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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Consider the equation  $x^2 + 4kx + 4(2k - 1) = 0$ , where  $k$  is a real parameter. Prove that the equation has a real solution for every real  $k$ . Solve the equation, if the sum of the squares of the roots is as small as possible.
2. Prove that the side of a rhombus is the geometric mean of the diagonals of the rhombus if and only if the acute angle of the rhombus is  $30^\circ$ .
3. Solve the following equations over the real numbers.
 

a)  $\sqrt[3]{15 + 2x} + \sqrt[3]{13 - 2x} = 4$

b)  $\log_x 3 \cdot \log_{\frac{x}{3}} 3 + \log_{\frac{x}{81}} 3 = 0$

c)  $\cos^4 x + \cos^4 \left(x - \frac{\pi}{4}\right) = \frac{1}{4}$
4. In the triangle  $ABC$ ,  $AC = 5.8$  and  $BC = 5$  units, and the altitude to the side  $AB$  is  $m_c = 4$  units. Find the radius of the circle circumscribed around the triangle, and the area of the triangle.
5. Prove that for every positive integer  $n$ 
  - a)  $36^n + 10 \cdot 3^n$  is divisible by 11;
  - b)  $4^{2n} - 3^{2n} - 7$  is divisible by 84.
6. a) Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = (ax + b)^2 + (cx + d)^2$ , where  $a, b, c, d$  are real and  $ac \neq 0$ . Where does  $f$  take its smallest value, and what is this smallest value?  
 b) What is the largest set on which  $g(x) = \frac{1}{1 + \tan x \cdot \tan 2x}$  is defined? Consider the function  $x \mapsto g(x)$  on this set. Does  $g$  take a largest or a smallest value? If so, where, and what is the extreme value?
7. Prove that the points  $A(1, -1)$ ,  $B(3, 3)$ , and  $C(4, 5)$  are on one line. In what ratio does the point  $B$  divide the segment  $AC$ ? What is the set of points  $P$  for which  $\frac{AP}{PC} = \frac{AB}{BC}$ , and what is its equation?
8. Solve the following equation over the pairs of real numbers.

$$\cos 2x + (1 - 2 \sin y) \cos x + 1 - \sin y = 0$$