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In honor of György Ágoston, university professor (pedagogy)

translated from: KoMaL, December 2002, page 523

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The terms of the sequence (a_n) are the positive integers that give remainder 5 when divided by 8, and the terms of the sequence (b_n) are the positive integers that give remainder 1 when divided by 6 (both in increasing order). Let S_n and T_n be the sums of the first n terms of (a_n) and (b_n) .

a) For which n is $\frac{S_n}{T_n} = \frac{25}{16}$?

b) What is the smallest n for which $S_n - T_n > 1000$?

Solution: $a_n = 5 + (n - 1) \cdot 8 = 8n - 3$ and $b_n = 1 + (n - 1) \cdot 6 = 6n - 5$, so

$$S_n = \frac{5 + (8n - 3)}{2} \cdot n = 4n^2 + n, \quad T_n = \frac{1 + (6n - 5)}{2} \cdot n = 3n^2 - 2n$$

a) $\frac{n(4n + 1)}{n(3n - 2)} = \frac{25}{16}$ gives $64n + 16 = 75n - 50$, so $n = 6$.

b) $S_n - T_n = n^2 + 3n > 1000$, and for $n > 0$ this means $n > \frac{-3 + \sqrt{4009}}{2} \approx 30.2$. The smallest such n is 31.

2. The position vectors of the points A and B are $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point of trisection of the segment OA closer to O is D , the point of trisection of the segment AB closer to A is E , and the lines BD and OE meet at P . Express the vectors $\overrightarrow{OP}$ and $\overrightarrow{PD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Solution: $\overrightarrow{OD} = \frac{1}{3}\mathbf{a}$, so $\overrightarrow{BD} = \frac{1}{3}\mathbf{a} - \mathbf{b}$, and $\overrightarrow{OE} = \frac{1}{3}(2\mathbf{a} + \mathbf{b})$. Since P is on both lines, $\overrightarrow{OP} = \alpha \cdot \frac{1}{3}(2\mathbf{a} + \mathbf{b})$ and $\overrightarrow{PD} = \beta \left(\frac{1}{3}\mathbf{a} - \mathbf{b} \right)$ for some real α, β , and $\overrightarrow{OD} = \overrightarrow{OP} + \overrightarrow{PD}$:

$$\frac{1}{3}\mathbf{a} = \frac{\alpha}{3}(2\mathbf{a} + \mathbf{b}) + \beta \left(\frac{1}{3}\mathbf{a} - \mathbf{b} \right), \quad (1 - 2\alpha - \beta)\mathbf{a} = (\alpha - 3\beta)\mathbf{b}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, both coefficients are 0: $2\alpha + \beta = 1$ and $\alpha = 3\beta$, so $\beta = \frac{1}{7}$ and $\alpha = \frac{3}{7}$. So

$$\overrightarrow{OP} = \frac{1}{7}(2\mathbf{a} + \mathbf{b}), \quad \overrightarrow{PD} = \frac{1}{21}\mathbf{a} - \frac{1}{7}\mathbf{b}$$

3. Solve the following equations over the real numbers.

a) $\sqrt{1 - 4x + 4x^2} + 2x - 1 = 0$ b) $x^{\log_3 x} = \sqrt[9]{81}$ c) $\sin 2x + 7 \cos 2x = 1$

Solution: a) $\sqrt{1 - 4x + 4x^2} = \sqrt{(1 - 2x)^2} = |1 - 2x|$, and $|1 - 2x| = 1 - 2x$ exactly when $1 - 2x \geq 0$. The solutions are all $x \leq \frac{1}{2}$.

b) Taking the base 3 logarithm of both sides, and using $\sqrt[9]{81} = 3^{\frac{4}{9}}$: $(\log_3 x)^2 = \frac{4}{9}$, so $\log_3 x = \pm \frac{2}{3}$. The solutions are $x_1 = \sqrt[3]{9}$ and $x_2 = \sqrt[3]{\frac{1}{9}}$.

c) $\sin 2x$ and $\cos 2x$ satisfy $\sin 2x = 1 - 7 \cos 2x$ and $\sin^2 2x + \cos^2 2x = 1$, so $50 \cos^2 2x - 14 \cos 2x = 0$. So $\cos 2x = 0$ and $\sin 2x = 1$, or $\cos 2x = 0.28$ and $\sin 2x = -0.96$. The solutions are

$$x = \frac{\pi}{4} + k\pi \quad \text{and} \quad x = -\frac{1}{2} \arccos 0.28 + k\pi \approx -0.64 + k\pi$$

(Note: the KoMaL solution gives $2x = 1.28 + 2k\pi$ in the second case; the sign is a slip, since $\sin 2x$ is negative there.)

4. Consider the regular square pyramid all of whose edges are equal, and the radius of whose inscribed sphere is $\varrho = 1$ unit. Find the length of the edges.

Solution: Consider the cross section through the apex and the midpoints of two opposite base edges. It is an isosceles triangle, and the circle inscribed in it is a great circle of the inscribed sphere. If the edge is a , then the base of this triangle is a , and its legs are the altitudes of the lateral faces (equilateral triangles), $\frac{a\sqrt{3}}{2}$. Its height m satisfies $m^2 = \frac{3a^2}{4} - \frac{a^2}{4} = \frac{a^2}{2}$, so $m = \frac{a\sqrt{2}}{2}$. Its area is $T = \frac{a^2\sqrt{2}}{4}$, and its semiperimeter is $s = \frac{a}{2} + \frac{a\sqrt{3}}{2}$. Since $\varrho = \frac{T}{s}$:

$$1 = \frac{\frac{a^2\sqrt{2}}{4}}{\frac{a}{2}(1 + \sqrt{3})}, \quad a = \sqrt{2} \left(1 + \sqrt{3}\right) \approx 3.86 \text{ units}$$

5. In the triangle ABC the angle at A is twice the angle at B , $6AB = 5BC$, and the area of the triangle is 62.5 times the sine of the angle at A . Find the sides of the triangle.

Solution: $\frac{AB}{BC} = \frac{5}{6}$, so let $AB = 5x$, $BC = 6x$, and $AC = b$. Let $\angle ABC = \beta$; then $\angle CAB = 2\beta$. Extend the side AB beyond A by $AD = b$. The angle CAB is an exterior angle of the isosceles triangle DAC , so $\angle CDA = \angle DCA = \beta$. So the triangles DBC and DCA (isosceles, each with two angles β) are similar, and

$$\frac{b + 5x}{6x} = \frac{6x}{b}, \quad b^2 + 5xb - 36x^2 = 0$$

so $b = 4x$ (since $b > 0$). By the condition on the area,

$$\frac{4x \cdot 5x}{2} \cdot \sin \alpha = 62.5 \cdot \sin \alpha, \quad x^2 = 6.25, \quad x = 2.5$$

The sides are $AB = 12.5$, $AC = 10$, and $BC = 15$ units.

6. Find the equation of the circle that passes through the point $P(-1, -1)$ and is tangent to the lines $x + 2y - 6 = 0$ and $2x - y + 3 = 0$.

Solution: The two lines are perpendicular and meet at $Q(0, 3)$. Let the center be $K(u, v)$ and the radius r . K is equally far from the two lines:

$$\frac{|u + 2v - 6|}{\sqrt{5}} = \frac{|2u - v + 3|}{\sqrt{5}}$$

so $u - 3v = -9$ or $3u + v = 3$. The point P is in the angle region of the bisector $3u + v = 3$ (on the other bisector the resulting quadratic equation has no real solution), so $v = 3 - 3u$. The distance of $K(u, 3 - 3u)$ from the line $2x - y + 3 = 0$ is $r = \frac{|5u|}{\sqrt{5}}$, so $r^2 = 5u^2$. Since P is on the circle,

$$(-1 - u)^2 + (-1 - (3 - 3u))^2 = 5u^2, \quad 5u^2 - 22u + 17 = 0$$

so $u = 1$ or $u = \frac{17}{5}$. There are two such circles:

$$(x - 1)^2 + y^2 = 5 \quad \text{and} \quad \left(x - \frac{17}{5}\right)^2 + \left(y + \frac{36}{5}\right)^2 = \frac{289}{5}$$

7. Solve the following equations over the real numbers, where a is a real parameter.

$$\text{a) } (a + 2) \cdot y + \frac{a}{y - 2} = 4a + 6 \quad \text{b) } (a + 2) \cdot 3^x + \frac{a}{3^x - 2} = 4a + 6$$

Solution: a) $y = 2$ cannot be a solution. Multiplying by $y - 2$ and rearranging:

$$(a + 2)y^2 - (6a + 10)y + 9a + 12 = 0$$

If $a = -2$, this is $2y - 6 = 0$, so the solution is $y = 3$. If $a \neq -2$, the roots are $y_1 = 3$ and $y_2 = \frac{3a + 4}{a + 2}$. y_1 is always a solution, and $y_2 \neq 2$ exactly when $a \neq 0$. So for $a = -2$ and $a = 0$ the only solution is $y = 3$; for every other a the solutions are 3 and $\frac{3a + 4}{a + 2}$.

b) With $y = 3^x$ this is the equation in a). $3^x = 3$ gives the solution $x = 1$ for every a . $3^x = \frac{3a + 4}{a + 2}$ gives another solution if $a \neq -2$, $\frac{3a + 4}{a + 2} > 0$, and $\frac{3a + 4}{a + 2} \neq 2$, that is, $x_2 = \log_3 \frac{3a + 4}{a + 2}$ is a solution if $a < -2$, or $-\frac{4}{3} < a < 0$, or $a > 0$.

8. Which pairs of real numbers satisfy the following equation?

$$x\sqrt{\log_x y} = y\sqrt{\log_y x}$$

Solution: We need $x > 0$, $x \neq 1$, $y > 0$, $y \neq 1$, and also $\log_x y \geq 0$ and $\log_y x \geq 0$. This holds if $x > 1$ and $y > 1$, or $0 < x < 1$ and $0 < y < 1$.

For such pairs, take the base x logarithm of both sides (an equivalent step). Using $\log_x y > 0$ and $\log_y x \cdot \log_x y = 1$:

$$\log_x x\sqrt{\log_x y} = \sqrt{\log_x y}, \quad \log_x y\sqrt{\log_y x} = \sqrt{\log_y x \cdot \log_x y} = \sqrt{\log_y x \cdot (\log_x y)^2} = \sqrt{\log_x y}$$

So the equation holds for every such pair: the solutions are the pairs with $x > 1$ and $y > 1$, and the pairs with $0 < x < 1$ and $0 < y < 1$.