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In honor of János Szendrei, mathematician, college professor

translated from: KoMaL, January 2003, page 22

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The sum of the legs of a right triangle is 10.5 units, and the radius of its inscribed circle is $\varrho = 1.5$ units. Find
 - a) the radius of the circumscribed circle of the triangle;
 - b) the area of the triangle;
 - c) the radii of the escribed circles of the triangle.

2. Solve the following equations over the real numbers.

$$\text{a) } \sqrt[3]{x + \frac{x}{x^3 - 1}} = x \cdot \sqrt[3]{\frac{x}{x^3 - 1}} \quad \text{b) } \frac{\log_x \frac{9}{4}}{\log_x \frac{27}{8}} = \frac{\frac{9}{4}}{\frac{27}{8}} \quad \text{c) } \tan(x + \pi) + \cot\left(\frac{3\pi}{2} - x\right) = 2 \tan x$$

3. Solve the following system of equations over the pairs of real numbers, where a is a real parameter.

$$x - y = 1 - a - a^2, \quad \log_x(2x^2 + 4x - y + 1) = 2$$

4. The shorter parallel side of the isosceles trapezoid $ABCD$ ($AB \parallel DC$) is $DC = 13$ units, its height is $m = CC_1 = 15.6$ units (C_1 is the foot of the perpendicular from C to the side AB), and the diagonal AC is perpendicular to the side BC . Find the length of the longer parallel side (AB), the legs (BC), and the diagonals (AC).
5. Which point of the parabola $y = x^2 + x + 1$ is closest to the line $y = 2x - 2$? What is this smallest distance?
6. In the triangle ABC , $AC = 8$, $BC = 24$, and the interior angle bisector from the vertex C is $CC_1 = 6\sqrt{2 - \sqrt{2}}$ units. Find the angle ACB , the area of the triangle, and the length of the side AB .
7. Consider the function $x \mapsto \frac{x}{x^2 + x + 1}$, $x \in \mathbb{R}$. Find the range of the function. Where does the function take its smallest and largest values?
8. Let a_n be the n th term and a_k the k th term of the arithmetic sequence (a_n) , S_n the sum of its first n terms and S_k the sum of its first k terms, and let $a_1 \neq 0$, $n \neq 1$, $k \neq 1$, and $n \neq k$. Prove that

$$\frac{S_n}{S_k} = \frac{n^2}{k^2} \quad \text{if and only if} \quad \frac{a_n}{a_k} = \frac{2n - 1}{2k - 1}$$