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In memory of János Radó, principal (my friend)

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following system of equations over the pairs of real numbers.

$$x^2 - y^2 = 2(x - y), \quad x^2 + y^2 = 5(x + y)$$

Solution: The first equation is $(x - y)(x + y - 2) = 0$.

If $x = y$: $2x^2 = 10x$, so $(0, 0)$ and $(5, 5)$.

If $x + y = 2$: $x^2 + (2 - x)^2 = 10$, that is, $x^2 - 2x - 3 = 0$, so $(3, -1)$ and $(-1, 3)$.

The solutions are $(0, 0)$, $(5, 5)$, $(3, -1)$, $(-1, 3)$.

2. The product of the first and fifth terms of a geometric sequence is 144, and the difference of the second and fourth terms is 18. Find the first term and the ratio of the sequence.

Solution: Let the first term be a and the ratio $q \neq 0$. Then $a \cdot aq^4 = (aq^2)^2 = 144$, so the third term is $aq^2 = \pm 12$, and $aq - aq^3 = aq^2 \left(\frac{1}{q} - q \right) = 18$.

If $aq^2 = 12$: $\frac{1}{q} - q = \frac{3}{2}$, $2q^2 + 3q - 2 = 0$, so $q = \frac{1}{2}$ ($a = 48$) or $q = -2$ ($a = 3$).

If $aq^2 = -12$: $q - \frac{1}{q} = \frac{3}{2}$, $2q^2 - 3q - 2 = 0$, so $q = 2$ ($a = -3$) or $q = -\frac{1}{2}$ ($a = -48$).

So $(a, q) = \left(48, \frac{1}{2} \right)$, $(3, -2)$, $(-3, 2)$, or $\left(-48, -\frac{1}{2} \right)$.

3. Find the equation of the circle that is tangent to the y -axis at the origin and is also tangent to the line $x + y = 1$.

Solution: The center is $(u, 0)$ and the radius is $|u|$. The circle is tangent to the line $x + y - 1 = 0$ if $\frac{|u - 1|}{\sqrt{2}} = |u|$, that

is, $u - 1 = \pm\sqrt{2}u$, so $u = \frac{1}{1 - \sqrt{2}} = -1 - \sqrt{2}$ or $u = \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1$. The circles are

$$x^2 + y^2 + (2 - 2\sqrt{2})x = 0 \quad \text{and} \quad x^2 + y^2 + (2 + 2\sqrt{2})x = 0$$

4. Find the values of the real parameter m for which the roots of the equation

$$2x^2 + (2 - 2m)x + (m^2 - 4m + 1) = 0$$

are real, and for these values of m find the largest and the smallest values of $x_1 \cdot x_2$, where x_1 and x_2 are the roots of the equation.

Solution: The roots are real if $(2 - 2m)^2 - 8(m^2 - 4m + 1) = -4(m^2 - 6m + 1) \geq 0$, that is, $3 - 2\sqrt{2} \leq m \leq 3 + 2\sqrt{2}$. By Viète's formulas

$$x_1 x_2 = \frac{m^2 - 4m + 1}{2} = \frac{(m - 2)^2 - 3}{2}$$

On this interval $m - 2$ runs from $1 - 2\sqrt{2}$ to $1 + 2\sqrt{2}$, so $0 \leq (m - 2)^2 \leq (1 + 2\sqrt{2})^2 = 9 + 4\sqrt{2}$. So the smallest value of $x_1 x_2$ is $-\frac{3}{2}$ (at $m = 2$), and the largest value is $3 + 2\sqrt{2}$ (at $m = 3 + 2\sqrt{2}$).

5. Let $x > 0$, $y > 0$, and $x + y = 4$. Prove that

$$\left(3 + \frac{1}{x}\right) \left(3 + \frac{1}{y}\right) \geq \left(\frac{7}{2}\right)^2$$

Solution: By the inequality between the arithmetic and geometric means, $\sqrt{xy} \leq \frac{x + y}{2} = 2$, so $xy \leq 4$. Then

$$\left(3 + \frac{1}{x}\right) \left(3 + \frac{1}{y}\right) = 9 + \frac{3(x + y) + 1}{xy} = 9 + \frac{13}{xy} \geq 9 + \frac{13}{4} = \frac{49}{4}$$

with equality for $x = y = 2$.

6. Two sides of a triangle are 12 and 24 units long, and the bisector of the angle between them is 8 units long. Find the angle between the two sides and the third side.

Solution: Let γ be the angle between the given sides. The bisector divides the triangle into two triangles; adding their areas:

$$\frac{1}{2} \cdot 12 \cdot 24 \sin \gamma = \frac{1}{2} (12 + 24) \cdot 8 \sin \frac{\gamma}{2}$$

With $\sin \gamma = 2 \sin \frac{\gamma}{2} \cos \frac{\gamma}{2}$ this gives $\cos \frac{\gamma}{2} = \frac{1}{2}$, so $\gamma = 120^\circ$. By the Law of Cosines, the third side is

$$\sqrt{144 + 576 + 288} = \sqrt{1008} = 12\sqrt{7} \approx 31.75 \text{ units}$$

7. A point P of the line of the side AB of the square $ABCD$ is 5 units from the vertex C and $\sqrt{17}$ units from the vertex D . What is the side of the square? Where is the point P ?

Solution: Let the side be a , and use coordinates: $A(0, 0)$, $B(a, 0)$, $C(a, a)$, $D(0, a)$, $P(x, 0)$. Then $x^2 + a^2 = 17$ and $(a - x)^2 + a^2 = 25$. Subtracting, $x = \frac{a^2 - 8}{2a}$, and substituting: $5a^4 - 84a^2 + 64 = 0$, so $a^2 = 16$ or $a^2 = \frac{4}{5}$.

If $a = 4$: $x = 1$, P is on the side AB , 1 unit from A .

If $a = \frac{2}{\sqrt{5}}$: $x = -\frac{9}{\sqrt{5}}$, P is on the extension of AB beyond A , $\frac{9}{\sqrt{5}} \approx 4.02$ units from A .

8. Solve the following equation over the pairs of real numbers.

$$4 \cos^2 x - 4 \sin x \cdot \sin y - 5 = 0$$

Solution: Using $\cos^2 x = 1 - \sin^2 x$, the equation becomes $4 \sin^2 x + 4 \sin x \sin y + 1 = 0$, that is,

$$(2 \sin x + \sin y)^2 + \cos^2 y = 0$$

So $\cos y = 0$ and $2 \sin x = -\sin y$.

If $y = \frac{\pi}{2} + 2k\pi$: $\sin x = -\frac{1}{2}$, so $x = -\frac{\pi}{6} + 2n\pi$ or $x = -\frac{5\pi}{6} + 2n\pi$.

If $y = -\frac{\pi}{2} + 2k\pi$: $\sin x = \frac{1}{2}$, so $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$.

(k and n are arbitrary integers.)