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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The diagonals of a parallelogram are 13 and 20 units long, and one of its sides is 10.5 units long. Find the area and the perimeter of the parallelogram.

Solution: The diagonals divide the parallelogram into four triangles of equal area. One of these triangles has sides  $\frac{13}{2}$ , 10, and  $\frac{21}{2}$ . By Heron's formula ( $t^2 = s(s-a)(s-b)(s-c)$ ), if the area of the parallelogram is  $T$ , then

$$\left(\frac{T}{4}\right)^2 = \frac{27}{2} \cdot \frac{14}{2} \cdot \frac{7}{2} \cdot \frac{6}{2}, \quad T^2 = (9 \cdot 7 \cdot 2)^2, \quad T = 126 \text{ square units}$$

In every parallelogram, the sum of the squares of the sides is equal to the sum of the squares of the diagonals. So if the other side is  $x$ , then

$$2x^2 + 2 \cdot 10.5^2 = 13^2 + 20^2, \quad x^2 = 174.25, \quad x = \frac{\sqrt{697}}{2} \approx 13.20$$

The perimeter is  $21 + \sqrt{697} \approx 47.40$  units.

2. Solve the following equation over the real numbers.

$$\log(x+9) + \log 4 = 2 - \frac{1}{2} \log(x-1)^2$$

(log denotes the base 10 logarithm.)

Solution: We need  $x > -9$  and  $x \neq 1$ . Since  $\frac{1}{2} \log(x-1)^2 = \log|x-1|$ , the equation is

$$\log(x+9) + \log|x-1| + \log 4 = \log 100, \quad (x+9)|x-1| = 25$$

If  $-9 < x < 1$ :  $x^2 + 8x - 9 = -25$ , that is,  $(x+4)^2 = 0$ , so  $x_1 = -4$ .

If  $x > 1$ :  $x^2 + 8x - 9 = 25$ , so  $x = -4 \pm 5\sqrt{2}$ , and only  $x_2 = -4 + 5\sqrt{2} \approx 3.07$  is greater than 1.

The solutions are  $x_1 = -4$  and  $x_2 = -4 + 5\sqrt{2}$ .

3. The vertices of a triangle are  $A(-2, 0)$ ,  $B(2, 0)$ , and  $C(0, 5)$ . Find the set of points  $P$  in the plane for which

$$PA^2 + PB^2 + PC^2 = 30$$

Solution: For  $P(x, y)$ :

$$(x+2)^2 + y^2 + (x-2)^2 + y^2 + x^2 + (y-5)^2 = 30, \quad x^2 + \left(y - \frac{5}{3}\right)^2 = \frac{16}{9}$$

The set of points is the circle with center  $\left(0, \frac{5}{3}\right)$  (the centroid of the triangle) and radius  $\frac{4}{3}$ .

4. For the geometric sequence  $(a_n)$ ,  $7(a_2 + a_3) + 2(a_1 + a_4) = 0$  and  $S_5 = 22$ . Find the first term and the ratio of the sequence.

Solution: Let the first term be  $a$  and the ratio  $q$ . Then

$$7(aq + aq^2) + 2(a + aq^3) = a(1+q)(7q + 2(1 - q + q^2)) = a(q+1)(2q^2 + 5q + 2) = 0$$

Since  $S_5 = 22$ ,  $a \neq 0$ , so  $q = -1$ ,  $q = -2$ , or  $q = -\frac{1}{2}$ .

Using  $S_5 = a \cdot \frac{q^5 - 1}{q - 1}$  for  $q \neq 1$ :

$$\begin{aligned} q = -1: \quad S_5 &= a = 22 \\ q = -2: \quad S_5 &= a \cdot \frac{(-2)^5 - 1}{-2 - 1} = 11a = 22, \text{ so } a = 2 \\ q = -\frac{1}{2}: \quad S_5 &= a \cdot \frac{\left(-\frac{1}{2}\right)^5 - 1}{-\frac{1}{2} - 1} = \frac{11}{16}a = 22, \text{ so } a = 32 \end{aligned}$$

So  $(a, q) = (22, -1)$ ,  $(2, -2)$ , or  $\left(32, -\frac{1}{2}\right)$ .

(Note: the KoMaL solution gives  $a = \frac{11}{16}$  in the third case; this is a slip.)

5. Find the exact value of  $\sin x$ , if  $\tan x + \cot x = -4$ .

Solution:  $\tan x + \cot x = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{2}{\sin 2x}$ , so  $\sin 2x = -\frac{1}{2}$ , and  $\cos 2x = \pm \frac{\sqrt{3}}{2}$ . Then

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{2 \mp \sqrt{3}}{4} = \left(\frac{\sqrt{6} \mp \sqrt{2}}{4}\right)^2$$

(since  $(\sqrt{6} \mp \sqrt{2})^2 = 8 \mp 4\sqrt{3}$ ). All four signs occur (for example, at  $x = \frac{7\pi}{12}, \frac{11\pi}{12}, -\frac{\pi}{12}, -\frac{5\pi}{12}$ ), so

$$\sin x = \pm \frac{\sqrt{6} + \sqrt{2}}{4} \quad \text{or} \quad \sin x = \pm \frac{\sqrt{6} - \sqrt{2}}{4}$$

6. Consider the function  $x \mapsto (p-1)x^2 + 2(p+2)x + p-2$ , where  $x$  is real and  $p$  is a real parameter. Find the value of  $p$  for which the largest value of the function is  $-6$ . Where does the function take this largest value?

Solution: If  $p = 1$ , the function is  $6x - 1$ , which has no largest value. If  $p \neq 1$ , the function has a largest value only if  $p < 1$ . Completing the square,

$$(p-1)x^2 + 2(p+2)x + p-2 = (p-1)\left(x + \frac{p+2}{p-1}\right)^2 + p-2 - \frac{(p+2)^2}{p-1}$$

so the largest value is  $p-2 - \frac{(p+2)^2}{p-1} = -6$ , that is,  $(p+4)(p-1) = (p+2)^2$ , so  $p = -8$ . The function takes its largest value at  $x_0 = -\frac{p+2}{p-1} = -\frac{2}{3}$ .

7. One root of the equation  $x^2 + px + q = 0$  is 4, and twice the other root is equal to the discriminant of the equation. Find  $p$  and  $q$ .

Solution: Let  $D$  be the discriminant. The roots are  $x_1 = 4$  and  $x_2 = \frac{D}{2}$ , so by Viète's formulas  $p = -\left(4 + \frac{D}{2}\right)$  and  $q = 4 \cdot \frac{D}{2} = 2D$ . Then

$$D = p^2 - 4q = \left(4 + \frac{D}{2}\right)^2 - 8D, \quad D^2 - 20D + 64 = 0$$

so  $D = 4$  or  $D = 16$ . The equation is  $x^2 - 6x + 8 = 0$  ( $p = -6, q = 8$ ) or  $x^2 - 12x + 32 = 0$  ( $p = -12, q = 32$ ).

8. A parabola whose axis is parallel to the  $y$ -axis is tangent to the  $x$ -axis and passes through the point  $A(3, 2)$ . A normal vector of the tangent line to the parabola at  $A$  is  $\mathbf{n}(4, 1)$ . Find the equation of the parabola.

Solution: Since the parabola is tangent to the  $x$ -axis, its equation is  $y = a(x-u)^2$ . It passes through  $A$ , so  $2 = a(3-u)^2$ . The tangent line at  $A$  is  $4x + y = 14$ , so the equation  $14 - 4x = a(x-u)^2$ , that is,

$$ax^2 - 2(au-2)x + au^2 - 14 = 0$$

has a double root, and its discriminant is 0:  $4(au-2)^2 - 4a(au^2-14) = 0$ , which gives  $a(7-2u) = -2$ , so  $a = \frac{2}{2u-7}$ . Then

$$2 = \frac{2}{2u-7} \cdot (3-u)^2, \quad (u-4)^2 = 0, \quad u = 4, \quad a = 2$$

The parabola is  $y = 2(x-4)^2$ .