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In memory of Professor Tibor Gallai

We recommend that students work through the problems of these practice sets with the help of a teacher, and that the problems be discussed in class or in a math club after they are solved.

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The sum of the first, third, and fifth terms of a geometric sequence is 126, and the sum of the second and fourth terms is 30 times the ratio. Find the first term and the ratio of the sequence.

Solution: Let the first term be a and the ratio q . Then $a(1 + q^2 + q^4) = 126$ and $aq + aq^3 = 30q$, that is, $q(a(1 + q^2) - 30) = 0$.

If $q = 0$ is allowed, then $a = 126$. If $q \neq 0$, then $a(1 + q^2) = 30$, and dividing the two equations:

$$\frac{1 + q^2 + q^4}{1 + q^2} = \frac{126}{30}, \quad 5q^4 - 16q^2 - 16 = 0$$

so $q^2 = 4$ ($q^2 = -\frac{4}{5}$ is impossible). So $q = 2$ or $q = -2$, and in both cases $a = 6$.

2. Prove that if $a > 0$, $b > 0$, $c > 0$, and $a + b + c = 3$, then

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} < 9$$

Solution: Since $a > 0$, $\sqrt{4a+1} < \sqrt{4a^2+4a+1} = 2a+1$. Similarly $\sqrt{4b+1} < 2b+1$ and $\sqrt{4c+1} < 2c+1$, so

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} < 2(a+b+c) + 3 = 9$$

3. In the rectangle $ABCD$, $AB = 20$ and $BC = 10$ units. There is a point P inside the rectangle from which both segments AB and AD are seen at a 90° angle. Find the distances of P from the vertices of the rectangle.

Solution: $\angle BPA + \angle DPA = 180^\circ$, so P is on the diagonal BD , and AP is the altitude of the right triangle ABD . Here $BD = \sqrt{500} = 10\sqrt{5}$. By the leg and altitude theorems

$$10^2 = PD \cdot BD, \quad 20^2 = PB \cdot BD, \quad PA^2 = PB \cdot PD$$

so $PD = 2\sqrt{5}$, $PB = 8\sqrt{5}$, and $PA = 4\sqrt{5}$. In the triangle PDC , the angle δ at D satisfies $\cos \delta = \frac{AB}{BD} = \frac{2}{\sqrt{5}}$, so by the Law of Cosines

$$PC^2 = (2\sqrt{5})^2 + 20^2 - 2 \cdot 2\sqrt{5} \cdot 20 \cdot \frac{2}{\sqrt{5}} = 260, \quad PC = 2\sqrt{65}$$

The distances are $PA = 4\sqrt{5} \approx 8.94$, $PB = 8\sqrt{5} \approx 17.89$, $PC = 2\sqrt{65} \approx 16.12$, and $PD = 2\sqrt{5} \approx 4.47$ units.

4. Solve the following equations over the real numbers.

a) $\log_3 (x-1)^2 + \log_3 (x+2)^2 = 2 \log_3 2$

b) $\log_3 (x-1)^2 + 2 \log_3 (x+2) = 2 \log_3 2$

c) $2 \log_3 (x-1) + \log_3 (x+2)^2 = 2 \log_3 2$

d) $2 \log_3 (x-1) + 2 \log_3 (x+2) = \log_3 4$

Solution: a) We need $x \neq 1$ and $x \neq -2$. The equation says $(x-1)^2 (x+2)^2 = 4$, that is, $(x-1)(x+2) = \pm 2$. From $x^2 + x - 2 = -2$ we get $x_1 = 0$ and $x_2 = -1$; from $x^2 + x - 2 = 2$ we get

$$x_3 = \frac{-1 + \sqrt{17}}{2} \approx 1.56, \quad x_4 = \frac{-1 - \sqrt{17}}{2} \approx -2.56$$

All four are solutions.

b) Here we need $x \neq 1$ and $x > -2$, so the solutions are x_1, x_2 , and x_3 .

c) and d) Here we need $x > 1$, so in both cases the only solution is $x_3 = \frac{-1 + \sqrt{17}}{2}$.

5. The sides of a triangle are a, b , and c , and its area is

$$T = \frac{\sqrt{3}}{4} (c^2 - a^2 - b^2)$$

Express the radius of the circle circumscribed around the triangle in terms of a and b .

Solution: Let γ be the angle opposite c . By the Law of Cosines $c^2 - a^2 - b^2 = -2ab \cos \gamma$, and $T = \frac{1}{2}ab \sin \gamma$, so

$$\frac{1}{2}ab \sin \gamma = -2 \cdot \frac{\sqrt{3}}{4}ab \cos \gamma, \quad \tan \gamma = -\sqrt{3}, \quad \gamma = 120^\circ$$

Then $c^2 = a^2 + b^2 - 2ab \cos 120^\circ = a^2 + ab + b^2$, and from $c = 2r \sin \gamma = \sqrt{3}r$ the radius is

$$r = \sqrt{\frac{a^2 + ab + b^2}{3}}$$

6. The roots of the equation $x^2 - a^2x - 2ab = 0$ are 2 greater than the roots of the equation $x^2 + 3ax - \frac{4}{3}b = 0$. Find the values of a and b , and the roots of the equations.

Solution: Let the roots of $x^2 + 3ax - \frac{4}{3}b = 0$ be x_1 and x_2 ; then the roots of $x^2 - a^2x - 2ab = 0$ are $x_1 + 2$ and $x_2 + 2$. By Viète's formulas

$$x_1 + x_2 = -3a, \quad x_1 x_2 = -\frac{4}{3}b, \quad (x_1 + 2) + (x_2 + 2) = a^2, \quad (x_1 + 2)(x_2 + 2) = -2ab$$

From the sums, $-3a + 4 = a^2$, so $a = 1$ or $a = -4$. From the products, $-\frac{4}{3}b - 6a + 4 = -2ab$, that is, $(3a - 2)(b - 3) = 0$, so $b = 3$.

If $a = 1, b = 3$: the equations are $x^2 + 3x - 4 = 0$ with roots -4 and 1 , and $x^2 - x - 6 = 0$ with roots -2 and 3 .

If $a = -4, b = 3$: the equations are $x^2 - 12x - 4 = 0$ with roots $6 \pm 2\sqrt{10}$, and $x^2 - 16x + 24 = 0$ with roots $8 \pm 2\sqrt{10}$.

7. Find the pairs of real numbers (x, y) with $0 \leq x \leq 2\pi$ and $0 \leq y \leq 2\pi$ that satisfy the following system of equations.

$$2 \cos^3 x + \sin y = 0, \quad 2 \sin^3 x + \cos y = 0$$

Solution: Write the equations as $2 \cos^3 x = -\sin y$ and $2 \sin^3 x = -\cos y$, square both, and add them:

$$4 (\sin^6 x + \cos^6 x) = 1$$

Since

$$\sin^6 x + \cos^6 x = (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x = 1 - \frac{3}{4} \sin^2 2x$$

we get $\sin^2 2x = 1$, so $x = \frac{\pi}{4} + \frac{k\pi}{2}$. In the interval $[0, 2\pi]$:

$$\text{If } x = \frac{\pi}{4}: \sin y = -\frac{1}{\sqrt{2}} \text{ and } \cos y = -\frac{1}{\sqrt{2}}, \text{ so } y = \frac{5\pi}{4}.$$

$$\text{If } x = \frac{3\pi}{4}: \sin y = \frac{1}{\sqrt{2}} \text{ and } \cos y = -\frac{1}{\sqrt{2}}, \text{ so } y = \frac{3\pi}{4}.$$

$$\text{If } x = \frac{5\pi}{4}: \sin y = \frac{1}{\sqrt{2}} \text{ and } \cos y = \frac{1}{\sqrt{2}}, \text{ so } y = \frac{\pi}{4}.$$

$$\text{If } x = \frac{7\pi}{4}: \sin y = -\frac{1}{\sqrt{2}} \text{ and } \cos y = \frac{1}{\sqrt{2}}, \text{ so } y = \frac{7\pi}{4}.$$

(Note: the KoMaL solution gives $y = \frac{5\pi}{4}$ in the second case; this is a slip.)

8. Two vertices of the triangle ABC are $A(7, 9)$ and $B(-1, 3)$, $BC = 4\sqrt{5}$ units, and the area of the triangle is 40 square units. Find the coordinates of the vertex C .

Solution: $AB = 10$, so the altitude from C is 8 units ($\frac{10 \cdot 8}{2} = 40$). The line AB has direction vector $(8, 6)$ and normal vector $(3, -4)$, so C is on a line $3x - 4y + k = 0$ that is 8 units from $B(-1, 3)$:

$$8 = \frac{|-3 - 12 + k|}{\sqrt{3^2 + 4^2}}, \quad k = 55 \text{ or } k = -25$$

C is also on the circle $(x+1)^2 + (y-3)^2 = 80$. The line $3x - 4y + 55 = 0$ meets this circle at $C_1(-9, 7)$ and $C_2(-2.6, 11.8)$, and the line $3x - 4y - 25 = 0$ meets it at $C_3(7, -1)$ and $C_4(0.6, -5.8)$. So there are four possible positions of C .