

author: Imre Rábai

In memory of Géza Fodor, mathematician, university professor

translated from: KoMaL, April 2002, page 199

Pdf file: KoMaL, April 2002, page 199

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The sides of a rectangle are 15.6 and 18 units long. We inscribe a rectangle in it, one side of which is three times as long as the other side, and which has exactly one vertex on each side of the given rectangle. Find the sides of the inscribed rectangle.

Solution: Let the shorter side of the inscribed rectangle be a ; then the other side is $3a$. The inscribed rectangle cuts off four right triangles, two congruent ones with hypotenuse a and two congruent ones with hypotenuse $3a$. These triangles are similar, so if the legs of the smaller one are x and y , then the legs of the larger one are $3x$ and $3y$. The sides of the given rectangle are

$$x + 3y = 15.6, \quad 3x + y = 18$$

so $x = 4.8$ and $y = 3.6$, and $a^2 = 4.8^2 + 3.6^2 = 36$, $a = 6$. The sides of the inscribed rectangle are 6 and 18 units.

2. a) Prove that in every triangle $\cot \alpha \sin \gamma + \cos \gamma = \frac{b}{a}$, where α is the angle opposite a and γ is the angle opposite c .
b) In a triangle $b = 2a$ and $\gamma = 60^\circ$. Find the other two angles of the triangle.

Solution: a) Since $\sin \beta = \sin (180^\circ - (\alpha + \gamma)) = \sin (\alpha + \gamma)$, by the Law of Sines

$$\frac{b}{a} = \frac{\sin \beta}{\sin \alpha} = \frac{\sin \alpha \cos \gamma + \cos \alpha \sin \gamma}{\sin \alpha} = \cos \gamma + \cot \alpha \sin \gamma$$

- b) Here $\frac{b}{a} = 2$, $\cos 60^\circ = \frac{1}{2}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$, so $2 = \frac{1}{2} + \frac{\sqrt{3}}{2} \cot \alpha$, so $\cot \alpha = \sqrt{3}$. So $\alpha = 30^\circ$ and $\beta = 90^\circ$.

3. For the geometric sequence (a_n) , $a_1 + a_3 + a_5 = 133$ and $\frac{1}{a_1} + \frac{1}{a_3} + \frac{1}{a_5} = \frac{133}{1296}$. Write down the first five terms of the sequence.

Solution: The conditions are $a_1 (1 + q^2 + q^4) = 133$ and

$$\frac{1}{a_1} + \frac{1}{a_1 q^2} + \frac{1}{a_1 q^4} = \frac{1 + q^2 + q^4}{a_1 q^4} = \frac{133}{1296}$$

Using the first equation, $\frac{133}{a_1^2 q^4} = \frac{133}{1296}$, so $(a_1 q^2)^2 = 36^2$. If $a_1 q^2 = -36$, the first equation gives $36q^4 + 169q^2 + 36 = 0$, which has no solution. If $a_1 q^2 = 36$, it gives $36q^4 - 97q^2 + 36 = 0$, so $q^2 = \frac{9}{4}$ ($a_1 = 16$) or $q^2 = \frac{4}{9}$ ($a_1 = 81$). The first five terms are

$$16, 24, 36, 54, 81, \text{ or } 16, -24, 36, -54, 81, \text{ or } 81, 54, 36, 24, 16, \text{ or } 81, -54, 36, -24, 16$$

4. We deposit 40 000 forints in a bank for 3 years at compound interest. In the first year the bank pays 10% interest; in the second year it increases the interest rate by $p\%$, and in the third year it increases it by another $p\%$ (that is, by p percentage points each time). This way, at the end of the third year, the bank pays 2939.20 forints more than it would have paid at 10% interest for all three years. Find the value of p .

Solution: By the conditions

$$40\,000 \cdot 1.1 \cdot \left(1 + \frac{10+p}{100}\right) \left(1 + \frac{10+2p}{100}\right) = 40\,000 \cdot 1.1^3 + 2939.20$$

so $\left(1.1 + \frac{p}{100}\right) \left(1.1 + \frac{2p}{100}\right) = 1.2768$. With $z = \frac{p}{100}$ this is $2z^2 + 3.3z - 0.0668 = 0$, and since $z > 0$, $z = 0.02$. So $p = 2$.

5. The vertex A of the quadrilateral $ABCD$ in the coordinate plane is at the origin, B is on the y -axis, C is in the first quadrant, and D is on the positive half of the x -axis. The equation of the line of the side BC is $2x + y = 12$. The side CD is perpendicular to the side BC . The area of the quadrilateral is 31 square units. Find the coordinates of the vertices B , C , and D .

Solution: B is on the y -axis, so $B(0, 12)$. Let $D(d, 0)$. The line CD is $x - 2y = d$, and its intersection with BC is

$$C\left(\frac{24+d}{5}, \frac{2(6-d)}{5}\right)$$

Since C is in the first quadrant, $d < 6$. The line BC meets the x -axis at $E(6, 0)$. The area of the triangle ABE is 36, and the area of the triangle CDE is $\frac{1}{2}(6-d) \cdot \frac{2}{5}(6-d)$, so

$$36 = 31 + \frac{1}{5}(d-6)^2, \quad (d-6)^2 = 25$$

so $d = 1$ or $d = 11$, and only $d = 1$ is possible. So $B(0, 12)$, $C(5, 2)$, $D(1, 0)$.

6. The radii of the bottom and top circles of a frustum of a cone are R and r ($R > r$). A plane parallel to the bases divides the frustum into two parts so that the frustum next to the larger base has one third of the volume of the original frustum. Express the radius of the cross section in terms of R and r .

Solution: Let the volume of the frustum be $3V$, the volume of the small cone that completes it to a full cone be V_1 , and the radius of the cross section be x . The part next to the larger base has volume V , so the cone cut off by the plane has volume $V_1 + 2V$. The volumes of similar solids are in the ratio of the cubes of corresponding lengths, so

$$1 + 3 \cdot \frac{V}{V_1} = \frac{R^3}{r^3}, \quad 1 + 2 \cdot \frac{V}{V_1} = \frac{x^3}{r^3}$$

Eliminating $\frac{V}{V_1} = \frac{R^3 - r^3}{3r^3}$: $x^3 = r^3 + \frac{2}{3}(R^3 - r^3)$, so

$$x = \sqrt[3]{\frac{r^3 + 2R^3}{3}}$$

7. Prove that the equation $(\sin x - \sqrt{3} \cos x) \cdot \cos 4x = 2$ has no solution over the real numbers.

Solution: Dividing by 2, and using $\frac{1}{2} = \cos \frac{\pi}{3}$ and $\frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}$, the equation is $\sin \left(x - \frac{\pi}{3}\right) \cdot \cos 4x = 1$. This holds only if $\sin \left(x - \frac{\pi}{3}\right) = \cos 4x = 1$ or $\sin \left(x - \frac{\pi}{3}\right) = \cos 4x = -1$.

In the first case $x = \frac{5\pi}{6} + 2k\pi$ and $4x = 2n\pi$, so $6x = 5\pi + 12k\pi = 3n\pi$, that is, $3n - 12k = 5$. This has no integer solution, since the left-hand side is divisible by 3 and the right-hand side is not.

In the second case $x = \frac{11\pi}{6} + 2k\pi$ and $4x = \pi + 2n\pi$, so $12x = 22\pi + 24k\pi = 3\pi + 6n\pi$, that is, $6n - 24k = 19$. This has no integer solution, since the left-hand side is even and the right-hand side is odd.

So the equation has no solution.

8. Consider the equation $4^x + 2(n+1) \cdot 2^x + n^2 = 8$, where n is an integer parameter.

a) For which n does the equation have two different real solutions?

b) For which n does the equation have exactly one solution?

Find the solutions too.

Solution: With $t = 2^x > 0$ the equation is $t^2 + 2(n+1)t + n^2 - 8 = 0$. One quarter of its discriminant is $(n+1)^2 - (n^2 - 8) = 2n + 9$.

a) We need two different positive roots t : $2n + 9 > 0$, the sum $-2(n+1) > 0$, and the product $n^2 - 8 > 0$. So $-\frac{9}{2} < n < -\sqrt{8}$, that is, $n = -4$ or $n = -3$.

If $n = -4$: $t^2 - 6t + 8 = 0$, $t = 4$ or $t = 2$, so $x_1 = 2$, $x_2 = 1$.

If $n = -3$: $t^2 - 4t + 1 = 0$, $t = 2 \pm \sqrt{3}$, so $x_3 = \log_2(2 + \sqrt{3})$, $x_4 = \log_2(2 - \sqrt{3})$.

b) There is exactly one solution if the quadratic has a double positive root, or one positive and one negative root. The first is impossible, since $2n + 9 = 0$ gives $n = -\frac{9}{2}$. The second happens if the product $n^2 - 8 < 0$ (then $2n + 9 > 0$ automatically), that is, $n = -2, -1, 0, 1, 2$. The positive root is $t = -(n+1) + \sqrt{2n+9}$:

$$n = -2: \quad 2^x = 1 + \sqrt{5}, \quad x_5 = \log_2(1 + \sqrt{5})$$

$$n = -1: \quad 2^x = \sqrt{7}, \quad x_6 = \log_2 \sqrt{7} = \frac{1}{2} \log_2 7$$

$$n = 0: \quad 2^x = 2, \quad x_7 = 1$$

$$n = 1: \quad 2^x = -2 + \sqrt{11}, \quad x_8 = \log_2(-2 + \sqrt{11})$$

$$n = 2: \quad 2^x = -3 + \sqrt{13}, \quad x_9 = \log_2(-3 + \sqrt{13})$$

(Note: the KoMaL solution gives $2^x = \sqrt{2}$ for $n = -1$; this is a slip, since for $n = -1$ the equation is $t^2 - 7 = 0$.)