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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The trapezoid  $ABCD$  has right angles at the vertices  $A$  and  $D$ , the diagonals  $AC$  and  $BD$  are perpendicular, and  $4AB = 9CD$ . Find the ratio of the lengths of the diagonals.

Solution: Let the vertices be  $A(0, 0)$ ,  $B(9a, 0)$ ,  $C(4a, m)$ ,  $D(0, m)$ . The vectors  $\overrightarrow{AC} = (4a, m)$  and  $\overrightarrow{BD} = (-9a, m)$  are perpendicular, so their dot product is  $-36a^2 + m^2 = 0$ . Then

$$\frac{BD^2}{AC^2} = \frac{81a^2 + m^2}{16a^2 + m^2} = \frac{117a^2}{52a^2} = \frac{9}{4}, \quad \frac{BD}{AC} = \frac{3}{2}$$

2. Solve the following equation, where  $p$  is a nonzero real parameter.

$$\frac{x-14}{2} + \frac{x-12}{4} + \frac{x-10}{6} = \frac{x-8}{8} + \frac{x-6}{10} + \frac{x-4}{12} + \frac{x-16}{p}$$

Solution: Subtract 3 from both sides, as 1 from each of the first three fractions on each side. Since  $\frac{x-14}{2} - 1 = \frac{x-16}{2}$ ,  $\frac{x-12}{4} - 1 = \frac{x-16}{4}$ , and so on,

$$(x-16) \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \frac{1}{p} \right) = (x-16) \left( \frac{73}{120} - \frac{1}{p} \right) = 0$$

If  $p = \frac{120}{73}$ , every real number is a solution; if  $p \neq \frac{120}{73}$ , the only solution is  $x = 16$ .

3. Find the equation of the circle that passes through the points  $A(0, 9)$  and  $B(7, 2)$  and is tangent to the  $x$ -axis.

Solution: The center is on the perpendicular bisector of  $AB$ , which is the line  $x - y = -2$ , and since the circle is tangent to the  $x$ -axis (and passes through points above it), the center is  $(r - 2, r)$ , where  $r$  is the radius. Its distance from  $A$  is  $r$ :

$$(r-2)^2 + (r-9)^2 = r^2, \quad r^2 - 22r + 85 = 0$$

so  $r = 5$  or  $r = 17$ . There are two such circles:

$$(x-3)^2 + (y-5)^2 = 25 \quad \text{and} \quad (x-15)^2 + (y-17)^2 = 289$$

4. The base of a pyramid is a trapezoid with three sides 6 units long and the fourth side 12 units long. The lateral edges of the pyramid make equal angles with the plane of the base. What is this angle, if the volume of the pyramid is 162 cubic units?

Solution: Let  $AB = 12$ ,  $BC = CD = DA = 6$ , and let  $E$  be the apex. The trapezoid is made up of three equilateral triangles with side 6 (the line through  $C$  parallel to  $AD$  cuts off a rhombus and an equilateral triangle). Since the lateral edges make equal angles with the base, they have equal projections, so the foot  $K$  of the height is equally far from the vertices of the base:  $K$  is the center of the circle circumscribed around the trapezoid, which is the midpoint of  $AB$ , and the radius is 6. The area of the base is  $3 \cdot \frac{36\sqrt{3}}{4} = 27\sqrt{3}$ , so

$$\frac{27\sqrt{3} \cdot m}{3} = 162, \quad m = 6\sqrt{3}$$

The angle  $\varphi$  between a lateral edge and the base satisfies  $\tan \varphi = \frac{6\sqrt{3}}{6} = \sqrt{3}$ , so  $\varphi = 60^\circ$ .

5. The lines that trisect an angle of a triangle divide the opposite side into segments of lengths 5, 6, and 9 units. Find the exact value of the cosine of this angle.

Solution: Let the angle at  $A$  be  $3\alpha$ , and let the trisectors meet  $BC$  at  $D$  and  $E$ , with  $BD = 5$ ,  $DE = 6$ ,  $EC = 9$ . By the angle bisector theorem (in the triangles  $ABE$  and  $ADC$ ),  $AB = 5x$ ,  $AE = 6x$ , and  $AD = 2y$ ,  $AC = 3y$ . Writing the area of the triangle  $ABE$  in two ways:

$$\frac{5x \cdot 2y \cdot \sin \alpha}{2} + \frac{2y \cdot 6x \cdot \sin \alpha}{2} = \frac{5x \cdot 6x \cdot \sin 2\alpha}{2}, \quad 11y = 30x \cos \alpha$$

Similarly, from the area of the triangle  $ADC$  we get  $5x = 2y \cos \alpha$ . Multiplying these:  $55xy = 60xy \cos^2 \alpha$ , so  $\cos \alpha = \sqrt{\frac{11}{12}}$ . Then

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha = \cos \alpha \left( 4 \cdot \frac{11}{12} - 3 \right) = \frac{2}{3} \sqrt{\frac{11}{12}} = \frac{\sqrt{33}}{9}$$

6. Find the smallest positive solution of the following equation.

$$2 \sin \frac{2002^{100} \cdot \pi}{-3} \cdot 5^{\frac{1}{\log_3 25}} = \left| \sqrt{x^2 - 2x + 1} - 4 \right|$$

Solution: 2002 is even and gives remainder 1 when divided by 3, so the same is true for  $2002^{100}$ ; so  $2002^{100}$  gives remainder 4 when divided by 6. So

$$2 \sin \left( -2002^{100} \cdot \frac{\pi}{3} \right) = 2 \sin \left( -\frac{4\pi}{3} \right) = \sqrt{3}, \quad 5^{\frac{1}{\log_3 25}} = 5^{\log_{25} 3} = 25^{\frac{1}{2} \log_{25} 3} = \sqrt{3}$$

So the left-hand side is 3. The right-hand side is  $||x - 1| - 4|$ , which is 3 if  $|x - 1| = 7$  or  $|x - 1| = 1$ , that is, at  $x = 8, -6, 2, 0$ . The smallest positive solution is  $x = 2$ .

7. For which pairs of positive primes  $p$  and  $q$  does the equation

$$(p^2 - pq - 2q^2)x^2 - 3q^2x - (2p + 0.5) = 0$$

have exactly one real root?

Solution: If  $p^2 - pq - 2q^2 = (p - 2q)(p + q) = 0$ , then  $p = 2q$  or  $p = -q$ , and neither is possible for primes. So the equation is quadratic, and we need the discriminant to be 0:

$$D = 9q^4 + 4(2p + 0.5)(p^2 - pq - 2q^2) = 9q^4 + 2(4p + 1)(p^2 - pq - 2q^2) = 0$$

The second term is even, so  $9q^4$  is even, and  $q = 2$ . Substituting:

$$144 + 2(4p + 1)(p^2 - 2p - 8) = 0, \quad 4p^3 - 7p^2 - 34p + 64 = 0$$

Here  $-7p^2$  must be even, so  $p = 2$ , and this is indeed a root:  $4p^3 - 7p^2 - 34p + 64 = (p - 2)(4p^2 + p - 32)$ . So  $p = q = 2$ , and then the equation is  $-8x^2 - 12x - 4.5 = 0$ , with the single root  $x = -\frac{3}{4}$ .

8. For which real numbers is the following expression defined?

$$\sqrt{\frac{2^{-x+1} - 0.5}{\log_x(x^2 + x)}}$$

Solution: We need  $x > 0$ ,  $x \neq 1$ ,  $\log_x(x^2 + x) \neq 0$ , and the fraction must be nonnegative. The numerator  $2^{1-x} - 2^{-1}$  is nonnegative exactly when  $1 - x \geq -1$ , that is,  $x \leq 2$ .

If  $x > 1$ , then  $x^2 + x > 1$ , so the denominator is positive, and we need  $x \leq 2$ :  $1 < x \leq 2$ .

If  $0 < x < 1$ , the denominator is positive exactly when  $x^2 + x < 1$ , that is,  $x < \frac{-1 + \sqrt{5}}{2}$ , and then the numerator is positive. If  $\frac{-1 + \sqrt{5}}{2} < x < 1$ , the denominator is negative and the numerator is positive, so the fraction is negative.

The expression is defined for  $0 < x < \frac{-1 + \sqrt{5}}{2}$  and for  $1 < x \leq 2$ .