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In memory of György Geréb, psychologist, college professor

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. One of the parallel sides of an isosceles trapezoid that has an inscribed circle is 10 units long, and the radius of the inscribed circle is $\varrho = 5\sqrt{3}$ units. What percent of the area of the trapezoid is the area of the inscribed circle?

Solution: If the parallel sides of an isosceles trapezoid with an inscribed circle are $2a$ and $2b$, then $\varrho^2 = ab$. (Indeed, by the equality of tangent segments each leg is $a + b$, so the height is $2\varrho = \sqrt{(a + b)^2 - (b - a)^2} = 2\sqrt{ab}$.) Here $2a = 10$ and $\varrho = 5\sqrt{3}$, so $75 = 5b$, $b = 15$, and the other parallel side is 30 units. The area of the circle is $T_1 = 75\pi$, and the area of the trapezoid is $T_2 = \frac{10 + 30}{2} \cdot 10\sqrt{3} = 200\sqrt{3}$. Since

$$\frac{T_1}{T_2} = \frac{\sqrt{3}\pi}{8} \approx 0.680$$

the area of the inscribed circle is about 68% of the area of the trapezoid.

2. The distance between two towns is 560 km. One of two cars covers this distance in 1 hour less than the other car, because its speed is 10 km/h greater. Find the speeds of the cars.

Solution: Let v be the speed of the slower car; then the faster car's speed is $v + 10$, and

$$\frac{560}{v} - \frac{560}{v + 10} = 1, \quad v^2 + 10v - 5600 = 0$$

so $v = 70$ (the other root, -80 , is meaningless). The speeds are 70 km/h and 80 km/h.

3. The points $P(b, -2)$ and $Q(16b, 6)$ are on the graph of the logarithm function $f : x \mapsto \log_a x$, $x > 0$ ($a > 0$, $a \neq 1$). Find a and b , and write down the equation of the line PQ .

Solution: We need $b > 0$. The conditions are $\log_a b = -2$ and $\log_a 16b = 6$, that is, $a^{-2} = b$ and $a^6 = 16b$. So $a^8 = 16$, $a = \sqrt{2}$, and $b = \frac{1}{2}$. So $P\left(\frac{1}{2}, -2\right)$ and $Q(8, 6)$, and the equation of the line PQ is $16x - 15y = 38$.

4. For which real numbers x is the following expression defined, and what is its range?

$$\frac{\sin 2x + 2 \sin x}{\sin 2x - 2 \sin x} \cdot \tan^2 \frac{x}{2}$$

Solution: The expression is defined if $\sin 2x - 2 \sin x = 2 \sin x (\cos x - 1) \neq 0$ and $\tan \frac{x}{2}$ is defined, that is, if $x \neq k\pi$ (k an integer). Then

$$\frac{\sin 2x + 2 \sin x}{\sin 2x - 2 \sin x} = \frac{2 \sin x (\cos x + 1)}{2 \sin x (\cos x - 1)} = \frac{\cos x + 1}{\cos x - 1}, \quad \tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}$$

so the expression is $\frac{\cos x + 1}{\cos x - 1} \cdot \frac{1 - \cos x}{1 + \cos x} = -1$ for every $x \neq k\pi$. The range is $\{-1\}$.

5. Find the equation of the circle of radius $\sqrt{8}$ that passes through the point $P(0, 2)$ and is externally tangent to the circle $x^2 + y^2 + 2x + 6y + 8 = 0$.

Solution: The given circle is $(x + 1)^2 + (y + 3)^2 = 2$, with center $K(-1, -3)$ and radius $\sqrt{2}$. Let the center of the circle we are looking for be $C(u, v)$. Since P is on the circle, and C is $\sqrt{2} + \sqrt{8} = 3\sqrt{2}$ units from K :

$$u^2 + (2 - v)^2 = 8, \quad (u + 1)^2 + (v + 3)^2 = 18$$

Subtracting, $u = 2 - 5v$, and substituting, $26v^2 - 24v = 0$. So $(u, v) = (2, 0)$ or $\left(-\frac{34}{13}, \frac{12}{13}\right)$. There are two such circles:

$$(x - 2)^2 + y^2 = 8 \quad \text{and} \quad \left(x + \frac{34}{13}\right)^2 + \left(y - \frac{12}{13}\right)^2 = 8$$

6. Solve the following inequality over the real numbers.

$$2\sqrt{\log_3 x} + \sqrt{\log_x 3} \geq 3$$

Solution: The expressions are defined if $x > 1$. Let $z = \sqrt{\log_3 x} > 0$; then $\sqrt{\log_x 3} = \sqrt{\frac{1}{\log_3 x}} = \frac{1}{z}$, and the inequality is $2z + \frac{1}{z} \geq 3$, that is, $2z^2 - 3z + 1 \geq 0$. For $z > 0$ this holds if $0 < z \leq \frac{1}{2}$ or $z \geq 1$, that is, $0 < \log_3 x \leq \frac{1}{4}$ or $\log_3 x \geq 1$. The solution is

$$1 < x \leq \sqrt[4]{3} \quad \text{or} \quad x \geq 3$$

(The KoMaL solution writes $1 < x < \sqrt[4]{3}$, but $x = \sqrt[4]{3}$ also satisfies the inequality, with equality.)

7. A convex hexagon is inscribed in a circle of radius $\sqrt{13}$ units. Three of its sides are $2a$ and the other three sides are $5a$ units long. Find the area of the hexagon.

Solution: The hexagon has a vertex B whose two neighbors A and C are at distances $AB = 2a$ and $BC = 5a$. Let O be the center of the circle. The central angles of the six sides add up to 360° , so the central angles of one side $2a$ and one side $5a$ add up to 120° . So the longer arc AC has central angle 240° , and the inscribed angle ABC is 120° . Also $AC = 2\sqrt{13} \sin 120^\circ = \sqrt{39}$. By the Law of Cosines in the triangle ABC

$$39 = 4a^2 + 25a^2 - 2 \cdot 2a \cdot 5a \cdot \cos 120^\circ = 39a^2$$

so $a = 1$. The altitudes of the triangles AOB and BOC from O are $m_1 = \sqrt{13-1} = 2\sqrt{3}$ and $m_2 = \sqrt{13 - \frac{25}{4}} = \frac{3\sqrt{3}}{2}$, so the area of the hexagon is

$$T = 3 \left(\frac{2 \cdot 2\sqrt{3}}{2} + \frac{5}{2} \cdot \frac{3\sqrt{3}}{2} \right) = \frac{69\sqrt{3}}{4} \approx 29.88 \text{ square units}$$

8. The first term of a sequence is $a_1 = 1$, and for $n \geq 1$

$$a_{n+1} = \left(1 - \frac{1}{(n+2)^2} \right) \cdot a_n$$

Write the n th term of the sequence and the product of its first n terms in closed form.

Solution: Multiplying the equations $a_2 = a_1 \left(1 - \frac{1}{3^2} \right)$, $a_3 = a_2 \left(1 - \frac{1}{4^2} \right)$, ..., $a_n = a_{n-1} \left(1 - \frac{1}{(n+1)^2} \right)$, and dividing by $a_1 a_2 \cdots a_{n-1} \neq 0$, using $1 - \frac{1}{k^2} = \left(1 - \frac{1}{k} \right) \left(1 + \frac{1}{k} \right)$:

$$a_n = \left(\frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n}{n+1} \right) \cdot \left(\frac{4}{3} \cdot \frac{5}{4} \cdots \frac{n+2}{n+1} \right) = \frac{2}{n+1} \cdot \frac{n+2}{3} = \frac{2}{3} \cdot \frac{n+2}{n+1}$$

(this also gives $a_1 = 1$). The product of the first n terms is

$$\left(\frac{2}{3} \cdot \frac{3}{2} \right) \left(\frac{2}{3} \cdot \frac{4}{3} \right) \left(\frac{2}{3} \cdot \frac{5}{4} \right) \cdots \left(\frac{2}{3} \cdot \frac{n+2}{n+1} \right) = \left(\frac{2}{3} \right)^n \cdot \frac{n+2}{2}$$

(Both statements can also be proved by induction.)