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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Simplify the following expression.

$$\left[\frac{1 + \sqrt[3]{x} + \sqrt[3]{x^2}}{1 - x} - \left(x^{-\frac{2}{3}} - x^{-\frac{1}{3}} \right)^{-1} \right]^{-1} \cdot (1 + \sqrt[3]{x})$$

Solution: Let $t = \sqrt[3]{x}$; the expression is defined for $t \neq 0$, $t \neq \pm 1$, that is, $x \neq 0$, $x \neq \pm 1$. Then

$$\frac{1 + t + t^2}{1 - t^3} = \frac{1}{1 - t}, \quad \left(\frac{1}{t^2} - \frac{1}{t} \right)^{-1} = \frac{t^2}{1 - t}$$

so the expression in the brackets is $\frac{1 - t^2}{1 - t} = 1 + t$, and the whole expression is $\frac{1}{1 + t} \cdot (1 + t) = 1$.

2. Find the digits x and y so that the following equality is true.

$$3 \cdot \overline{xxxxx} = \overline{yxxxxx} - y$$

Solution: Let $a = \overline{xxxxx} = 11\,111x$. Since $\overline{yxxxxx} = 100\,000y + a$, the equation is $3a = 100\,000y + a - y$, that is, $2a = 99\,999y = 9y \cdot 11\,111$. On the other hand $2a = 2x \cdot 11\,111$, so $2x = 9y$. Since y is the first digit of a six-digit number, $y \neq 0$, so $x = 9$ and $y = 2$. (Indeed, $3 \cdot 99\,999 = 299\,997 = 299\,999 - 2$.)

3. Consider the rhombus $ABCD$. The circle of radius R circumscribed around the triangle ABD passes through the center of the circle inscribed in the triangle BCD . Express the area of the rhombus in terms of R .

Solution: Let O be the center of the circle inscribed in the triangle BCD , and let α be the angle of the rhombus at A (and at C). In the triangle BCD , $\angle BOD = 180^\circ - \frac{\angle B + \angle D}{2} = 90^\circ + \frac{\alpha}{2}$. The points A and O are on opposite sides of BD , and $ABOD$ is a cyclic quadrilateral, so

$$\alpha + 90^\circ + \frac{\alpha}{2} = 180^\circ, \quad \alpha = 60^\circ$$

So the triangles ABD and BCD are equilateral. If the side of the rhombus is a , then $a = 2R \sin 60^\circ = \sqrt{3}R$, and the area of the rhombus is

$$T = \frac{a^2 \sqrt{3}}{2} = \frac{3\sqrt{3}}{2} R^2$$

4. For which value of the parameter k will one root of the equation

$$(k^2 - 5k + 3)x^2 + (3k - 1)x + 2 = 0$$

be twice the other root?

Solution: The equation is quadratic if $k^2 - 5k + 3 \neq 0$ (otherwise it has at most one root), and it has real roots if $D = (3k - 1)^2 - 8(k^2 - 5k + 3) \geq 0$; we will check these conditions at the end. If x_1 and x_2 are the roots, the condition holds exactly when

$$P = (2x_1 - x_2)(2x_2 - x_1) = 5x_1x_2 - 2(x_1^2 + x_2^2) = 9x_1x_2 - 2(x_1 + x_2)^2$$

is zero. By Viète's formulas

$$P = \frac{18}{k^2 - 5k + 3} - \frac{2(3k - 1)^2}{(k^2 - 5k + 3)^2} = \frac{2(26 - 39k)}{(k^2 - 5k + 3)^2}$$

so $P = 0$ exactly when $k = \frac{2}{3}$. Then the leading coefficient is $\frac{1}{9} \neq 0$ and the discriminant is $\frac{1}{9} > 0$; the equation is $\frac{1}{9}x^2 + x + 2 = 0$, with roots -6 and -3 .

5. In the triangle ABC the angle at B is $\beta = 60^\circ$. We draw the angle bisectors from A and C ; they meet at O , and they meet the opposite sides at D and E . Show that $OD = OE$.

Solution: The angle AOE is an exterior angle of the triangle ACO , so

$$\angle AOE = \frac{\alpha}{2} + \frac{\gamma}{2} = \frac{180^\circ - \beta}{2} = 60^\circ$$

So $\angle EOD = 180^\circ - 60^\circ = 120^\circ$, and $\angle EBD + \angle EOD = 180^\circ$: the quadrilateral $BDOE$ is cyclic. In its circumscribed circle, the chords DO and EO are seen from B at equal angles of 30° (since BO bisects the angle at B), so these two chords are equal: $OD = OE$.

6. Solve the following inequality over the real numbers.

$$0.2^{\log_{0.7} \frac{2x+1}{2x-3}} < 1$$

Solution: The logarithm is defined if $\frac{2x+1}{2x-3} > 0$, that is, $x < -\frac{1}{2}$ or $x > \frac{3}{2}$. A power of a positive base less than 1 is less than 1 if and only if the exponent is positive, so we need $\log_{0.7} \frac{2x+1}{2x-3} > 0$. The base 0.7 logarithm function is decreasing, so this holds exactly when

$$\frac{2x+1}{2x-3} < 0.7^0 = 1, \quad \frac{4}{2x-3} < 0, \quad x < \frac{3}{2}$$

Together with the domain, the solution is $x < -\frac{1}{2}$.

7. In a tetrahedron two opposite edges are x units long, and the other edges are 1 unit long. Express the volume of the tetrahedron as a function of x . For which x is the volume largest? What is the largest volume?

Solution: Let the tetrahedron be $ABCD$ with $AB = CD = x$, and let M and N be the midpoints of AB and CD . AN and BN are medians, and so also altitudes, of the isosceles triangles ACD and BCD , so CD is perpendicular to the plane ABN , and so to the line NM in it. Similarly $NM \perp AB$. So

$$V = \frac{1}{3}T_{ABN} \cdot CD = \frac{1}{6} \cdot AB \cdot CD \cdot MN$$

By the Pythagorean Theorem $MN^2 = BN^2 - BM^2 = \left(1 - \frac{x^2}{4}\right) - \frac{x^2}{4} = 1 - \frac{x^2}{2}$, so

$$V(x) = \frac{1}{12}x^2\sqrt{4 - 2x^2}, \quad 0 < x < \sqrt{2}$$

V is largest where $144V^2 = x^4(4 - 2x^2)$ is largest. By the inequality between the arithmetic and geometric means,

$$x^2 \cdot x^2 \cdot (4 - 2x^2) \leq \left(\frac{x^2 + x^2 + 4 - 2x^2}{3}\right)^3 = \left(\frac{4}{3}\right)^3$$

with equality if $x^2 = 4 - 2x^2$, that is, $x^2 = \frac{4}{3}$. So the volume is largest for $x = \frac{2}{\sqrt{3}}$, and the largest volume is

$$V_{\max} = \frac{2\sqrt{3}}{27}.$$

8. Prove that the medians AA_1 and BB_1 of the triangle ABC are perpendicular if and only if, with the usual notation,

$$\cot \gamma = 2(\cot \alpha + \cot \beta)$$

Solution: Let T be the area of the triangle. By the Law of Cosines and $2T = bc \sin \alpha$: $\cos \alpha = \frac{-a^2 + b^2 + c^2}{2bc}$ and $\sin \alpha = \frac{2T}{bc}$, so $\cot \alpha = \frac{-a^2 + b^2 + c^2}{4T}$, and similarly for β and γ . So

$$2(\cot \alpha + \cot \beta) - \cot \gamma = 2 \cdot \frac{2c^2}{4T} - \frac{a^2 + b^2 - c^2}{4T} = \frac{1}{4T}(5c^2 - a^2 - b^2)$$

On the other hand, let S be the centroid and $s_c = CC_1$ the median from C . By Thales' theorem, the angle ASB is a right angle if and only if $SC_1 = \frac{c}{2}$, and since $SC_1 = \frac{s_c}{3}$, this means $c^2 - \frac{4}{9}s_c^2 = 0$. By the median formula $4s_c^2 = 2a^2 + 2b^2 - c^2$, so

$$c^2 - \frac{4}{9}s_c^2 = c^2 - \frac{2a^2 + 2b^2 - c^2}{9} = \frac{2}{9}(5c^2 - a^2 - b^2)$$

So both the perpendicularity of the medians AA_1 , BB_1 and the equation $\cot \gamma = 2(\cot \alpha + \cot \beta)$ are equivalent to $a^2 + b^2 = 5c^2$, which proves the statement.