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In memory of László Párkányi, teacher

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. In one year, a factory increased its production by 26.5%. This increase came from two sources: the number of workers increased, and the production per worker increased. The percent increase of the production per worker was 1.5 times the percent increase of the number of workers. By what percent did each of them increase?

Solution: If the number of workers increased by $2p\%$, then the production per worker increased by $3p\%$, so

$$\left(1 + \frac{2p}{100}\right) \left(1 + \frac{3p}{100}\right) = 1.265$$

With $x = \frac{p}{100}$ this is $6x^2 + 5x - 0.265 = 0$, so $x = 0.05$ (the other root is negative). So $p = 5$: the number of workers increased by 10%, and the production per worker by 15%.

2. In the triangle ABC , $AB = 10$ units, $A(0, 6)$, the point B is on the x -axis, and the centroid of the triangle is $S(4, 6)$. Find the area of the triangle.

Solution: Let $B(b, 0)$. Then $AB^2 = b^2 + 6^2 = 100$, so $b = 8$ or $b = -8$. The centroid is the average of the vertices, so $C = 3S - A - B = (12 - b, 12)$.

If $B(8, 0)$, then $C(4, 12)$; with $\overrightarrow{AB} = (8, -6)$ and $\overrightarrow{AC} = (4, 6)$ the area is $\frac{1}{2} |8 \cdot 6 - (-6) \cdot 4| = 36$.

If $B(-8, 0)$, then $C(20, 12)$; with $\overrightarrow{AB} = (-8, -6)$ and $\overrightarrow{AC} = (20, 6)$ the area is $\frac{1}{2} |-8 \cdot 6 - (-6) \cdot 20| = 36$.

In both cases the area is 36 square units.

3. For which real numbers x are the values of the following expressions positive?

a) $\frac{x^4 - 16}{x^2 - 4}$ b) $|x + 2| - |x - 2| - x$ c) $1 + \log_2 \sin 2x$

Solution: a) We need $x \neq 2$ and $x \neq -2$, and then $\frac{x^4 - 16}{x^2 - 4} = x^2 + 4 > 0$. So the expression is positive for every $x \neq \pm 2$.

b) If $x > 2$, the expression is $4 - x > 0$, so $2 < x < 4$. If $-2 \leq x \leq 2$, it is $2x - x = x > 0$, so $0 < x \leq 2$. If $x < -2$, it is $-4 - x > 0$, so $x < -4$. The expression is positive if $x < -4$ or $0 < x < 4$.

c) We need $\sin 2x > 0$ and $\log_2 \sin 2x > -1$, that is, $\sin 2x > \frac{1}{2}$ (the function $\log_2$ is increasing). So $\frac{\pi}{6} + 2k\pi < 2x < \frac{5\pi}{6} + 2k\pi$:

$$\frac{\pi}{12} + k\pi < x < \frac{5\pi}{12} + k\pi, \quad k \text{ is an integer}$$

4. The sum of the first n terms of the sequence (a_n) is $S_n = 2n^2 + 3n$ for every positive integer n . The n th term of the sequence (b_n) is $b_n = 4n - 1$.

a) Prove that (a_n) and (b_n) are arithmetic sequences. Write the n th term of (a_n) as a function of n .

b) Express the sum of the first n terms of (b_n) in terms of n .

c) Find the sum of the first n terms of the sequence $c_n = a_n + b_n$.

Solution: a) $a_1 = S_1 = 5$, and for $n \geq 2$

$$a_n = S_n - S_{n-1} = 2n^2 + 3n - (2(n-1)^2 + 3(n-1)) = 4n + 1$$

This formula gives $a_1 = 5$ too, so $a_n = 4n + 1$ for every n , and $a_n - a_{n-1} = 4$: (a_n) is arithmetic. Also $b_n - b_{n-1} = 4$, so (b_n) is arithmetic as well.

b) $b_1 = 3$, so the sum is $\frac{3 + (4n - 1)}{2} \cdot n = 2n^2 + n$.

c) $c_n = (4n + 1) + (4n - 1) = 8n$, so the sum is $8(1 + 2 + \dots + n) = 8 \cdot \frac{n(n+1)}{2} = 4n^2 + 4n$.

5. Solve the following equation over the real numbers.

$$\cos x - \sin x = \frac{\cos 2x}{1 + \sin 2x}$$

Solution: We need $\sin 2x \neq -1$. Since $\cos 2x = (\cos x - \sin x)(\cos x + \sin x)$ and $1 + \sin 2x = (\cos x + \sin x)^2$, the equation is

$$\cos x - \sin x = \frac{\cos x - \sin x}{\cos x + \sin x}, \quad (\cos x - \sin x)(\cos x + \sin x - 1) = 0$$

If $\cos x - \sin x = 0$, then $\tan x = 1$, and $x = \frac{\pi}{4} + n\pi$.

If $\cos x + \sin x = 1$, then dividing by $\sqrt{2}$, $\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, so $x = 2k\pi$ or $x = \frac{\pi}{2} + 2k\pi$.

In all of these cases $\sin 2x \neq -1$, so the solutions are $x = \frac{\pi}{4} + n\pi$, $x = 2k\pi$, and $x = \frac{\pi}{2} + 2k\pi$ (n and k are arbitrary integers).

6. One angle of a triangle is 75° , and the altitude from the vertex of this angle divides the opposite side into parts of lengths $\sqrt{6}$ and $\sqrt{2}$. Find the other two angles of the triangle.

Solution: Let h be the altitude. It divides the 75° angle into two parts, φ (next to the part $\sqrt{6}$) and ψ (next to the part $\sqrt{2}$), with $\tan \varphi = \frac{\sqrt{6}}{h}$ and $\tan \psi = \frac{\sqrt{2}}{h}$. Since $\tan 75^\circ = 2 + \sqrt{3}$,

$$\tan(\varphi + \psi) = \frac{\frac{\sqrt{6} + \sqrt{2}}{h}}{1 - \frac{2\sqrt{3}}{h^2}} = 2 + \sqrt{3}, \quad (2 + \sqrt{3})h^2 - (\sqrt{6} + \sqrt{2})h - 2\sqrt{3}(2 + \sqrt{3}) = 0$$

The positive root of this equation is $h = \sqrt{6}$ (the product of the roots is negative, so there is only one positive root). Then $\varphi = 45^\circ$ and $\psi = 30^\circ$, so the angles of the triangle at the ends of the opposite side are $90^\circ - 45^\circ = 45^\circ$ and $90^\circ - 30^\circ = 60^\circ$.

The other two angles are 45° and 60° .

7. Find the values of the real parameter p for which the roots of the equation $x^2 + 4px + 2p^2 + 3p - 1 = 0$ satisfy the following conditions.

- a) The two real roots are equal. b) The two real roots are reciprocals of each other.
c) One real root is twice the other real root. d) The product of the real roots is as small as possible.

Solution: The roots are real if and only if the discriminant is nonnegative:

$$D = 16p^2 - 4(2p^2 + 3p - 1) = 8(p - 1) \left(p - \frac{1}{2}\right) \geq 0, \quad \text{that is, } p \leq \frac{1}{2} \text{ or } p \geq 1$$

a) $D = 0$ for $p = \frac{1}{2}$ (the equation is $x^2 + 2x + 1 = 0$, $x_1 = x_2 = -1$) and for $p = 1$ (the equation is $x^2 + 4x + 4 = 0$, $x_1 = x_2 = -2$).

b) We need $x_1x_2 = 2p^2 + 3p - 1 = 1$, so $p = -2$ or $p = \frac{1}{2}$. For $p = -2$ the equation is $x^2 - 8x + 1 = 0$, with roots $4 \pm \sqrt{15}$; for $p = \frac{1}{2}$ the double root -1 is its own reciprocal.

c) If $x_2 = 2x_1$, then $3x_1 = -4p$ and $2x_1^2 = 2p^2 + 3p - 1$, so $\frac{32p^2}{9} = 2p^2 + 3p - 1$, that is, $14p^2 - 27p + 9 = 0$, and $p = \frac{3}{2}$ or $p = \frac{3}{7}$ (both give $D > 0$). For $p = \frac{3}{2}$ the equation is $x^2 + 6x + 8 = 0$, with roots -2 and -4 ; for $p = \frac{3}{7}$ it is $x^2 + \frac{12}{7}x + \frac{32}{49} = 0$, with roots $-\frac{4}{7}$ and $-\frac{8}{7}$.

d) $x_1x_2 = 2p^2 + 3p - 1 = 2\left(p + \frac{3}{4}\right)^2 - \frac{17}{8}$. Since $p = -\frac{3}{4} < \frac{1}{2}$ is allowed, the product is smallest for $p = -\frac{3}{4}$, and then $x_1x_2 = -\frac{17}{8}$.

(Note: in c), the KoMaL solution prints the constant term as $\frac{32}{7}$ instead of $\frac{32}{49}$; this is a misprint.)

8. A pan of pastry is cut into pieces with straight cuts parallel to the sides of the pan, each running from side to side. The pieces along the edges are called “bad”, and the others “good”. How many cuts should we make so that the pastry can be divided into servings of two “good” pieces and one “bad” piece each? How many servings can be made this way?

Solution: Let the numbers of cuts in the two directions be k and n , with $k \leq n$ (both must be positive, otherwise every piece is “bad”). There are $(k + 1)(n + 1)$ pieces, of which $(k - 1)(n - 1)$ are “good”, and $2(k + 1) + 2(n - 1) = 2k + 2n$ are “bad”. We need twice as many “good” pieces as “bad” ones:

$$(k - 1)(n - 1) = 4(k + n), \quad kn - 5k - 5n + 1 = 0, \quad (k - 5)(n - 5) = 24$$

If $k - 5$ and $n - 5$ were negative, then $k - 5 \geq -4$ would force $n - 5 \leq -6$, which is impossible. So $(k - 5, n - 5)$ is $(1, 24)$, $(2, 12)$, $(3, 8)$, or $(4, 6)$:

$k = 6, n = 29$: 140 “good” and 70 “bad” pieces, 70 servings;

$k = 7, n = 17$: 96 “good” and 48 “bad” pieces, 48 servings;

$k = 8, n = 13$: 84 “good” and 42 “bad” pieces, 42 servings;

$k = 9, n = 11$: 80 “good” and 40 “bad” pieces, 40 servings.

So the number of cuts is 35, 24, 21, or 20 (split between the two directions as above), and the number of servings is 70, 48, 42, or 40.

(Note: the KoMaL solution gives $n = 9$ and $n = 7$ in the first two cases; these are misprints for $n = 29$ and $n = 17$.)