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In honor of the Juhász Gyula Teacher Training College, where I earned my first degree and where I first taught

translated from: KoMaL, March 2000, pages 152-153

Pdf file: KoMaL, March 2000, pages 152-153

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equation over the real numbers.

$$\sqrt{x^2 - 3x + 6} + \sqrt{x^2 - 3x + 3} = 3$$

Solution: Let $y = \sqrt{x^2 - 3x + 6}$ and $z = \sqrt{x^2 - 3x + 3}$ ($y > 0, z \geq 0$). Then $y + z = 3$ and $y^2 - z^2 = 3$, so $y - z = \frac{3}{y + z} = 1$. So $y = 2, z = 1$, and $x^2 - 3x + 3 = 1$, that is, $x^2 - 3x + 2 = 0$. The solutions are $x = 1$ and $x = 2$.

2. We deposit one million forints in a bank at an annual compound interest rate of 14%. After 3 years we withdraw one million forints. Then the interest rate is lowered to 12%. After this, at least how many full years do we have to wait for our money to grow to 600 000 forints?

Solution: After the withdrawal, $(1.14^3 - 1) \cdot 1\,000\,000 = 481\,544$ forints remain in the bank. After n more years we have

$$481\,544 \cdot 1.12^n \geq 600\,000, \quad 1.12^n \geq 1.246\dots, \quad n \geq \frac{\log 1.246\dots}{\log 1.12} \approx 1.94$$

We have to wait two full years. (Then there will be 604 049 forints in the bank.)

3. The center of a semicircle that is tangent to both legs of a right triangle is on the hypotenuse. What is the radius of the semicircle, if its center divides the hypotenuse into segments of lengths 1.8 and 2.4 units?

Solution: Draw the perpendiculars from the center of the semicircle to the legs. Their lengths are equal to the radius r , and they cut the triangle into a square and two similar right triangles, whose hypotenuses are 1.8 and 2.4. So the ratio of similarity is 3 : 4. In the smaller right triangle, the legs are r and $\frac{3}{4}r$, so its hypotenuse is $\frac{5}{4}r = 1.8$, and the radius of the semicircle is

$$r = \frac{4 \cdot 1.8}{5} = 1.44 \text{ units}$$

4. The equation of the line of the diagonal BD of the rhombus $ABCD$ is $2x - y = 3$, the side of the rhombus is 5 units long, and $A(6, 4)$. Find the coordinates of the other vertices and the area of the rhombus.

Solution: The points of the line BD are $(t, 2t - 3)$. B and D are the points of this line that are 5 units from A :

$$(t - 6)^2 + (2t - 7)^2 = 25, \quad t^2 - 8t + 12 = 0$$

so $t = 6$ or $t = 2$. So $B(6, 9)$ and $D(2, 1)$ (or the other way around). The diagonals of a rhombus bisect each other, so $C = B + D - A = (2, 6)$. The diagonals are $AC = \sqrt{20}$ and $BD = \sqrt{80}$, so the area is $\frac{\sqrt{20} \cdot \sqrt{80}}{2} = 20$ square units.

5. Solve the following system of equations over the pairs of real numbers.

$$\frac{x}{y} - \frac{y}{x} = \frac{8}{3}, \quad \log_2(x + y) - \log_2(x - y) = 1$$

Solution: We need $x \neq 0$, $y \neq 0$, $x + y > 0$, and $x - y > 0$. The second equation says $x + y = 2(x - y)$, that is, $x = 3y$. With $u = \frac{x}{y}$, the first equation is $3u^2 - 8u - 3 = 0$, so $u = 3$ or $u = -\frac{1}{3}$, and $u = 3$ is consistent with $x = 3y$. Then $x - y = 2y > 0$, so $y > 0$. The solutions are

$$x = 3t, \quad y = t, \quad t > 0$$

6. The base edge of a square pyramid is $2a$ and its height is a units. A cube is inscribed in the pyramid: its bottom face is on the base of the pyramid, and the vertices of its top face are on the lateral edges of the pyramid. How many times the volume of the cube is the volume of the pyramid?

Solution: Let the edge of the cube be x . Cut the pyramid with the plane through two opposite lateral edges. The cut is an isosceles triangle with base $2a\sqrt{2}$ and height a , and the cut of the cube is a rectangle with base $x\sqrt{2}$ and height x , inscribed in this triangle. By similar triangles

$$\frac{x\sqrt{2}}{2a\sqrt{2}} = \frac{a - x}{a}, \quad x = \frac{2}{3}a$$

The volume of the pyramid is $\frac{4a^2 \cdot a}{3} = \frac{4}{3}a^3$, the volume of the cube is $\frac{8}{27}a^3$, and $\frac{4}{3}a^3 : \frac{8}{27}a^3 = \frac{9}{2}$. The volume of the pyramid is 4.5 times the volume of the cube.

7. Prove that for every value of the real parameter p the equation $x|x - 2p| + p = 0$ has exactly one real root.

Solution: If $p = 0$, the equation is $x|x| = 0$, and its only solution is $x = 0$.

If $p > 0$, then $x|x - 2p| = -p < 0$, so only $x < 0$ can be a solution, and then $x - 2p < 0$. The equation becomes $x(2p - x) + p = 0$, that is, $x^2 - 2px - p = 0$. Its discriminant is $4p^2 + 4p > 0$, and the product of its roots is $-p < 0$, so it has one negative and one positive root. Only the negative root is a solution.

If $p < 0$, then only $x > 0$ can be a solution, and then $x - 2p > 0$. The equation becomes $x(x - 2p) + p = 0$, that is, $x^2 - 2px + p = 0$. Its discriminant is $4p^2 - 4p > 0$, and the product of its roots is $p < 0$, so exactly one root is positive, and only that root is a solution.

8. Let a , b , and c be the sides of a triangle. Prove that

$$\frac{1}{4} < \frac{ab + bc + ca}{(a + b + c)^2} \leq \frac{1}{3}$$

Solution: Multiplying by $(a + b + c)^2 > 0$ and expanding, the inequalities are equivalent to

$$ab + bc + ca \leq a^2 + b^2 + c^2 < 2ab + 2bc + 2ca$$

By the triangle inequality $0 \leq |a - b| < c$, $0 \leq |b - c| < a$, and $0 \leq |c - a| < b$. Squaring:

$$0 \leq a^2 + b^2 - 2ab < c^2, \quad 0 \leq b^2 + c^2 - 2bc < a^2, \quad 0 \leq c^2 + a^2 - 2ca < b^2$$

Adding these, $0 \leq 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca < a^2 + b^2 + c^2$, and rearranging gives exactly the two inequalities above.