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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. In the sequence (a_n) , $a_n = 2a_{n+1} - a_{n+2}$ for every positive integer n , and $a_2 + a_6 = 8$. Find the sum of the first 7 terms of the sequence.

Solution: For every positive integer n , $a_{n+2} - a_{n+1} = a_{n+1} - a_n$, so the sequence is arithmetic. So $a_1 + a_7 = a_2 + a_6 = 8$, and the sum of the first 7 terms is

$$7 \cdot \frac{a_1 + a_7}{2} = 28$$

2. The circles k_1 and k_2 , both of radius 4 units, are tangent to each other at the point D . The line of the common diameter of the circles meets k_1 at D and A . A line through D that makes a 60° angle with AD meets k_1 at C and k_2 at B ($C \neq D$, $B \neq D$). Find the area of the triangle ABC .

Solution: By Thales' theorem $\angle ACD = 90^\circ$. In the right triangle ADC the hypotenuse is $AD = 8$ and $\angle ADC = 60^\circ$, so $CD = 4$ and $AC = 4\sqrt{3}$ units. Similarly, in k_2 the chord DB makes a 60° angle with a diameter of length 8, so $DB = 4$. C and B are on opposite sides of D , so $BC = 8$, and the triangle ABC has a right angle at C . Its area is

$$T = \frac{8 \cdot 4\sqrt{3}}{2} = 16\sqrt{3} \text{ square units}$$

3. Solve the following system of equations, where x and y are real numbers.

$$x + 2y + 2\sqrt{x + 2y} = 15, \quad \sqrt{2y - 4x} - 1 = \frac{2}{\sqrt{2y - 4x}}$$

Solution: Let $a = \sqrt{x + 2y} \geq 0$ and $b = \sqrt{2y - 4x} > 0$. Then $a^2 + 2a - 15 = 0$ and $b^2 - b - 2 = 0$, so $a = 3$ and $b = 2$. So $x + 2y = 9$ and $2y - 4x = 4$, and the only solution is $x = 1$, $y = 4$.

4. In the triangle ABC , $\angle BAC = 60^\circ$. The angle bisector through the vertex A meets the side BC at a point D with $\frac{BD}{CD} = \frac{1}{4}$. Find the other two angles of the triangle.

Solution: By the angle bisector theorem $\frac{AB}{AC} = \frac{BD}{CD} = \frac{1}{4}$. Let $\angle ACB = \gamma$; then $\angle ABC = 120^\circ - \gamma$, and by the Law of Sines

$$\frac{1}{4} = \frac{\sin \gamma}{\sin (120^\circ - \gamma)}, \quad \frac{\sqrt{3}}{2} \cos \gamma + \frac{1}{2} \sin \gamma = 4 \sin \gamma, \quad \tan \gamma = \frac{\sqrt{3}}{7}$$

So $\angle ACB \approx 13.90^\circ$ and $\angle ABC \approx 106.10^\circ$.

5. Among the isosceles trapezoids with perimeter 200 cm that have two angles of 45° , which one has the largest area? What is this area?

Solution: Let the height of the trapezoid be x cm and the shorter parallel side y cm. Then the longer parallel side is $2x + y$, and the legs are $x\sqrt{2}$ cm. The area is $T = \frac{2x + y}{2} \cdot x = (x + y)x$. By the condition $2x + 2y + 2x\sqrt{2} = 200$, so $x + y = 100 - x\sqrt{2}$, and

$$T(x) = x(100 - x\sqrt{2}) = \sqrt{2}(25\sqrt{2})^2 - \sqrt{2}(x - 25\sqrt{2})^2$$

where $0 < y = 100 - x(1 + \sqrt{2})$, that is, $0 < x < 100(\sqrt{2} - 1)$. Since $25\sqrt{2}$ is in this interval, the area is largest for $x = 25\sqrt{2}$ cm, and the largest area is $T = 1250\sqrt{2} \approx 1768$ cm². This is the area of the trapezoid with parallel sides $50 - 25\sqrt{2}$ cm and $50 + 25\sqrt{2}$ cm and legs 50 cm.

6. Solve the following inequalities over the real numbers.

$$\text{a) } \log_{|x-1|}(x+6) + \log_{\frac{1}{|x-1|}} x < 0 \quad \text{b) } \log_{|x-1|}(x+6) + \log_{\frac{1}{|x-1|}} x \geq 0$$

Solution: The expressions are defined if $x > 0$, $|x - 1| > 0$, and $|x - 1| \neq 1$, that is, $0 < x < 1$ or $1 < x < 2$ or $x > 2$. Since $\log_{\frac{1}{a}} b = -\log_a b$, the inequalities are

$$\text{a) } \log_{|x-1|}(x+6) < \log_{|x-1|} x; \quad \text{b) } \log_{|x-1|}(x+6) \geq \log_{|x-1|} x.$$

If $0 < |x - 1| < 1$, that is, $0 < x < 1$ or $1 < x < 2$, the logarithm function is decreasing, so a) means $x + 6 > x$, which is true for all these x . If $|x - 1| > 1$, that is, $x > 2$, the logarithm function is increasing, so a) means $x + 6 < x$, which is never true.

So the solutions of a) are $0 < x < 1$ and $1 < x < 2$, and similarly the solutions of b) are all $x > 2$.

7. The base edge of a right prism with a square base is a , and its lateral edge is b units long, where a and b are positive integers. We paint the surface of the prism, and then cut the prism into unit cubes. Among these cubes, there are 112 that have exactly one face painted. Find the surface area and the volume of the original prism.

Solution: If $a \leq 2$ or $b = 1$, then no small cube has exactly one painted face, so $a > 2$ and $b \geq 2$. The small cubes along the edges of the prism have at least two painted faces, so the number of cubes with exactly one painted face is

$$2(a-2)^2 + 4(a-2)(b-2) = 112, \quad (a-2)(a+2b-6) = 56$$

Here $a - 2 \leq a + 2b - 6$ (since $b \geq 2$), and the two factors have the same parity. 56 can be written in two such ways: $2 \cdot 28$ and $4 \cdot 14$. If $a - 2 = 2$, then $a = 4$ and $b = 15$; if $a - 2 = 4$, then $a = 6$ and $b = 7$. Both satisfy the conditions.

If $a = 4$, $b = 15$: the surface area is $2a^2 + 4ab = 272$ square units, and the volume is $a^2b = 240$ cubic units. If $a = 6$, $b = 7$: the surface area is 240 square units, and the volume is 252 cubic units.

8. For which values of the real parameter k are the four different real roots of the equation

$$x^4 - (k + 3)x^2 + 3k = 0$$

four consecutive terms of an arithmetic sequence?

Solution: The equation is quadratic in x^2 : $(x^2 - 3)(x^2 - k) = 0$, so $x^2 = 3$ or $x^2 = k$. It has four different real roots if $k > 0$, $k \neq 3$, and the roots are $\pm\sqrt{3}$, $\pm\sqrt{k}$. In increasing order:

If $k > 3$: $-\sqrt{k}$, $-\sqrt{3}$, $\sqrt{3}$, $\sqrt{k}$, and the condition is $2\sqrt{3} = \sqrt{k} - \sqrt{3}$, that is, $\sqrt{k} = 3\sqrt{3}$, $k = 27$.

If $0 < k < 3$: $-\sqrt{3}$, $-\sqrt{k}$, $\sqrt{k}$, $\sqrt{3}$, and the condition is $2\sqrt{k} = \sqrt{3} - \sqrt{k}$, that is, $\sqrt{k} = \frac{\sqrt{3}}{3}$, $k = \frac{1}{3}$.

So $k = 27$ or $k = \frac{1}{3}$.