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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time.

There are 180 minutes to solve and present all problems.

1. Solve the following equation over the real numbers.

$$\frac{2x+2}{7} = \frac{(x^2-x-6)(x+1)}{x^2+2x-3}$$

Solution: Since $x^2+2x-3 \neq 0$, we need $x \neq -3$ and $x \neq 1$. Factoring the quadratic expressions:

$$\frac{2x+2}{7} = \frac{(x-3)(x+2)(x+1)}{(x+3)(x-1)}$$

Multiplying by $7(x+3)(x-1)$: $2(x+3)(x-1)(x+1) = 7(x-3)(x+2)(x+1)$. Rearranging and factoring:

$$0 = (x+1)(5x^2-11x-36) = (x+1)(x-4)(5x+9)$$

The solutions are $x_1 = -1$, $x_2 = 4$, and $x_3 = -\frac{9}{5}$.

2. For which integers a is the value of the following expression also an integer?

$$\left(\frac{a+1}{1-a} + \frac{a-1}{a+1} - \frac{4a^2}{a^2-1} \right) : \left(\frac{2}{a^3+a^2} - \frac{2-2a+2a^2}{a^2} \right)$$

Solution: The denominators must not be zero, so $a \neq -1, 0, 1$. Then

$$\frac{a+1}{1-a} + \frac{a-1}{a+1} - \frac{4a^2}{a^2-1} = \frac{-(a+1)^2 + (a-1)^2 - 4a^2}{(a-1)(a+1)} = \frac{-4a(a+1)}{(a-1)(a+1)} = \frac{-4a}{a-1}$$

$$\frac{2}{a^3+a^2} - \frac{2-2a+2a^2}{a^2} = \frac{2 - (2-2a+2a^2)(a+1)}{a^2(a+1)} = \frac{-2a^3}{a^2(a+1)} = \frac{-2a}{a+1}$$

(the divisor is not zero), so the expression is

$$\frac{-4a}{a-1} \cdot \frac{a+1}{-2a} = \frac{2a+2}{a-1} = 2 + \frac{4}{a-1}$$

This is an integer if $a-1$ is a divisor of 4: $a-1 = -4, -2, -1, 1, 2, 4$, so $a = -3, -1, 0, 2, 3, 5$. Because of the conditions, the answer is $a = -3, 2, 3$, or 5 .

3. The second coordinates of the three points $A(1, a)$, $B(3, b)$, $C(4, c)$ are

$$a = -\frac{\sin 39^\circ + \sin 13^\circ}{\sin 26^\circ \cdot \cos 13^\circ}, \quad b = \sqrt{10^{2+\log 25}}, \quad c = \left(\frac{1}{\sqrt{5}-2}\right)^3 - \left(\frac{1}{\sqrt{5}+2}\right)^3$$

(log is the base 10 logarithm). Are these three points on one line?

Solution: Using $\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$:

$$a = -\frac{2 \sin 26^\circ \cos 13^\circ}{\sin 26^\circ \cos 13^\circ} = -2, \quad b = \sqrt{10^{\log 100 + \log 25}} = \sqrt{10^{\log 2500}} = \sqrt{2500} = 50$$

Since $\frac{1}{\sqrt{5}-2} = \sqrt{5}+2$ and $\frac{1}{\sqrt{5}+2} = \sqrt{5}-2$:

$$c = (\sqrt{5}+2)^3 - (\sqrt{5}-2)^3 = 2(3 \cdot 5 \cdot 2 + 2^3) = 76$$

So the points are $A(1, -2)$, $B(3, 50)$, $C(4, 76)$, and $\overrightarrow{AB} = (2, 52) = 2 \cdot (1, 26) = 2 \cdot \overrightarrow{BC}$. So A, B, C are on one line.

4. Which of the following two options is better for someone who wants to save money?

I. The bank pays 20% interest per year, and the yearly inflation is 15%.

II. The bank pays 12% interest per year, and the yearly inflation is 7%.

Solution: In case I, if we deposit x forints, after one year we have $1.2x$ forints. Taking inflation into account, this is worth $\frac{1.2x}{1.15} \approx 1.0435x$ forints.

In case II, after one year we have $1.12x$ forints, which is worth $\frac{1.12x}{1.07} \approx 1.0467x$ forints.

So option II is better.

5. The first four terms of an arithmetic sequence of integers are a_1, a_2, a_3, a_4 . Show that $1 \cdot a_1^2 + 2 \cdot a_2^2 + 3 \cdot a_3^2 + 4 \cdot a_4^2$ can be written as the sum of two perfect squares.

Solution: Let the first term be a and the difference d (both integers). Then

$$a^2 + 2(a+d)^2 + 3(a+2d)^2 + 4(a+3d)^2 = 10a^2 + 40ad + 50d^2 = (a+5d)^2 + (3a+5d)^2$$

6. Let ABC be an acute triangle. The circle with diameter AC meets the line of the altitude from B at D and E , and the circle with diameter AB meets the line of the altitude from C at F and G . Show that the points D, E, F, G are on one circle.

Solution: Let A' be the foot of the altitude from A , and M the orthocenter. By Thales' theorem, both circles pass through A and A' (since $\angle AA'C = \angle AA'B = 90^\circ$), so AA' is a common chord of the two circles, and M is on this chord. By the intersecting chords theorem,

$$MD \cdot ME = MA \cdot MA' \quad \text{and} \quad MF \cdot MG = MA \cdot MA'$$

So $MD \cdot ME = MF \cdot MG$, where M is between D and E and between F and G . By the converse of the intersecting chords theorem, D, E, F, G are on one circle.

7. The base of a right prism is the triangle ABC with $AB = 21$ cm, $BC = 20$ cm, and $CA = 13$ cm. We mark the points A' , B' , C' on the lateral edges so that $AA' = 5$ cm, $BB' = 25$ cm, and $CC' = 4$ cm. What is the angle between the plane of the triangle $A'B'C'$ and the plane of the triangle ABC ?

Solution: By Heron's formula, the area of the triangle ABC is

$$t_{ABC} = \sqrt{27(27-20)(27-13)(27-21)} = \sqrt{15876} = 126 \text{ cm}^2$$

By the Pythagorean Theorem,

$$A'B' = \sqrt{21^2 + 20^2} = 29 \text{ cm}, \quad B'C' = \sqrt{20^2 + 21^2} = 29 \text{ cm}, \quad C'A' = \sqrt{13^2 + 1^2} = \sqrt{170} \text{ cm}$$

The triangle $A'B'C'$ is isosceles; its altitude to the base $C'A'$ and its area are

$$m = \sqrt{29^2 - \frac{170}{4}} = \sqrt{798.5} \text{ cm}, \quad t_{A'B'C'} = \frac{\sqrt{170} \cdot \sqrt{798.5}}{2} = \frac{\sqrt{135745}}{2} \text{ cm}^2$$

The triangle ABC is the orthogonal projection of $A'B'C'$ onto the base plane, so if the angle of the planes is α , then $t_{ABC} = t_{A'B'C'} \cdot \cos \alpha$:

$$\cos \alpha = \frac{252}{\sqrt{135745}} \approx 0.6840, \quad \alpha \approx 46.8^\circ$$

8. Let $f(x) = 2x^6 - 3x^4 + x^2$. Prove that $f(\sin \alpha) + f(\cos \alpha) = 0$.

Solution: $f(x) = x^2(2x^4 - 3x^2 + 1) = x^2(x^2 - 1)(2x^2 - 1)$. Since $\sin^2 \alpha - 1 = -\cos^2 \alpha$ and $\cos^2 \alpha - 1 = -\sin^2 \alpha$,

$$\begin{aligned} f(\sin \alpha) + f(\cos \alpha) &= -\sin^2 \alpha \cos^2 \alpha (2\sin^2 \alpha - 1) - \cos^2 \alpha \sin^2 \alpha (2\cos^2 \alpha - 1) \\ &= -\sin^2 \alpha \cos^2 \alpha (2\sin^2 \alpha + 2\cos^2 \alpha - 2) = 0 \end{aligned}$$

because $\sin^2 \alpha + \cos^2 \alpha = 1$.