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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equation over the real numbers.

$$\log_{x-1}(x-2) \cdot \log_{x-2}(x-3) \cdot \log_{x-3}(x-1) = 1$$

Solution: Only positive numbers have logarithms, so $x-2 > 0$, $x-3 > 0$, $x-1 > 0$; and the bases must be different from 1, so $x-1 \neq 1$, $x-2 \neq 1$, $x-3 \neq 1$. So $x > 3$ and $x \neq 4$. Changing to base $x-1$:

$$\log_{x-1}(x-2) \cdot \frac{\log_{x-1}(x-3)}{\log_{x-1}(x-2)} \cdot \frac{\log_{x-1}(x-1)}{\log_{x-1}(x-3)} = \log_{x-1}(x-1) = 1$$

So the equation is an identity on its domain: the solutions are all $x > 3$, $x \neq 4$.

2. The edges of a rectangular box (of three different lengths) are in the ratio 1 : 2 : 3. We decreased every edge by 4 cm, and so the surface area decreased by 1056 cm². By how much did the volume of the box decrease?

Solution: If the edges of the original box are a , $2a$, $3a$, its surface area is $A_1 = 2(2a^2 + 3a^2 + 6a^2) = 22a^2$. The new box has edges $a-4$, $2a-4$, $3a-4$, and its surface area is

$$A_2 = 2((a-4)(2a-4) + (a-4)(3a-4) + (2a-4)(3a-4)) = 22a^2 - 96a + 96$$

So $A_1 - A_2 = 96a - 96 = 1056$, and $a = 12$ cm. The original box has edges 12, 24, 36 cm and volume 10 368 cm³; the new box has edges 8, 20, 32 cm and volume 5120 cm³. The volume decreased by 5248 cm³.

3. Solve the following system of equations over the pairs of integers.

$$\frac{20}{x+y+2} - \frac{6}{3x+y-5} = 1, \quad \frac{12}{x+y} + \frac{21}{3x+y} = 7$$

Solution: The denominators cannot be 0. Let $a = x+y$ and $b = 3x+y$. Multiplying the first equation by $(a+2)(b-5)$ and the second by ab :

$$-a + 18b - 102 = ab, \quad 12b + 21a = 7ab$$

From these, $a = \frac{57b-357}{14}$, and substituting, $19b^2 - 184b + 357 = 0$, whose roots are $b = 7$ and $b = \frac{51}{19}$. The second gives $a = -\frac{102}{7}$, and then x and y are not integers. So $b = 7$ and $a = 3$: $x+y = 3$ and $3x+y = 7$, so $x = 2$, $y = 1$. This pair satisfies all the conditions, so it is the only solution.

4. Show that the sides of a triangle satisfy $a^2b - bc^2 + c^3 = a^2c + b^2c - b^3$ if and only if the triangle is a right triangle or an isosceles triangle.

Solution: Rearranging and factoring:

$$a^2b - a^2c + b^3 - b^2c - bc^2 + c^3 = a^2(b - c) + b^2(b - c) - c^2(b - c) = (b - c)(a^2 + b^2 - c^2) = 0$$

This product is 0 if and only if $b = c$ or $a^2 + b^2 = c^2$, that is, the triangle is isosceles ($b = c$) or a right triangle (with the right angle opposite c).

5. The roots of the equation $x^2 - 6x + 4 = 0$ are the lengths of two sides of a rectangle, in centimeters. The axial cross sections of two noncongruent cylinders are congruent to this rectangle. Find the sum of the surface areas and the sum of the volumes of the two cylinders.

Solution: We do not need to compute the roots. The discriminant is positive, and the product (4) and the sum (6) of the roots are positive, so the equation has two different positive roots x_1 and x_2 . One cylinder has base diameter x_1 and height x_2 , the other has base diameter x_2 and height x_1 . Using $x_1 + x_2 = 6$ and $x_1x_2 = 4$:

$$\begin{aligned} A_1 + A_2 &= 2\left(\frac{x_1}{2}\right)\pi\left(\frac{x_1}{2} + x_2\right) + 2\left(\frac{x_2}{2}\right)\pi\left(\frac{x_2}{2} + x_1\right) \\ &= \pi\frac{(x_1 + x_2)^2 + 2x_1x_2}{2} = 22\pi \approx 69.1 \text{ cm}^2 \end{aligned}$$

$$V_1 + V_2 = \left(\frac{x_1}{2}\right)^2\pi x_2 + \left(\frac{x_2}{2}\right)^2\pi x_1 = \frac{\pi}{4}x_1x_2(x_1 + x_2) = 6\pi \approx 18.8 \text{ cm}^3$$

6. Four siblings inherited a plot of land in the shape of a convex quadrilateral. They divided it into four quadrilaterals by joining the midpoints of opposite sides. The first three siblings received plots of 360, 720, and 900 m². How large a plot did the fourth sibling receive?

Solution: (KoMaL did not publish a solution: the problem turned out to be harder than intended, and it was moved to the KoMaL points competition. The following solution is ours.)

Let the quadrilateral be $ABCD$ with area T , let K, L, M, N be the midpoints of AB, BC, CD, DA , and let O be the intersection of KM and LN . $KLMN$ is a parallelogram, so O is the midpoint of both KM and LN , and O is inside $ABCD$. Let t_A, t_B, t_C, t_D be the areas of the four parts at the vertices A, B, C, D .

Since K is the midpoint of AB , the triangles AKO and KBO have equal areas; call this area k . Similarly let l, m, n be the equal areas at L, M, N . Then $t_A = k + n, t_B = k + l, t_C = l + m, t_D = m + n$, so

$$t_A + t_C = t_B + t_D = \frac{T}{2}$$

the opposite parts have equal sums. So the fourth plot is $360 + 900 - 720 = 540$, or $360 + 720 - 900 = 180$, or $720 + 900 - 360 = 1260$ m², depending on which plots are opposite. But not all of these are possible.

The triangle KBL is similar to ABC with ratio $\frac{1}{2}$, so $T_{ABC} = 4T_{KBL}$. The parallelogram $KLMN$ has area $\frac{T}{2}$, so the triangle KLO has area $\frac{T}{8}$, and $T_{KBL} = t_B - \frac{T}{8}$. So $T_{ABC} = 4t_B - \frac{T}{2} = 4t_B - (t_B + t_D) = 3t_B - t_D$. Since the quadrilateral is convex, $T_{ABC} > 0$, so $t_D < 3t_B$, and similarly

$$t_B < 3t_D, \quad t_A < 3t_C, \quad t_C < 3t_A$$

Of two opposite parts, the larger is less than three times the smaller. If the fourth plot were 180, it would be opposite the plot of $900 > 3 \cdot 180$; if it were 1260, it would be opposite the plot of 360, and $1260 > 3 \cdot 360$. So the fourth plot is 540 m² (opposite the plot of 720, while 360 and 900 are opposite). This is possible: for example, in the quadrilateral with vertices $(0, 0), (2, 0), (1, 9), (-1, 1)$, the four parts have areas 2, 4, 5, 3, and enlarging it by the factor $\sqrt{180}$ gives parts of 360, 720, 900, 540 m².

The fourth sibling received 540 m².

7. Prove that if $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\alpha + \beta + \gamma = \frac{\pi}{2}$, then

$$\tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha = 1$$

Solution: Since $\gamma = \frac{\pi}{2} - (\alpha + \beta)$, $\tan \gamma = \cot(\alpha + \beta) = \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta}$ (all angles are acute). So

$$\begin{aligned} \tan \alpha \tan \beta + \tan \gamma (\tan \beta + \tan \alpha) &= \tan \alpha \tan \beta + \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta} (\tan \alpha + \tan \beta) \\ &= \tan \alpha \tan \beta + 1 - \tan \alpha \tan \beta = 1 \end{aligned}$$

8. Solve the following equation over the real numbers.

$$x^2 + \frac{9x^2}{(x+3)^2} = 40$$

Solution: We need $x \neq -3$. Subtracting $\frac{6x^2}{x+3}$ from both sides, the left-hand side becomes a perfect square:

$$x^2 - \frac{6x^2}{x+3} + \frac{9x^2}{(x+3)^2} = \left(x - \frac{3x}{x+3}\right)^2 = \left(\frac{x^2}{x+3}\right)^2$$

so the equation is

$$\left(\frac{x^2}{x+3}\right)^2 + 6 \cdot \frac{x^2}{x+3} - 40 = 0$$

This is a quadratic equation in $\frac{x^2}{x+3}$, with roots 4 and -10 . From $\frac{x^2}{x+3} = 4$: $x^2 - 4x - 12 = 0$, so $x_1 = 6$ and $x_2 = -2$. From $\frac{x^2}{x+3} = -10$: $x^2 + 10x + 30 = 0$, which has no real solution. The solutions are $x = 6$ and $x = -2$.