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In memory of Ottó Steinfeld, mathematician

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. In the triangle ABC , $AC = 5.8$ and $BC = 5$ units. The altitude to the side AB is $CT = 4$ units. Find the area of the triangle.

Solution: Let T be the foot of the perpendicular from C to the line AB . By the Pythagorean Theorem

$$BT^2 = 5^2 - 4^2 = 3^2, \quad AT^2 = 5.8^2 - 4^2 = 4.2^2$$

There are two such triangles: if T is between A and B , then $AB = 4.2 + 3 = 7.2$; if not, then $AB = 4.2 - 3 = 1.2$ units. The area is $\frac{7.2 \cdot 4}{2} = 14.4$ or $\frac{1.2 \cdot 4}{2} = 2.4$ square units.

2. Solve the following system of equations over the pairs of real numbers.

$$x^2 = 2y + 8, \quad y^2 = 2x + 8$$

Solution: Subtracting the second equation from the first: $x^2 - y^2 = 2(y - x)$, that is, $(x - y)(x + y + 2) = 0$.

If $x = y$: $x^2 - 2x - 8 = 0$, so $(4, 4)$ and $(-2, -2)$.

If $x + y + 2 = 0$: $x^2 = 2(-2 - x) + 8$, that is, $x^2 + 2x - 4 = 0$, so $(-1 + \sqrt{5}, -1 - \sqrt{5})$ and $(-1 - \sqrt{5}, -1 + \sqrt{5})$.

3. In the triangle ABC the bisector of the interior angle at C meets the side AB at D . Find the angle of the triangle at C and the length of the side AB , if $AC = 5$, $BC = 12$, and $CD = \frac{60\sqrt{2}}{17}$ units.

Solution: Let γ be the angle at C . The bisector divides the triangle into two triangles; adding their areas:

$$\frac{1}{2} \cdot 5 \cdot 12 \sin \gamma = \frac{1}{2} (5 + 12) \cdot \frac{60\sqrt{2}}{17} \sin \frac{\gamma}{2}$$

With $\sin \gamma = 2 \sin \frac{\gamma}{2} \cos \frac{\gamma}{2}$ this gives $\cos \frac{\gamma}{2} = \frac{\sqrt{2}}{2}$, so $\gamma = 90^\circ$, and $AB = \sqrt{5^2 + 12^2} = 13$ units.

4. Solve the following inequalities over the real numbers.

a) $\sqrt{4x - x^2} > 6 - 2x$ b) $\log_{\frac{3}{4}} \log_{18} (x^2 - 63) \geq 0$

Solution: a) We need $4x - x^2 \geq 0$, that is, $0 \leq x \leq 4$. If $6 - 2x < 0$, that is, $3 < x \leq 4$, the inequality holds. If $0 \leq x \leq 3$, both sides are nonnegative, and squaring gives an equivalent inequality:

$$4x - x^2 > 36 - 24x + 4x^2, \quad 5x^2 - 28x + 36 < 0, \quad 2 < x < 3.6$$

so $2 < x \leq 3$ also works. The solution is $2 < x \leq 4$.

b) We need $x^2 - 63 > 0$ and $\log_{18} (x^2 - 63) > 0$. Since $\log_{\frac{3}{4}}$ is decreasing and $\log_{18}$ is increasing, the inequality is

$$0 < \log_{18} (x^2 - 63) \leq 1, \quad 1 < x^2 - 63 \leq 18, \quad 64 < x^2 \leq 81$$

The solution is $-9 \leq x < -8$ or $8 < x \leq 9$.

5. In the rectangle $ABCD$, $BC = 3AB$. The midpoints of the shorter sides are $E \left(\frac{7}{2}, -\frac{3}{2} \right)$ and $F \left(-\frac{11}{2}, \frac{3}{2} \right)$. Find the coordinates of the vertices.

Solution: Let E be the midpoint of AB and F the midpoint of DC . Then $\overrightarrow{FE} = (9, -3)$, and $|\overrightarrow{FE}| = BC = 3\sqrt{10}$, so $AB = \sqrt{10}$. The vectors $\overrightarrow{EA}, \overrightarrow{EB}, \overrightarrow{FD}, \overrightarrow{FC}$ are perpendicular to $\overrightarrow{FE}$, and their length is $\frac{\sqrt{10}}{2} = \frac{1}{6} |\overrightarrow{FE}|$, so

$$\overrightarrow{EA} = \overrightarrow{FD} = \left(\frac{1}{2}, \frac{3}{2} \right), \quad \overrightarrow{EB} = \overrightarrow{FC} = \left(-\frac{1}{2}, -\frac{3}{2} \right)$$

The vertices are $A(4, 0)$, $B(3, -3)$, $C(-6, 0)$, $D(-5, 3)$ (up to the labeling).

6. Find the positive integers a , b , and c for which

$$ab = a + 2b + 33$$

and the segments of lengths a , b , and c form a triangle whose three sides are all divisible by 3.

Solution: From the equation $a(b - 1) = 2b + 33$, so $b \neq 1$ and

$$a = \frac{2(b - 1) + 35}{b - 1} = 2 + \frac{35}{b - 1}$$

So $b - 1$ is a positive divisor of 35: $(a, b) = (37, 2)$, $(9, 6)$, $(7, 8)$, or $(3, 36)$. Both a and b must be divisible by 3, so $(a, b) = (9, 6)$ or $(3, 36)$. By the triangle inequality, $3 < c < 15$ in the first case and $33 < c < 39$ in the second, and c is divisible by 3. The solutions are

$$(a, b, c) = (9, 6, 6), (9, 6, 9), (9, 6, 12), (3, 36, 36)$$

7. What are the possible values of $2p - 2q$, if p and q are real numbers such that the roots of the equation $x^2 - px + q = 0$ are real, and the roots x_1 and x_2 satisfy $x_1^2 + x_2^2 = 4$?

Solution: By Viète's formulas $x_1 + x_2 = p$ and $x_1x_2 = q$, so $x_1^2 + x_2^2 = p^2 - 2q = 4$, and $2q = p^2 - 4$. The roots are real if

$$p^2 - 4q = p^2 - 2(p^2 - 4) = 8 - p^2 \geq 0, \quad -2\sqrt{2} \leq p \leq 2\sqrt{2}$$

and $2p - 2q = 2p - (p^2 - 4) = 5 - (p - 1)^2$. On this interval $p - 1$ runs from $-1 - 2\sqrt{2}$ to $-1 + 2\sqrt{2}$, so $0 \leq (p - 1)^2 \leq (1 + 2\sqrt{2})^2 = 9 + 4\sqrt{2}$. So

$$-4 - 4\sqrt{2} \leq 2p - 2q \leq 5$$

8. Solve the following equation, where p is a real parameter.

$$\frac{x - 2p}{x + 2p} = \frac{x - p}{x - 2p} - \frac{8p^2 - 3x}{x^2 - 4p^2}$$

Solution: We need $x \neq 2p$ and $x \neq -2p$. Multiplying by $x^2 - 4p^2$ and rearranging:

$$(x - 2p)^2 = (x - p)(x + 2p) - (8p^2 - 3x), \quad (5p + 3)x = 14p^2$$

If $p = -\frac{3}{5}$, there is no solution. Otherwise $x_0 = \frac{14p^2}{5p + 3}$, and it is a solution if $x_0 \neq \pm 2p$:

$$\frac{14p^2}{5p + 3} = 2p \iff p = 0 \text{ or } p = \frac{3}{2}, \quad \frac{14p^2}{5p + 3} = -2p \iff p = 0 \text{ or } p = -\frac{1}{4}$$

Summary: if p is $-\frac{3}{5}$, 0 , $\frac{3}{2}$, or $-\frac{1}{4}$, there is no solution; for every other p the only solution is $x = \frac{14p^2}{5p + 3}$.