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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The midpoints of the sides of a convex quadrilateral are the vertices of a rhombus. The side of the rhombus is 6 units, and one of its angles is 30° . Find the area of the quadrilateral. (How many such quadrilaterals are there?)

Solution: The sides of the midpoint quadrilateral are midlines of triangles formed by the sides and diagonals of the quadrilateral, so they are parallel to the diagonals and half as long. So the diagonals of the quadrilateral are both 12 units long, and the angle between them is 30° . The area of the quadrilateral is

$$T = \frac{12 \cdot 12 \cdot \sin 30^\circ}{2} = 36 \text{ square units}$$

There are infinitely many such quadrilaterals.

2. Solve the following inequality over the real numbers.

$$\log_x (-2x^2 - 2x + 4) \geq \log_x (4 - 3x)$$

Solution: The expressions are defined if $x > 0$, $x \neq 1$, $-2x^2 - 2x + 4 > 0$ (that is, $-2 < x < 1$), and $4 - 3x > 0$. So $0 < x < 1$. For such bases the logarithm function is decreasing, so the inequality is

$$-2x^2 - 2x + 4 \leq 4 - 3x, \quad 2x \left(x - \frac{1}{2} \right) \geq 0$$

The solution is $\frac{1}{2} \leq x < 1$.

3. The sides a, b, c_1 of one triangle satisfy

$$\frac{(a+b)^2 - c_1^2}{ab} = 2 - \sqrt{2}$$

and the sides a, b, c_2 of another triangle satisfy

$$\frac{c_2^2 - (a-b)^2}{ab} = 2 + \sqrt{2}$$

Is it true that the two triangles have the same area?

Solution: Let γ_1 and γ_2 be the angles opposite c_1 and c_2 . By the Law of Cosines, $c_1^2 = a^2 + b^2 - 2ab \cos \gamma_1$, so the first condition is

$$\frac{2ab + 2ab \cos \gamma_1}{ab} = 2 - \sqrt{2}, \quad \cos \gamma_1 = -\frac{\sqrt{2}}{2}, \quad \gamma_1 = 135^\circ$$

Similarly the second condition is

$$\frac{2ab - 2ab \cos \gamma_2}{ab} = 2 + \sqrt{2}, \quad \cos \gamma_2 = -\frac{\sqrt{2}}{2}, \quad \gamma_2 = 135^\circ$$

So the two triangles have two equal sides and equal included angles, so they are congruent, and of course their areas are equal. Yes, it is true.

4. Solve the following equation over the real numbers.

$$\sqrt{x+4+2\sqrt{x+3}} - \sqrt{2x+3-4\sqrt{2x-1}} = 3$$

Solution: The equation is defined if $x \geq \frac{1}{2}$. Notice that

$$x+4+2\sqrt{x+3} = (\sqrt{x+3}+1)^2 \quad \text{and} \quad 2x+3-4\sqrt{2x-1} = (\sqrt{2x-1}-2)^2$$

so the equation is $\sqrt{x+3}+1 - |\sqrt{2x-1}-2| = 3$, that is, $\sqrt{x+3} - |\sqrt{2x-1}-2| = 2$.

If $\sqrt{2x-1} \geq 2$, that is, $x \geq \frac{5}{2}$: $\sqrt{x+3} = \sqrt{2x-1}$, so $x_1 = 4$, and this is indeed a solution.

If $\sqrt{2x-1} < 2$, that is, $\frac{1}{2} \leq x < \frac{5}{2}$: $\sqrt{x+3} + \sqrt{2x-1} = 4$. Squaring twice, $x^2 - 104x + 208 = 0$, and the root in this interval is $x_2 = 52 - 8\sqrt{39} \approx 2.04$.

The solutions are $x_1 = 4$ and $x_2 = 52 - 8\sqrt{39}$.

5. Consider the function

$$x \mapsto f(x) = 3\sqrt{3} \sin^2 x + 6 \sin x \cos x - 3\sqrt{3} \cos^2 x, \quad x \in \mathbb{R}$$

Find the largest and the smallest value of the function, and the values of x where these extreme values are taken.

Solution: Using $\sin^2 x - \cos^2 x = -\cos 2x$ and $2 \sin x \cos x = \sin 2x$:

$$f(x) = 3 \sin 2x - 3\sqrt{3} \cos 2x = 6 \left(\frac{1}{2} \sin 2x - \frac{\sqrt{3}}{2} \cos 2x \right) = 6 \sin \left(2x - \frac{\pi}{3} \right)$$

The largest value of f is 6, at $x = \frac{5\pi}{12} + k\pi$, and the smallest value is -6 , at $x = \frac{11\pi}{12} + k\pi$ (k is an arbitrary integer).

6. Four numbers are four consecutive terms of an arithmetic sequence. If we add 1 to the first number, subtract 6 from the second, subtract 4 from the third, and add 4 to the fourth, then we get four consecutive terms of a geometric sequence. What are the four numbers?

Solution: Let the terms of the geometric sequence be a, aq, aq^2, aq^3 . Then the four terms of the arithmetic sequence are $a - 1, aq + 6, aq^2 + 4, aq^3 - 4$. In an arithmetic sequence, twice each term (except the first) is the sum of its neighbors, so

$$a - 1 + aq^2 + 4 = 2(aq + 6) \quad \text{and} \quad aq + 6 + aq^3 - 4 = 2(aq^2 + 4)$$

that is, $a(q - 1)^2 = 9$ and $aq(q - 1)^2 = 6$. So $q = \frac{2}{3}$ and $a = 81$. The four numbers are 80, 60, 40, 20.

7. The circle k is tangent to the x -axis, and is tangent to the line $e: 3x - 4y + 45 = 0$ at the point of e whose y -coordinate is $y_0 = 9$. Find the equation of the circle k .

Solution: The point of tangency on e is $E(x_0, 9)$ with $3x_0 - 36 + 45 = 0$, so $E(-3, 9)$. The line e meets the x -axis at $A(-15, 0)$, and $AE = \sqrt{12^2 + 9^2} = 15$. The two tangent segments from A to the circle are equal, so the point of tangency on the x -axis is $F_1(0, 0)$ or $F_2(-30, 0)$. The center is on the line through E perpendicular to e , which is $4x + 3y = 15$, and on the vertical line through F_1 or F_2 . So the center is $(0, 5)$ or $(-30, 45)$, and there are two circles:

$$x^2 + (y - 5)^2 = 25 \quad \text{and} \quad (x + 30)^2 + (y - 45)^2 = 2025$$

8. For which value of the real parameter a is the sum of the squares of the real roots of the equation $x^2 - 2x + (a - 1)^2 = 0$ smallest, and for which value is it largest? What are the smallest and largest values?

Solution: The roots are real if the discriminant is nonnegative: $4 - 4(a - 1)^2 \geq 0$, that is, $|a - 1| \leq 1$, so $0 \leq a \leq 2$. By Viète's formulas $x_1 + x_2 = 2$ and $x_1x_2 = (a - 1)^2$, so

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = 4 - 2(a - 1)^2$$

Since $0 \leq (a - 1)^2 \leq 1$, we get $2 \leq x_1^2 + x_2^2 \leq 4$. The smallest value is 2, at $a = 0$ and $a = 2$, and the largest value is 4, at $a = 1$.