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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equations over the real numbers.

$$\text{a) } \frac{(x-1)(x^3+x^2+x+1)}{(x+1)(x^3-x^2+x-1)} = 1 \quad \text{b) } \sqrt{\tan^2 x} = -\frac{1}{\cot x} \quad \text{c) } \frac{\log_x 7}{\log_x 3} = \frac{\log_3 5}{\log_7 5}$$

Solution: a) Since

$$(x-1)(x^3+x^2+x+1) = (x-1)(x+1)(x^2+1) = (x+1)(x^3-x^2+x-1)$$

every real number except -1 and 1 is a solution.

b) The equation is defined if $\tan x$ and $\cot x$ are defined and $\cot x \neq 0$, that is, if $x \neq \frac{k\pi}{2}$. Then the equation says $|\tan x| = -\tan x$, which holds exactly when $\tan x \leq 0$. The solutions are

$$\frac{\pi}{2} + n\pi < x < \pi + n\pi, \quad n \text{ is an integer}$$

c) Every positive real number except 1 is a solution, because changing the base of the logarithms,

$$\frac{\log_x 7}{\log_x 3} = \log_3 7 \quad \text{and} \quad \frac{\log_3 5}{\log_7 5} = \frac{\frac{1}{\log_5 3}}{\frac{1}{\log_5 7}} = \frac{\log_5 7}{\log_5 3} = \log_3 7$$

2. The diagonals of a trapezoid are perpendicular. Its parallel sides are 13 and 39 units long, and one of its legs is $\sqrt{369}$ units long. Find the other leg, the area, and the height of the trapezoid.

Solution: The diagonals divide the trapezoid into four right triangles. The triangles at the parallel sides are similar, with ratio $3 : 1$. Let the segments of the diagonals near the shorter parallel side be x and y ; then the other segments are $3x$ and $3y$. So $x^2 + y^2 = 13^2$, and for the given leg $x^2 + 9y^2 = 369$, so $y^2 = 25$, $x^2 = 144$, and $x = 12$, $y = 5$. The other leg is $\sqrt{9x^2 + y^2} = \sqrt{1321} \approx 36.35$ units.

The diagonals are $4x = 48$ and $4y = 20$ units long, and they are perpendicular, so the area is $T = \frac{48 \cdot 20}{2} = 480$ square units. For the height m : $\frac{13 + 39}{2} \cdot m = 480$, so $m = \frac{240}{13} \approx 18.46$ units.

3. Find the values of the parameter m for which the equation

$$(m+1)x^2 + (2m+1)x - 2 = 0$$

has two different real roots, both less than -1 .

Solution: For two different roots we need $m+1 \neq 0$ and a positive discriminant:

$$D = (2m+1)^2 + 8(m+1) = (2m+3)^2 > 0, \quad \text{that is, } m \neq -\frac{3}{2}$$

The roots are $x_1 = -2 < -1$ and $x_2 = \frac{1}{m+1}$. We need $\frac{1}{m+1} < -1$, that is, $-1 < m+1 < 0$, so $-2 < m < -1$.

So the equation has two different roots less than -1 if $-2 < m < -\frac{3}{2}$ or $-\frac{3}{2} < m < -1$.

4. The first term of a sequence is 1, the sixth term is 51, and the sum of the first three terms is 23; the differences of neighboring terms are consecutive terms of an arithmetic sequence. Find the first three terms of the sequence.

Solution: Let the first five terms of the sequence of differences be $a, a+d, a+2d, a+3d, a+4d$. Then the first six terms of the sequence are $1, 1+a, 1+2a+d, 1+3a+3d, 1+4a+6d, 1+5a+10d$. By the conditions $3+3a+d = 23$ and $1+5a+10d = 51$, so $a = 6$ and $d = 2$. The first three terms are 1, 7, 15.

5. Two sides of a triangle, a and c , are given, with $\frac{c}{2} < a < c$, and

$$\frac{\sin \alpha}{\sin \beta} = \frac{\cos(\alpha + \gamma)}{\cos(\beta + \gamma)}$$

where α is the angle opposite a , γ is the angle opposite c , and β is the angle opposite the third side b . Express the side b in terms of a and c .

Solution: Since $\alpha + \gamma = 180^\circ - \beta$ and $\beta + \gamma = 180^\circ - \alpha$, we have $\cos(\alpha + \gamma) = -\cos \beta$ and $\cos(\beta + \gamma) = -\cos \alpha$. So

$$\frac{\sin \alpha}{\sin \beta} = \frac{-\cos \beta}{-\cos \alpha}, \quad \sin \alpha \cos \alpha = \sin \beta \cos \beta, \quad \sin 2\alpha = \sin 2\beta$$

This holds exactly when $2\alpha = 2\beta$, that is, $\alpha = \beta$, and then $b = a$; or $2\alpha + 2\beta = 180^\circ$, that is, $\gamma = 90^\circ$, and then $b = \sqrt{c^2 - a^2}$. (Both triangles exist because $\frac{c}{2} < a < c$.)

6. Two circles are externally tangent to each other. We rotate the segment of one of their common external tangents between the points of tangency around the line through the centers of the circles. The lateral surface area of the resulting frustum of a cone is 576π square units, and the radius of one of the circles is 16 units. What is the radius of the other circle?

Solution: Let the centers be O_1, O_2 , the radii R_1, R_2 , and the point of tangency of the circles E . Let the endpoints of the external tangent segment be E_1 and E_2 , with $E_1E_2 = a$, and let their projections onto the line O_1O_2 be T_1, T_2 . The common tangent of the circles at E meets E_1E_2 at F . The radii of the bases of the frustum are $r_1 = E_1T_1$ and $r_2 = E_2T_2$, and its lateral surface area is $P = (r_1 + r_2)a\pi$.

By the equality of tangent segments, $FE_1 = FE = FE_2 = \frac{a}{2}$. The triangle O_1FO_2 has a right angle at F (since FO_1 and FO_2 bisect the supplementary angles E_1FE and $EF E_2$), and FE is its altitude, so $\left(\frac{a}{2}\right)^2 = O_1E \cdot EO_2 = R_1R_2$, that is, $a^2 = 4R_1R_2$. The quadrilateral $E_1T_1T_2E_2$ is a trapezoid, and its midline is EF (it is perpendicular to O_1O_2 and passes through the midpoint F of E_1E_2), so $\frac{r_1 + r_2}{2} = \frac{a}{2}$, that is, $r_1 + r_2 = a$.

So $P = a^2\pi = 4R_1R_2\pi$. With $P = 576\pi$ and $R_1 = 16$: $576 = 64R_2$, so the other radius is $R_2 = 9$ units.

7. The circle k is tangent to the y -axis, and is tangent to the line $3x - 4y = 48$ at the point E of the line with x -coordinate $x_1 = 8$. Find the equation of the circle.

Solution: The y -coordinate of E satisfies $24 - 4y_1 = 48$, so $E(8, -6)$. The circle is tangent to the y -axis, so if its center is $K(u, v)$, its equation is $(x - u)^2 + (y - v)^2 = u^2$. K is on the line through E perpendicular to the given line, which is $4x + 3y = 14$. So

$$(8 - u)^2 + (-6 - v)^2 = u^2 \quad \text{and} \quad 4u + 3v = 14$$

from which $u = 5, v = -2$ or $u = 20, v = -22$. There are two such circles:

$$(x - 5)^2 + (y + 2)^2 = 25 \quad \text{and} \quad (x - 20)^2 + (y + 22)^2 = 400$$

8. Find the range of the expression

$$\frac{4}{4 \cdot 2^{2x} - 8 \cdot 2^{x-1} + \frac{11}{3}}$$

for $-2 \leq x \leq 1$.

Solution: Rewriting the expression:

$$\frac{4}{4 \cdot 2^{2x} - 8 \cdot 2^{x-1} + \frac{11}{3}} = \frac{1}{(2^x)^2 - 2^x + \frac{11}{12}} = \frac{1}{\left(2^x - \frac{1}{2}\right)^2 + \frac{2}{3}}$$

If $-2 \leq x \leq 1$, then $\frac{1}{4} \leq 2^x \leq 2$, so $-\frac{1}{4} \leq 2^x - \frac{1}{2} \leq \frac{3}{2}$, and $0 \leq \left(2^x - \frac{1}{2}\right)^2 \leq \frac{9}{4}$. So the denominator is between $\frac{2}{3}$ and $\frac{35}{12}$, and

$$\frac{12}{35} \leq \frac{1}{\left(2^x - \frac{1}{2}\right)^2 + \frac{2}{3}} \leq \frac{3}{2}$$

Since the expression is continuous on the interval, it takes every value in between: the range is $\left[\frac{12}{35}, \frac{3}{2}\right]$.