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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Find the sides of a right triangle, if its area is $T = 43.2$ square units and the radius of its inscribed circle is $\varrho = 2.4$ units.

Solution: Let the legs be a and b , the hypotenuse c , and the perimeter $2s$. Then $ab = 2T = 86.4$, and $s = \frac{T}{\varrho} = 18$, so $2s = 36$.

Since the two tangent segments from each vertex are equal, the hypotenuse is $c = a + b - 2\varrho = a + b - 4.8$. So $2(a + b) - 4.8 = 36$, and $a + b = 20.4$. From $a + b = 20.4$ and $ab = 86.4$, the legs are 14.4 and 6 units, and the hypotenuse is $c = 20.4 - 4.8 = 15.6$ units.

2. Prove that the equation

$$2x^2 - 2(b + c)x + (ab + bc + ca - a^2) = 0$$

has a real solution for all real numbers a, b, c . When are the two solutions of the equation equal, and what is the solution then?

Solution: The discriminant of the equation is

$$D = 4(b + c)^2 - 4 \cdot 2 \cdot (ab + bc + ca - a^2) = 4((a - b)^2 + (a - c)^2) \geq 0$$

for all a, b, c , so there is always a real solution. The two solutions are equal if and only if $D = 0$, that is, $a = b = c$.

Then $x_1 = x_2 = \frac{b + c}{2} = a$.

3. The parallel sides of an isosceles trapezoid are a and b ($a > b$), and the diagonals make an angle α ($0^\circ < \alpha < 90^\circ$) with the parallel sides. Express the radius of the circle circumscribed around the trapezoid in terms of a, b , and α .

Solution: Let e be the diagonal, c the leg, and r the radius. The projection of a diagonal onto the longer base is $\frac{a + b}{2}$,

so $e \cos \alpha = \frac{a + b}{2}$. In the triangle formed by the longer base, a diagonal, and a leg, the angle opposite the leg is α , so $c = 2r \sin \alpha$, and by the Law of Cosines $c^2 = e^2 + a^2 - 2ea \cos \alpha = e^2 + a^2 - a(a + b) = e^2 - ab$. So

$$c^2 = \left(\frac{a + b}{2 \cos \alpha} \right)^2 - ab = \frac{a^2 + b^2 + 2ab - 4ab \cos^2 \alpha}{4 \cos^2 \alpha} = \frac{a^2 + b^2 - 2ab \cos 2\alpha}{4 \cos^2 \alpha}$$

and the radius of the circumscribed circle is

$$r = \frac{c}{2 \sin \alpha} = \frac{\sqrt{a^2 + b^2 - 2ab \cos 2\alpha}}{2 \sin 2\alpha}$$

4. The first term of a geometric sequence is 3 more than the ratio of the sequence, and there is a positive integer n for which the sum of the first n terms is 35, and the product of the first term and the n th term is 100. Find the first term, the ratio, and n .

Solution: By the conditions $a_1 = q + 3$, $S_n = 35$, and $a_1 \cdot a_n q^{n-1} = 100$. If $q = 1$, then $a_1 = 4$, every term is 4, and $a_1 \cdot a_n = 16 \neq 100$. So $q \neq 1$, $q - 1 = a_1 - 4$, and $a_1 q^n = \frac{100q}{a_1} = \frac{100}{a_1} (a_1 - 3)$. From $S_n = \frac{a_1 q^n - a_1}{q - 1} = 35$:

$$35(a_1 - 4) = \frac{100}{a_1}(a_1 - 3) - a_1, \quad 36a_1^2 - 240a_1 + 300 = 0, \quad a_1 = 5 \text{ or } a_1 = \frac{5}{3}$$

If $a_1 = 5$, then $q = 2$, and $25 \cdot 2^{n-1} = 100$ gives $n = 3$ (indeed $5 + 10 + 20 = 35$). If $a_1 = \frac{5}{3}$, then $q = -\frac{4}{3}$, and $\frac{25}{9} \left(-\frac{4}{3}\right)^{n-1} = 100$ has no solution n . So $a_1 = 5$, $q = 2$, and $n = 3$.

5. Find the equation of the line that passes through the point $P(5, 3)$ and meets the lines $3x + 2y = 16$ and $3x + 2y = 11$ at points whose x -coordinates differ by 1.

Solution: The line $x = 5$ through P does not work (both intersection points have x -coordinate 5). Every other line through P has an equation $y - 3 = m(x - 5)$, and $m \neq -\frac{3}{2}$, otherwise the line is parallel to the given lines. This line meets the first given line at $x_1 = \frac{10 + 10m}{3 + 2m}$ and the second at $x_2 = \frac{5 + 10m}{3 + 2m}$. The condition is

$$|x_1 - x_2| = \frac{5}{|3 + 2m|} = 1$$

so $m = 1$ or $m = -4$. There are two such lines: $x - y = 2$ and $4x + y = 23$.

6. Solve the following equation over the pairs of real numbers.

$$x^2 - 4x \sin(xy) + 4 = 0$$

Solution: Completing the square, and using $4 - 4 \sin^2(xy) = 4 \cos^2(xy)$:

$$(x - 2 \sin(xy))^2 + 4 \cos^2(xy) = 0$$

This holds if and only if $\cos(xy) = 0$ and $x = 2 \sin(xy)$.

If $\sin(xy) = 1$: $x = 2$ and $2y = \frac{\pi}{2} + 2n\pi$, so $y = \frac{\pi}{4} + n\pi$.

If $\sin(xy) = -1$: $x = -2$ and $-2y = \frac{3\pi}{2} + 2k\pi$, so $y = -\frac{3\pi}{4} + k\pi$.

The solutions are $\left(2, \frac{\pi}{4} + n\pi\right)$ and $\left(-2, -\frac{3\pi}{4} + k\pi\right)$, where n and k are arbitrary integers.

7. The lateral edges of a pyramid with a rectangular base are all equal, with length c , and the sides of the rectangle are a and b . At what distance from the apex should we place a plane parallel to the base so that the plane divides the pyramid into two parts of equal volume? Express the area of the cross section in terms of a , b , and c .

Solution: Let the height of the pyramid be m and the required distance x . The plane cuts off a pyramid similar to the original one, with ratio $\frac{x}{m}$, so its volume is $\left(\frac{x}{m}\right)^3$ times the original volume, and $\left(\frac{x}{m}\right)^3 = \frac{1}{2}$, that is, $x = \frac{m}{\sqrt[3]{2}}$.

The foot of the height is the center of the rectangle, so using the cross section through a diagonal of the base, $m^2 = c^2 - \frac{a^2 + b^2}{4}$ (the pyramid exists if $4c^2 > a^2 + b^2$). So

$$m = \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}, \quad x = \frac{\sqrt{4c^2 - a^2 - b^2}}{2\sqrt[3]{2}}$$

The cross section is a rectangle similar to the base, with ratio $\frac{1}{\sqrt[3]{2}}$, so its area is $t = \frac{ab}{\sqrt[3]{4}}$.

8. What are the possible values of $q - p$, if p and q are real numbers such that the roots of the equation $x^2 - px - q = 0$ are real, and the roots satisfy $x_1^2 + x_2^2 = \frac{9}{2}$?

Solution: The roots are real if $p^2 + 4q \geq 0$. By Viète's formulas $x_1 + x_2 = p$ and $x_1x_2 = -q$, so

$$\frac{9}{2} = x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = p^2 + 2q, \quad q = \frac{9}{4} - \frac{1}{2}p^2$$

Then $p^2 + 4q = 9 - p^2 \geq 0$ exactly when $-3 \leq p \leq 3$, and

$$q - p = \frac{9}{4} - \frac{1}{2}p^2 - p = \frac{11}{4} - \frac{1}{2}(p + 1)^2$$

On this interval $-2 \leq p + 1 \leq 4$, so $0 \leq (p + 1)^2 \leq 16$, and

$$-\frac{21}{4} \leq q - p \leq \frac{11}{4}$$

Note: In KoMaL, the condition was printed as $x_1^2 + x_2^2 = \frac{q}{2}$; the published solution corrected it to $\frac{9}{2}$.