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In honor of Gergő (Grünwald) Gézáné, mathematician, university assistant professor

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The sides of the triangle ABC are $AB = 20$, $BC = 13$, and $CA = 11$ units. Find the area of the triangle, and the radii of its inscribed and circumscribed circles.

Solution: The perimeter is $2s = 44$, so by Heron's formula $T^2 = 22 \cdot 2 \cdot 11 \cdot 9 = (11 \cdot 2 \cdot 3)^2$, and $T = 66$ square units.

The radius of the inscribed circle is $\varrho = \frac{T}{s} = \frac{66}{22} = 3$ units, and the radius of the circumscribed circle is

$$r = \frac{abc}{4T} = \frac{13 \cdot 11 \cdot 20}{4 \cdot 66} = \frac{65}{6} \approx 10.83 \text{ units}$$

(Note: KoMaL prints the triangle as $ABCD$, an obvious misprint; and its solution uses 22 instead of 20 in this last computation, getting $\frac{143}{12}$, which is a slip.)

2. Solve the following equation over the real numbers.

$$\left(x^2 + x + 1 + \frac{2}{x-1}\right) \left(x^2 - x + 1 - \frac{2}{x+1}\right) = x^4 + x^2 + 1$$

Solution: The equation is not defined for $x = 1$ and $x = -1$. Otherwise

$$\begin{aligned} \left(x^2 + x + 1 + \frac{2}{x-1}\right) \left(x^2 - x + 1 - \frac{2}{x+1}\right) &= \frac{x^3 - 1 + 2}{x-1} \cdot \frac{x^3 + 1 - 2}{x+1} = \frac{x^3 + 1}{x+1} \cdot \frac{x^3 - 1}{x-1} \\ &= (x^2 - x + 1)(x^2 + x + 1) = (x^2 + 1)^2 - x^2 = x^4 + x^2 + 1 \end{aligned}$$

So every real number except -1 and 1 is a solution.

3. The first term of the arithmetic sequence (b_n) is $b_1 = 3$ and its difference is $d_1 = 4$; the first term of the arithmetic sequence (c_k) is $c_1 = 2$ and its difference is $d_2 = 7$. The common terms of the two sequences form the sequence (a_m) . Express the sum of the first m terms of (a_m) in terms of m .

Solution: $b_n = 4n - 1$ and $c_k = 7k - 5$. For a common term $4n - 1 = 7k - 5$, that is, $4n + 4 = 7k$, where n and k are positive integers. So $k = 4m$ and $n = 7m - 1$, and the common terms are $a_m = 4(7m - 1) - 1 = 28m - 5$ (m a positive integer). The first term is $a_1 = 23$, so

$$S_m = \frac{m}{2} (23 + 28m - 5) = 14m^2 + 9m$$

4. Two sides of a triangle are b and c units long, and the area of the triangle is $T = a^2 - (b - c)^2$. Express the side a in terms of b and c .

Solution: Let α be the angle opposite a . Since $T = \frac{bc \sin \alpha}{2}$ and $a^2 = b^2 + c^2 - 2bc \cos \alpha$:

$$\frac{bc \sin \alpha}{2} = 2bc - 2bc \cos \alpha, \quad \sin \alpha = 4(1 - \cos \alpha), \quad 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = 8 \sin^2 \frac{\alpha}{2}$$

so $\tan \frac{\alpha}{2} = \frac{1}{4}$. Using $\cos \alpha = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$, we get $\cos \alpha = \frac{15}{17}$, and

$$a = \sqrt{b^2 + c^2 - \frac{30}{17}bc}$$

5. Solve the following system of equations over the pairs of real numbers.

$$3^{1+2 \log_3(y-x)} = 48, \quad 2 \log_5(2y - x - 12) = \log_5(y - x) + \log_5(y + x)$$

Solution: We need $y - x > 0$, $y + x > 0$, and $2y - x - 12 > 0$. The equations are equivalent to

$$3(y - x)^2 = 48 \quad \text{and} \quad (2y - x - 12)^2 = y^2 - x^2$$

From the first, $y - x = 4$ (since $y - x > 0$). Substituting $y = x + 4$ into the second: $(x - 4)^2 = 8x + 16$, that is, $x^2 - 16x = 0$, so $x = 0$ or $x = 16$. For $x = 0$, $y = 4$, and $2y - x - 12 = -4 < 0$, so this is not a solution. For $x = 16$, $y = 20$, and this pair is indeed a solution. The only solution is $(16, 20)$.

6. Solve the following equation over the real numbers.

$$2 \sin 2x = 2\sqrt{6} \sin \left(x - \frac{\pi}{4}\right) + 3$$

Solution: Since $2\sqrt{6} \sin \left(x - \frac{\pi}{4}\right) = 2\sqrt{6} \cdot \frac{\sin x - \cos x}{\sqrt{2}} = 2\sqrt{3}(\sin x - \cos x)$, the equation is

$$4 \sin x \cos x - 2\sqrt{3} \sin x + 2\sqrt{3} \cos x - 3 = 0, \quad (2 \sin x + \sqrt{3})(2 \cos x - \sqrt{3}) = 0$$

So $\sin x = -\frac{\sqrt{3}}{2}$ or $\cos x = \frac{\sqrt{3}}{2}$. The solutions are

$$x = -\frac{\pi}{3} + 2n\pi, \quad x = -\frac{2\pi}{3} + 2n\pi, \quad x = \frac{\pi}{6} + 2n\pi, \quad x = -\frac{\pi}{6} + 2n\pi, \quad n \text{ is an integer}$$

7. The line $x + y = 15$ meets the x -axis at A and the y -axis at B . Find the equation of the line that is perpendicular to the line $3x + y = 2002$, meets the segment OB at C and the segment AB at D , and for which the area of the quadrilateral $OADC$ is 58.5 square units, where O is the origin.

Solution: The lines perpendicular to $3x + y = 2002$ have equations $y = -\frac{1}{3}x + b$, so $C(0, b)$ with $0 < b < 15$. The x -coordinate of D satisfies $x + \frac{1}{3}x + b = 15$, so $x = \frac{3}{4}(15 - b)$. Since $A(15, 0)$ and $B(0, 15)$, the area of the triangle AOB is $\frac{15^2}{2} = 112.5$, so the area of the triangle BCD is $112.5 - 58.5 = 54$. Also, this area is $\frac{1}{2}(15 - b) \cdot \frac{3}{4}(15 - b)$, so

$$\frac{3}{8}(15 - b)^2 = 54, \quad (15 - b)^2 = 144$$

so $b = 3$ or $b = 27$, and only $b = 3$ is possible. The line is $y = -\frac{1}{3}x + 3$, that is, $x - 3y = -9$.

8. Prove that if $x^2 + y^2 = 18$ for a pair of real numbers (x, y) , then $|x + y| \leq 6$. For which pairs does equality hold?

Solution: For all real x and y , $2xy \leq x^2 + y^2$, so

$$|x + y| = \sqrt{x^2 + y^2 + 2xy} \leq \sqrt{2(x^2 + y^2)} = 6$$

Equality holds exactly when $x = y$, that is, $x^2 = 9$: for the pairs $(3, 3)$ and $(-3, -3)$.