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In memory of Zsigmond Kokits, mathematician, associate professor

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equations over the real numbers.

a) $\frac{x-3}{x-2} + \frac{10}{x+2} = \frac{44}{4-x^2}$

b) $\frac{x-13}{x-2} + \frac{10}{x+2} = \frac{44}{4-x^2}$

c) $\frac{x-4}{x-2} + \frac{10}{x+2} = \frac{44}{4-x^2}$

d) $\frac{x-8}{x-2} + \frac{10}{x+2} = \frac{44}{4-x^2}$

Solution: The equations are not defined for $x = 2$ and $x = -2$. Multiplying both sides by $x^2 - 4$ and rearranging:

a) $x^2 + 9x + 18 = 0$, so the solutions are $x_1 = -6$ and $x_2 = -3$.

b) $x^2 - x - 2 = 0$, so the solution is $x = -1$ ($x = 2$ is not a solution).

c) $x^2 + 8x + 16 = 0$, so the solution is $x = -4$ (a double root).

d) $x^2 + 4x + 8 = 0$, so there is no solution.

2. One side of a rectangle is four times as long as the other side. Find the ratio of the area of the rectangle to the area of the quadrilateral bounded by the angle bisectors of the rectangle, and the ratio of the diagonal of the rectangle to a side of this quadrilateral.

Solution: The angle bisectors of a rectangle bound a square. If the sides of the rectangle are a and $4a$, then the diagonal of the square is the difference of the sides, $3a$, so the side of the square is $b = \frac{3a}{\sqrt{2}}$ and its area is $\frac{9a^2}{2}$. The diagonal of the rectangle is $a\sqrt{17}$. So

$$\frac{4a^2}{\frac{9a^2}{2}} = \frac{8}{9} \quad \text{and} \quad \frac{a\sqrt{17}}{\frac{3a}{\sqrt{2}}} = \frac{\sqrt{34}}{3}$$

3. Find the smallest and the largest value of the expression $f(x) = \frac{3x^2 + 11x - 34}{x^2 + 4x - 12}$ on the interval $[-4, -1]$.

Solution: On this interval the denominator $(x - 2)(x + 6)$ is not zero, and

$$f(x) = \frac{(3x^2 + 12x - 36) + (-x + 2)}{x^2 + 4x - 12} = 3 - \frac{x - 2}{(x - 2)(x + 6)} = 3 - \frac{1}{x + 6}$$

If $-4 \leq x \leq -1$, then $2 \leq x + 6 \leq 5$, so $\frac{1}{5} \leq \frac{1}{x + 6} \leq \frac{1}{2}$, and

$$f(-4) = \frac{5}{2} \leq 3 - \frac{1}{x + 6} \leq \frac{14}{5} = f(-1)$$

The smallest value is $\frac{5}{2}$, and the largest value is $\frac{14}{5}$.

Note: In KoMaL, the numerator is printed as $3x^3 + 11x - 34$, but the published solution works with $3x^2 + 11x - 34$, so the printed x^3 is most likely a misprint.

4. Solve the following equation over the real numbers.

$$3 \tan \left(x + \frac{3\pi}{2} \right) = \tan 2x$$

Solution: If $x = \frac{\pi}{2} + n\pi$, then both sides are defined and both are 0 (since $\tan \left(x + \frac{3\pi}{2} \right) = \tan(2\pi + n\pi) = 0$ and $\tan 2x = \tan(\pi + 2n\pi) = 0$), so these are solutions. (Note that the identity $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$ cannot be used for these values.)

Now let $x \neq \frac{\pi}{2} + n\pi$. The left-hand side is defined only for $x \neq k\pi$, and then $\tan \left(x + \frac{3\pi}{2} \right) = \tan \left(x + \frac{\pi}{2} \right) = -\cot x = -\frac{1}{\tan x}$. So

$$-\frac{3}{\tan x} = \frac{2 \tan x}{1 - \tan^2 x}, \quad \tan^2 x = 3$$

so $\tan x = \sqrt{3}$ or $\tan x = -\sqrt{3}$. The solutions are

$$x = \frac{\pi}{2} + n\pi, \quad x = \frac{\pi}{3} + k\pi, \quad x = -\frac{\pi}{3} + m\pi$$

where n, k, m are arbitrary integers.

5. Prove that for every positive integer n

$$S_n = 4 \cdot 6 + 5 \cdot 13 + \dots + (n + 3)(7n - 1) = \frac{n}{6} (14n^2 + 81n + 49)$$

Solution: We use the well-known formulas

$$1 + 2 + \dots + n = \frac{n(n + 1)}{2} \quad \text{and} \quad 1^2 + 2^2 + \dots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

Since $(k + 3)(7k - 1) = 7k^2 + 20k - 3$,

$$\begin{aligned} S_n &= 7(1^2 + 2^2 + \dots + n^2) + 20(1 + 2 + \dots + n) - 3n = \frac{7n(n + 1)(2n + 1)}{6} + 10n(n + 1) - 3n \\ &= \frac{n}{6} (14n^2 + 21n + 7 + 60n + 60 - 18) = \frac{n}{6} (14n^2 + 81n + 49) \end{aligned}$$

(The statement can also be proved by induction.)

6. a) The discriminant of the equation $x^2 + px + q = 0$ (p, q real) is $D \geq 0$, and the roots of the equation are x_1 and x_2 . Prove that $(x_1 - x_2)^2 = D$.
- b) The real roots of the equation $x^2 - 2ax + 2a^2 - 3a = 0$ (a real) are x_1 and x_2 . For which value of the parameter a is $(x_1 - x_2)^2$ largest? What is this largest value?

Solution: a) By Viète's formulas $x_1 + x_2 = -p$ and $x_1x_2 = q$, so $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = p^2 - 4q = D$.

b) $D = 4a^2 - 4(2a^2 - 3a) = -4a(a - 3)$, so the roots are real if $0 \leq a \leq 3$. By part a)

$$(x_1 - x_2)^2 = D = -4a^2 + 12a = 9 - 4\left(a - \frac{3}{2}\right)^2 \leq 9$$

The largest value of $(x_1 - x_2)^2$ is 9, at $a = \frac{3}{2}$.

7. The vertices of the triangle ABC are $A(0, 16)$, $B(0, 0)$, and $C(3, 0)$. Find the equation of the line that passes through the point $P_0(-6, 0)$ and divides the triangle into two parts of equal area.

Solution: The line is not parallel to the y -axis, so its equation is $y = m(x + 6)$. It meets the y -axis at $D(0, 6m)$, and the line AC at $E(x_0, y_0)$. The area of the triangle ABC is 24, so the area of the triangle ADE must be 12, that is, $AD \cdot x_0 = 24$, where $AD = 16 - 6m$. The equation of the line AC is $16x + 3y = 48$, so

$$16x_0 + 3(mx_0 + 6m) = 48, \quad x_0 = 6 \cdot \frac{8 - 3m}{16 + 3m}$$

and

$$(16 - 6m) \cdot 6 \cdot \frac{8 - 3m}{16 + 3m} = 24, \quad 9m^2 - 54m + 32 = 0$$

so $m = \frac{2}{3}$ or $m = \frac{16}{3}$. For $m = \frac{16}{3}$ the point $D(0, 32)$ is not on the side AB , so this does not work. The line is

$$y = \frac{2}{3}x + 4$$

(it meets AB at $(0, 4)$ and AC at $(2, \frac{16}{3})$).

8. a) Prove that if $xy + yz + zx = 1$ (x, y, z real), then $x^2 + y^2 + z^2 \geq 1$.
- b) Consider the rectangular boxes whose space diagonal is 2 units long. Which of these boxes has the largest surface area, and what is this largest surface area?

Solution: a) Since

$$x^2 + y^2 + z^2 - (xy + yz + zx) = \frac{1}{2} \left((x - y)^2 + (y - z)^2 + (z - x)^2 \right) \geq 0$$

we have $x^2 + y^2 + z^2 \geq xy + yz + zx = 1$, with equality for $x = y = z$.

b) If the edges of the box are x, y, z , then $x^2 + y^2 + z^2 = 4$, and by the inequality in part a), the surface area is

$$A = 2(xy + yz + zx) \leq 2(x^2 + y^2 + z^2) = 8$$

with equality for $x = y = z$. So the largest surface area is 8 square units, for the cube (with edge $\frac{2}{\sqrt{3}}$ units, since $3x^2 = 4$).