Problem Set - 2003 (October 2003)

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.
The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.
In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following systems of equations over the ordered pairs of real numbers.

   a) $(x+1)\,y=0,\ \ \left(x^2-1\right)(y+1)=0$ b) $x^2+xy=2,\ \ y^2-2xy=5$

2. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=2x^2-2x+4$. Prove the identity $f(k+2)=f(k)+8k+4$.

3. Find the values of the parameter a for which one root of the following equations is twice the other root.

   1) $x^2+3ax+2a^2=0$ 2) $x^2-3ax+2a^2-1=0$

   3) $x^2-3(a+1)x+2a^2+6a=0$ 4) $x^2+ax-2a-4=0$

4. In the coordinate plane, e and f are two parallel lines, and k is a circle between them. Reflecting k in the line e we get the circle k_1: $x^2+y^2-4x-12y+39=0$, and reflecting k in the line f we get the circle k_2: $x^2+y^2-16x+12y+99=0$. Find the point with respect to which k_1 and k_2 are centrally symmetric, and the equation of the line with respect to which k_1 and k_2 are symmetric. Find the distance between the lines e and f.

5. The base of a pyramid with four lateral faces is the rhombus $ABCD$. The perpendicular projection of the apex E onto the plane of the base is the intersection point F of the diagonals of the rhombus. Find the volume and the surface area of the pyramid, if the diagonals of the rhombus are $AC=10$ cm and $BD=18$ cm, and the height of the pyramid is $EF=12$ cm.

6. Solve the following inequalities over the real numbers.

   a) $\dfrac{\log_3 x}{\left(\log_{\frac{1}{3}} x+3\right)\left(\log_{\frac{1}{3}} x-1\right)}<0$ b) $\dfrac{\log_{\frac{1}{3}}^2 x+2\log_{\frac{1}{3}} x-3}{\log_3 x}\geq 0$

7. a) Prove that the sequence (a_n) is arithmetic if and only if $a_{n+1}=2a_n-a_{n-1}$ for every natural number $n>1$.

   b) The sequence (a_n) is arithmetic with difference d. There are real numbers p and q such that $a_{n+1}=2pa_n-qa_{n-1}$ for every natural number $n>1$. Find the possible values of p and q, if (a_n) is (i) not constant; (ii) constant with $a_1\neq 0$; (iii) constant with $a_1=0$.

8. From the point P of a circle we draw the chords $PA=12$ cm and $PB=16$ cm. The distance of the midpoint F of the chord PA from the line PB is $2\sqrt{3}$ cm. Find the radius of the circle.

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