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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Find the four-digit number $\overline{abcd}$ for which

$$\overline{abcd} + \overline{abc} + \overline{ab} + \overline{a} = 2003$$

Solution: Writing the numbers by place value:

$$1000a + 100b + 10c + d + 100a + 10b + c + 10a + b + a = 1111a + 111b + 11c + d = 2003$$

where a, b, c, d are digits and $a \neq 0$. So $a = 1$, and $111b + 11c + d = 892$. Since $11c + d \leq 108$, $b = 8$, and then $11c + d = 4$, so $c = 0$, $d = 4$. The number is 1804, and indeed $1804 + 180 + 18 + 1 = 2003$.

2. The diagonals of a trapezoid are perpendicular, one of them is 5 units long, and the height of the trapezoid is 4 units. What is its area?

Solution: Let the diagonals be $e = 5$ and f , the parallel sides a and c , and the height $m = 4$. Since the diagonals are perpendicular,

$$T = \frac{e \cdot f}{2} = \frac{(a + c) \cdot m}{2}, \quad \frac{5f}{2} = 2(a + c), \quad a + c = \frac{5}{4}f$$

Translating one diagonal along the parallel sides we get a right triangle with legs e and f and hypotenuse $a + c$, so $(a + c)^2 = e^2 + f^2$, that is, $\frac{25}{16}f^2 = 25 + f^2$, and $f = \frac{20}{3}$. The area is

$$T = \frac{5 \cdot \frac{20}{3}}{2} = \frac{50}{3} \text{ square units}$$

3. Solve the following equation over the pairs of real numbers.

$$|\sin x + \cos x| = \sqrt{2}(y^2 + 2y + 2)$$

Solution: Dividing both sides by $\sqrt{2}$:

$$\left| \frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x \right| = \left| \sin \left(x + \frac{\pi}{4} \right) \right| = (y + 1)^2 + 1$$

The left-hand side is at most 1 and the right-hand side is at least 1, so both are 1: $y = -1$ and $\sin \left(x + \frac{\pi}{4} \right) = \pm 1$, that is, $x = \frac{\pi}{4} + k\pi$. The solutions are $\left(\frac{\pi}{4} + k\pi, -1 \right)$, where k is an integer.

4. The sides of a triangle enclosing the angle α are $\sin \alpha$ and $\cos \alpha$ units long, and the third side is $\sqrt{1 - \sin \alpha \cdot \cos \alpha}$. Is it true that the area of the triangle is a rational number?

Solution: The sides must be positive, so $0^\circ < \alpha < 90^\circ$. By the Law of Cosines

$$1 - \sin \alpha \cos \alpha = \sin^2 \alpha + \cos^2 \alpha - 2 \sin \alpha \cos^2 \alpha, \quad \sin \alpha \cos \alpha (2 \cos \alpha - 1) = 0$$

so $\cos \alpha = \frac{1}{2}$, and $\alpha = 60^\circ$. The sides are $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, and $\sqrt{1 - \frac{\sqrt{3}}{4}}$, and these satisfy the triangle inequalities, so such a triangle exists. Its area is

$$T = \frac{\frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \sin 60^\circ}{2} = \frac{3}{16}$$

which is rational. So the statement is true.

5. Find the values of the real parameter a for which the domain of the following expression is the empty set.

$$\sqrt{(a^2 - 2a - 3) \cdot x^2 + (2a - 3) \cdot x + 1}$$

Solution: For $x = 0$ the expression under the square root is 1, so $x = 0$ is always in the domain. So there is no such a . (Another way: the domain is empty exactly when $(a^2 - 2a - 3)x^2 + (2a - 3)x + 1 < 0$ for every real x . This needs $a^2 - 2a - 3 = (a + 1)(a - 3) < 0$, that is, $-1 < a < 3$, and a negative discriminant, $(2a - 3)^2 - 4(a^2 - 2a - 3) = 21 - 4a < 0$, that is, $a > \frac{21}{4}$; these cannot hold at the same time.)

6. In an increasing geometric sequence, the sum of the first and the n th term is 66, the product of the second and the $(n - 1)$ th term is 128, and the sum of the first n terms is 126. Write down the first n terms of the sequence.

Solution: The ratio $q \neq 1$, and

$$a_1 + a_1 q^{n-1} = 66, \quad a_1 q \cdot a_1 q^{n-2} = 128, \quad a_1 \cdot \frac{q^n - 1}{q - 1} = 126$$

From the second equation $a_1 q^{n-1} = \frac{128}{a_1}$, and substituting into the first, $a_1^2 - 66a_1 + 128 = 0$, so $a_1 = 64$ or $a_1 = 2$.

If $a_1 = 64$, then $q^{n-1} = \frac{1}{32}$, which is impossible in an increasing sequence. If $a_1 = 2$, then $q^{n-1} = 32$, so $q^n = 32q$, and the third equation gives $2 \cdot \frac{32q - 1}{q - 1} = 126$, so $q = 2$ and $n = 6$. The first n terms are 2, 4, 8, 16, 32, 64.

7. Solve the following equation over the real numbers, where a and b are real numbers with $a \neq 0$, $b \neq 0$, and $|a| \neq |b|$.

$$\frac{a + bx}{a - bx} - \frac{a - bx}{a + bx} = \frac{4ab}{a^2 - b^2}$$

Solution: We need $x \neq \frac{a}{b}$ and $x \neq -\frac{a}{b}$. Bringing the left-hand side to a common denominator:

$$\frac{4abx}{a^2 - b^2x^2} = \frac{4ab}{a^2 - b^2}, \quad x(a^2 - b^2) = a^2 - b^2x^2, \quad b^2x^2 + (a^2 - b^2)x - a^2 = 0$$

(dividing by $4ab \neq 0$). This factors as $(x - 1)(b^2x + a^2) = 0$, so $x_1 = 1$ and $x_2 = -\frac{a^2}{b^2}$. Neither is excluded:

$1 \neq \pm \frac{a}{b}$ because $|a| \neq |b|$, and $-\frac{a^2}{b^2} = \pm \frac{a}{b}$ would mean $a = \mp b$. So the solutions are $x_1 = 1$ and $x_2 = -\frac{a^2}{b^2}$.

8. Find the set of real numbers that satisfy both of the following inequalities.

$$(1) \quad x^{\log_{36}(x+6) - \log_6 x} > 1 \qquad (2) \quad \sin\left(\cos x + \frac{1}{2}\right) \leq 0$$

Solution: Inequality (1) makes sense only for $x > 0$. Since $\log_6 x = \log_{36} x^2$, the exponent is $\log_{36}(x+6) - \log_{36} x^2$. If $x > 1$, the exponential function with base x is increasing, so (1) means $\log_{36}(x+6) > \log_{36} x^2$, that is, $x+6 > x^2$, $(x+2)(x-3) < 0$, so $1 < x < 3$.

If $0 < x < 1$, (1) means $x+6 < x^2$, which has no solution in $(0, 1)$. For $x = 1$ the inequality does not hold. So the solution of (1) is $1 < x < 3$.

Inequality (2) is defined for every real x . Since $-\frac{1}{2} \leq \cos x + \frac{1}{2} \leq \frac{3}{2}$, it holds exactly when $-\pi \leq \cos x + \frac{1}{2} \leq 0$, that is, $\cos x \leq -\frac{1}{2}$:

$$\frac{2\pi}{3} + 2n\pi \leq x \leq \frac{4\pi}{3} + 2n\pi$$

Both inequalities hold if $\frac{2\pi}{3} \leq x < 3$.