - \frac{1}{x-3}} = 1 - \frac{1}{\frac{x-3}{x-3} - \frac{1}{x-3}} \\ &= 1 - \frac{1}{\frac{x-3-1}{x-3}} = 1 - \frac{1}{\frac{x-4}{x-3}} = 1 - \frac{1}{1} \cdot \frac{x-3}{x-4} = 1 - \frac{x-3}{x-4} \\ &= \frac{x-4}{x-4} - \frac{x-3}{x-4} = \frac{(x-4)-(x-3)}{x-4} = \frac{x-4-x+3}{x-4} = \boxed{\frac{-1}{x-4}} \end{aligned}$$

$$8. \frac{a^{-1} - b^{-1}}{a^{-2} - b^{-2}}$$

Solution: This does not even look like a complex fraction, but it will, once we re-write it positive exponents. Recall that $a^{-n} = \frac{1}{a^n}$. So our problem can be re-written as

$$\frac{a^{-1} - b^{-1}}{a^{-2} - b^{-2}} = \frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}}$$

Method 1.

$$\frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}} = \frac{\frac{b-a}{ab}}{\frac{b^2-a^2}{a^2b^2}} = \frac{b-a}{ab} \cdot \frac{a^2b^2}{b^2-a^2} = \frac{(b-a)ab}{b^2-a^2} = \frac{(b-a)ab}{(b-a)(b+a)} = \frac{\cancel{(b-a)}ab}{\cancel{(b-a)}(b+a)} = \frac{ab}{b+a} = \boxed{\frac{ab}{a+b}}$$

Method 2.

$$\frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}} = \frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}} \cdot \frac{a^2b^2}{a^2b^2} = \frac{a^2b^2\left(\frac{1}{a} - \frac{1}{b}\right)}{a^2b^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)} = \frac{ab^2 - a^2b}{b^2 - a^2} = \frac{ab\cancel{(b-a)}}{\cancel{(b-a)}(b+a)} = \frac{ab(b-a)}{(b-a)(b+a)} = \boxed{\frac{ab}{a+b}}$$

$$9. \frac{x^2 - 2y^{-3}}{x^{-1} + 3y^2}$$

Solution: Method 1.

$$\frac{x^2 - 2y^{-3}}{x^{-1} + 3y^2} = \frac{x^2 - 2\frac{1}{y^3}}{\frac{1}{x} + 3y^2} = \frac{\frac{x^2}{1} - \frac{2}{y^3}}{\frac{1}{x} + \frac{3y^2}{1}} = \frac{\frac{x^2y^3}{y^3} - \frac{2}{y^3}}{\frac{1}{x} + \frac{3xy^2}{x}} = \frac{\frac{x^2y^3 - 2}{y^3}}{\frac{1 + 3xy^2}{x}} = \frac{x^2y^3 - 2}{y^3} \cdot \frac{x}{1 + 3xy^2} = \boxed{\frac{x(x^2y^3 - 2)}{y^3(1 + 3xy^2)}}$$

Method 2.

$$\frac{x^2 - 2y^{-3}}{x^{-1} + 3y^2} = \frac{x^2 - 2\frac{1}{y^3}}{\frac{1}{x} + 3y^2} = \frac{\frac{x^2}{1} - \frac{2}{y^3}}{\frac{1}{x} + \frac{3y^2}{1}} \cdot \frac{xy^3}{xy^3} = \frac{xy^3\left(\frac{x^2}{1} - \frac{2}{y^3}\right)}{xy^3\left(\frac{1}{x} + \frac{3y^2}{1}\right)} = \frac{x^3y^3 - 2x}{y^3 + 3xy^5} = \boxed{\frac{x(x^2y^3 - 2)}{y^3(1 + 3xy^2)}}$$

$$10. (x - y^{-2})^{-3}$$

Solution: Method 1.

$$\begin{aligned} (x - y^{-2})^{-3} &= \frac{1}{(x - y^{-2})^3} = \frac{1}{\left(x - \frac{1}{y^2}\right)^3} = \frac{1}{\left(\frac{xy^2}{y^2} - \frac{1}{y^2}\right)^3} = \frac{1}{\left(\frac{xy^2 - 1}{y^2}\right)^3} = \frac{1}{\frac{(xy^2 - 1)^3}{(y^2)^3}} \\ &= 1 \cdot \frac{(y^2)^3}{(xy^2 - 1)^3} = \boxed{\frac{y^6}{(xy^2 - 1)^3}} \end{aligned}$$

Solutions for 21.3 – Functions

1. Which of the following are functions?

a) Let the domain be the set of students enrolled in this course. To each student, let us assign the number of minutes they spent studying yesterday for this class.

Solution: Since there is a unique number assigned to each student, this is a function.

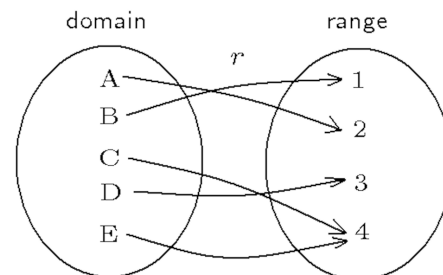
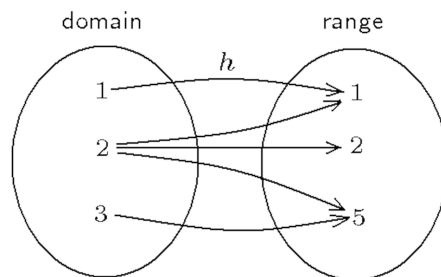
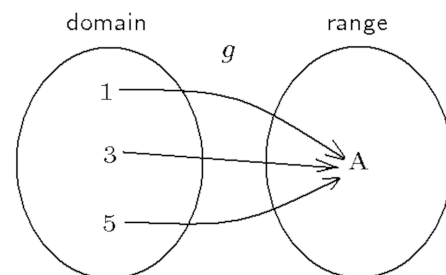
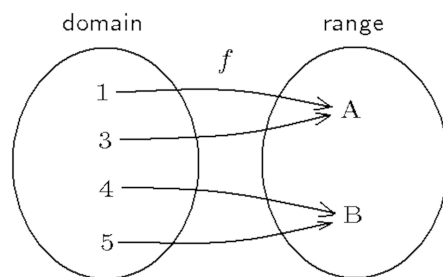
b) Let the domain be the set of students enrolled in this course. To each student, let us assign their uncle.

Solution: Most likely this is not a function. If there is just one student in the class who has more than one uncle, then this assignment fails to be a function.

c) Let the domain be the set $D = \{0, 1, 4, 9\}$. To each number d in the domain, let us assign a number whose square is d . For example, $f(9) = 3$ because $3^2 = 9$.

Solution: This is not a function because there are more than one values assigned to some elements in the domain. For example, $(9) = 3$ and $f(9) = -3$ because both 3 and -3 results in 9 when squared.

2. Which of the following are functions?



Solution: f is a function because to each element of the domain, only one thing is assigned. Indeed, $f(1) = A$, $f(3) = A$, $f(4) = B$, and $f(5) = B$. The fact that $f(1) = f(3)$ and $f(4) = f(5)$ is OK for a function. This is the 'allowed kind' of 'branching'.

g is also a function because to each element of the domain, only one thing is assigned. g is also called a constant function because all assigned values are the same.

h is not a function because there are several values assigned to 2.

r is also a function because to each element of the domain, only one thing is assigned.

3. The following relations are given in form of ordered pairs. Which of the following are functions?

a) $f = \{(1, 4), (2, 7), (4, 1), (5, 7), (7, 7)\}$

This is a function because to each element of the domain, $D = \{1, 2, 4, 5, 7\}$, only one thing is assigned.

b) $g = \{(1, 1), (-2, 4), (2, 4), (3, 9), (-3, 9)\}$

This is a function because to each element of the domain, $D = \{-3, -2, 1, 2, 3\}$, only one thing is assigned.

c) $h = \{(1, 1), (3, 5), (5, 3), (5, 5)\}$

This is not a function because two numbers are assigned to 5.

4. Define f by its domain being $\mathbb{R}$, the set of all real numbers, and its assignment being $f(x) = -x^2 + 7$. Compute or simplify each of the following expressions.

a) $f(3)$

Solution: $f(-3) = -(-3)^2 + 7 = -9 + 7 = \boxed{-2}$

b) $f(\sqrt{6})$

Solution: $f(\sqrt{6}) = -(\sqrt{6})^2 + 7 = -6 + 7 = \boxed{1}$

c) $f(5)$

Solution: $f(5) = -(5)^2 + 7 = -25 + 7 = \boxed{-18}$

d) $f(2x)$

Solution: $f(2x) = -(2x)^2 + 7 = \boxed{-4x^2 + 7}$

e) $f(a-1)$

Solution: $f(a-1) = -(a-1)^2 + 7 = -(a^2 - 2a + 1) + 7 = -a^2 + 2a - 1 + 7 = \boxed{-a^2 + 2a + 6}$

5. Define f as follows. The domain is $\mathbb{R}$, and the assignment is $f(t) = t^2 - t + 1$. Compute or simplify each of the following expressions.

a) $f(0)$

Solution: $f(0) = 0^2 - 0 + 1 = \boxed{1}$

b) $f(-1)$

Solution: $f(-1) = (-1)^2 - (-1) + 1 = \boxed{3}$

c) $f(5)$

Solution: $f(5) = (5)^2 - (5) + 1 = \boxed{21}$

d) $f(-5)$

Solution: $f(-5) = (-5)^2 - (-5) + 1 = \boxed{31}$

e) $f(2t)$

Solution: $f(2t) = (2t)^2 - (2t) + 1 = \boxed{4t^2 - 2t + 1}$

6. Define f as follows. The domain is $\mathbb{R}$, and the assignment is $f(x) = 2x^2 - 5$. Compute or simplify each of the following expressions.

a) $f(2)$

Solution: $f(2) = 2 \cdot 2^2 - 5 = 2 \cdot 4 - 5 = 8 - 5 = \boxed{3}$

b) $f(3)$

Solution: $f(3) = 2 \cdot 3^2 - 5 = 2 \cdot 9 - 5 = 18 - 5 = \boxed{13}$

c) $f(5)$

Solution: $f(5) = 2 \cdot 5^2 - 5 = 2 \cdot 25 - 5 = 50 - 5 = \boxed{45}$

d) $f(2) + f(3)$

Solution: We have just computed the values of $f(2)$ and $f(3)$. $f(2) + f(3) = 3 + 13 = \boxed{16}$

e) $f(2+3)$

Solution: $f(2+3) = f(5)$ which we have already computed and found to be $\boxed{45}$.

This problem illustrates the fact that most of the time $f(a) + f(b)$ and $f(a+b)$ are different.

7. In each case, a pair of functions is given. Are they equal or not?

a) $f = \{(-2, 5), (0, 1), (2, 5), (3, 10)\}$ and g is defined by domain $\{-2, 0, 2, 3\}$ and assignment $g(a) = a^2 + 1$.

Solution: these functions are the same because they have the same domain and the same assignment.

So, $f = g$.

b) f is given by the table below

element of domain	assigned element by f
1	0
2	-3
3	-10
5	-24

and g is defined by domain $\{1, 2, 3, 5\}$ and assignment $g(x) = -x^2 + 1$.

Solution: these functions are not the same because they do not have the same assignment: $f(3) = -10$ and $g(-3) = -8$.

So, $f \neq g$.

c) $f = \{(0, -3), (1, 2), (2, 7), (3, 12)\}$ and g is defined by domain $\mathbb{R}$ and assignment $g(m) = 5m - 3$.

Solution: these functions are not the same because they do not have the same domain. The domain of f is quite small, it contains only four numbers. Meanwhile, the domain of g is $\mathbb{R}$, the set of all real numbers. So, $f \neq g$.

Solutions for 22.1 – Rational Expressions 2

Perform the indicated operations and simplify.

1. a) $\frac{3x}{x-2} - \frac{x+4}{x-2}$

Solution: This is a subtraction of fractions. The denominators are the same, the only difficulty is that we are subtracting expressions instead of numbers. The second pair of parentheses is essential.

$$\frac{3x}{x-2} - \frac{x+4}{x-2} = \frac{(3x) - (x+4)}{x-2} = \frac{3x - x - 4}{x-2} = \frac{2x - 4}{x-2} = \frac{2(x-2)}{x-2} = \frac{\cancel{2(x-2)}}{\cancel{(x-2)}} = \boxed{2}$$

2. $\frac{10}{x-y} - \frac{5}{y-x}$

Solution: This problem is much easier when we realize that the denominators are opposites of each other, since the opposite of $y-x$ is $-1(y-x) = -y+x = x-y$

$$\frac{10}{x-y} - \frac{5}{y-x} = \frac{10}{x-y} - \frac{-1 \cdot 5}{-1 \cdot (y-x)} = \frac{10}{x-y} - \frac{-5}{x-y} = \frac{10 - (-5)}{x-y} = \boxed{\frac{15}{x-y}}$$

3. $\frac{1}{x-y} - \frac{1}{x+y}$

Solution: We first bring the two fractions to the common denominator, which is $(x+y)(x-y) = x^2 - y^2$

$$\frac{1}{x-y} - \frac{1}{x+y} = \frac{1 \cdot (x+y)}{(x-y)(x+y)} - \frac{1 \cdot (x-y)}{(x+y)(x-y)} = \frac{x+y}{(x-y)(x+y)} - \frac{x-y}{(x-y)(x+y)}$$

We are now ready to subtract. We need to be extremely careful to subtract the ENTIRE expression, and not just its first term. To do that, we need to insert parentheses around these expressions.

$$\frac{(x+y) - (x-y)}{(x-y)(x+y)} = \frac{x+y-x+y}{(x-y)(x+y)} = \frac{2y}{(x-y)(x+y)} \quad \text{or} \quad \frac{2y}{x^2 - y^2}$$

The final answer can be presented as $\boxed{\frac{2y}{(x-y)(x+y)} \text{ or } \frac{2y}{x^2 - y^2}}$. They are both equally correct.

4. $\frac{2}{p-5} - \frac{p+11}{p^2-2p-15}$

Solution: The denominator of the second fraction factors. $p^2 - 2p - 15 = (p+3)(p-5)$. We will now bring these fractions to the common denominator.

$$\frac{2}{p-5} - \frac{p+11}{p^2-2p-15} = \frac{2(p+3)}{(p-5)(p+3)} - \frac{p+11}{(p-5)(p+3)}$$

We can now perform the subtraction of polynomials in the numerator

$$\frac{2(p+3) - (p+11)}{(p-5)(p+3)} = \frac{2p+6-p-11}{(p-5)(p+3)} = \frac{p-5}{(p-5)(p+3)} = \frac{\cancel{p-5}}{\cancel{(p-5)}(p+3)} = \boxed{\frac{1}{p+3}}$$

$$5. \frac{x^2 - 5x + 78}{18x + x^2 - 208} - \frac{x}{x + 26}$$

Solution: The denominator of the first fraction factors. $x^2 + 18x - 208 = (x + 26)(x - 8)$. We will now bring these fractions to the common denominator.

$$\frac{x^2 - 5x + 78}{x^2 + 18x - 208} - \frac{x}{x + 26} = \frac{x^2 - 5x + 78}{(x + 26)(x - 8)} - \frac{x(x - 8)}{(x + 26)(x - 8)}$$

We can now perform the subtraction of polynomials in the numerator

$$\frac{(x^2 - 5x + 78) - x(x - 8)}{(x + 26)(x - 8)} = \frac{x^2 - 5x + 78 - x^2 + 8x}{(x + 26)(x - 8)} = \frac{3x + 78}{(x + 26)(x - 8)}$$

After we factor out the greatest common factor from the numerator, the result can be simplified.

$$\frac{3x + 78}{(x + 26)(x - 8)} = \frac{3(x + 26)}{(x + 26)(x - 8)} = \boxed{\frac{3}{x - 8}}$$

$$6. \left[\left(\frac{1}{b} - b \right) \div \left(1 - \frac{1}{b} \right) \right] (1 + b)$$

Solution: Let us perform the subtractions first.

$$\frac{1}{b} - b = \frac{1}{b} - \frac{b}{1} = \frac{1}{b} - \frac{b \cdot b}{1 \cdot b} = \frac{1}{b} - \frac{b^2}{b} = \frac{1 - b^2}{b} \quad \text{and} \quad 1 - \frac{1}{b} = \frac{1}{1} - \frac{1}{b} = \frac{1 \cdot b}{1 \cdot b} - \frac{1}{b} = \frac{b}{b} - \frac{1}{b} = \frac{b - 1}{b}$$

$$\text{Thus } \left[\left(\frac{1}{b} - b \right) \div \left(1 - \frac{1}{b} \right) \right] (1 + b) = \left[\frac{1 - b^2}{b} \div \frac{b - 1}{b} \right] (1 + b)$$

We re-write the division as multiplication by the reciprocal, and then cancel.

$$\left(\frac{1 - b^2}{\cancel{b}} \cdot \frac{\cancel{b}}{b - 1} \right) (1 + b) = \frac{1 - b^2}{b - 1} (1 + b)$$

This expression can be further simplified, since $1 - b^2 = (1 + b)(1 - b)$ via the difference of squares theorem, and $1 - b$ and $b - 1$ are opposites.

$$\frac{1 - b^2}{b - 1} (1 + b) = \frac{(1 - b)(1 + b)}{b - 1} (1 + b) = \frac{-1(b - 1)(1 + b)}{b - 1} (1 + b) = \frac{-1(\cancel{b - 1})(1 + b)}{\cancel{b - 1}} (1 + b) = \boxed{-(b + 1)^2}$$

Solutions for 22.2 – Rational Equations

Solve each of the given equations.

$$1. \frac{3x}{2x-1} = 6$$

Solution: We will clear the denominator by multiplying both sides by $2x-1$.

$$\begin{aligned} \frac{3x}{2x-1} &= 6 && \text{multiplied by } 2x-1 \\ 3x &= 6(2x-1) && \text{distribute} \\ 3x &= 12x-6 && \text{subtract } 3x \\ 0 &= 9x-6 && \text{add 6} \\ 6 &= 9x && \text{divide by 9} \\ \frac{6}{9} &= x \implies x = \frac{2}{3} \end{aligned}$$

$$\text{We check: if } x = \frac{2}{3}, \text{ then LHS} = \frac{3 \cdot \frac{2}{3}}{2 \cdot \frac{2}{3} - 1} = \frac{2}{\frac{4}{3} - 1} = \frac{2}{\frac{4}{3} - \frac{3}{3}} = \frac{2}{\frac{1}{3}} = 2 \cdot \frac{3}{1} = 6 = \text{RHS} \checkmark$$

and so our solution, $x = \frac{2}{3}$ is correct.

$$2. \frac{5}{x+7} - \frac{1}{6} = \frac{8}{4x+28}$$

Solution: It is important to notice that we do not need to multiply $x+7$ by $4x+28$, because $4x+28 = 4(x+7)$. This is important because if we multiply by more than what is needed, we might end up with a cubic equation that we cannot solve.

$$\begin{aligned} \frac{5}{x+7} - \frac{1}{6} &= \frac{8}{4x+28} && \text{factor } 4x+28 \\ \frac{5}{x+7} - \frac{1}{6} &= \frac{8}{4(x+7)} && \text{multiply by } 12(x+7) \\ 12(x+7) \left(\frac{5}{x+7} - \frac{1}{6} \right) &= 12(x+7) \left(\frac{8}{4(x+7)} \right) \\ 12(x+7) \cdot \frac{5}{x+7} - 12(x+7) \cdot \frac{1}{6} &= 12(x+7) \left(\frac{8}{4(x+7)} \right) && \text{cancel out terms} \\ \cancel{12(x+7)} \cdot \frac{5}{\cancel{x+7}} - 12(x+7) \cdot \frac{1}{6} &= \cancel{4} \cdot \cancel{3(x+7)} \left(\frac{8}{\cancel{4(x+7)}} \right) \\ 60 - 2(x+7) &= 24 \\ 60 - 2x - 14 &= 24 \\ -2x + 46 &= 24 \\ -2x &= -22 \\ x &= 11 \end{aligned}$$

We check: if $x = 11$, then

$$\begin{aligned} \text{LHS} &= \frac{5}{11+7} - \frac{1}{6} = \frac{5}{18} - \frac{1}{6} = \frac{5}{18} - \frac{3}{18} = \frac{2}{18} = \frac{1}{9} \\ \text{RHS} &= \frac{8}{4(11)+28} = \frac{8}{44+28} = \frac{8}{72} = \frac{1}{9} \checkmark \end{aligned}$$

Thus our solution, $x = 11$ is correct.

$$3. \frac{x}{x+1} + \frac{4}{x+7} = \frac{5x-1}{x^2+8x+7}$$

Solution: It is important to notice that we do not need to multiply by $(x+7)(x+1)(x^2+8x+7)$, because $x^2+8x+7 = (x+7)(x+1)$. This is important because if we multiply by more than what is needed, we might end up with a 4th or 5th degree equation that we cannot solve.

$$\begin{aligned} \frac{x}{x+1} + \frac{4}{x+7} &= \frac{5x-1}{x^2+8x+7} \\ \frac{x}{x+1} + \frac{4}{x+7} &= \frac{5x-1}{(x+7)(x+1)} && \text{multiply by } (x+7)(x+1) \\ (x+7)(x+1) \left(\frac{x}{x+1} + \frac{4}{x+7} \right) &= (x+7)(x+1) \frac{5x-1}{(x+7)(x+1)} \\ (x+7)(x+1) \frac{x}{x+1} + (x+7)(x+1) \frac{4}{x+7} &= (x+7)(x+1) \frac{5x-1}{(x+7)(x+1)} && \text{cancel out terms} \\ (x+7)\cancel{(x+1)} \frac{x}{\cancel{x+1}} + \cancel{(x+7)}(x+1) \frac{4}{\cancel{x+7}} &= \cancel{(x+7)}\cancel{(x+1)} \frac{5x-1}{\cancel{(x+7)}\cancel{(x+1)}} \\ x(x+7) + 4(x+1) &= 5x-1 \\ x^2 + 7x + 4x + 4 &= 5x-1 \\ x^2 + 11x + 4 &= 5x-1 \\ x^2 + 6x + 4 &= -1 \\ x^2 + 6x + 5 &= 0 \\ (x+5)(x+1) &= 0 \implies x_1 = -5 \quad x_2 = -1 \end{aligned}$$

We check: if $x = -5$, then

$$\begin{aligned} \text{LHS} &= \frac{-5}{-5+1} + \frac{4}{-5+7} = \frac{-5}{-4} + \frac{4}{-2} = \frac{5}{4} + 2 = \frac{5}{4} + \frac{8}{4} = \frac{13}{4} \\ \text{RHS} &= \frac{5(-5)-1}{(-5)^2+8(-5)+7} = \frac{-26}{25-40+7} = \frac{-26}{-8} = \frac{13}{4} \checkmark \end{aligned}$$

Thus $x = -5$ works. And if $x = -1$, then

$$\text{LHS} = \frac{-1}{-1+1} + \frac{4}{-1+7} = \frac{-1}{0} + \frac{4}{6} = \text{undefined}$$

Since $x = -1$ is not even in the domain of the first line, it cannot be a solution. So our only solution is $\boxed{x = -5}$.

$$4. \frac{1}{4y} - \frac{1}{2} = \frac{1}{y}$$

Solution:

$$\begin{aligned} \frac{1}{4y} - \frac{1}{2} &= \frac{1}{y} && \text{multiply by } 4y \\ 4y \left(\frac{1}{4y} - \frac{1}{2} \right) &= 4y \cdot \frac{1}{y} \\ 4y \cdot \frac{1}{4y} - 4y \cdot \frac{1}{2} &= 4 \\ 1 - 2y &= 4 && \text{subtract 1} \\ -2y &= 3 && \text{divide by } -2 \\ y &= -\frac{3}{2} \end{aligned}$$

We check: if $y = -\frac{3}{2}$, then $\text{LHS} = \frac{1}{4 \left(-\frac{3}{2} \right)} - \frac{1}{2} = \frac{1}{-6} - \frac{1}{2} = -\frac{1}{6} - \frac{3}{6} = -\frac{4}{6} = -\frac{2}{3}$ and

$$\text{RHS} = \frac{1}{-\frac{3}{2}} = -\frac{2}{3} \checkmark. \text{ Thus our solution, } \boxed{y = -\frac{3}{2}} \text{ is correct.}$$

$$5. \frac{3}{x-5} - \frac{1}{x+3} = \frac{10}{x^2 - 2x - 15}$$

Solution: Let us first factor the quadratic denominator. $x^2 - 2x - 15 = (x+3)(x-5)$. Therefore, we only need to multiply both sides by $(x+3)(x-5)$.

$$\begin{aligned} \frac{3}{x-5} - \frac{1}{x+3} &= \frac{10}{x^2 - 2x - 15} \\ \frac{3}{x-5} - \frac{1}{x+3} &= \frac{10}{(x+3)(x-5)} && \text{multiply by } (x+3)(x-5) \\ (x+3)(x-5) \left(\frac{3}{x-5} - \frac{1}{x+3} \right) &= (x+3)(x-5) \frac{10}{(x+3)(x-5)} \\ (x+3)(x-5) \frac{3}{x-5} - (x+3)(x-5) \frac{1}{x+3} &= (x+3)(x-5) \frac{10}{(x+3)(x-5)} \\ \cancel{(x+3)} \cancel{(x-5)} \frac{3}{\cancel{x-5}} - \cancel{(x+3)} \cancel{(x-5)} \frac{1}{\cancel{x+3}} &= \cancel{(x+3)} \cancel{(x-5)} \frac{10}{\cancel{(x+3)} \cancel{(x-5)}} && \text{cancel terms} \\ 3(x+3) - (x-5) &= 10 \\ 3x+9-x+5 &= 10 \\ 2x+14 &= 10 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

We check: if $x = -2$, then $\text{LHS} = \frac{3}{-2-5} - \frac{1}{-2+3} = \frac{3}{-7} - \frac{1}{1} = -\frac{3}{7} - \frac{7}{7} = -\frac{10}{7}$ and

$$\text{RHS} = \frac{10}{(-2)^2 - 2(-2) - 15} = \frac{10}{4+4-15} = \frac{10}{-7} = -\frac{10}{7} \checkmark \quad \text{Thus our solution, } \boxed{x = -2} \text{ is correct.}$$

$$6. \frac{x}{x-7} - \frac{3}{x+3} = \frac{4x+42}{x^2 - 4x - 21}$$

Solution:

$$\begin{aligned} \frac{x}{x-7} - \frac{3}{x+3} &= \frac{4x+42}{x^2 - 4x - 21} \\ \frac{x}{x-7} - \frac{3}{x+3} &= \frac{4x+42}{(x-7)(x+3)} && \text{multiply by } (x-7)(x+3) \\ (x-7)(x+3) \left(\frac{x}{x-7} - \frac{3}{x+3} \right) &= (x-7)(x+3) \frac{4x+42}{(x-7)(x+3)} \\ (x-7)(x+3) \frac{x}{x-7} - (x-7)(x+3) \frac{3}{x+3} &= (x-7)(x+3) \frac{4x+42}{(x-7)(x+3)} && \text{cancel out terms} \\ \cancel{(x-7)} \cancel{(x+3)} \frac{x}{\cancel{x-7}} - \cancel{(x-7)} \cancel{(x+3)} \frac{3}{\cancel{x+3}} &= \cancel{(x-7)} \cancel{(x+3)} \frac{4x+42}{\cancel{(x-7)} \cancel{(x+3)}} \\ x(x+3) - 3(x-7) &= 4x+42 \\ x^2 + 3x - 3x + 21 &= 4x+42 \\ x^2 - 4x - 21 &= 0 \\ (x+3)(x-7) &= 0 \implies x_1 = -3 \quad x_2 = 7 \end{aligned}$$

We check: if $x = 7$, then $\text{LHS} = \frac{7}{7-7} - \frac{3}{7+3} = \frac{7}{0} - \frac{3}{10} = \text{undefined}$

Thus $x = 7$ does not work, it is an extraneous solution. And if $x = -3$, then

$$\text{LHS} = \frac{-3}{-3-7} + \frac{3}{-3+3} = \frac{-3}{-10} + \frac{3}{0} = \text{undefined}$$

Since $x = -3$ is not even in the domain of the first line, it cannot be a solution. So it is also an extraneous solution. So this equation has no solution.

Solutions for 23.1 – Radical Equations

1. $\sqrt{3x-2} = x$

Solution:

$$\begin{aligned}\sqrt{3x-2} &= x && \text{square} \\ 3x-2 &= x^2 && \text{reduce one side to zero} \\ 0 &= x^2 - 3x + 2 && \text{factor} \\ 0 &= (x-2)(x-1) \implies x_1 = 2 \text{ and } x_2 = 1\end{aligned}$$

We check: if $x = 2$, then $\text{LHS} = \sqrt{3(2)-2} = \sqrt{4} = 2 = \text{RHS} \checkmark$

and if $x = 1$, then $\text{LHS} = \sqrt{3(1)-2} = \sqrt{1} = 1 = \text{RHS} \checkmark$ Therefore, 1 and 2 are solutions of the equation.

2. $x + 2\sqrt{2x+16} = -2$

Solution:

$$\begin{aligned}x + 2\sqrt{2x+16} &= -2 && \text{add 2 and subtract } 2\sqrt{2x+16} \\ x + 2 &= -2\sqrt{2x+16} && \text{square both sides} \\ (x+2)^2 &= \left(-2\sqrt{2x+16}\right)^2 \\ x^2 + 4x + 4 &= 4(2x+16) \\ x^2 + 4x + 4 &= 8x + 64 && \text{subtract } (8x + 64) \\ x^2 - 4x - 60 &= 0 && \text{factor} \\ (x-10)(x+6) &= 0\end{aligned}$$

$$x_1 = 10 \text{ and } x_2 = -6$$

We check: if $x = 10$, then

$$\text{LHS} = 10 + 2\sqrt{2 \cdot 10 + 16} = 10 + 2\sqrt{36} = 10 + 12 = 22 \quad \text{and} \quad \text{RHS} = -2 \quad \text{LHS} \neq \text{RHS}$$

Thus $x = 10$ is NOT a solution. If $x = -6$, then

$$\text{LHS} = -6 + 2\sqrt{2(-6) + 16} = -6 + 2\sqrt{4} = -6 + 2 \cdot 2 = -2 = \text{RHS} \checkmark$$

Thus the only solution is -6.

3. $10 + \sqrt{4x-7} = 7$

Solution:

$$\begin{aligned}10 + \sqrt{4x-7} &= 7 && \text{subtract 10} \\ \sqrt{4x-7} &= -3\end{aligned}$$

Since the square root of no real number is negative, there is no real solution.

4. $5 + \sqrt{x+15} = x$

Solution:

$$\begin{array}{ll}
 5 + \sqrt{x+15} &= x && \text{subtract 5} \\
 \sqrt{x+15} &= x-5 && \text{square both sides} \\
 x+15 &= (x-5)^2 && \text{FOIL right hand side} \\
 x+15 &= x^2-10x+25 && \text{reduce one side to zero} \\
 0 &= x^2-11x+10 && \text{factor} \\
 0 &= (x-1)(x-10) \implies x_1 = 1 \text{ and } x_2 = 10
 \end{array}$$

We check: If $x = 1$, then

$$\text{LHS} = 5 + \sqrt{1+15} = 5 + \sqrt{16} = 5 + 4 = 9 \quad \text{and} \quad \text{RHS} = 1 \quad \text{RHS} \neq \text{LHS}$$

Thus $x = 1$ is NOT a solution. If $x = 10$, then

$$\text{LHS} = 5 + \sqrt{10+15} = 5 + \sqrt{25} = 5 + 5 = 10 = \text{RHS} \checkmark$$

Thus $\boxed{x = 10}$ is the only solution.

5. $2\sqrt{x-1} = x-4$

Solution:

$$\begin{array}{ll}
 2\sqrt{x-1} &= x-4 && \text{square both sides} \\
 4(x-1) &= (x-4)^2 && \text{FOIL, distribute} \\
 4x-4 &= x^2-8x+16 && \text{reduce one side to zero} \\
 0 &= x^2-12x+20 && \text{factor} \\
 0 &= (x-2)(x-10) \implies x_1 = 2 \text{ and } x_2 = 10
 \end{array}$$

We check: If $x = 2$, then

$$\text{LHS} = 2\sqrt{2-1} = 2\sqrt{1} = 2(1) = 2 \quad \text{and} \quad \text{RHS} = 2-4 = -2 \quad \text{RHS} \neq \text{LHS}$$

Thus $x = 2$ is NOT a solution. If $x = 10$, then

$$\text{LHS} = 2\sqrt{10-1} = 2\sqrt{9} = 2(3) = 6 \quad \text{and} \quad \text{RHS} = 10-4 = 6 = \text{RHS} \checkmark$$

Thus $\boxed{x = 10}$ is the only solution.

6. $x = 2\sqrt{x+8}$

$$\begin{array}{ll}
 \text{Solution:} & x = 2\sqrt{x+8} \quad \text{square both sides} && x^2 - 4x - 32 = 0 \\
 & x^2 = (2\sqrt{x+8})^2 && (x-8)(x+4) = 0 \\
 & x^2 = 4(x+8) && x_1 = 8 \text{ and } x_2 = -4
 \end{array}$$

We check: If $x = 8$, then $\text{LHS} = 8$ and $\text{RHS} = 2\sqrt{8+8} = 2\sqrt{16} = 2 \cdot 4 = 8 \checkmark$.

Thus $x = 8$ is indeed a solution. If $x = -4$, then

$$\text{RHS} = 2\sqrt{-4+8} = 2\sqrt{4} = 2 \cdot 2 = 4 = \text{LHS} \quad \text{RHS} \neq \text{LHS}$$

Thus $x = -4$ is NOT a solution. The only solution is $\boxed{x = 8}$.

7. $5\sqrt{x} + 1 = 3\sqrt{x} + 17$

Solution:

$$\begin{array}{rcll}
 5\sqrt{x} + 1 & = & 3\sqrt{x} + 17 & \text{subtract } 3\sqrt{x} \\
 2\sqrt{x} + 1 & = & 17 & \text{subtract 1} \\
 2\sqrt{x} & = & 16 & \text{divide by 2} \\
 \sqrt{x} & = & 8 & \text{square both sides} \\
 x & = & 64 &
 \end{array}$$

We check: If $x = 64$, then

$$\text{LHS} = 5\sqrt{64} + 1 = 5(8) + 1 = 41 \quad \text{and} \quad \text{RHS} = 3\sqrt{64} + 17 = 3(8) + 17 = 24 + 17 = 41 \quad \checkmark$$

Thus $\boxed{64}$ is indeed a solution.

8. $\sqrt{2x+5} + 5 = x$

Solution:

$$\begin{array}{rcll}
 \sqrt{2x+5} + 5 & = & x & \text{subtract 5} \\
 \sqrt{2x+5} & = & x - 5 & \text{square} \\
 2x + 5 & = & x^2 - 10x + 25 & \text{reduce one side to zero} \\
 0 & = & x^2 - 12x + 20 & \text{factor} \\
 0 & = & (x-2)(x-10) & \implies x_1 = 2 \quad \text{and} \quad x_2 = 10
 \end{array}$$

We check: If $x = 2$, then

$$\begin{array}{lcl}
 \text{LHS} & = & \sqrt{2(2)+5} + 5 = \sqrt{4+5} + 5 = \sqrt{9} + 5 = 3 + 5 = 8 \quad \text{and} \quad \text{RHS} = 2 \\
 \text{RHS} & \neq & \text{LHS}
 \end{array}$$

Thus $x = 2$ is NOT a solution. If $x = 10$, then

$$\text{LHS} = \sqrt{2(10)+5} + 5 = \sqrt{20+5} + 5 = \sqrt{25} + 5 = 5 + 5 = 10 = \text{RHS} \quad \checkmark$$

Thus $\boxed{x = 10}$ is the only solution.

9. $2\sqrt{2x+29} = x + 12$

Solution:

$$\begin{array}{rcll}
 2\sqrt{2x+29} & = & x + 12 & \text{square both sides} \\
 (2\sqrt{2x+29})^2 & = & (x+12)^2 & \\
 4(2x+29) & = & (x+12)^2 & \\
 8x + 116 & = & x^2 + 24x + 144 & \text{reduce one side to zero} \\
 0 & = & x^2 + 16x + 28 & \\
 0 & = & (x+14)(x+2) & \implies x_1 = -14 \quad \text{and} \quad x_2 = -2
 \end{array}$$

We check: if $x = -14$, then

$$\text{LHS} = 2\sqrt{2(-14)+29} = 2\sqrt{1} = 2 \quad \text{and} \quad \text{RHS} = -14 + 12 = -2 \quad \text{LHS} \neq \text{RHS}$$

Since the two sides are not equal, $x = -14$ is NOT a solution. If $x = -2$, then

$$\text{LHS} = 2\sqrt{2(-2)+29} = 2\sqrt{25} = 2 \cdot 5 = 10 \quad \text{and} \quad \text{RHS} = -2 + 12 = 10 \quad \checkmark$$

Thus $\boxed{-2}$ is the only solution.

10. $\sqrt{x+10}+10=x$

Solution:

$$\begin{aligned}\sqrt{x+10}+10 &= x && \text{subtract 10} \\ \sqrt{x+10} &= x-10 && \text{square} \\ x+10 &= (x-10)^2 \\ x+10 &= x^2-20x+100 && \text{reduce one side to zero} \\ 0 &= x^2-21x+90 && \text{factor} \\ 0 &= (x-6)(x-15) \implies x_1=6 \text{ and } x_2=15\end{aligned}$$

We check: if $x=6$, then

$$\text{LHS} = \sqrt{6+10}+10 = \sqrt{16}+10 = 4+10 = 14 \quad \text{and} \quad \text{RHS} = 6 \quad \text{LHS} \neq \text{RHS}$$

If $x=15$, then

$$\text{LHS} = \sqrt{15+10}+10 = \sqrt{25}+10 = 5+10 = 15 = \text{RHS} \checkmark$$

since $x=6$ doesn't work, the only solution is 15.