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In memory of István Pásztor, mathematician, associate professor

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The population of a town is now 86 000. The population grows by 5% every year. What was the population 3 years ago? By what percent did the population grow in these three years?

Solution: If the population three years ago was x , then

$$x \cdot 1.05^3 = 86\,000, \quad x \cdot 1.157625 = 86\,000, \quad x \approx 74\,290$$

The population was about 74 290, and it grew by about 15.76%.

2. Solve the following system of equations over the pairs of real numbers.

$$x^{\log_3 y} + 2 \cdot y^{\log_3 x} = 27, \quad \log_3 y - \log_3 x = 1$$

Solution: We need $x > 0$ and $y > 0$. The second equation says $\log_3 y = \log_3 3x$, so $y = 3x$. Substituting, and using $(3x)^{\log_3 x} = 3^{\log_3 x} \cdot x^{\log_3 x} = x \cdot x^{\log_3 x}$:

$$x^{1+\log_3 x} + 2 \cdot x \cdot x^{\log_3 x} = 27, \quad x \cdot x^{\log_3 x} = 9$$

Taking the base 3 logarithm of both sides: $\log_3 x + (\log_3 x)^2 = 2$, so $\log_3 x = 1$ or $\log_3 x = -2$. The solutions are $(3, 9)$ and $\left(\frac{1}{9}, \frac{1}{3}\right)$, and both pairs are indeed solutions.

3. On each side of a square with side $2\sqrt{3}$ units, we construct an isosceles triangle inside the square, whose angle between the legs is 120° . Find the area of the quadrilateral whose vertices are the vertices of these triangles inside the square.

Solution: By symmetry, the quadrilateral is a square. The altitude of each isosceles triangle to its base is $\sqrt{3} \tan 30^\circ = 1$ unit, so the diagonals of the new square are $2\sqrt{3} - 2$ units long, and its area is

$$T = \frac{(2\sqrt{3} - 2)^2}{2} = 8 - 4\sqrt{3} \approx 1.07 \text{ square units}$$

4. The hypotenuse of the right triangle ABC is $AB = 2\sqrt{3}$ units. The midpoint of the hypotenuse is C_1 , and the midpoint of the leg BC is A_1 . The median CC_1 is perpendicular to the median AA_1 . Find the lengths of the legs.

Solution: Let S be the centroid. Then the triangles ASC and CSA_1 have right angles at S . Let $CA_1 = A_1B = a$ and $AC = b$. By Thales' theorem $CC_1 = \sqrt{3}$, so $CS = \frac{2\sqrt{3}}{3}$. Let $SA_1 = x$; then $AS = 2x$. By the Pythagorean Theorem

$$4a^2 + b^2 = 12, \quad a^2 + b^2 = (3x)^2, \quad x^2 + \frac{4}{3} = a^2, \quad 4x^2 + \frac{4}{3} = b^2$$

From the last three, $9x^2 = 5x^2 + \frac{8}{3}$, so $x^2 = \frac{2}{3}$, $a^2 = 2$, and $b^2 = 4$ (and indeed $4a^2 + b^2 = 12$). The legs are $AC = b = 2$ units and $BC = 2a = 2\sqrt{2}$ units.

(Note: the KoMaL solution gives $AC = \sqrt{2}$ in its last sentence; this is a slip, since its own computation gives $b = 2$.)

5. The areas of two faces of a rectangular box are 12 and 36 square units, and its space diagonal is 13 units long. Find the surface area and the volume of the box.

Solution: Let the edges from one vertex be a, b, c , with $ab = 12$, $bc = 36$, and $a^2 + b^2 + c^2 = 169$. Then $c = 3a$ and $b = \frac{12}{a}$, so

$$10a^2 + \frac{144}{a^2} = 169, \quad 10a^4 - 169a^2 + 144 = 0, \quad a^2 = 16 \text{ or } a^2 = \frac{9}{10}$$

There are two such boxes. The first has edges $a = 4$, $b = 3$, $c = 12$: its surface area is $2(12 + 36 + 48) = 192$ square units and its volume is 144 cubic units. The second has edges $a = \frac{3}{\sqrt{10}}$, $b = 4\sqrt{10}$, $c = \frac{9}{\sqrt{10}}$: its surface area is $2(12 + 36 + 2.7) = 101.4$ square units and its volume is $\frac{108}{\sqrt{10}} \approx 34.15$ cubic units.

6. For which natural numbers n are exactly two of the following statements true?

- a) $120n - 4n^2 - 899 > 0$ b) $n + 1$ is divisible by 7 c) $n^2 - 1$ is divisible by 7

Solution: Statement a) is true if $4n^2 - 120n + 899 < 0$, that is, $14.5 < n < 15.5$, so $n = 15$. Statement b) is true if $n = 7k - 1$ (k a positive integer). Statement c) is true if $n^2 - 1 = (n - 1)(n + 1)$ is divisible by 7, that is, if $n = 7k + 1$ or $n = 7k - 1$.

a) and b) are never true together, since 15 is not of the form $7k - 1$. a) and c) are both true for $n = 15$, and then b) is false. If b) is true, then c) is also true, since $n^2 - 1 = (7k - 1)^2 - 1 = 7k(7k - 2)$, and then a) is false.

Exactly two statements are true if $n = 15$ or $n = 7k - 1$ (k a positive integer).

7. Find the values of the parameter α in the interval $[0, 2\pi]$ for which the roots of the equation $(2\sin\alpha + 1)x^2 - 4x + 4\sin\alpha - 2 = 0$ have opposite signs.

Solution: A quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) has two real roots of opposite signs if and only if $ac < 0$: then the discriminant $b^2 - 4ac$ is positive, and the product of the roots, $\frac{c}{a}$, is negative. (If $c = 0$ or $ac > 0$, then a root is 0, or the roots have the same sign.) Here $\sin\alpha \neq -\frac{1}{2}$, and the condition is

$$(2\sin\alpha + 1)(4\sin\alpha - 2) < 0, \quad -\frac{1}{2} < \sin\alpha < \frac{1}{2}$$

In the interval $[0, 2\pi]$:

$$0 \leq \alpha < \frac{\pi}{6}, \quad \text{or} \quad \frac{5\pi}{6} < \alpha < \frac{7\pi}{6}, \quad \text{or} \quad \frac{11\pi}{6} < \alpha \leq 2\pi$$

8. How many lines pass through the point $B(4, 3)$ that meet the x -axis at a point with an integer x -coordinate, and meet the positive half of the y -axis at a point whose y -coordinate is a prime? Find the equations of these lines.

Solution: If the line meets the x -axis at $A(a, 0)$ (a an integer) and the y -axis at $C(0, p)$ (p a prime), then its equation is $\frac{x}{a} + \frac{y}{p} = 1$, that is, $px + ay = pa$. The point $B(4, 3)$ is on the line, so

$$4p + 3a = ap, \quad (a - 4)(p - 3) = 12, \quad a - 4 = \frac{12}{p - 3}$$

So $p - 3$ is a divisor of 12, and p is a prime: $p = 2$ ($a = -8$), $p = 5$ ($a = 10$), or $p = 7$ ($a = 7$). There are three such lines:

$$x - 4y = -8, \quad x + 2y = 10, \quad x + y = 7$$