

author: Gyula Orosz

translated from: KoMaL, January 2002, pages 30-31

Pdf file: KoMaL, January 2002, pages 30-31

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The real numbers a, b, c, x, y, z satisfy $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. What is the value of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$?

Solution: None of the numbers is 0. Let $u = \frac{x}{a}$, $v = \frac{y}{b}$, $w = \frac{z}{c}$. Then the conditions are $u + v + w = 1$ and $\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 0$, and we need $u^2 + v^2 + w^2$. Since

$$\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = \frac{uv + vw + wu}{uvw} = 0$$

we have $uv + vw + wu = 0$, so

$$u^2 + v^2 + w^2 = (u + v + w)^2 - 2(uv + vw + wu) = 1$$

The value is 1. (Such numbers exist: for example $u = v = \frac{2}{3}$, $w = -\frac{1}{3}$.)

2. Solve the equation $4 \cdot \log_x 2 = 7 + 2 \cdot \log_2 x$ over the real numbers.

Solution: We need $x > 0$ and $x \neq 1$. Changing to base 2, the equation is $\frac{4}{\log_2 x} = 7 + 2 \log_2 x$. With $a = \log_2 x$, this is $2a^2 + 7a - 4 = 0$, so $a = \frac{1}{2}$ or $a = -4$. The solutions are $x_1 = \sqrt{2}$ and $x_2 = 2^{-4} = \frac{1}{16}$, and both satisfy the original equation.

3. One root of the equation $x^3 + x^2 - 14x + m = 0$ is 3. Find the value of the real parameter m and the other two roots.

Solution: Substituting $x = 3$: $27 + 9 - 42 + m = 0$, so $m = 6$. Factoring out $x - 3$:

$$x^3 + x^2 - 14x + 6 = (x - 3)(x^2 + 4x - 2)$$

so the other two roots are $-2 + \sqrt{6}$ and $-2 - \sqrt{6}$.

4. In the triangle ABC the altitude from B meets the line AC at D , and the altitude from C meets the line AB at E . How long is the segment DE , if $\angle CAB = 70^\circ$, $AB = 12$ cm, and $AC = 10$ cm?

Solution: Let the angles of the triangle be $\alpha = 70^\circ$, β , γ , and the side $BC = a$. The quadrilateral $BCDE$ is cyclic, because the side BC is seen at a right angle from both D and E . Opposite angles of a cyclic quadrilateral add up to 180° , so $\angle DEA = \gamma$, and similarly $\angle EDA = \beta$. So the triangles ABC and ADE are similar, and

$$\frac{DE}{BC} = \frac{AE}{AC} = \cos \alpha$$

(from the right triangle AEC). So $DE = a \cos \alpha$. By the Law of Cosines

$$a^2 = 10^2 + 12^2 - 2 \cdot 10 \cdot 12 \cdot \cos 70^\circ, \quad a \approx 12.72 \text{ cm}$$

and $DE = a \cos 70^\circ \approx 4.35$ cm.

5. Solve the inequality $\cos 2x - 3 \cos x + 2 \leq 0$ over the real numbers.

Solution: Since $\cos 2x = 2 \cos^2 x - 1$, the inequality is $2 \cos^2 x - 3 \cos x + 1 \leq 0$, that is,

$$2(\cos x - 1) \left(\cos x - \frac{1}{2} \right) \leq 0$$

so $\frac{1}{2} \leq \cos x \leq 1$. The solution is

$$-\frac{\pi}{3} + 2k\pi \leq x \leq \frac{\pi}{3} + 2k\pi, \quad k \text{ is an integer}$$

6. At most how many different positive primes can be given so that the sum of any three of them is also a prime?

Solution: Four such primes can be found, for example 7, 11, 13, 23 (the sums are 31, 41, 43, 47), or 19, 23, 37, 41. We show that five is impossible: from any five integers we can choose three whose sum is divisible by 3 (and since the sum of three different primes is greater than 3, it is not a prime).

Integers give remainders 0, 1, or 2 when divided by 3. If three of the numbers have the same remainder, their sum is divisible by 3. If three of the numbers have three different remainders, their sum is also divisible by 3. So to avoid this, we can use at most two remainder classes, with at most two numbers from each: at most four numbers. So the answer is 4.

7. Which of the right circular cylinders inscribed in a sphere of radius R has the largest volume? What is the largest possible volume?

Solution: Let the radius of the base of the cylinder be r and its height m . In the cross section through the axis, the diagonal of the rectangle is a diameter of the sphere, so $4R^2 = 4r^2 + m^2$, and $V = r^2 \pi m = \frac{\pi}{4} (4R^2 - m^2) m$. So

$$\frac{32V^2}{\pi^2} = (4R^2 - m^2) (4R^2 - m^2) \cdot 2m^2$$

By the inequality between the arithmetic and geometric means for the positive numbers $4R^2 - m^2$, $4R^2 - m^2$, $2m^2$ (whose sum is the constant $8R^2$):

$$\sqrt[3]{\frac{32V^2}{\pi^2}} \leq \frac{8R^2}{3}, \quad V^2 \leq \frac{16\pi^2 R^6}{27}, \quad V \leq \frac{4\pi R^3}{3\sqrt{3}}$$

Equality holds if $4R^2 - m^2 = 2m^2$, that is, $m = \frac{2}{\sqrt{3}}R$ and $r = \sqrt{\frac{2}{3}}R$. The largest volume is $\frac{4\pi R^3}{3\sqrt{3}}$ (this is $\frac{1}{\sqrt{3}}$ times the volume of the sphere).

8. Consider the parabola $y = x^2$ and the point $A(9, 8)$. At which points P of the parabola is the tangent line perpendicular to the line AP ?

Solution: Let $P(p, p^2)$. A line $y - p^2 = m(x - p)$ through P is tangent to the parabola if the equation $x^2 - mx - p^2 + pm = 0$ has a double root, that is, $m^2 + 4p^2 - 4pm = (m - 2p)^2 = 0$. So the slope of the tangent at P is $2p$, and $\mathbf{v}(1, 2p)$ is a direction vector of the tangent. It must be perpendicular to $\overrightarrow{AP} = (p - 9, p^2 - 8)$:

$$\mathbf{v} \cdot \overrightarrow{AP} = p - 9 + 2p(p^2 - 8) = 2p^3 - 15p - 9 = (p - 3)(2p^2 + 6p + 3) = 0$$

So $p = 3$ or $p = \frac{-3 \pm \sqrt{3}}{2}$. There are three such points:

$$P_1(3, 9), \quad P_2\left(\frac{-3 + \sqrt{3}}{2}, 3 - \frac{3\sqrt{3}}{2}\right) \approx (-0.63, 0.40), \quad P_3\left(\frac{-3 - \sqrt{3}}{2}, 3 + \frac{3\sqrt{3}}{2}\right) \approx (-2.37, 5.60)$$

(Note: the KoMaL solution gives 6.00 for the second coordinate of P_3 ; this is a slip.)