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translated from: KoMaL, March 2002, page 149

Pdf file: KoMaL, March 2002, page 149

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Consider the equation $x^2 + 4kx + 4(2k - 1) = 0$, where k is a real parameter. Prove that the equation has a real solution for every real k . Solve the equation, if the sum of the squares of the roots is as small as possible.

Solution: The discriminant is $D = 16k^2 - 16(2k - 1) = (4(k - 1))^2 \geq 0$ for every real k , so there is always a real solution. The roots are $x_1 = -2$ and $x_2 = -4k + 2 = -4\left(k - \frac{1}{2}\right)$, so

$$x_1^2 + x_2^2 = 4 + 16\left(k - \frac{1}{2}\right)^2$$

This is smallest for $k = \frac{1}{2}$, and then $x_1 = -2$, $x_2 = 0$.

2. Prove that the side of a rhombus is the geometric mean of the diagonals of the rhombus if and only if the acute angle of the rhombus is 30° .

Solution: Let the side be a , the diagonals e and f , and the acute angle α . The area of the rhombus is $T = a^2 \sin \alpha$ and also $T = \frac{ef}{2}$. So $a^2 = ef$ if and only if $a^2 \sin \alpha = \frac{a^2}{2}$, that is, $\sin \alpha = \frac{1}{2}$, and since α is acute, this means $\alpha = 30^\circ$.

3. Solve the following equations over the real numbers.

$$\text{a) } \sqrt[3]{15+2x} + \sqrt[3]{13-2x} = 4 \quad \text{b) } \log_x 3 \cdot \log_{\frac{x}{3}} 3 + \log_{\frac{x}{81}} 3 = 0 \quad \text{c) } \cos^4 x + \cos^4 \left(x - \frac{\pi}{4}\right) = \frac{1}{4}$$

Solution: a) Let $u = \sqrt[3]{15+2x}$ and $v = \sqrt[3]{13-2x}$. Then $u + v = 4$ and $u^3 + v^3 = 28$. Since $(u + v)^3 = u^3 + v^3 + 3uv(u + v)$, we get $64 = 28 + 12uv$, so $uv = 3$, that is, $(15 + 2x)(13 - 2x) = 27$, which gives $x^2 + x - 42 = 0$. The solutions are $x_1 = 6$ and $x_2 = -7$, and both are indeed solutions. (Note: the KoMaL solution gives $x_2 = 7$; this is a slip.)

b) We need $x > 0$ and $x \neq 1, 3, 81$. With $t = \log_3 x$, using $\log_a b = \frac{1}{\log_b a}$:

$$\frac{1}{t} \cdot \frac{1}{t-1} + \frac{1}{t-4} = 0, \quad t - 4 + t(t-1) = 0, \quad t^2 = 4$$

So $\log_3 x = 2$ or $\log_3 x = -2$, and the solutions are $x_1 = 9$ and $x_2 = \frac{1}{9}$.

c) Using $\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$ and $\cos\left(2x - \frac{\pi}{2}\right) = \sin 2x$:

$$\left(\frac{1 + \cos 2x}{2}\right)^2 + \left(\frac{1 + \sin 2x}{2}\right)^2 = \frac{1}{4}, \quad 1 + 2\cos 2x + \cos^2 2x + 1 + 2\sin 2x + \sin^2 2x = 1$$

that is, $\sin 2x + \cos 2x = -1$. This holds if $\sin 2x = -1$ and $\cos 2x = 0$, or $\sin 2x = 0$ and $\cos 2x = -1$ (since $\sin^2 2x + \cos^2 2x = 1$). The solutions are $x = \frac{3\pi}{4} + k\pi$ and $x = \frac{\pi}{2} + n\pi$ (k, n arbitrary integers).

4. In the triangle ABC , $AC = 5.8$ and $BC = 5$ units, and the altitude to the side AB is $m_c = 4$ units. Find the radius of the circle circumscribed around the triangle, and the area of the triangle.

Solution: The area is $T = \frac{c \cdot m_c}{2}$ and also $T = \frac{abc}{4r}$, where r is the radius of the circumscribed circle. So

$$\frac{4c}{2} = \frac{5.8 \cdot 5 \cdot c}{4r}, \quad r = \frac{29}{8} = 3.625 \text{ units}$$

There are two such triangles. Let β be the angle at B ; then $\sin \beta = \frac{4}{5}$, so $\cos \beta = \pm \frac{3}{5}$. By the Law of Cosines $5.8^2 = 5^2 + c^2 \mp 2 \cdot 5 \cdot c \cdot \frac{3}{5}$, so $c = 7.2$ or $c = 1.2$, and the area is $\frac{7.2 \cdot 4}{2} = 14.4$ or $\frac{1.2 \cdot 4}{2} = 2.4$ square units.

(Note: the KoMaL solution gives $r = 7.25$; this is a slip.)

5. Prove that for every positive integer n

a) $36^n + 10 \cdot 3^n$ is divisible by 11;

b) $4^{2n} - 3^{2n} - 7$ is divisible by 84.

Solution: a) $36^n + 10 \cdot 3^n = 3^n(12^n + 10) = 3^n((12^n - 1) + 11)$, and $12^n - 1$ is divisible by $12 - 1 = 11$. (In fact the expression is divisible by 33.)

b) We use that $a^n - b^n$ is divisible by $a - b$. Then

$$\begin{aligned} 4^{2n} - 3^{2n} - 7 &= (16^n - 9^n) - 7 \quad \text{is divisible by 7,} \\ 4^{2n} - 3^{2n} - 7 &= (16^n - 1) - 9^n - 6 \quad \text{is divisible by 3,} \\ 4^{2n} - 3^{2n} - 7 &= (16^n - 8) - (9^n - 1) \quad \text{is divisible by 8.} \end{aligned}$$

Since 3, 7, 8 are pairwise relatively prime, the expression is divisible by $3 \cdot 7 \cdot 8 = 168$, so also by 84.

Note: In KoMaL, part b) was printed as $4^{2n} - 3^{3n} - 7$, which is false already for $n = 1$; KoMaL later corrected it to $4^{2n} - 3^{2n} - 7$.

6. a) Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = (ax + b)^2 + (cx + d)^2$, where a, b, c, d are real and $ac \neq 0$. Where does f take its smallest value, and what is this smallest value?

b) What is the largest set on which $g(x) = \frac{1}{1 + \tan x \cdot \tan 2x}$ is defined? Consider the function $x \mapsto g(x)$ on this set. Does g take a largest or a smallest value? If so, where, and what is the extreme value?

Solution: a) $f(x) = (a^2 + c^2)x^2 + 2(ab + cd)x + b^2 + d^2$, and completing the square,

$$f(x) = (a^2 + c^2) \left(x + \frac{ab + cd}{a^2 + c^2} \right)^2 + \frac{(ad - bc)^2}{a^2 + c^2}$$

The smallest value is $\frac{(ad - bc)^2}{a^2 + c^2}$, at $x_0 = -\frac{ab + cd}{a^2 + c^2}$.

b) $g(x)$ is defined if $\tan x$ and $\tan 2x$ are defined and $1 + \tan x \tan 2x \neq 0$. The last condition holds whenever the first two do (see below), so the domain is all real numbers except $x = \frac{\pi}{2} + k\pi$ and $x = \frac{\pi}{4} + \frac{n\pi}{2}$. On this set

$$g(x) = \frac{1}{1 + \frac{\sin x}{\cos x} \cdot \frac{\sin 2x}{\cos 2x}} = \frac{\cos x \cos 2x}{\cos x \cos 2x + \sin x \sin 2x} = \frac{\cos x \cos 2x}{\cos x} = \cos 2x$$

The largest value of g is 1, at $x = k\pi$. g does not take a smallest value: $\cos 2x = -1$ only for $x = \frac{\pi}{2} + n\pi$, which is not in the domain.

7. Prove that the points $A(1, -1)$, $B(3, 3)$, and $C(4, 5)$ are on one line. In what ratio does the point B divide the segment AC ? What is the set of points P for which $\frac{AP}{PC} = \frac{AB}{BC}$, and what is its equation?

Solution: $\overrightarrow{AB} = (2, 4)$ and $\overrightarrow{BC} = (1, 2)$, so $\overrightarrow{AB} = 2\overrightarrow{BC}$: A, B, C are on one line, and B divides AC in the ratio 2 : 1.

For $P(x, y)$ the condition is $AP = 2 \cdot PC$, which (since both sides are nonnegative) is equivalent to $AP^2 = 4 \cdot PC^2$:

$$(x - 1)^2 + (y + 1)^2 = 4 \left((x - 4)^2 + (y - 5)^2 \right), \quad (x - 5)^2 + (y - 7)^2 = 20$$

The set is the circle with center $(5, 7)$ and radius $2\sqrt{5}$.

Note: In KoMaL, the last condition was printed as $\frac{AP}{PB} = \frac{AB}{BC}$, and KoMaL later corrected it to $\frac{AP}{PC}$. With the original wording, the set is the circle $\left(x - \frac{11}{3}\right)^2 + \left(y - \frac{13}{3}\right)^2 = \frac{80}{9}$.

8. Solve the following equation over the pairs of real numbers.

$$\cos 2x + (1 - 2 \sin y) \cos x + 1 - \sin y = 0$$

Solution: Since $\cos 2x + 1 = 2 \cos^2 x$, the equation is

$$2 \cos^2 x + (1 - 2 \sin y) \cos x - \sin y = (2 \cos x + 1)(\cos x - \sin y) = 0$$

If $\cos x = -\frac{1}{2}$: $x = \frac{2\pi}{3} + 2k\pi$ or $x = \frac{4\pi}{3} + 2k\pi$, and y is any real number.

If $\cos x = \sin y = \cos\left(\frac{\pi}{2} - y\right)$: $x = \frac{\pi}{2} - y + 2k\pi$ or $x = y - \frac{\pi}{2} + 2k\pi$. So the solutions are $\left(\frac{\pi}{2} - t + 2k\pi, t\right)$ and $\left(t - \frac{\pi}{2} + 2k\pi, t\right)$, where t is any real number.

(k is an arbitrary integer.)