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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Simplify A , B , and C , and put them in increasing order.

$$A = \tan(0 \cdot 12^\circ) + \tan(1 \cdot 12^\circ) + \tan(2 \cdot 12^\circ) + \dots + \tan(15 \cdot 12^\circ), \quad B = \sqrt[3]{6 \cdot \sqrt{3} - 10}$$

$$C = \log_2 \sqrt{\frac{2}{a} \cdot \sqrt[3]{\frac{a}{2}} \cdot \sqrt[4]{\frac{a^8}{32}}}, \quad \text{where } a = \sqrt{7} - 1$$

Solution: The first and last terms of A are 0, and the other terms cancel in pairs, because for $k = 1, 2, \dots, 7$

$$\tan(k \cdot 12^\circ) = -\tan(180^\circ - k \cdot 12^\circ) = -\tan((15 - k) \cdot 12^\circ)$$

So $A = 0$.

Let us look for $6\sqrt{3} - 10$ as a cube $(t + \sqrt{3})^3 = (t^3 + 9t) + (3t^2 + 3)\sqrt{3}$. From $3t^2 + 3 = 6$ and $t^3 + 9t = -10$ we get $t = -1$, so $B = \sqrt[3]{(\sqrt{3} - 1)^3} = \sqrt{3} - 1 \approx 0.73$.

For every $a > 0$:

$$\sqrt[4]{\frac{a^8}{32}} = \frac{a^2}{2^{5/4}}, \quad \sqrt[3]{\frac{a}{2} \cdot \frac{a^2}{2^{5/4}}} = \frac{a}{2^{3/4}}, \quad C = \log_2 \sqrt{\frac{2}{a} \cdot \frac{a}{2^{3/4}}} = \log_2 2^{1/8} = \frac{1}{8}$$

(independently of the value of a). The increasing order is $A < C < B$.

2. In the convex pentagon $ABCDE$, F is the midpoint of AE , H is the midpoint of FD , J is the midpoint of HC , G is the midpoint of BE , I is the midpoint of GD , and K is the midpoint of IC . Find the ratio $JK : AB$.

Solution: FG is a midline in the triangle ABE , HI is a midline in the triangle FGD , and JK is a midline in the triangle HIC . So $AB = 2 \cdot FG = 4 \cdot HI = 8 \cdot JK$, and $JK : AB = 1 : 8$.

3. Show that the following equation has exactly two integer roots.

$$\sqrt{1 - \frac{x^2 - 2000x + 1999}{x - 1}} + \sqrt{-1 + \frac{x^2 - 1999x + 1998}{x - 1}} = 1$$

Solution: Since $x^2 - 2000x + 1999 = (x - 1999)(x - 1)$ and $x^2 - 1999x + 1998 = (x - 1998)(x - 1)$, for $x \neq 1$ the equation is

$$\sqrt{2000 - x} + \sqrt{x - 1999} = 1$$

The square roots require $1999 \leq x \leq 2000$. Squaring, $1 + 2\sqrt{(2000 - x)(x - 1999)} = 1$, so $x = 1999$ or $x = 2000$, and both of these are solutions. So the equation has exactly two roots, and both are integers.

4. The parabola $y^2 - 6y - x + 11 = 0$ and a circle have three common points, which are the vertices of an equilateral triangle. Find the equation of the circle, if its center is on the axis of the parabola.

Solution: The parabola is $(y - 3)^2 = x - 2$, with vertex $V(2, 3)$ and axis $y = 3$. Both curves are symmetric about this axis, so the common points that are not on the axis come in pairs. So with three common points, one of them is the vertex V , and the other two are $P(b^2 - 6b + 11, b)$ and its reflection, with $b > 3$. The angle of the triangle at V is 60° , so VP makes a 30° angle with the axis:

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{b - 3}{(b - 3)^2}, \quad b = 3 + \sqrt{3}, \quad P(5, 3 + \sqrt{3})$$

The side of the triangle is $VP = \sqrt{9 + 3} = \sqrt{12}$, its height is $\frac{\sqrt{3}}{2}\sqrt{12} = 3$, and the radius of the circumscribed circle is $\frac{2}{3} \cdot 3 = 2$. So the center is $(4, 3)$, and the circle is

$$(x - 4)^2 + (y - 3)^2 = 4$$

(Check: with $u = (y - 3)^2 = x - 2$ the equation of the circle becomes $(u - 2)^2 + u = 4$, that is, $u(u - 3) = 0$, so there are exactly three common points.)

5. Find the lattice points of the coordinate plane whose coordinates satisfy both of the following conditions.

$$x^3 - x^2y - xy + y^2 = 0, \quad x^2 + y^2 - 4x - 2y - 4 \leq 0$$

Solution: The conditions can be written as $(x^2 - y)(x - y) = 0$ and $(x - 2)^2 + (y - 1)^2 \leq 9$. So we are looking for the lattice points that are on the parabola $y = x^2$ or on the line $y = x$, and in the closed disk with center $(2, 1)$ and radius 3. These points are

$$(-1, 1), (0, 0), (1, 1), (2, 4), (2, 2), (3, 3)$$

6. In one year, the revenue of a shopping center increased by 54%. The larger revenue was the result of an increase in the number of shoppers and an increase in the average spending per shopper. By what percent did the number of shoppers increase, if the average spending per shopper increased by four times as many percent as the number of shoppers?

Solution: If the number of shoppers increased by $p\%$, then the spending per shopper increased by $4p\%$, so

$$\left(1 + \frac{p}{100}\right) \left(1 + \frac{4p}{100}\right) = 1.54, \quad 4p^2 + 500p - 5400 = 0$$

The only positive root is $p = 10$. The number of shoppers increased by 10% (and the spending per shopper by 40%).

7. The real parameters a and c have different signs, and $a + c \neq b$. Consider the equation

$$(a^2 + b^2 + c^2) \cdot x^2 - 2b(a + c) \cdot x + 2ac = 0$$

Show that the equation has two real roots, and the larger root is in the interval $(0, 1)$.

Solution: One quarter of the discriminant is

$$\frac{D}{4} = b^2(a + c)^2 - 2ac(a^2 + b^2 + c^2) = b^2(a^2 + c^2) - 2ac(a^2 + c^2) = (b^2 - 2ac)(a^2 + c^2)$$

Since $ac < 0$, both factors are positive, so $D > 0$ and there are two different real roots. Let $f(x)$ be the left-hand side. Then $f(0) = 2ac < 0$ and

$$f(1) = a^2 + b^2 + c^2 - 2ab - 2bc + 2ac = (a - b + c)^2 > 0$$

because $a + c \neq b$. So one root is in the interval $(0, 1)$. The product of the roots is $\frac{2ac}{a^2 + b^2 + c^2} < 0$, so the other root is negative, and the root in $(0, 1)$ is the larger one.

8. The hypotenuse AB of the right triangle ABC is 1 unit long. We draw the squares $ACDE$ and $ABFG$ outward on the sides AC and AB . What is the angle of the triangle ABC at A , if the area of the quadrilateral $ECBG$ is as large as possible?

Solution: Let the angle at A be α ; then $AC = \cos \alpha$ and $BC = \sin \alpha$. The angle EAG is $360^\circ - 90^\circ - 90^\circ - \alpha = 180^\circ - \alpha$. The quadrilateral is made up of the triangles ACE , ABG , ABC , and AEG :

$$T = \frac{\cos^2 \alpha}{2} + \frac{1}{2} + \frac{\cos \alpha \sin \alpha}{2} + \frac{\cos \alpha \sin (180^\circ - \alpha)}{2} = \frac{3}{4} + \frac{2 \sin 2\alpha + \cos 2\alpha}{4}$$

Let φ be the acute angle with $\tan \varphi = \frac{1}{2}$ ($\varphi \approx 26.57^\circ$). Then $\sin \varphi = \frac{1}{\sqrt{5}}$, $\cos \varphi = \frac{2}{\sqrt{5}}$, and

$$2 \sin 2\alpha + \cos 2\alpha = \sqrt{5} (\sin 2\alpha \cos \varphi + \cos 2\alpha \sin \varphi) = \sqrt{5} \sin (2\alpha + \varphi)$$

This is largest when $2\alpha + \varphi = 90^\circ$, that is, $\tan 2\alpha = 2$, $2\alpha \approx 63.43^\circ$, and $\alpha \approx 31.72^\circ$. (Then the area is $\frac{3 + \sqrt{5}}{4} \approx 1.31$ square units.)