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In honor of János Szendrei, mathematician, college professor

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The sum of the legs of a right triangle is 10.5 units, and the radius of its inscribed circle is $\varrho = 1.5$ units. Find
 - a) the radius of the circumscribed circle of the triangle;
 - b) the area of the triangle;
 - c) the radii of the escribed circles of the triangle.

Solution: Let the legs be a and b , and the hypotenuse $c = 2r$, where r is the radius of the circumscribed circle. By the equality of tangent segments from a point to a circle:

a) $2r = c = (a - \varrho) + (b - \varrho) = 10.5 - 3 = 7.5$, so $r = 3.75$ units.

b) Since $c^2 = a^2 + b^2 = (a + b)^2 - 2ab$ and $ab = 2T$: $7.5^2 = 10.5^2 - 4T$, so $T = 13.5$ square units.

c) The radii of the escribed circles are $\varrho_a = \frac{T}{s - a}$, $\varrho_b = \frac{T}{s - b}$, $\varrho_c = \frac{T}{s - c}$, where s is the semiperimeter. From $ab = 27$ and $a + b = 10.5$, the legs are 4.5 and 6. Since $2s = 10.5 + 7.5$, $s = 9$, so $\varrho_a = 3$, $\varrho_b = 4.5$, and $\varrho_c = 9$ units.

2. Solve the following equations over the real numbers.

$$\text{a) } \sqrt[3]{x + \frac{x}{x^3 - 1}} = x \cdot \sqrt[3]{\frac{x}{x^3 - 1}} \quad \text{b) } \frac{\log_x \frac{9}{4}}{\log_x \frac{27}{8}} = \frac{\frac{9}{4}}{\frac{27}{8}} \quad \text{c) } \tan(x + \pi) + \cot\left(\frac{3\pi}{2} - x\right) = 2 \tan x$$

Solution: a) The equation is defined for $x \neq 1$, and every such number is a solution, because both sides are identically equal to $\sqrt[3]{\frac{x^4}{x^3 - 1}}$: $x + \frac{x}{x^3 - 1} = \frac{x^4}{x^3 - 1}$ and $x \cdot \sqrt[3]{\frac{x}{x^3 - 1}} = \sqrt[3]{\frac{x^3 \cdot x}{x^3 - 1}}$.

b) The equation is defined for $x > 0$, $x \neq 1$, and every such number is a solution, since

$$\frac{\log_x \frac{9}{4}}{\log_x \frac{27}{8}} = \log_{(\frac{3}{2})^3} \left(\frac{3}{2}\right)^2 = \frac{2}{3} \quad \text{and} \quad \frac{\frac{9}{4}}{\frac{27}{8}} = \frac{2}{3}$$

c) The equation is defined for $x \neq \frac{\pi}{2} + n\pi$, and on this set it is an identity, since $\tan(x + \pi) + \cot\left(\frac{3\pi}{2} - x\right) = \tan x + \cot\left(\frac{\pi}{2} - x\right) = 2 \tan x$. So every $x \neq \frac{\pi}{2} + n\pi$ is a solution.

3. Solve the following system of equations over the pairs of real numbers, where a is a real parameter.

$$x - y = 1 - a - a^2, \quad \log_x (2x^2 + 4x - y + 1) = 2$$

Solution: We need $x > 0$, $x \neq 1$, and $2x^2 + 4x - y + 1 > 0$. From the first equation $y = x + a^2 + a - 1$, and from the second $2x^2 + 4x - y + 1 = x^2$, that is, $y = x^2 + 4x + 1$. So

$$x^2 + 3x - a^2 - a + 2 = 0, \quad x_1 = a - 1, \quad x_2 = -a - 2$$

If $x_1 = a - 1$, we need $a - 1 > 0$ and $a - 1 \neq 1$, that is, $a > 1$ and $a \neq 2$. Then $y_1 = a^2 + 2a - 2$, and this pair is indeed a solution, since then $2x^2 + 4x - y + 1 = (a - 1)^2 > 0$.

If $x_2 = -a - 2$, we need $-a - 2 > 0$ and $-a - 2 \neq 1$, that is, $a < -2$ and $a \neq -3$. Then $y_2 = a^2 - 3$, and this pair is also a solution, since then $2x^2 + 4x - y + 1 = (a + 2)^2 > 0$.

So for $a > 1$, $a \neq 2$ the solution is $(a - 1, a^2 + 2a - 2)$; for $a < -2$, $a \neq -3$ the solution is $(-a - 2, a^2 - 3)$; for other values of a there is no solution.

4. The shorter parallel side of the isosceles trapezoid $ABCD$ ($AB \parallel DC$) is $DC = 13$ units, its height is $m = CC_1 = 15.6$ units (C_1 is the foot of the perpendicular from C to the side AB), and the diagonal AC is perpendicular to the side BC . Find the length of the longer parallel side (AB), the legs (BC), and the diagonals (AC).

Solution: Let $C_1B = x$; then $AC_1 = x + 13$. In the right triangle ACB , by the altitude theorem, $x(x + 13) = 15.6^2$, so $x = 10.4$ units, and $AB = 13 + 2 \cdot 10.4 = 33.8$ units. By the Pythagorean Theorem

$$BC = DA = \sqrt{10.4^2 + 15.6^2} = \frac{26\sqrt{13}}{5} \approx 18.75 \text{ units}, \quad AC = BD = \sqrt{23.4^2 + 15.6^2} = \frac{39\sqrt{13}}{5} \approx 28.12 \text{ units}$$

5. Which point of the parabola $y = x^2 + x + 1$ is closest to the line $y = 2x - 2$? What is this smallest distance?

Solution: The closest point is the point P where a line parallel to the given line, $y = 2x + b$, is tangent to the parabola. This happens when the equation $x^2 + x + 1 = 2x + b$, that is, $x^2 - x + 1 - b = 0$, has a double root: $1 - 4(1 - b) = 0$, so $b = \frac{3}{4}$. Then $\left(x - \frac{1}{2}\right)^2 = 0$, and the point of tangency is $P\left(\frac{1}{2}, \frac{7}{4}\right)$. The smallest distance is the distance of P from the line $2x - y - 2 = 0$:

$$\frac{\left|1 - \frac{7}{4} - 2\right|}{\sqrt{5}} = \frac{11}{4\sqrt{5}} \approx 1.23 \text{ units}$$

6. In the triangle ABC , $AC = 8$, $BC = 24$, and the interior angle bisector from the vertex C is $CC_1 = 6\sqrt{2 - \sqrt{2}}$ units. Find the angle ACB , the area of the triangle, and the length of the side AB .

Solution: Let $\angle ACB = 2\gamma$. The line through A parallel to CC_1 meets the line BC at D . Since $\angle DAC = \angle ACC_1 = \gamma$ and $\angle ADC = \angle C_1CB = \gamma$, the triangle ADC is isosceles, $DC = AC = 8$, and $AD = 16 \cos \gamma$. The triangles DAB and CC_1B are similar, so

$$\frac{16 \cos \gamma}{6\sqrt{2 - \sqrt{2}}} = \frac{8 + 24}{24}, \quad \cos \gamma = \frac{1}{2}\sqrt{2 - \sqrt{2}}$$

so $\gamma = 67.5^\circ$ and $\angle ACB = 135^\circ$. The area is $T = \frac{8 \cdot 24 \cdot \sin 135^\circ}{2} = 48\sqrt{2}$ square units, and by the Law of Cosines

$$AB^2 = 8^2 + 24^2 + 2 \cdot 8 \cdot 24 \cdot \frac{\sqrt{2}}{2} = 640 + 192\sqrt{2}, \quad AB \approx 30.19 \text{ units}$$

7. Consider the function $x \mapsto \frac{x}{x^2 + x + 1}$, $x \in \mathbb{R}$. Find the range of the function. Where does the function take its smallest and largest values?

Solution: The function takes the value k if the equation $\frac{x}{x^2 + x + 1} = k$ has a real solution. Since $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$ for every real x , this is equivalent to $kx^2 + (k - 1)x + k = 0$, which has a real solution if and only if $k = 0$ or the discriminant is nonnegative:

$$D = (k - 1)^2 - 4k^2 = (1 - 3k)(1 + k) \geq 0, \quad -1 \leq k \leq \frac{1}{3}$$

The range is the interval $\left[-1, \frac{1}{3}\right]$. The smallest value, -1 , is taken at $x = -1$, and the largest value, $\frac{1}{3}$, at $x = 1$.

8. Let a_n be the n th term and a_k the k th term of the arithmetic sequence (a_n) , S_n the sum of its first n terms and S_k the sum of its first k terms, and let $a_1 \neq 0$, $n \neq 1$, $k \neq 1$, and $n \neq k$. Prove that

$$\frac{S_n}{S_k} = \frac{n^2}{k^2} \quad \text{if and only if} \quad \frac{a_n}{a_k} = \frac{2n - 1}{2k - 1}$$

Solution: Let d be the difference. If the first condition holds, then

$$\frac{\frac{n}{2}(2a_1 + (n - 1)d)}{\frac{k}{2}(2a_1 + (k - 1)d)} = \frac{n^2}{k^2}$$

and rearranging, $2a_1(k - n) = d(k - n)$, so (since $k \neq n$) $d = 2a_1$. Then

$$\frac{a_n}{a_k} = \frac{a_1 + (n - 1) \cdot 2a_1}{a_1 + (k - 1) \cdot 2a_1} = \frac{2n - 1}{2k - 1}$$

since $a_1 \neq 0$. Conversely, if $\frac{a_1 + (n - 1)d}{a_1 + (k - 1)d} = \frac{2n - 1}{2k - 1}$, then rearranging gives $d = 2a_1$ again (since $k \neq n$), and so

$$\frac{S_n}{S_k} = \frac{\frac{n}{2}(2a_1 + (n - 1) \cdot 2a_1)}{\frac{k}{2}(2a_1 + (k - 1) \cdot 2a_1)} = \frac{n^2}{k^2}$$