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translated from: KoMaL, November 2003, page 462

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equation and inequalities over the real numbers.

a) $|x - 2| + |x + 3| = |(x - 2) + (x + 3)|$ b) $|x - 2| + |x + 3| > |2x + 1|$ c) $|x - 2| + |x + 3| < |2x + 1|$

Solution: For real numbers, $|a| + |b| = |a + b|$ exactly when $ab \geq 0$, $|a| + |b| > |a + b|$ exactly when $ab < 0$, and $|a| + |b| < |a + b|$ never holds. With $a = x - 2$, $b = x + 3$ (and $a + b = 2x + 1$):

a) $(x - 2)(x + 3) \geq 0$, that is, $x \leq -3$ or $x \geq 2$.

b) $(x - 2)(x + 3) < 0$, that is, $-3 < x < 2$.

c) There is no solution.

2. The diagonals of an isosceles trapezoid are perpendicular. The area of the trapezoid is 16 cm^2 . Find the height, the midline, and the diagonals of the trapezoid. Can the sides be determined from these data?

Solution: Since the diagonals are perpendicular (and equal), they make 45° angles with the parallel sides. So the height of the trapezoid is equal to its midline, and the diagonal is $\sqrt{2}$ times the height. If the height is m , the midline k , and the diagonal e , then $k = m$, $e = m\sqrt{2}$, and the area is $T = km = m^2$. So $m^2 = 16$, $m = 4 \text{ cm}$, $k = 4 \text{ cm}$, and $e = 4\sqrt{2} \text{ cm}$. There are infinitely many such trapezoids, so the sides cannot be determined.

3. Which triples of numbers (x, y, z) satisfy the system of equations $x + y = 4$, $xy = z^2 + 2z + 5$?

Solution: From the first equation $y = 4 - x$, so $x(4 - x) = (z + 1)^2 + 4$, that is,

$$(x - 2)^2 + (z + 1)^2 = 0$$

This holds exactly when $x = 2$ and $z = -1$, and then $y = 2$. The only solution is $(2, 2, -1)$.

4. Solve the following systems of equations.

a) $x + y = \frac{\pi}{2}, \sin x + \cos y = \sqrt{2}$ b) $2 \cos x = \sqrt{2}, \sin x + \cos y = \sqrt{2}$

Solution: a) From the first equation $y = \frac{\pi}{2} - x$, so $\cos y = \sin x$, and $2 \sin x = \sqrt{2}$, $\sin x = \frac{\sqrt{2}}{2}$. The solutions are

$$\left(\frac{\pi}{4} + 2n\pi, \frac{\pi}{4} - 2n\pi\right) \quad \text{and} \quad \left(\frac{3\pi}{4} + 2k\pi, -\frac{\pi}{4} - 2k\pi\right), \quad n, k \text{ integers}$$

b) From the first equation $x = \frac{\pi}{4} + 2k\pi$ or $x = -\frac{\pi}{4} + 2k\pi$. If $x = \frac{\pi}{4} + 2k\pi$, then $\sin x = \frac{\sqrt{2}}{2}$, so $\cos y = \frac{\sqrt{2}}{2}$, and $y = \pm \frac{\pi}{4} + 2n\pi$. If $x = -\frac{\pi}{4} + 2k\pi$, then $\cos y = \sqrt{2} + \frac{\sqrt{2}}{2} > 1$, which is impossible. The solutions are

$$\left(\frac{\pi}{4} + 2k\pi, \frac{\pi}{4} + 2n\pi\right) \quad \text{and} \quad \left(\frac{\pi}{4} + 2k\pi, -\frac{\pi}{4} + 2n\pi\right), \quad k, n \text{ integers}$$

5. We divide the sum of the first n positive integers that give remainder 2 when divided by 3 by the sum of the first n positive integers that give remainder 1 when divided by 4. Find n , if the quotient is $\frac{8}{9}$.

Solution: The n th positive integer with remainder 2 when divided by 3 is $3n - 1$, and the n th positive integer with remainder 1 when divided by 4 is $4n - 3$. The two sums are $\frac{2 + (3n - 1)}{2} \cdot n$ and $\frac{1 + (4n - 3)}{2} \cdot n$, and their quotient is $\frac{3n + 1}{4n - 2}$. From $\frac{3n + 1}{4n - 2} = \frac{8}{9}$, $27n + 9 = 32n - 16$, so $n = 5$.

6. Find the values of the real parameter m for which the roots of the equation $2x^2 + 2mx + m^2 - 1 = 0$ are real, and for these values of m find the smallest and largest values of $f(m) = 2(x_1 + x_2 + x_1x_2 + 1)$, where x_1 and x_2 are the solutions of the equation.

Solution: The discriminant is $D = 4m^2 - 8(m^2 - 1) = 4(2 - m^2)$, so the roots are real if $-\sqrt{2} \leq m \leq \sqrt{2}$. Since $x_1 + x_2 = -m$ and $x_1x_2 = \frac{m^2 - 1}{2}$,

$$f(m) = 2\left(-m + \frac{m^2 - 1}{2} + 1\right) = (m - 1)^2$$

For $-\sqrt{2} \leq m \leq \sqrt{2}$, $-1 - \sqrt{2} \leq m - 1 \leq -1 + \sqrt{2}$, so $0 \leq (m - 1)^2 \leq (1 + \sqrt{2})^2$. The smallest value of f is 0, at $m = 1$, and the largest value is $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$, at $m = -\sqrt{2}$.

7. Prove that the angle γ opposite the side c of the triangle with sides a, b, c is 120° if and only if

$$4s(s - c) = ab, \quad \text{where} \quad s = \frac{1}{2}(a + b + c)$$

Solution: Since $2s = a + b + c$ and $2s - 2c = a + b - c$,

$$4s(s - c) = (a + b + c)(a + b - c) = (a + b)^2 - c^2 = a^2 + b^2 + 2ab - c^2$$

So $4s(s - c) = ab$ if and only if $c^2 = a^2 + b^2 + ab$. By the Law of Cosines $c^2 = a^2 + b^2 - 2ab \cos \gamma$, so this holds if and only if $-2 \cos \gamma = 1$, that is, $\cos \gamma = -\frac{1}{2}$, which means $\gamma = 120^\circ$.

8. The line g perpendicular to the vector $\mathbf{n}(-1, 1)$ meets the y -axis at C , and the parabola $y = (x - 2)^2 + 1$ at the points A and B . Find the equation of the line g and the coordinates of the points A and B , if $|\vec{AB}| = 3 \cdot |\vec{AC}|$.

Solution: The equation of g is $-x + y = c$, that is, $y = x + c$, so $C(0, c)$. Let the x -coordinates of A and B be x_1 and x_2 ; then $A(x_1, x_1 + c)$ and $B(x_2, x_2 + c)$, and x_1, x_2 are the roots of $(x - 2)^2 + 1 = x + c$, that is, $x^2 - 5x + 5 - c = 0$. By the condition $AB^2 = 9 \cdot AC^2$:

$$2(x_2 - x_1)^2 = 9 \cdot 2x_1^2, \quad x_2 - x_1 = \pm 3x_1$$

so $x_2 = 4x_1$ or $x_2 = -2x_1$. Since $x_1 + x_2 = 5$ and $x_1x_2 = 5 - c$:

If $x_2 = 4x_1$: $5x_1 = 5$, $x_1 = 1$, $x_2 = 4$, and $4 = 5 - c$, so $c = 1$.

If $x_2 = -2x_1$: $-x_1 = 5$, $x_1 = -5$, $x_2 = 10$, and $-50 = 5 - c$, so $c = 55$.

If $c = 1$, the line is $y = x + 1$, with $A_1(1, 2)$, $B_1(4, 5)$. If $c = 55$, the line is $y = x + 55$, with $A_2(-5, 50)$, $B_2(10, 65)$. These are indeed solutions.